AN EXTENSION OF BITTNER AND URBAN'S THEOREM

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ABSTRACT. A class of Besov spaces are characterized by the quadratic and cubic Hermite multiwavelets (K. Bittner and K. Urban, On interpolatory divergence-free wavelets, *Math. Comp.*, **76** (2007), 903-929). That characterization has a limitation, because of the regularity restriction of the Hermite splines. In this paper, we extend Bittner and Urban's theorem by using B-spline wavelets with weak duals introduced in the paper: R. Q. Jia, J. Z. Wang and D. X. Zhou, Compactly supported wavelet bases for Sobolev spaces, *Appl. Comput. Harmon. Anal.*, **15** (2003), 224-241.

1. INTRODUCTION

It is important in both theory and application to characterize function spaces by using wavelet bases. In particular, the characterization for Besov spaces interests mathematicians since many fundamental spaces (e.g., Sobolev, Hölder, Lipschitz spaces and etc.) are special examples of those spaces ([5]). In [1], Bittner and Urban study the following Besov spaces and the related vector Besov spaces:

Let $0 < p, q \leq \infty$ and $s > 0$,

$$BB_{p,q}^s(\mathbb{R}) := \{f \in L_p(\mathbb{R}) : |f|_{BB_{p,q}^s(\mathbb{R})} < \infty\}$$

with the Besov (semi-)norm $\|f\|_{BB_{p,q}^s(\mathbb{R})} := \|f\|_{L_p(\mathbb{R})} + |f|_{BB_{p,q}^s(\mathbb{R})}$. To define $|f|_{BB_{p,q}^s(\mathbb{R})}$, we need the classical difference operator $\Delta_h : \Delta_h f(\cdot) := f(\cdot + h) + f(\cdot)$, as well as $\Delta_h^M f := \Delta_h(\Delta_h^{M-1} f)$ for a positive integer $M > 1$. Moreover,

$$\omega_p^M(f, 2^{-j}) := \sup_{|h| \leq 2^{-j}} \|\Delta_h^M f(\cdot)\|_{L_p(\mathbb{R})}$$

denotes the M -th order smooth modulus of a function f . Then $|f|_{BB_{p,q}^s(\mathbb{R})}$ is defined as

$$|f|_{BB_{p,q}^s(\mathbb{R})} := \|(2^{js} \omega_p^M(f, 2^{-j}))_{j \in \mathbb{Z}}\|_{\ell_q},$$

where $M \geq [s] + 1$ and $[s]$ stands for the largest integer less than or equal to s .

Based on the quadratic or cubic Hermite multiwavelets, the authors of [1] characterize $BB_{p,q}^s$ by using the sequence norms

$$\|a\|_{\ell_p} \quad \text{and} \quad \|b\|_{\ell_{p,q}^s} := \|(2^{n(s+\frac{1}{2}-\frac{1}{p})} \| (b_{n,k})_{k \in \mathbb{Z}} \|_{\ell_p})_{n \geq n_0}\|_{\ell_q}$$

Received by the editor August 11, 2009 and, in revised form, July 18, 2011.

2010 *Mathematics Subject Classification.* Primary 42C40, 35Q30, 41A15.

Key words and phrases. Wavelet characterization, Besov spaces, completeness.

This work is supported by the National Natural Science Foundation of China (No. 10871012) and the Natural Science Foundation of Beijing (No. 1082003).

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for $a = (a_k)_{k \in \mathbb{Z}} \in \ell_p$ and $b = (b_{n,k})_{n \geq n_0, k \in \mathbb{Z}} \in \ell_{p,q}^s$, where $\ell_{p,q}^s := \{(c_{n,k})_{n \geq n_0, k \in \mathbb{Z}} : \|(2^{n(s+\frac{1}{2}-\frac{1}{p})}\|(c_{n,k})_{k \in \mathbb{Z}}\|_{\ell_p})_{n \geq n_0}\|_{\ell_q} < +\infty\}$. However, due to the regularity restrictions of the Hermite splines, their characterization requires $\frac{1}{p} < s < \min\{3, 1 + \frac{1}{p}\}$ in the quadratic case and $1 + \frac{1}{p} < s < \min\{4, 2 + \frac{1}{p}\}$ in the cubic case.

This current paper removes that restriction of s by using B-spline wavelets with weak duals introduced in [9]. Because we establish some differential relations of those wavelets ([11]), divergence-free wavelets can be constructed. However, the result of this paper cannot be used to construct interpolating divergence-free wavelets, since it is shown in [1] that multi-wavelets are needed for that purpose.

Let $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ be the set of positive integers, the set of integers and the set of real numbers, respectively. Throughout this paper, we use $A \lesssim B$ to abbreviate that A is bounded by a constant multiple of B , $A \gtrsim B$ is defined as $B \lesssim A$ and $A \sim B$ means $A \lesssim B$ and $B \lesssim A$. Let $L_p(\Omega)$ denote the classical Lebesgue spaces for a Lebesgue measurable set $\Omega \subseteq \mathbb{R}$ and $1 \leq p \leq \infty$. Write

$$\langle f, g \rangle = \int_{\Omega} f(t) \overline{g(t)} dt$$

for $f \in L_p(\Omega)$, $g \in L_{p'}(\Omega)$ with $\frac{1}{p} + \frac{1}{p'} = 1$. Similarly, $\ell_p := \{a = (a_k)_{k \in \mathbb{Z}} : \sum_k |a_k|^p < \infty\}$. For a Lebesgue measurable function f , the support of f means the closure of $\{x \in \mathbb{R} : f(x) \neq 0\}$ (denoted by $\text{Supp}(f)$), which is well defined up to a set of measure 0. As usual, ϕ_r stands for the B-spline scaling function of order r , i.e., r convolutions of the characteristic function on the interval $[0, 1]$,

$$\phi_r(\cdot) := \underbrace{\chi_{[0,1]} * \chi_{[0,1]} * \cdots * \chi_{[0,1]}}_{r \text{ functions}}(\cdot).$$

Here and after, we always define $f_{n,k}(\cdot) := 2^{\frac{n}{2}} f(2^n \cdot - k)$ unless we specifically state otherwise. Let $\varphi := \phi_r$, $\tilde{\varphi} := \phi_{\theta}$ be B-spline functions with $r + \theta$ even and $V_n = \text{clos}_{L_2} \text{span}\{\varphi_{n,k}\}_{k \in \mathbb{Z}}$, $\tilde{V}_n = \text{clos}_{L_2} \text{span}\{\tilde{\varphi}_{n,k}\}_{k \in \mathbb{Z}}$. Assume that the corresponding wavelet pair $\psi, \tilde{\psi}$ is given by

$$(1.1) \quad \psi(x) = \sum_{k \in \mathbb{Z}} (-1)^k \overline{\mu(1-k)} \varphi(2x-k), \quad \tilde{\psi}(x) = \sum_{k \in \mathbb{Z}} (-1)^k \overline{\nu(1-k)} \tilde{\varphi}(2x-k)$$

for $\mu(k) = \langle \tilde{\varphi}(x), \varphi(2x-k) \rangle$ and $\nu(k) = \langle \varphi(x), \tilde{\varphi}(2x-k) \rangle$. Then the reference [9] shows the following:

- (1) $\langle \psi(\cdot), \tilde{\varphi}(\cdot-k) \rangle = \langle \varphi(\cdot), \tilde{\psi}(\cdot-k) \rangle = \langle \psi_{j,k}, \tilde{\psi}_{j',k'} \rangle = 0$ for $j \neq j'$ and $k, k' \in \mathbb{Z}$;
- (2) $V_{n+1} = V_n \oplus W_n$ and $\tilde{V}_{n+1} = \tilde{V}_n \oplus \tilde{W}_n$. Here, $W_n = \text{clos}_{L_2} \text{span}\{\psi_{n,k}\}$, $\tilde{W}_n = \text{clos}_{L_2} \text{span}\{\tilde{\psi}_{n,k}\}$. Furthermore, $\{\psi_{n,k}\}_{n \geq n_0, k \in \mathbb{Z}}$ is a Riesz basis of $L_2(\mathbb{R})$;
- (3) $L_2(\mathbb{R}) = V_n \oplus \tilde{V}_n^{\perp}$ and the projection operator P_n (with range V_n and null space $\tilde{V}_n^{\perp}$) satisfies $\|P_n\|_{L_2} \leq C$, independent of n .

From (3), we find easily that

$$(1.2) \quad P_n f(x) := \sum_{k \in \mathbb{Z}} a_{n,k} \varphi_{n,k}(x),$$

where $a_n = (a_{n,k})_k \in \ell_2$ is the unique solution for $(a_n * \lambda)(k) = \langle f, \tilde{\varphi}_{n,k} \rangle$ with $\lambda(k) = \langle \varphi(\cdot), \tilde{\varphi}(\cdot-k) \rangle$. By (2), $Q_n f(x) := P_{n+1} f - P_n f$ can be written as

$$Q_n f(x) = \sum_{k \in \mathbb{Z}} b_{n,k} \psi_{n,k}(x),$$

where $b_n = (a_{n,k})_k \in \ell_2$ is the unique solution for $(b_n * \tau)(k) = \langle Q_n f, \tilde{\psi}_{n,k} \rangle$ with $\tau(k) = \langle \psi(\cdot), \tilde{\psi}(\cdot - k) \rangle$.

Since both φ and ψ have compact support, we extend the above operators from $L_2(\mathbb{R})$ to $L_p(\mathbb{R})$ for $p \geq 1$ in section 3. The purpose of this paper is to prove the following main result:

Theorem 1.1. *Let $\varphi = \phi_r$, $\tilde{\varphi} = \phi_\theta$ with $r + \theta$ even and the corresponding wavelets ψ , $\tilde{\psi}$ be given in (1.1). Then for $1 \leq p \leq \infty$, $0 < q < \infty$ and $0 < s < r - 1 + \frac{1}{p}$, each $f \in BB_{p,q}^s$ has the expansion*

$$(1.3) \quad f = \sum_{k \in \mathbb{Z}} a_{n_0,k} \varphi_{n_0,k} + \sum_{n=n_0}^{\infty} \sum_{k \in \mathbb{Z}} b_{n,k} \psi_{n,k}$$

in $BB_{p,q}^s$, where a_{n_0} and b_n are the unique solution of $(a_{n_0} * \lambda)(k) = \langle f, \tilde{\varphi}_{n_0,k} \rangle$ and $(b_n * \tau)(k) = \langle f, \tilde{\psi}_{n,k} \rangle$, respectively. Moreover, there exist positive constants C_1, C_2 (depending on r and θ) such that

$$(1.4) \quad \begin{aligned} C_1(\|a_{n_0}\|_{\ell_p} + \|b\|_{\ell_{p,q}^s}) &\leq \sum_{k \in \mathbb{Z}} a_{n_0,k} \varphi_{n_0,k} + \sum_{n=n_0}^{\infty} \sum_{k \in \mathbb{Z}} b_{n,k} \psi_{n,k} \|_{BB_{p,q}^s} \\ &\leq C_2(\|a_{n_0}\|_{\ell_p} + \|b\|_{\ell_{p,q}^s}). \end{aligned}$$

Since for each $s > 0$, there always exists a positive integer r such that $s < r - 1 + \frac{1}{p}$, our characterization covers all finite $s > 0$. Of course, our functions φ and ψ have large supports. In the next section, we conclude $\|\sum_{k \in \mathbb{Z}} a_{n_0,k} \varphi_{n_0,k} + \sum_{n=n_0}^{\infty} \sum_{k \in \mathbb{Z}} b_{n,k} \psi_{n,k}\|_{BB_{p,q}^s} \lesssim \|a_{n_0}\|_{\ell_p} + \|b\|_{\ell_{p,q}^s}$, which is an easy exercise in the characterization of function spaces. Section 3 is devoted to showing $\|a_{n_0}\|_{\ell_p} + \|b\|_{\ell_{p,q}^s} \lesssim \|f\|_{BB_{p,q}^s}$ for $f \in BB_{p,q}^s$, $(a_{n_0} * \lambda)(k) = \langle f, \tilde{\varphi}_{n_0,k} \rangle$ and $(b_n * \tau)(k) = \langle f, \tilde{\psi}_{n,k} \rangle$. In the last part, we prove that (1.3) holds in $BB_{p,q}^s$ for each $f \in BB_{p,q}^s$.

2. UPPER BOUND

The estimation for the upper bound is usually much easier than that of the lower bound. In this section, we establish the upper bound in (1.4), for which we need the following simple lemma.

Lemma 2.1. *If $1 \leq p \leq \infty$, $0 < q \leq \infty$, then the B-spline $\phi_r \in BB_{p,q}^\sigma$ for $\sigma < r - 1 + \frac{1}{p}$.*

Proof. Since r is a positive integer and $\sigma < r$, $[\sigma] + 1 \leq r$ and one can take $M = r$, in order to show $\phi_r \in BB_{p,q}^\sigma$, i.e., $|\phi_r|_{BB_{p,q}^\sigma} = \|(2^{j\sigma} \omega_p^r(\phi_r, 2^{-j}))_{j \in \mathbb{Z}}\|_{\ell_q} < \infty$. Clearly, $\omega_p^r(\phi_r, 2^{-j}) \leq C \|\phi_r\|_p$ and $\|(2^{j\sigma} \omega_p^r(\phi_r, 2^{-j}))_{j \leq 0}\|_{\ell_q} \lesssim \|\phi_r\|_p$. Then it is sufficient to show that

$$(2.1) \quad \|(2^{j\sigma} \omega_p^r(\phi_r, 2^{-j}))_{j \geq 0}\|_{\ell_q} < \infty.$$

Since $\Delta_h^r f(\cdot) := \Delta(\Delta_h^{r-1} f)(\cdot) = \int_0^1 h(\Delta_h^{r-1} f)'(\cdot + sh) ds = h \int_0^1 \Delta_h^{r-1} f'(\cdot + sh) ds$, one knows that

$$\|\Delta_h^r f\|_p \leq |h| \int_0^1 \|\Delta_h^{r-1} f'(\cdot + sh)\|_p ds = |h| \int_0^1 \|\Delta_h^{r-1} f'(\cdot)\|_p ds$$

by the integral Minkowski inequality. Moreover, the definition of $\omega_p^r(f, t)$ implies that

$$\omega_p^r(\phi_r, 2^{-j}) \leq 2^{-j} \omega_p^{r-1}(\phi_r', 2^{-j}) \leq 2^{-j(r-1)} \omega_p^1(\phi_r^{(r-1)}, 2^{-j}).$$

On the other hand, $\|\phi_1(\cdot + h) - \phi_1(\cdot)\|_p \lesssim |h|^{\frac{1}{p}}$ shows $\omega_p^1(\phi_1, 2^{-j}) \lesssim 2^{-\frac{j}{p}}$. Note that $\phi_r'(\cdot) = \phi_{r-1}(\cdot) - \phi_{r-1}(\cdot - 1)$ and $\phi_r^{(r-1)}$ is a linear combination of $\phi_1(\cdot - k)$. Then, $\omega_p^r(\phi_r, 2^{-j}) \lesssim 2^{-j(r-1+\frac{1}{p})}$ and $2^{j\sigma}\omega_p^r(\phi_r, 2^{-j}) \lesssim 2^{-j(r-1+\frac{1}{p}-\sigma)}$. This with the assumption $\sigma < r - 1 + \frac{1}{p}$ leads to (2.1), and the proof is completed. $\square$

To establish the upper bounds, we need the following result:

Lemma 2.2 ([1, Lemma 3.4]). *Let $0 < p, q \leq \infty$ and $\phi \in BB_{p,q}^\sigma$ be compactly supported. Then for $0 < s < \sigma$,*

$$\begin{aligned} \left\| \sum_{k \in \mathbb{Z}} a_{n,k} \phi(2^n \cdot - k) \right\|_{BB_{p,q}^s} &\lesssim 2^{n(s-\frac{1}{p})} \|(a_{n,k})_k\|_{\ell_p}, \\ \left\| \sum_{n \geq n_0} \sum_{k \in \mathbb{Z}} a_{n,k} \phi(2^n \cdot - k) \right\|_{BB_{p,q}^s} &\lesssim \|(2^{n(s-\frac{1}{p})} \|(a_{n,k})_{k \in \mathbb{Z}}\|_{\ell_p})_{n \geq n_0}\|_{\ell_q}. \end{aligned}$$

Note that $\psi(\cdot)$ is a linear combination of $\varphi(\cdot - k)$. Then the following result follows from Lemma 2.1 and Lemma 2.2.

Theorem 2.1. *Let $\varphi = \phi_r$ and ψ be the wavelet given in Section 1. If $1 \leq p \leq \infty$, $0 < q \leq \infty$, $0 < s < r - 1 + \frac{1}{p}$. Then, for $a_{n_0} \in \ell_p$ and $b \in \ell_{p,q}^s$,*

$$\left\| \sum_{k \in \mathbb{Z}} a_{n_0,k} \varphi_{n,k}(\cdot) + \sum_{n \geq n_0} \sum_{k \in \mathbb{Z}} b_{n,k} \psi_{n,k}(\cdot) \right\|_{BB_{p,q}^s} \lesssim \|a_{n_0}\|_{\ell_p} + \|b\|_{\ell_{p,q}^s}.$$

3. LOWER BOUND

This part is devoted to the establishment of the lower bound in (1.4). As in [2], we shall end it by establishing the relation between projection operators and the modulus of smoothness. The following lemma is frequently used. We include a short proof for completeness, although similar versions appear in some other references (with slightly different assumptions).

Lemma 3.1. *Let $p \geq 1$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $\varphi \in L_{p'}(\mathbb{R}) \cap L_p(\mathbb{R})$ with compact support. If $f \in L_p(\mathbb{R})$ and $c = (c_k) \in \ell_p$, then*

$$(3.1) \quad \|(\langle f, \varphi_{n,k} \rangle)\|_{\ell_p} \lesssim 2^{n(\frac{1}{2} - \frac{1}{p'})} \|f\|_p, \quad \text{and} \quad \left\| \sum_k c_k \varphi_{n,k} \right\|_p \lesssim 2^{n(\frac{1}{2} - \frac{1}{p})} \|c\|_{\ell_p}.$$

In particular, when $\varphi = \phi_r$,

$$(3.2) \quad \left\| \sum_k c_k \varphi_{n,k} \right\|_p \sim 2^{n(\frac{1}{2} - \frac{1}{p})} \|c\|_{\ell_p}.$$

Proof. One proves the lemma for $p \in (1, \infty)$; similar arguments lead to the cases for $p = 1, \infty$. Assume $\text{Supp } \varphi \subseteq [a, b]$, then

$$|\langle f(\cdot), \varphi(\cdot - k) \rangle|^p \leq \int_a^b |f(x + k)|^p dx \left(\int_a^b |\varphi(x)|^{p'} dx \right)^{\frac{p}{p'}} \lesssim \int_{a+k}^{b+k} |f(x)|^p dx$$

and $\|(\langle f(\cdot), \varphi(\cdot - k) \rangle)\|_{\ell_p} \lesssim \|f\|_p$. In general,

$$\begin{aligned} \|(\langle f(\cdot), \varphi_{n,k}(\cdot) \rangle)_k\|_{\ell_p} &= 2^{-\frac{n}{2}} \|(\langle f(2^{-n} \cdot), \varphi(\cdot - k) \rangle)_k\|_{\ell_p} \lesssim 2^{-\frac{n}{2}} \|f(2^{-n} \cdot)\|_p \\ &= 2^{-\frac{n}{2}} 2^{\frac{n}{p}} \|f\|_p = 2^{n(\frac{1}{2} - \frac{1}{p})} \|f\|_p, \end{aligned}$$

which is the first inequality in (3.1). Note that

$$\left\| \sum_k c_k \varphi_{n,k}(x) \right\|_p^p = 2^{\frac{np}{2}-n} \left\| \sum_k c_k \varphi(x-k) \right\|_p^p$$

and

$$\begin{aligned} \left\| \sum_k c_k \varphi(x-k) \right\|_p^p &= \sum_{k'} \int_{k'}^{k'+1} \left| \sum_{k=k'+1-\lceil b \rceil}^{k'-\lfloor a \rfloor} c_k \varphi(x-k) \right|^p dx \\ &\lesssim \sum_{k'} \int_{k'}^{k'+1} \sum_{k=k'+1-\lceil b \rceil}^{k'-\lfloor a \rfloor} |c_k|^p |\varphi(x-k)|^p dx \lesssim \|c\|_{\ell_p}^p. \end{aligned}$$

Then $\left\| \sum_k c_k \varphi_{n,k} \right\|_p \lesssim 2^{n(\frac{1}{2}-\frac{1}{p})} \|c\|_{\ell_p}$, which is the second part of (3.1). Here, $\lfloor s \rfloor$ ($\lceil s \rceil$) denote the largest (smallest) integer less (greater) than or equal to s .

In particular, for $\varphi = \phi_r$, because $\{\phi_r(x-k) : k'-r+1 \leq k \leq k'\}$ is linearly independent on $[k', k'+1]$, the classical functional analysis tells us (two norms are always equivalent on r -dimensional linear spaces)

$$\left\| \sum_k c_k \phi_r(x-k) \right\|_{L_p([k', k'+1])} \sim \|(c_k)_{k=k'-r+1}^{k'}\|_{\ell_p},$$

where the constant depends only on r . Moreover,

$$\begin{aligned} \left\| \sum_k c_k \varphi(x-k) \right\|_{L_p}^p &= \sum_{k' \in \mathbb{Z}} \left\| \sum_k c_k \varphi(x-k) \right\|_{L_p([k', k'+1])}^p \\ &\sim \sum_{k' \in \mathbb{Z}} \|(c_k)_{k=k'-r+1}^{k'}\|_{\ell_p}^p \sim \|c\|_{\ell_p}^p. \end{aligned}$$

Finally,

$$\left\| \sum_k c_k \varphi_{n,k}(x) \right\|_p^p = 2^{n(\frac{p}{2}-1)} \left\| \sum_k c_k \varphi(x-k) \right\|_p^p \sim 2^{n(\frac{p}{2}-1)} \|c\|_{\ell_p}^p$$

and (3.2) results. $\square$

To extend the projection operators P_n and Q_n (introduced in the first section) from $L_2(\mathbb{R})$ to $L_p(\mathbb{R})$ for $p \geq 1$, we use the following lemma:

Lemma 3.2. *Let $\varphi = \phi_r$, $\tilde{\varphi} = \phi_\theta$ with $r + \theta$ even, let ψ and $\tilde{\psi}$ be the corresponding wavelets given in (1.1). Define $V_n := \{\sum_k a_k \varphi_{n,k} : (a_k) \in \ell_p\}$ and $W_n := \{\sum_k b_k \psi_{n,k} : (b_k) \in \ell_p\}$ in the sense of $L_p(\mathbb{R})$. Then*

$$V_n \bigoplus W_n = V_{n+1},$$

where “ $\bigoplus$ ” denotes the direct sum.

Proof. By Lemma 3.1, both V_n and W_n are well-defined subspaces of $L_p(\mathbb{R})$. Note that $f \in V_0$ or W_0 is equivalent to $f(2^n \cdot) \in V_n$ or W_n . Then it is sufficient to show

$$(3.3) \quad V_0 \bigoplus W_0 = V_1.$$

Since φ has compact support and $\varphi(x) = \sum_{l=0}^r h_l \varphi(2x-l)$, one has $\sum_k a_k \varphi(x-k) = \sum_{l=0}^r h_l \sum_k a_k \varphi(2x-2k-l) \in V_1$ and $V_0 \subseteq V_1$. Similarly, $W_0 \subseteq W_1$ and therefore $V_0 + W_0 \subseteq V_1$.

Next, one shows $V_1 \subseteq V_0 + W_0$:

According to the proof of Lemma 4.3 in [9], there exist $(c_l)_{l \in \mathbb{Z}}, (d_l)_{l \in \mathbb{Z}}, (c'_l)_{l \in \mathbb{Z}}, (d'_l)_{l \in \mathbb{Z}} \in \ell_2(\mathbb{Z})$ such that

$$(3.4) \quad \varphi(2x) = \sum_l c_l \varphi(x-l) + \sum_l d_l \psi(x-l)$$

and

$$(3.5) \quad \varphi(2x-1) = \sum_l c'_l \varphi(x-l) + \sum_l d'_l \psi(x-l).$$

By combining (3.4) with $\langle \varphi(x), \tilde{\psi}(x-l) \rangle = \langle \psi(x), \tilde{\varphi}(x-l) \rangle = 0$, one has

$$\langle \varphi(2x), \tilde{\varphi}(x-l') \rangle = (c * \lambda)(l'), \quad \langle \varphi(2x), \tilde{\psi}(x-l') \rangle = (d * \tau)(l'),$$

where $\lambda(k) := \langle \varphi(\cdot), \tilde{\varphi}(\cdot - k) \rangle$ and $\tau(k) := \langle \psi(\cdot), \tilde{\psi}(\cdot - k) \rangle$. Because $\hat{\lambda}(\xi) := \sum_k \lambda(k) e^{ik\xi} \neq 0$ for each $\xi \in \mathbb{R}$ ([9], Lemma 2.2) and $\lambda(k) = 0$ except for finitely many k due to the compactness of φ and $\tilde{\varphi}$, one gets

$$\hat{\eta}(\xi) := \frac{1}{\hat{\lambda}(\xi)} = \sum_k \eta(k) e^{ik\xi}$$

with $(\eta(k))_k \in \ell_1$, thanks to Wiener's Lemma (see [10]). Note that both φ and $\tilde{\varphi}$ have compact supports. Then $(\langle \varphi(2x), \tilde{\varphi}(x-l') \rangle)_{l'} \in \ell_1$. Moreover, $c = \eta * (\langle \varphi(2x), \tilde{\varphi}(x-l') \rangle) \in \ell_1$. Similarly, since $\hat{\tau}(\xi) \neq 0$ ([9], Lemma 4.2), one sees that $d \in \ell_1$. Hence, $c, d \in \ell_p$ for $p \geq 1$ and $\varphi(2x) \in V_0 + W_0$. The same arguments (with (3.5)) lead to $\varphi(2x-1) \in V_0 + W_0$.

In general, let $f \in V_1$ and $f(x) = \sum_k a_k \varphi(2x-2k) + \sum_k b_k \varphi(2x-2k-1)$ with $(a_k) \in \ell_p, (b_k) \in \ell_p$. Then, by (3.4),

$$\begin{aligned} \sum_k a_k \varphi(2x-2k) &= \sum_k a_k \sum_l c_l \varphi(x-k-l) + \sum_k a_k \sum_l d_l \psi(x-k-l) \\ &= \sum_l (a * c)(l) \varphi(x-l) + \sum_l (a * d)(l) \psi(x-l). \end{aligned}$$

Since $c, d \in \ell_1$ and $a \in \ell_p$, $(a * c) \in \ell_p$, $(a * d) \in \ell_p$ and $\sum_k a_k \varphi(2x-2k) \in V_0 + W_0$. Analogously, by (3.5), $\sum_k b_k \varphi(2x-2k-1) \in V_0 + W_0$. Hence, $f \in V_0 + W_0$, which implies $V_1 \subseteq V_0 + W_0$.

To get (3.3), it remains to conclude $a_k = b_k = 0$, when $(a_k), (b_k) \in \ell_p$ and

$$\sum_k a_k \varphi(x-k) + \sum_k b_k \psi(x-k) = 0.$$

Clearly, $(a * \lambda)(l) = (b * \tau)(l) = 0$ for each $l \in \mathbb{Z}$, and furthermore, $\hat{a}(\xi) \hat{\lambda}(\xi) = \hat{b}(\xi) \hat{\tau}(\xi) = 0$. Note that since $\hat{\lambda}(\xi) \neq 0, \hat{\tau}(\xi) \neq 0$ for each $\xi \in \mathbb{R}$, one has $\hat{a}(\xi) = \hat{b}(\xi) = 0$ and $a = b = 0$. This completes the proof of (3.3) and finally Lemma 3.2 follows. $\square$

Let $f \in L_p(\mathbb{R})$. Then $c_n := (\langle f, \tilde{\varphi}_{n,k} \rangle)_k \in \ell_p$ due to Lemma 3.1. From the proof of Lemma 3.2, $(a_n * \lambda)(k) = c_n(k)$ has the unique solution $a_n \in \ell_p$. Now, we define

$$P_n f(x) = \sum_{k \in \mathbb{Z}} a_{n,k} \varphi_{n,k}(x) \in V_n,$$

which is formally the same as in (1.2). Moreover, the proof of Lemma 3.2 shows

$$(3.6) \quad \frac{1}{\|\lambda\|_{\ell_1}} \|c_n\|_{\ell_p} \leq \|a_n\|_{\ell_p} \leq \|\eta\|_{\ell_1} \|c_n\|_{\ell_p}.$$

Define $Q_n f := P_{n+1} f - P_n f$. Then there exists a unique $(b_{n,k})_k \in \ell_p$ such that

$$Q_n f = \sum_k b_{n,k} \psi_{n,k} \in W_n$$

according to Lemma 3.2. Here, $d_n(k) := \langle Q_n f, \tilde{\psi}_{n,k} \rangle$ satisfies $(b_n * \tau)(k) = d_n(k)$ and

$$(3.7) \quad \frac{1}{\|\tau\|_{\ell_1}} \|d_n\|_{\ell_p} \leq \|b_n\|_{\ell_p} \leq \|\sigma\|_{\ell_1} \|d_n\|_{\ell_p}$$

with $\hat{\sigma}(\xi) := \frac{1}{\tilde{\tau}(\xi)}$. In order to prove Lemma 3.4, we need the next lemma, which can be found in [7] (Theorem 3.2) and [8]:

Lemma 3.3. *Let $\varphi = \phi_r$. Then there exist a continuous, compactly supported function sequence $\{\gamma_{n,k}\}_{n \geq n_0, k \in \mathbb{Z}}$ (not necessarily generated by one function) such that for $f \in L_p$, $D_n f(x) = \sum_{k \in \mathbb{Z}} \langle f, \gamma_{n,k} \rangle \varphi_{n,k}(x)$ satisfies*

$$\|f - D_n f\|_p \lesssim \omega_p^r(f, 2^{-n}).$$

Now, we show some properties of P_n , which will be used in the proof of Theorem 3.1 and Theorem 4.1.

Lemma 3.4. *If $f \in L_p(\mathbb{R})$, then*

- (i) $P_n P_l f = P_l P_n f = P_l f$, when $l \leq n$;
- (ii) $\|P_n f\|_p \leq C \|f\|_p$ for some C (independent of f and n);
- (iii) $\|f - P_n f\|_p \lesssim \omega_p^r(f, 2^{-n})$.

Proof. (i) It is easy to see $P_n v_n = v_n$ for $v_n \in V_n$; in fact, assume $v_n = \sum_k a'_{n,k} \varphi_{n,k}$ and $P_n v_n = \sum_k a_{n,k} \varphi_{n,k}$ with $(a'_{n,k})_k, (a_{n,k})_k \in \ell_p$, then

$$(a_n * \lambda)(l) = \langle v_n, \tilde{\varphi}_{n,l} \rangle = \langle \sum_k a'_{n,k} \varphi_{n,k}, \tilde{\varphi}_{n,l} \rangle = (a'_n * \lambda)(l)$$

by the definition of P_n . Furthermore, $\hat{a}'_n(\xi) \lambda(\xi) = \hat{a}_n(\xi) \lambda(\xi)$, which implies $a_n = a'_n$ by $\hat{\lambda}(\xi) \neq 0$ for each $\xi \in \mathbb{R}$. This shows $P_n v_n = v_n$ for $v_n \in V_n$. Then $P_n P_l f = P_l f$, when $l \leq n$. To show $P_l P_n f = P_l f$, one observes

$$P_n f = P_l f + \sum_{j=l}^{n-1} Q_j f$$

from the definition of Q_n . On the other hand, it is easy to see that $P_l g = 0$ for $g \in W_j$ with $j \geq l$, since $c_l(k) = \langle g, \tilde{\varphi}_{l,k} \rangle = 0$. Hence,

$$P_l P_n f = P_l P_l f + \sum_{j=l}^{n-1} P_l(Q_j f) = P_l f + \sum_{j=l}^{n-1} P_l(Q_j f) = P_l f.$$

(ii) By combining (3.6) with (3.1), one has

$$\|P_n f\|_p \lesssim 2^{n(\frac{1}{2} - \frac{1}{p})} \|a_n\|_{\ell_p} \leq C 2^{n(\frac{1}{2} - \frac{1}{p})} \|c_n\|_{\ell_p}.$$

Clearly, Lemma 3.1 shows that $\|c_n\|_{\ell_p} := \|(\langle f, \tilde{\varphi}_{n,k} \rangle)_k\|_{\ell_p} \lesssim 2^{n(\frac{1}{2} - \frac{1}{p'})} \|f\|_p$ with $\frac{1}{p} + \frac{1}{p'} = 1$. Then, $\|P_n f\|_p \leq C \|f\|_p$.

(iii) Since $D_n f \in V_n$, $P_n D_n f = D_n f$ and $\|D_n f - P_n f\|_p = \|P_n D_n f - P_n f\|_p \leq C\|f - D_n f\|_p$ due to (ii). Moreover,

$$\|f - P_n f\|_p \leq \|f - D_n f\|_p + \|D_n f - P_n f\|_p \lesssim \|f - D_n f\|_p \lesssim \omega_p^r(f, 2^{-n}),$$

where the last inequality comes from Lemma 3.3. $\square$

Now, we are ready to give the lower bounds:

Theorem 3.1. *Let $f \in BB_{p,q}^s$ for $s > 0$ with $1 \leq p \leq \infty$, $0 < q \leq \infty$, $(a_{n_0} * \lambda)(k) = \langle f, \tilde{\varphi}_{n_0,k} \rangle$ and $(b_n * \tau)(k) = \langle Q_n f, \tilde{\psi}_{n,k} \rangle$ for $n \geq n_0$. Then*

$$\|a_{n_0}\|_{\ell_p} + \|b\|_{\ell_{p,q}^s} \lesssim \|f\|_{BB_{p,q}^s}.$$

Proof. By (3.6) and Lemma 3.1, one knows that $\|a_{n_0}\|_{\ell_p} \lesssim \|(\langle f, \tilde{\varphi}_{n_0,k} \rangle)_k\|_{\ell_p} \leq 2^{n_0(\frac{1}{2} - \frac{1}{p'})} \|f\|_p$ and therefore $\|a_{n_0}\|_{\ell_p} \lesssim \|f\|_{BB_{p,q}^s}$. In order to show $\|b\|_{\ell_{p,q}^s} \lesssim \|f\|_{BB_{p,q}^s}$, it is sufficient to conclude $\|d\|_{\ell_{p,q}^s} \lesssim \|f\|_{BB_{p,q}^s}$, due to (3.7). According to (3.1), one gets

$$(3.8) \quad \|(d_{n,k})_k\|_{\ell_p} := \|(\langle Q_n f(x), \tilde{\psi}_{n,k}(\cdot) \rangle)_k\|_{\ell_p} \lesssim \|Q_n f(x)\|_{L_p} 2^{n(\frac{1}{2} - \frac{1}{p'})}.$$

Using (iii) in Lemma 3.4, one has

$$(3.9) \quad \begin{aligned} \|Q_n f\|_p &:= \|P_{n+1} f(x) - P_n f(x)\|_p \leq \|f - P_{n+1} f\|_p \\ &+ \|f - P_n f\|_p \lesssim \omega_p^r(f, 2^{-n}). \end{aligned}$$

Moreover, (3.8) and (3.9) imply that $\|(d_{n,k})_k\|_{\ell_p} \lesssim 2^{n(\frac{1}{2} - \frac{1}{p'})} \omega_p^r(f, 2^{-n})$. Finally,

$$\|d\|_{\ell_{p,q}^s} := \|(2^{n(s + \frac{1}{2} - \frac{1}{p'})} \|(d_{n,k})_k\|_{\ell_p})_{n \geq n_0}\|_{\ell_q} \lesssim \|(2^{ns} \omega_p^r(f, 2^{-n}))_{n \geq n_0}\|_{\ell_q} \leq \|f\|_{BB_{p,q}^s}.$$

This completes the proof. $\square$

4. COMPLETENESS

In this section, we prove the completeness of $\{\varphi_{n_0,k}, \psi_{n,k}, k \in \mathbb{Z}, n \geq n_0\}$ in $BB_{p,q}^s$. First, we give a simple property for $P_l f$. A similar result can be found in [2] in which $P_l f$ is defined by biorthogonal scaling functions φ and $\tilde{\varphi}$ with $\langle \varphi(\cdot), \tilde{\varphi}(\cdot - k) \rangle = \delta_{k,0}$.

Lemma 4.1. *For $1 \leq p \leq \infty$ and $f \in L_p(\mathbb{R})$, $\omega_p^r(P_l f, 2^{-n}) \lesssim 2^{(l-n)(r-1+\frac{1}{p})} \|P_l f\|_p$ for $l \leq n$.*

Proof. It is sufficient to show Lemma 4.1 only for $1 \leq p < \infty$, the case for $p = \infty$ is similar. By $P_l f(x) = \sum_{k \in \mathbb{Z}} a_{l,k} \varphi_{l,k}(x)$, one knows that

$$\|\Delta_h^r P_l f\|_p^p = \left\| \sum_{k \in \mathbb{Z}} a_{l,k} \Delta_h^r \varphi_{l,k}(x) \right\|_p^p = 2^{l(\frac{p}{2}-1)} \int \left| \sum_{k \in \mathbb{Z}} a_{l,k} \Delta_{2^l h}^r \varphi(x - k) \right|^p dx.$$

Because $\varphi = \phi_r$ has compact support, there exist fixed $L_1 < L_2$ such that

$$\begin{aligned} \int \left| \sum_{k \in \mathbb{Z}} a_{l,k} \Delta_{2^l h}^r \varphi(x - k) \right|^p dx &= \sum_{k' \in \mathbb{Z}} \int_{k'}^{k'+1} \left| \sum_{k=k'+L_1}^{k'+L_2} a_{l,k} \Delta_{2^l h}^r \varphi(x - k) \right|^p dx \\ &\lesssim \sum_{k' \in \mathbb{Z}} \int_{k'}^{k'+1} \sum_{k=k'+L_1}^{k'+L_2} |a_{l,k}|^p |\Delta_{2^l h}^r \varphi(x - k)|^p dx \end{aligned}$$

for $|2^l h| \leq 1$. Note that $2^l 2^{-n} \leq 1$ for $l \leq n$, and the proof of Lemma 2.1 implies $\|\Delta_h^r \varphi\|_p \lesssim |h|^{r-1+\frac{1}{p}}$ for $\varphi = \phi_r$. Then $\|\Delta_h^r P_l f\|_p \lesssim 2^{l(\frac{1}{2}-\frac{1}{p})} |2^l h|^{r-1+\frac{1}{p}} \|(a_{l,k})_k\|_{\ell_p}$. Now, it follows that

$$\omega_p^r(P_l f, 2^{-n}) \lesssim 2^{l(\frac{1}{2}-\frac{1}{p})} 2^{(l-n)(r-1+\frac{1}{p})} \|(a_{l,k})_k\|_{\ell_p}.$$

On the other hand, (3.2) tells us that $\|P_l f\|_p \sim 2^{l(\frac{1}{2}-\frac{1}{p})} \|(a_{l,k})_k\|_{\ell_p}$. Hence, $\omega_p^r(P_l f, 2^{-n}) \lesssim 2^{(l-n)(r-1+\frac{1}{p})} \|P_l f\|_p$ holds. $\square$

To prove Theorem 4.1, we need the discrete Hardy inequality:

Lemma 4.2 ([4, Lemma 4.2]). *Given two positive sequences $a = (a_j)_{j \in \mathbb{Z}}$ and $b = (b_j)_{j \in \mathbb{Z}}$, if there exist $C_0 > 0, \mu > 0$ such that $b_j \leq C_0 (\sum_{r=j}^\infty a_r^\mu)^\frac{1}{\mu}$, Then $\|b\|_{\ell_q^s} \lesssim \|a\|_{\ell_q^s}$ holds for each $s > 0$ and $0 < q \leq +\infty$. Here $\ell_q^s := \{(a_j)_j : (2^{js} a_j)_j \in \ell_q\}$ and $\|(a_j)_j\|_{\ell_q^s} = \|(2^{js} a_j)_j\|_{\ell_q}$.*

Theorem 4.1. *Let $f \in BB_{p,q}^s$ with $1 \leq p \leq \infty, 0 < q < \infty$ and $s \in (0, r-1+\frac{1}{p})$. Then $f = P_{n_0} f + \sum_{n \geq n_0} Q_n f$ holds in $BB_{p,q}^s$.*

Proof. Motivated by [3], one introduces

$$DB_{p,q}^s(\mathbb{R}) := \{f \in L_p(\mathbb{R}) : \|f\|_{DB_{p,q}^s(\mathbb{R})} := \|P_{n_0} f\|_{L_p(\mathbb{R})} + |f|_{DB_{p,q}^s} < \infty\},$$

where $|f|_{DB_{p,q}^s} = \|(2^{ns} \|f - P_n f\|_{L_p(\mathbb{R})})_{n \geq n_0}\|_{\ell_q}$. By Lemma 3.4 (i), $f \in DB_{p,q}^s$ implies $P_l f \in DB_{p,q}^s$ for $l \geq n_0$. Similarly, $P_{n_0}(P_l f - f) = 0$ and

$$\begin{aligned} \|P_l f - f\|_{DB_{p,q}^s} &= \|(2^{ns} \|(P_l f - f) - P_n(P_l f - f)\|_p)_{n \geq n_0}\|_{\ell_q} \\ &\lesssim \|(2^{ns} \|(P_n f - f)\|_p)_{n \geq l}\|_{\ell_q}. \end{aligned}$$

Therefore, $\lim_{l \rightarrow +\infty} \|P_l f - f\|_{DB_{p,q}^s} = 0$, due to $f \in DB_{p,q}^s$ (Here, $q < \infty$ is needed). Now, it is sufficient to prove

$$(4.1) \quad \|f\|_{DB_{p,q}^s} \sim \|f\|_{BB_{p,q}^s},$$

to conclude Theorem 4.1.

First, assume $f \in BB_{p,q}^s$, then $2^{ns} \omega_p^r(f, 2^{-n}) \leq |f|_{BB_{p,q}^s}$. By Lemma (3.4), (iii), one knows $\|f - P_n f\|_p \leq \omega_p^r(f, 2^{-n})$. Hence $\lim_{n \rightarrow +\infty} \|f - P_n f\|_p = 0$ for $1 \leq p \leq \infty$. Moreover, $\|f - P_n f\|_p = \|\sum_{j=n}^\infty Q_j f\|_p \leq \sum_{j=n}^\infty \|Q_j f\|_p$. This with Lemma 4.2 gives

$$\|(2^{ns} \|f - P_n f\|_p)_{n \geq n_0}\|_{\ell_q} \lesssim \|(2^{ns} \|Q_n f\|_p)_{n \geq n_0}\|_{\ell_q}.$$

On the other hand, $\|Q_n f\|_p \lesssim 2^{n(\frac{1}{2}-\frac{1}{p})} \|b_n\|_{\ell_p}$ and $\|P_{n_0} f\|_p \lesssim 2^{n_0(\frac{1}{2}-\frac{1}{p})} \|a_{n_0}\|_{\ell_p}$ due to Lemma 3.1, where $Q_n f(x) = \sum_k b_{n,k} \psi_{n,k}$ and $P_n f(x) = \sum_k a_{n,k} \varphi_{n,k}$ are defined before Lemma 3.3. Hence,

$$\|(2^{ns} \|f - P_n f\|_p)_{n \geq n_0}\|_{\ell_q} \lesssim \|b\|_{\ell_{p,q}^s}$$

and $\|f\|_{DB_{p,q}^s} \lesssim (\|a_{n_0}\|_{\ell_p} + \|b\|_{\ell_{p,q}^s})$. Then $\|f\|_{DB_{p,q}^s} \lesssim \|f\|_{BB_{p,q}^s}$ follows from Theorem 3.1.

Next, one needs to show $\|f\|_{BB_{p,q}^s} \lesssim \|f\|_{DB_{p,q}^s}$ for $f \in DB_{p,q}^s$, in order to obtain (4.1).

Since $\|f\|_p \leq \|P_{n_0}f\|_p + \|f - P_{n_0}f\|_p \leq \|f\|_{DB_{p,q}^s}$, one has

$$\|(2^{ns}\omega_p^r(f, 2^{-n}))_{n \leq n_0}\|_{\ell_q} \lesssim \|(2^{ns}\|f\|_p)_{n \leq n_0}\|_{\ell_q} \lesssim \|f\|_p \leq \|f\|_{DB_{p,q}^s}.$$

Therefore, it remains to prove

$$(4.2) \quad \|(2^{ns}\omega_p^r(f, 2^{-n}))_{n \geq n_0}\|_{\ell_q} \lesssim \|f\|_{DB_{p,q}^s}.$$

Note that $P_{l+1}f - P_l f = P_{l+1}(f - P_l f)$ and $f = P_{n_0}f + \sum_{l=n_0}^{n-1} P_{l+1}(f - P_l f) + (f - P_n f)$. Then $\omega_p^r(f, 2^{-n}) \leq \omega_p^r(P_{n_0}f, 2^{-n}) + \sum_{l=n_0}^{n-1} \omega_p^r(P_{l+1}(f - P_l f), 2^{-n}) + \omega_p^r(f - P_n f, 2^{-n})$. Furthermore, it follows that

$$\begin{aligned} \omega_p^r(f, 2^{-n}) &\lesssim 2^{-n(r-1+\frac{1}{p})} \|P_{n_0}f\|_p + \sum_{l=n_0}^n 2^{(r-1+\frac{1}{p})(l-n)} \|f - P_l f\|_p \\ &= 2^{-n(r-1+\frac{1}{p})} \|P_{n_0}f\|_p + \sum_{l=-n}^{-n_0} 2^{(r-1+\frac{1}{p})(-l-n)} \|f - P_{-l} f\|_p \end{aligned}$$

from Lemma 4.1 and (ii) in Lemma 3.4. Clearly,

$$\|(2^{ns}2^{-n(r-1+\frac{1}{p})}\|P_{n_0}f\|_p)_{n \geq n_0}\|_{\ell_q} \lesssim \|P_{n_0}f\|_p$$

because of $s < r - 1 + \frac{1}{p}$. On the other hand, $\|(2^{ns}[\sum_{l=-n}^{-n_0} 2^{(r-1+\frac{1}{p})(-l-n)} \|f - P_{-l} f\|_p])_{n \geq n_0}\|_{\ell_q} \lesssim \|(2^{ns}\|f - P_n f\|_p)_{n \geq n_0}\|_{\ell_q}$, due to Lemma 4.2. Hence,

$$\|(2^{ns}\omega_p^r(f, 2^{-n}))_{n \geq n_0}\|_{\ell_q} \lesssim \|P_{n_0}f\|_p + \|(2^{ns}\|f - P_n f\|_p)_{n \geq n_0}\|_{\ell_q} := \|f\|_{DB_{p,q}^s}.$$

Now (4.2) results and the proof is completed. $\square$

Remark 4.1. From Theorem 4.1 and $\langle \psi_{n,k}, \tilde{\psi}_{n',k'} \rangle = 0$ for $n \neq n'$, we know that $\langle Q_n f, \tilde{\psi}_{n,k} \rangle = \langle f, \tilde{\psi}_{n,k} \rangle$ for $f \in BB_{p,q}^s$. Moreover, Theorems 2.1, 3.1 and 4.1 imply Theorem 1.1, which is the main result of this paper.

ACKNOWLEDGEMENT

The authors would like to thank Professor Gilbert Walter for his many suggestions.

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