IRREDUCIBILITY CRITERION FOR ALGEBROID CURVES

TAKAFUMI SHIBUTA

ABSTRACT. The purpose of this paper is to give an algorithm for deciding the irreducibility of reduced algebroid curves over any algebraically closed field without using resolution of singularities. To do this, we introduce a new notion of local tropical variety which is a straightforward extension of tropism introduced by Maurer, and we prove an analogue of the the fundamental theorem of tropical geometry.

1. INTRODUCTION

Throughout this paper, k is a perfect field of arbitrary characteristic, and K is the algebraic closure of k . An *algebroid curve* over k is a Noetherian ring A such that:

- (1) A is a complete local ring.
- (2) A is equidimensional of Krull dimension one (i.e., $\dim A/\mathfrak{p} = 1$ for all $\mathfrak{p} \in \text{Ass } A$).
- (3) A has a coefficient field k .

If A is domain, we say that A is *irreducible*, and if A has no nilpotent element, we say that A is *reduced*. By Cohen's theorem, A is isomorphic to $k[[\mathbf{x}]]/I$ for some $k[[\mathbf{x}]] = k[[x_1, \dots, x_r]]$ and $I \subset k[[\mathbf{x}]]$. The purpose of this paper is to give an algorithm for deciding the irreducibility of a reduced algebroid curve over K without using resolution of singularities. For plane algebroid curves over an algebraic closed field of characteristic zero, Abhyankar [1] has given an irreducibility criterion for bivariate power series, and Kuo [16] has presented a simple algorithm for deciding irreducibility of bivariate power series.

In Section 3, we will give a criterion for an algebroid curve $k[[\mathbf{x}]]/I$ to be irreducible. To do this, we introduce a new notion of *local tropical variety* in Section 2. Local tropical variety is a straightforward extension of *tropism* introduced by Maurer [17], and it is an analogue of tropical variety in formal power series rings (see [3], [9], [20], [22] for tropical varieties). We denote by $\mathbb{N} = \{0, 1, 2, 3, \dots\}$, $\mathbb{N}_+$, and $\mathbb{R}_+$, the set of non-negative integers, positive integers, and positive real numbers, respectively. We set $\tilde{\mathbb{N}}_+ = \mathbb{N}_+ \cup \{\infty\}$ and $\tilde{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{\infty\}$. For $\mathbf{w} = (w_1, \dots, w_r) \in \tilde{\mathbb{N}}_+^r$, we set $\gcd(\mathbf{w}) = \gcd(w_i \mid w_i \neq \infty)$. The local tropical variety $\mathcal{T}_{\text{loc}}(I)$ of $I \subset k[[\mathbf{x}]]$ is defined as the set of weight vectors $\mathbf{w} \in \tilde{\mathbb{R}}_+^r$ such that the initial ideal of I with respect to $\mathbf{w}$ contains no monomial (Definition 2.3). We say that $\mathbf{w} \in \tilde{\mathbb{N}}_+^r$ is a tropism of I if $\mathbf{w} \in \mathcal{T}_{\text{loc}}(I)$ and $\gcd(\mathbf{w}) = 1$. We set $\boldsymbol{\omega}(I) = (\text{int}(x_1; I), \dots, \text{int}(x_r; I)) \in \tilde{\mathbb{N}}_+^r$ for $I \subset k[[\mathbf{x}]]$ where $\text{int}(x_i; I) = \dim_k k[[\mathbf{x}]]/\langle x_i, I \rangle$ (Definition 3.1). We give irreducibility criteria for algebroid curves in terms of local tropical varieties.

Received by the editor February 23, 2011 and in revised form, February 25, 2011 and August 16, 2011.

2010 *Mathematics Subject Classification.* Primary 14H50, 14Q05; Secondary 14H20, 13F25.

©2012 American Mathematical Society
 Reverts to public domain 28 years from publication

Theorem 1 (Theorem 3.7). *Let $I \subset k[[\mathbf{x}]]$ be an equidimensional ideal of dimension one. Then the following hold.*

- (1) *If $\mathcal{T}_{\text{loc}}(I) \neq \widetilde{\mathbb{R}}_+ \cdot \boldsymbol{\omega}(I)$ (e.g. $\text{in}_{\boldsymbol{\omega}(I)}(I)$ contains monomials, or I has at least two tropisms), then I is not prime.*
- (2) *If $\boldsymbol{\omega}(I)$ is a tropism of I , then I is prime.*
- (3) *Let $\{\mathbf{w}_1, \dots, \mathbf{w}_l\} \subset \widetilde{\mathbb{N}}_+^r$ be the set of the tropisms of I . If $\boldsymbol{\omega}(I) \neq (\infty, \dots, \infty)$ and $\boldsymbol{\omega}(I) = \mathbf{w}_1 + \dots + \mathbf{w}_l$, then there exist prime ideals $\mathfrak{p}_j \subset k[[\mathbf{x}]]$ for $1 \leq j \leq l$ such that $\boldsymbol{\omega}(\mathfrak{p}_j) = \mathbf{w}_j$ and $I = \bigcap_{j=1}^l \mathfrak{p}_j$.*

In Section 5, an algorithm for deciding the irreducibility of reduced algebroid curve (Algorithm 5.5) is given. For a reduced algebroid curve $K[[x_1, \dots, x_r]]/I$, it computes an ideal $J \subset K[[x_1, \dots, x_{r'}]]$, $r' \geq r$, such that $K[[x_1, \dots, x_r]]/I \cong K[[x_1, \dots, x_{r'}]]/J$, and the irreducibility of $K[[x_1, \dots, x_{r'}]]/J$ can be determined by Theorem 1, that is, if I is prime, then $\boldsymbol{\omega}(J)$ is a tropism of J , and if I is not prime, then J has at least two tropisms. This algorithm based on the above irreducibility criterion and an algorithm for computing the value-semigroup of an irreducible algebroid curve (Algorithm 4.18). Algorithm 4.18 is an improved version of Hefez–Hernandes’s algorithm [12], [13]. On smaller examples, our algorithms can be carried out by hand.

One can decide irreducibility of an reduced algebroid curve using resolution of singularities; a reduced algebroid curve is irreducible if and only if the strict transform does not bifurcate. Our method determines the irreducibility of algebroid curves more directly, and it is easier to follow the computation in some cases (see Remark 5.11). We will give some examples of curves with parameters in the exponent of the defining equations (see Examples 3.13, 4.21, 4.22, 5.7, 4.25, 5.9, 5.10, and Proposition 5.8). The resolutions of singularity with parameters in the exponent is not easy to compute in general.

We use Gröbner basis and standard basis for computation. The readers refer to [5], [6], and [11] for the computational theories. We denote by $\text{LT}_{\prec}(f)$ the leading term of $f \in K[\mathbf{x}]$ (resp. $f \in K[[\mathbf{x}]]$) with respect to a local ordering (resp. a global ordering) $\prec$ (see [11], Definition 1.2.4 for the definitions of global ordering and local ordering).

2. LOCAL TROPICAL VARIETIES

2.1. Definition of local tropical varieties. In this section, we introduce a notion of local tropical varieties, and prove its basic properties.

For $\mathbf{x} = (x_1, \dots, x_r)$ and $\mathbf{a} = (a_1, \dots, a_r)$, we use multi-index notation $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \dots x_r^{a_r}$. Let L be any field, and $L[[\mathbf{x}]] = L[[x_1, \dots, x_r]]$ a formal power series ring over L . We set $\widetilde{\mathbb{N}}_+ = \mathbb{N}_+ \cup \{\infty\}$, and $\widetilde{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{\infty\}$ as in the introduction.

Definition 2.1. Let $\mathbf{w} = (w_1, \dots, w_r) \in \widetilde{\mathbb{R}}_+^r$. For $f = \sum_{\mathbf{a} \in \mathbb{N}^r} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}} \in L[[\mathbf{x}]]$, $c_{\mathbf{a}} \in L$, we define the *order* of f with respect to $\mathbf{w}$ as

$$\text{ord}_{\mathbf{w}}(f) = \min\{\mathbf{w} \cdot \mathbf{a} \mid c_{\mathbf{a}} \neq 0\} \in \widetilde{\mathbb{R}}_+ \cup \{0\},$$

where $\mathbf{w} \cdot \mathbf{a} = \sum w_i a_i \in \widetilde{\mathbb{R}}_+$ with the convention $\infty \cdot 0 = 0$. We set $\text{ord}_{\mathbf{w}}(0) = \infty$. If $\text{ord}_{\mathbf{w}}(f) < \infty$, we define the *initial form* of f as

$$\text{in}_{\mathbf{w}}(f) = \sum_{\mathbf{w} \cdot \mathbf{a} = \text{ord}_{\mathbf{w}}(f)} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}} \in L[x_i \mid w_i \neq \infty].$$

If $\text{ord}_{\mathbf{w}}(f) = \infty$, we set $\text{in}_{\mathbf{w}}(f) = 0$.

For an ideal $I \subset L[[\mathbf{x}]]$, we call $\text{in}_{\mathbf{w}}(I) = \langle \text{in}_{\mathbf{w}}(f) \mid f \in I \rangle \subset L[x_i \mid w_i \neq \infty]$ the *initial ideal* of I with respect to $\mathbf{w}$. We say that $\{f_1, \dots, f_n\} \subset I$ is a *standard basis* of I with respect to $\mathbf{w}$ if $\text{in}_{\mathbf{w}}(I) = \langle \text{in}_{\mathbf{w}}(f_1), \dots, \text{in}_{\mathbf{w}}(f_n) \rangle$.

For $g = \sum_{\mathbf{a} \in \mathbb{N}^r} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}} \in L[\mathbf{x}]$ and $\mathbf{w} = (w_1, \dots, w_r) \in \mathbb{R}_+^r$, we set $\deg_{\mathbf{w}}(g) = \max\{\mathbf{w} \cdot \mathbf{a} \mid c_{\mathbf{a}} \neq 0\}$. We say that a polynomial or an ideal of $L[\mathbf{x}]$ is $\mathbf{w}$ -homogeneous if it is homogeneous with respect to the grading defined by $\deg_{\mathbf{w}}$.

Example 2.2. Let $\mathbf{w} = (1, 2, \infty)$. Then $\text{ord}_{\mathbf{w}}(x^2 + y + xy + z) = \min\{2, 2, 3, \infty\} = 2$, $\text{in}_{\mathbf{w}}(x^2 + y + xy + z) = x^2 + y$, and $\text{ord}_{\mathbf{w}}(z + xz) = \infty$, $\text{in}_{\mathbf{w}}(z + xz) = 0$.

Let $\mathbf{w} \in \mathbb{R}_+^r$, and let $\prec$ be a local ordering such that $\mathbf{x}^{\mathbf{a}} \prec \mathbf{x}^{\mathbf{b}}$ for any $\mathbf{a}, \mathbf{b} \in \mathbb{N}^r$ with $\mathbf{w} \cdot \mathbf{a} > \mathbf{w} \cdot \mathbf{b}$. Then a standard basis of $I \subset K[[\mathbf{x}]]$ with respect to $\prec$ is also a standard basis of I with respect to $\mathbf{w}$. If I is generated by polynomials, then a standard basis of I with respect to a local order is computable (see [11], Section 1.7).

Now, we define local tropical varieties. In this paper, we consider only the constant coefficient case. Our definition is different from Touda's definition ([24] Definition 4.7). For $\mathbf{w} = (w_1, \dots, w_r) \in \tilde{\mathbb{N}}_+^r$, $\mathbf{w} \neq (\infty, \dots, \infty)$, we set $\text{gcd}(\mathbf{w}) = \text{gcd}\{w_i \mid w_i \neq \infty\}$. We say that $\mathbf{w}$ is *primitive* if $\text{gcd}(\mathbf{w}) = 1$.

Definition 2.3. Let $I \subset L[[\mathbf{x}]]$ be an ideal. We call

$$\mathcal{T}_{\text{loc}}(I) = \{\mathbf{w} \in \tilde{\mathbb{R}}_+^r \mid \text{in}_{\mathbf{w}}(I) \text{ contains no monomial}\}$$

the *local tropical variety* of I . We say that an element $\mathbf{w} \in \mathcal{T}_{\text{loc}}(I) \cap \tilde{\mathbb{N}}_+^r$ is a *tropism* of I if $\mathbf{w}$ is primitive.

Note that $\text{in}_{\mathbf{w}}(I)$ contains no monomial if and only if $\text{in}_{\mathbf{w}}(I) : (x_1 \cdots x_r)^{\infty} \neq L[\mathbf{x}]$. It is also equivalent to the existence of a zero $(\alpha_1, \dots, \alpha_r) \in V_{\overline{L}}(\text{in}_{\mathbf{w}}(I))$ such that $\alpha_i \neq 0$ for all $i = 1, \dots, r$, where $\overline{L}$ is an algebraic closure of L . In particular, if $\text{in}_{\mathbf{w}}(I)$ is a binomial ideal of form $\langle \mathbf{x}^{\mathbf{a}_j} - \mathbf{x}^{\mathbf{b}_j} \mid j = 1, \dots, n \rangle$ for some $\mathbf{a}_j, \mathbf{b}_j \in \mathbb{N}^r$, then $\mathbf{w} \in \mathcal{T}_{\text{loc}}(I)$ since $(1, \dots, 1) \in V_L(\text{in}_{\mathbf{w}}(I))$. The topological closure of $\mathcal{T}_{\text{loc}}(I) \cap \mathbb{R}_+^r$ in $\mathbb{R}^r$ is a rational polyhedral complex ([2] Theorem 2.0.4). If I is generated by polynomials, then $\mathcal{T}_{\text{loc}}(I)$ is computable [4].

Similarly to the usual varieties, local tropical varieties satisfy the following properties.

Lemma 2.4. Let $I, J, I_1, \dots, I_l \subset L[[\mathbf{x}]]$ be ideals. Then the following hold.

- (1) If $I \subset J$, then $\mathcal{T}_{\text{loc}}(I) \supset \mathcal{T}_{\text{loc}}(J)$.
- (2) $\sqrt{\text{in}_{\mathbf{w}}(I)} = \sqrt{\text{in}_{\mathbf{w}}(\sqrt{I})}$ for any $\mathbf{w} \in \tilde{\mathbb{R}}_+^r$. In particular, $\mathcal{T}_{\text{loc}}(I) = \mathcal{T}_{\text{loc}}(\sqrt{I})$.
- (3) $\mathcal{T}_{\text{loc}}(\bigcap_{i=1}^l I_i) = \bigcup_{i=1}^l \mathcal{T}_{\text{loc}}(I_i)$.
- (4) $\mathcal{T}_{\text{loc}}(I) = \bigcap_{f \in I} \mathcal{T}_{\text{loc}}(f)$.

Proof. (1) is trivial.

(2) Since $I \subset \sqrt{I}$, $\sqrt{\text{in}_{\mathbf{w}}(I)} \subset \sqrt{\text{in}_{\mathbf{w}}(\sqrt{I})}$. To prove the converse, let $f \in \text{in}_{\mathbf{w}}(\sqrt{I})$. Then there exists $g \in \sqrt{I}$ such that $\text{in}_{\mathbf{w}}(g) = f$. Since $g^n \in I$ for some $n \in \mathbb{N}$, $f^n = \text{in}_{\mathbf{w}}(g^n) \in \text{in}_{\mathbf{w}}(I)$. Thus $f \in \sqrt{\text{in}_{\mathbf{w}}(I)}$. Hence $\text{in}_{\mathbf{w}}(\sqrt{I}) \subset \sqrt{\text{in}_{\mathbf{w}}(I)}$, and thus $\sqrt{\text{in}_{\mathbf{w}}(\sqrt{I})} \subset \sqrt{\text{in}_{\mathbf{w}}(I)}$.

(3) Since $\bigcap I_i \subset I_i$ for all i , we have $\mathcal{T}_{\text{loc}}(\bigcap I_i) \supset \bigcup_i \mathcal{T}_{\text{loc}}(I_i)$. Let $\mathbf{w} \notin \bigcup_i \mathcal{T}_{\text{loc}}(I_i)$. For each i , there exists $f_i \in I_i$ such that $\text{in}_{\mathbf{w}}(f_i)$ is a monomial. Since $\prod_i f_i \in \prod I_i \subset \bigcap I_i$, we conclude that $\text{in}_{\mathbf{w}}(\bigcap I_i)$ contains a monomial in $\text{in}_{\mathbf{w}}(\prod_i f_i) = \prod_i \text{in}_{\mathbf{w}}(f_i)$.

(4) The inclusion $\bigcap_{f \in I} \mathcal{T}_{\text{loc}}(f) \subset \mathcal{T}_{\text{loc}}(I)$ is trivial. To prove the inverse inclusion, take $\mathbf{w} \in \mathcal{T}_{\text{loc}}(I)$. Since $\text{in}_{\mathbf{w}}(I) \subset L[x_i \mid w_i \neq \infty]$ is $\mathbf{w}$ -homogeneous, and monomials are homogeneous, it follows that $\text{in}_{\mathbf{w}}(I)$ contains no monomial if and only if $\text{in}_{\mathbf{w}}(f)$ is not a monomial for all $f \in I$. Thus $\mathcal{T}_{\text{loc}}(I) \subset \bigcap_{f \in I} \mathcal{T}_{\text{loc}}(f)$. $\square$

By Lemma 2.4, $\mathcal{T}_{\text{loc}}(I) = \bigcup_{\mathfrak{P} \in \text{Min } I} \mathcal{T}_{\text{loc}}(\mathfrak{P})$ where $\text{Min } I$ is the set of the minimal associated prime ideals of I .

2.2. Fundamental theorem of local tropical geometry. As in Introduction, k is a perfect field of arbitrary characteristic, and K is the algebraic closure of k . We will prove an analogue of the the fundamental theorem of tropical geometry (see [20], Theorem 2.1, [8], Theorem 4.2, [14], Theorem 2.13 for the fundamental theorem of tropical geometry). We use the theory of affinoid algebras similarly to the proof of the fundamental theorem of tropical geometry in [8].

We set

$$\mathbb{K} = K((t^{\mathbb{R}})) = \left\{ \sum_{a \in \mathbb{R}} c_a t^a \mid c_a \in K \text{ and } \{a : c_a \neq 0\} \text{ is well-ordered} \right\}.$$

Then $\mathbb{K}$ is a field in the natural way. We define the valuation ord_t on $\mathbb{K}$ by $\text{ord}_t(\sum_{a \in \mathbb{R}} c_a t^a) = \min\{a \mid c_a \neq 0\}$. For $\xi = \sum_{a \in \mathbb{R}} c_a t^a$, we define $\text{in}_t(\xi) = c_b t^b$ where $b = \text{ord}_t(\xi)$. For $\boldsymbol{\xi} = (\xi_1, \dots, \xi_r) \in \mathbb{K}^{\oplus r}$, we set $\text{ord}_t(\boldsymbol{\xi}) = (\text{ord}_t(\xi_1), \dots, \text{ord}_t(\xi_r))$ and $\text{in}_t(\boldsymbol{\xi}) = (\text{in}_t(\xi_1), \dots, \text{in}_t(\xi_r))$. Fix $1 < e \in \mathbb{R}_+$. We define $|\xi| := e^{-\text{ord}_t(\xi)}$ for $\xi \in \mathbb{K}$, and call it the *non-archimedean absolute value* of ξ . Then $\mathbb{K}$ is complete with respect to the metric induced by $|\cdot|$. We denote by $\mathbb{K}^{\circ} := \{\xi \in \mathbb{K} \mid \text{ord}_t(\xi) \geq 0\} = \{\xi \in \mathbb{K} \mid |\xi| \leq 1\}$, and $\mathbb{K}^{\circ\circ} := \{\xi \in \mathbb{K} \mid \text{ord}_t(\xi) > 0\} = \{\xi \in \mathbb{K} \mid |\xi| < 1\}$, the valuation ring of ord_t , and its unique maximal ideal, respectively. Then the residue field and $\mathbb{K}^{\circ}/\mathbb{K}^{\circ\circ}$ is isomorphic to K . Since K is algebraically closed and $|\mathbb{K}^{\times}| := \{|\xi| \mid \xi \in \mathbb{K}^{\times}\} = e^{\mathbb{R}} = \mathbb{R}_+$, $\mathbb{K}$ is also algebraically closed (see [10], Section 1.1. The field $\mathbb{K}$ coincides with the field $k((\Gamma))$ in loc. cit., Example 1.1.3 with $\Gamma = e^{\mathbb{R}}$). We denote the *Tate algebra* by

$$T_r := \mathbb{K}\langle \mathbf{x} \rangle = \left\{ \sum_{\mathbf{a} \in \mathbb{N}^r} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}} \in \mathbb{K}[[\mathbf{x}]] \mid c_{\mathbf{a}} \in \mathbb{K}, \text{ord}_t(c_{\mathbf{a}}) \rightarrow \infty \text{ as } |\mathbf{a}| \rightarrow \infty \right\}$$

where $|\mathbf{a}| = a_1 + \dots + a_r$ for $\mathbf{a} = (a_1, \dots, a_r)$. We extend ord_t to a function on T_r by setting $\text{ord}_t(\sum_{\mathbf{a} \in \mathbb{N}^r} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}) = \min\{\text{ord}_t(c_{\mathbf{a}}) \mid c_{\mathbf{a}} \neq 0\}$. We set $T_r^{\circ} := \{f \in T_r \mid \text{ord}_t(f) \geq 0\} = \{\sum_{\mathbf{a} \in \mathbb{N}^r} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}} \mid c_{\mathbf{a}} \in \mathbb{K}^{\circ}, \text{ord}_t(c_{\mathbf{a}}) \rightarrow \infty \text{ as } |\mathbf{a}| \rightarrow \infty\}$, and $T_r^{\circ\circ} := \{f \in T_r \mid \text{ord}_t(f) > 0\} = \mathbb{K}^{\circ\circ} T_r^{\circ}$. Then $\overline{T}_r := T_r^{\circ}/T_r^{\circ\circ}$ is canonically isomorphic to $K[\mathbf{x}]$.

For a ring R containing a field L , $\text{Max}(R)$ denote the set of the maximal ideals of a ring R , and $\text{Sp}_L(R)$ denotes the set of L -algebra homomorphisms $R \rightarrow L$. A quotient ring $A = T_r/J$ of T_r is called an *affinoid algebra* (over $\mathbb{K}$). Since $\mathbb{K}$ is algebraically closed and by [10, Theorem 3.2.1 (5)], $\text{Max}(T_r) = \{\langle x_1 - \xi_1, \dots, x_r - \xi_r \rangle \mid \xi_i \in \mathbb{K}^{\circ}\}$, and thus we can identify $\text{Max}(A)$ with $\text{Sp}_{\mathbb{K}}(A)$ in this case. For

$f + J \in A$, we set

$$\begin{aligned} \text{ord}_t^A(f + J) &:= \sup\{\text{ord}_t(g) \mid g \in f + J\}, \\ \text{ord}_t^{sp}(f + J) &:= \inf\{\text{ord}_t(\gamma(f)) \mid \gamma \in \text{Sp}_{\mathbb{K}}(A)\} \\ &= \inf\{\text{ord}_t(f(\xi_1, \dots, \xi_r)) \mid J \subset \langle x_1 - \xi_1, \dots, x_r - \xi_r \rangle \in \text{Max}(T_r)\} \end{aligned}$$

corresponding to the *quotient norm* $\|f + J\|_A = e^{-\text{ord}_t^A(f + J)}$ and the *spectral semi-norm* $\|f + J\|_{sp} = e^{-\text{ord}_t^{sp}(f + J)}$, respectively. The quotient norm is actually a norm, and the spectral semi-norm is a norm if A is reduced.

Lemma 2.5. $\text{ord}_t^{sp}(f + J) \geq \text{ord}_t^A(f + J)$, and $\text{ord}_t^{sp}(f + J) = \lim_{n \rightarrow \infty} \frac{1}{n} \text{ord}_t^A(f^n + J)$.

Proof. The assertions follows from the definitions and [10], Corollary 3.4.6. $\square$

We set $A^\circ = \{f + J \in A \mid \text{ord}_t^{sp}(f + J) \geq 0\}$, $A^{\circ\circ} = \{f + J \in A \mid \text{ord}_t^{sp}(f + J) > 0\}$, and $\overline{A} = A^\circ/A^{\circ\circ}$. Then A° is a subring of A , $A^{\circ\circ}$ is an ideal of A° , and $\overline{A}$ is a finitely generated K -algebra. Let $\phi : A \rightarrow B$ be a morphism between affinoid algebras over $\mathbb{K}$. Since $\{\gamma \circ \phi \mid \gamma \in \text{Sp}_{\mathbb{K}}(B)\} \subset \text{Sp}_{\mathbb{K}}(A)$, we have $\phi(A^\circ) \subset B^\circ$ and $\phi(A^{\circ\circ}) \subset B^{\circ\circ}$. Thus ϕ induces morphisms $\phi^\circ : A^\circ \rightarrow B^\circ$ and $\overline{\phi} : \overline{A} \rightarrow \overline{B}$. Note that we can identify $\text{Max}(\overline{A})$ with $\text{Sp}_K(\overline{A})$ since K is algebraically closed.

Theorem 2.6 ([10] Theorem 3.5.3). (i) *Let $\phi : A \rightarrow B$ be a morphism between affinoid algebras over $\mathbb{K}$. Then the following properties are equivalent:*

- (1) ϕ is finite.
- (2) $\phi^\circ : A^\circ \rightarrow B^\circ$ is integral.
- (3) $\overline{\phi} : \overline{A} \rightarrow \overline{B}$ is finite.

(ii) *The natural map $\text{Sp}_{\mathbb{K}}(A) \rightarrow \text{Sp}_K(\overline{A})$, $\gamma \mapsto \overline{\gamma}$, is surjective for any affinoid algebra A over $\mathbb{K}$.*

The natural surjection $\mathbb{K}^\circ \rightarrow K \cong \mathbb{K}^\circ/\mathbb{K}^{\circ\circ}$, as well as all naturally induced maps, are denoted τ (e.g. $\tau : T_r^\circ \rightarrow \overline{T}_r \cong K[\mathbf{x}]$, and $\tau : \mathbb{K}^\circ[\mathbf{x}] \rightarrow K[\mathbf{x}]$). For simplicity of notation, the image of x_i in quotient rings of T_r or $K[\mathbf{x}]$ is also denoted x_i .

Proposition 2.7. *Let $A = T_r/J$ be an affinoid algebra over $\mathbb{K}$. Then the map $\text{Sp}_{\mathbb{K}}(A) \rightarrow \text{Sp}_K(\overline{T}_r/\tau(J \cap T_r^\circ))$, $\gamma \mapsto (x_i \mapsto \tau(\gamma(x_i)))$ is surjective.*

Proof. Let $p : T_r \rightarrow A$, $f \mapsto f + J$, be the natural surjection. We claim that $\tau(J \cap T_r^\circ) \subset \text{Ker } \overline{p} = \sqrt{\tau(J \cap T_r^\circ)}$. Since $\tau(J \cap T_r^\circ) = (J \cap T_r^\circ + T_r^{\circ\circ})/T_r^{\circ\circ}$, $\text{Ker } \overline{p} = (p^\circ)^{-1}(A^{\circ\circ})/T_r^{\circ\circ}$, and $\sqrt{\tau(J \cap T_r^\circ)} = \sqrt{J \cap T_r^\circ + T_r^{\circ\circ}}/T_r^{\circ\circ} \supset \sqrt{J \cap T_r^\circ + T_r^{\circ\circ}}/T_r^{\circ\circ}$, it is enough to show that $J \cap T_r^\circ + T_r^{\circ\circ} \subset (p^\circ)^{-1}(A^{\circ\circ}) \subset \sqrt{J \cap T_r^\circ + T_r^{\circ\circ}}$ to prove the claim. Since $\text{ord}_t^{sp}(f + J) \geq \text{ord}_t^A(f + J) > 0$ for $f \in J \cap T_r^\circ + T_r^{\circ\circ}$, the inclusion $J \cap T_r^\circ + T_r^{\circ\circ} \subset (p^\circ)^{-1}(A^{\circ\circ})$ is satisfied. To prove the inclusion $(p^\circ)^{-1}(A^{\circ\circ}) \subset \sqrt{J \cap T_r^\circ + T_r^{\circ\circ}}$, take $f \in (p^\circ)^{-1}(A^{\circ\circ})$. Then $\text{ord}_t^{sp}(f + J) > 0$ as $f + J = p^\circ(f) \in A^{\circ\circ}$. Since $\text{ord}_t^{sp}(f + J) = \lim_{n \rightarrow \infty} \frac{1}{n} \text{ord}_t^A(f^n + J)$ by Lemma 2.5, there exists n such that $\text{ord}_t^A(f^n + J) > 0$. Thus $\text{ord}_t(f^n + g) > 0$ for some $g \in J$. As $f^n \in T_r^\circ$ and $f^n + g \in T_r^{\circ\circ}$, we have $g \in T_r^{\circ\circ}$. This shows that $f^n = -g + (f^n + g) \in J \cap T_r^\circ + T_r^{\circ\circ}$. Thus $(p^\circ)^{-1}(A^{\circ\circ}) \subset \sqrt{J \cap T_r^\circ + T_r^{\circ\circ}}$.

By the claim, we have $\sqrt{\text{Ker } \overline{p}} = \sqrt{\tau(J \cap T_r^\circ)}$. Hence, by Hilbert's Nullstellensatz, the map $\text{Sp}_K(\overline{T}_r/\text{Ker } \overline{p}) \rightarrow \text{Sp}_K(\overline{T}_r/\tau(J \cap T_r^\circ))$ induced by $\overline{T}_r/\tau(J \cap T_r^\circ) \rightarrow \overline{T}_r/\text{Ker } \overline{p}$ is bijective. By Theorem 2.6 (i), $\overline{T}_r/\text{Ker } \overline{p} \hookrightarrow \overline{A}$ is finite, and thus

$\mathrm{Sp}_K(\overline{A}) \rightarrow \mathrm{Sp}_K(\overline{T}_r / \mathrm{Ker} \overline{p})$ is surjective. By Theorem 2.6 (ii), $\mathrm{Sp}_{\mathbb{K}}(A) \rightarrow \mathrm{Sp}_K(\overline{A})$ is surjective. Combining all, we conclude that the map $\mathrm{Sp}_{\mathbb{K}}(A) \rightarrow \mathrm{Sp}_K(\overline{T}_r / \tau(J \cap T_r^\circ))$ is surjective. $\square$

Let $\mathbf{w} = (w_1, \dots, w_r) \in \mathbb{R}_+^r$. Then $f(t^{w_1}x_1, \dots, t^{w_r}x_r) \in T_r$ for any $f \in K[\mathbf{x}]$. Thus we can define a K -algebra homomorphism

$$\psi_{\mathbf{w}} : K[\mathbf{x}] \rightarrow T_r, f(x_1, \dots, x_r) \mapsto f(t^{w_1}x_1, \dots, t^{w_r}x_r).$$

Remark 2.8. When we consider $\mathrm{ord}_{\mathbf{w}}(f)$ and $\mathrm{in}_{\mathbf{w}}(f) \in \mathbb{K}[\mathbf{x}]$ for $f \in \mathbb{K}[\mathbf{x}]$, we regard $\mathbb{K}$ as constant coefficient, namely, for $\xi \in \mathbb{K}$ and $\mathbf{a} \in \mathbb{N}^r$, $\mathrm{ord}_{\mathbf{w}}(\xi \mathbf{x}^{\mathbf{a}}) = \mathbf{w} \cdot \mathbf{a}$, not $\mathbf{w} \cdot \mathbf{a} + \mathrm{ord}_t(\xi)$.

Proposition 2.9. *Let $\mathbf{w} = (w_1, \dots, w_r) \in \mathbb{R}_+^r$. Let $I \subset K[\mathbf{x}]$ be an ideal, and $J := \langle \psi_{\mathbf{w}}(I) \rangle \subset T_r$. Then $\tau(J \cap T_r^\circ) = \mathrm{in}_{\mathbf{w}}(I)$ as an ideal of $\overline{T}_r \cong K[\mathbf{x}]$.*

Proof. Let $f_1, \dots, f_n \in I$ be a standard basis of I with respect to $\mathbf{w}$, that is, $\mathrm{in}_{\mathbf{w}}(I) = \langle \mathrm{in}_{\mathbf{w}}(f_1), \dots, \mathrm{in}_{\mathbf{w}}(f_n) \rangle$. For $f \in K[\mathbf{x}]$, we define $\tilde{f}^{(\mathbf{w})} := t^{-\mathrm{ord}_{\mathbf{w}}(f)} \psi_{\mathbf{w}}(f) \in T_r^\circ$. Then $\mathrm{in}_{\mathbf{w}}(\tilde{f}^{(\mathbf{w})}) = \tau(\tilde{f}^{(\mathbf{w})}) = \mathrm{in}_{\mathbf{w}}(f)$. As $\mathrm{in}_{\mathbf{w}}(f_i) = \tau(\tilde{f}_i^{(\mathbf{w})}) \in \tau(J \cap T_r^\circ)$, we have $\mathrm{in}_{\mathbf{w}}(I) \subset \tau(J \cap T_r^\circ)$.

We will prove the converse inclusion. Let $\widehat{\psi}_{\mathbf{w}} : \mathbb{K}[\mathbf{x}] \rightarrow \mathbb{K}[\mathbf{x}]$ be a $\mathbb{K}$ -algebra automorphism of $\mathbb{K}[\mathbf{x}]$ which sends $G(x_1, \dots, x_r)$ to $G(t^{w_1}x_1, \dots, t^{w_r}x_r)$. Then $J\mathbb{K}[\mathbf{x}] = \widehat{\psi}_{\mathbf{w}}(I\mathbb{K}[\mathbf{x}])$, and $\mathrm{in}_{\mathbf{w}}(J\mathbb{K}[\mathbf{x}]) = \mathrm{in}_{\mathbf{w}}(I\mathbb{K}[\mathbf{x}]) = \mathrm{in}_{\mathbf{w}}(I)\mathbb{K}[\mathbf{x}]$. Note that $\mathrm{in}_{\mathbf{w}}(I)\mathbb{K}[\mathbf{x}] \cap \mathbb{K}^\circ[\mathbf{x}] = \mathrm{in}_{\mathbf{w}}(I)\mathbb{K}^\circ[\mathbf{x}]$ since $\mathbb{K}^\circ[\mathbf{x}]$ is a direct summand of $\mathbb{K}[\mathbf{x}]$ as a $K[\mathbf{x}]$ -module, and $\mathrm{in}_{\mathbf{w}}(I) \subset K[\mathbf{x}]$.

Take $g \in J \cap T_r^\circ$. We set $g_0 = g$ and construct $g_j \in J \cap T_r^\circ$ inductively on j as following: As $\mathrm{in}_{\mathbf{w}}(g_j) \in \mathrm{in}_{\mathbf{w}}(J\mathbb{K}[\mathbf{x}]) \cap \mathbb{K}^\circ[\mathbf{x}] = \mathrm{in}_{\mathbf{w}}(I)\mathbb{K}[\mathbf{x}] \cap \mathbb{K}^\circ[\mathbf{x}] = \mathrm{in}_{\mathbf{w}}(I)\mathbb{K}^\circ[\mathbf{x}]$, there exist $\mathbf{w}$ -homogeneous polynomials $h_i^{(j)} \in \mathbb{K}^\circ[\mathbf{x}]$, $1 \leq i \leq n$, such that $\mathrm{in}_{\mathbf{w}}(g_j) = \sum_{i=1}^n h_i^{(j)} \mathrm{in}_{\mathbf{w}}(f_i)$ and $\mathrm{ord}_{\mathbf{w}}(g_j) = \deg_{\mathbf{w}}(h_i^{(j)}) + \mathrm{ord}_{\mathbf{w}}(f_i)$. We set $g_{j+1} = g_j - \sum_{i=1}^n h_i^{(j)} \tilde{f}_i^{(\mathbf{w})}$. Then $\mathrm{ord}_{\mathbf{w}}(g_{j+1}) > \mathrm{ord}_{\mathbf{w}}(g_j)$, and

$$\tau(g_{j+1}) = \tau(g_j) - \sum \tau(h_i^{(j)}) \mathrm{in}_{\mathbf{w}}(f_i) = \tau(g_j) - \tau(\mathrm{in}_{\mathbf{w}}(g_j)) = \tau(g_j - \mathrm{in}_{\mathbf{w}}(g_j))$$

for all j . For a polynomial $p(x) = \sum_{\mathbf{a} \in \mathbb{N}^r} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}} \in K[\mathbf{x}]$, we set $\mathrm{supp}_{\mathbf{x}}(p(x)) := \{\mathbf{a} \mid c_{\mathbf{a}} \neq 0\}$. Let $d := \max\{\mathbf{w} \cdot \mathbf{a} \mid \mathbf{a} \in \mathrm{supp}_{\mathbf{x}}(\tau(g))\}$. Since $\{\mathbf{w} \cdot \mathbf{a} \mid \mathbf{a} \in \mathbb{N}^r, \mathbf{w} \cdot \mathbf{a} \leq d\}$ is a finite set, and $\mathrm{ord}_{\mathbf{w}}(g_{j+1}) > \mathrm{ord}_{\mathbf{w}}(g_j)$ for all j , we have $\mathrm{ord}_{\mathbf{w}}(g_m) > d$ for some $m \in \mathbb{N}$. Thus $\mathbf{w} \cdot \mathbf{a} > d$ for any $\mathbf{a} \in \mathrm{supp}_{\mathbf{x}}(\tau(g_m))$. On the other hand, $\mathrm{supp}_{\mathbf{x}}(\tau(g_{j+1})) \subset \mathrm{supp}_{\mathbf{x}}(\tau(g_j))$ for all j since $\tau(g_{j+1}) = \tau(g_j - \mathrm{in}_{\mathbf{w}}(g_j))$. Thus $\mathbf{w} \cdot \mathbf{a} \leq d$ for any $\mathbf{a} \in \mathrm{supp}_{\mathbf{x}}(\tau(g_m)) \subset \mathrm{supp}_{\mathbf{x}}(\tau(g))$. This is possible only if $\mathrm{supp}_{\mathbf{x}}(\tau(g_m)) = \emptyset$, that is, $\tau(g_m) = 0$. Since $\tau(g_j) - \tau(g_{j+1}) = \sum \tau(h_i^{(j)}) \mathrm{in}_{\mathbf{w}}(f_i) \in \mathrm{in}_{\mathbf{w}}(I)$, we conclude that $\tau(g) = \sum_{i=0}^{m-1} (\tau(g_j) - \tau(g_{j+1})) \in \mathrm{in}_{\mathbf{w}}(I)$. Therefore, $\tau(J \cap T_r^\circ) \subset \mathrm{in}_{\mathbf{w}}(I)$. $\square$

From now on, we write $\mathcal{M} = \mathbb{K}^{\circ\circ}$ for simplicity of notation. For $\boldsymbol{\xi} = (\xi_1, \dots, \xi_r) \in \mathcal{M}^{\oplus r}$ and $f(\mathbf{x}) \in K[\mathbf{x}]$, the substitution $f(\boldsymbol{\xi}) \in \mathbb{K}^\circ$ and the subring $K[\boldsymbol{\xi}] \subset \mathbb{K}^\circ$ are well-defined. Hence we can define the following.

Definition 2.10. Let $I \subset K[\mathbf{x}]$ be a subset (e.g. an ideal of $K[\mathbf{x}]$, or $k[\mathbf{x}]$). We set

$$V_{\mathcal{M}}(I) = \{\boldsymbol{\xi} = (\xi_1, \dots, \xi_r) \in \mathcal{M}^{\oplus r} \mid f(\boldsymbol{\xi}) = 0 \text{ for all } f \in I\}.$$

Theorem 2.11. *Let $(w_1, \dots, w_r) \in \mathbb{R}_+^r$ and $(\alpha_1, \dots, \alpha_r) \in V_K(\text{in}_{\mathbf{w}}(I))$. Then there exists $\xi = (\xi_1, \dots, \xi_r) \in V_{\mathcal{M}}(I)$ such that $\text{in}_t(\xi_i) = \alpha_i t^{w_i}$ if $\alpha_i \neq 0$, and $\text{ord}_t(\xi_i) > w_i$ otherwise.*

Proof. Let $\psi_{\mathbf{w}} : K[[\mathbf{x}]] \rightarrow T_r, f(x_1, \dots, x_r) \mapsto f(t^{w_1}x_1, \dots, t^{w_r}x_r)$, and $J = \langle \psi_{\mathbf{w}}(I) \rangle \subset T_r$. By Proposition 2.7 and Proposition 2.9, the map $\text{Sp}_{\mathbb{K}}(T_r/J) \rightarrow \text{Sp}_K(K[\mathbf{x}]/\text{in}_{\mathbf{w}}(I)), \gamma \mapsto (x_i \mapsto \tau(\gamma(x_i)))$, is surjective. Hence there exists $\eta = (\eta_1, \dots, \eta_r) \in (\mathbb{K}^\circ)^{\oplus r}$ such that $\tau(\eta_i) = \alpha_i$. Let $\xi = (t^{w_1}\eta_1, \dots, t^{w_r}\eta_r) \in \mathcal{M}^{\oplus r}$. Then ξ satisfies the desired conditions. $\square$

Remark 2.12. One can prove the same result as Theorem 2.11 for $\mathbf{w} \in \mathbb{Q}_+^r$ replacing $\mathbb{K} = K((t^{\mathbb{R}}))$ with $K((t^{\mathbb{Q}}))$ (or $K((t^*)) = \bigcup_{n \in \mathbb{N}_+} K((t^{1/n}))$ if $\text{char } K = 0$).

The next theorem is an analogue of the the fundamental theorem of tropical geometry.

Theorem 2.13. *Let $I \subset K[[\mathbf{x}]]$ be an ideal. Then*

$$\mathcal{T}_{\text{loc}}(I) = \text{ord}_t(V_{\mathcal{M}}(I)).$$

Proof. There is a natural one-to-one correspondence between $\mathcal{T}_{\text{loc}}(I + \langle x_i \rangle / \langle x_i \rangle) \subset \widetilde{\mathbb{R}}_+^{r-1}$ (resp. $V_{\mathcal{M}}(I + \langle x_i \rangle / \langle x_i \rangle)$) and $\{(w_1, \dots, w_r) \in \mathcal{T}_{\text{loc}}(I) \mid w_i = \infty\}$ (resp. $\{(\xi_1, \dots, \xi_r) \in V_{\mathcal{M}}(I) \mid \xi_i = 0\}$). Hence it is enough to show that $\mathcal{T}_{\text{loc}}(I) \cap \mathbb{R}_+^r = \text{ord}_t(V_{\mathcal{M}}(I)) \cap \mathbb{R}_+^r$.

First, we prove that $\mathcal{T}_{\text{loc}}(I) \supset \text{ord}_t(V_{\mathcal{M}}(I))$. Let $\xi = (\xi_1, \dots, \xi_r) \in V_{\mathcal{M}}(I)$ with $\xi_i \neq 0$ for all i , and set $\mathbf{w} = \text{ord}(\xi)$. Let $f \in I$, and $f_0 = \text{in}_{\mathbf{w}}(f) \in K[\mathbf{x}]$. Since the lowest order terms appearing in the expansion of $f(\xi)$ is $f_0(\text{in}_t(\xi))$ which should also be zero. As $\text{in}_t(\xi_i)$'s are monomials, $f_0(\text{in}_t(\xi)) = 0$ is possible only if f_0 is not a monomial. Thus $\mathbf{w} \in \mathcal{T}_{\text{loc}}(I)$.

To prove the converse inclusion, take $\mathbf{w} = (w_1, \dots, w_r) \in \mathcal{T}_{\text{loc}}(I) \cap \mathbb{R}_+^r$. As $\text{in}_{\mathbf{w}}(I)$ contains no monomial, there exists $\alpha \in V_K(\text{in}_{\mathbf{w}}(I)) \cap (K^\times)^r$. Thus, by Theorem 2.11, there exists $\xi = (\xi_1, \dots, \xi_r) \in V_{\mathcal{M}}(I)$ such that $\mathbf{w} = \text{ord}_t(\xi) \in \text{ord}_t(V_{\mathcal{M}}(I))$. $\square$

3. IRREDUCIBILITY CRITERION FOR ALGEBROID CURVES

In this section, we give an irreducibility criterion for algebroid curves over the perfect field k in terms of local tropical varieties.

Definition 3.1. Let $I \subset k[[\mathbf{x}]]$ be an equidimensional ideal of dimension one, and $f \in k[[\mathbf{x}]]$. We define the *intersection number* of f and I as

$$\text{int}(f; I) = \ell_{k[[\mathbf{x}]]}(k[[\mathbf{x}]]/(I + \langle f \rangle)) = \dim_k k[[\mathbf{x}]]/(I + \langle f \rangle) \in \widetilde{\mathbb{N}}_+ \cup \{0\},$$

where $\ell_R(M)$ denotes the length of an R -module M . We set

$$\omega(I) = (\text{int}(x_1; I), \dots, \text{int}(x_r; I)) \in \widetilde{\mathbb{N}}_+^r.$$

As $A = k[[\mathbf{x}]]/I$ is Cohen-Macaulay, $\text{int}(f; I)$ coincides with the multiplicity $e(f; A)$. Since $\dim_k k[[\mathbf{x}]]/\langle I, f \rangle = \dim_k k[\mathbf{x}]/\text{in}_{\mathbf{w}}(\langle I, f \rangle) = \dim_k k[\mathbf{x}]/\text{in}_{\prec}(\langle I, f \rangle)$ for any $\mathbf{w} \in \mathbb{R}_+^r$, and any local ordering $\prec$, one can compute $\text{int}(f; I)$ by computing a standard basis of $\langle I, f \rangle$.

Lemma 3.2. *Let $I \subset k[[\mathbf{x}]]$ be an equidimensional ideal of dimension one, and set $A = k[[\mathbf{x}]]/I$. Then $\text{int}(f; I) = \sum_{\mathfrak{p} \in \text{Ass } I} \ell(A_{\mathfrak{p}}) \cdot \text{int}(f; \mathfrak{p})$ where $\text{Ass } I$ is the set of the associated primes of I . In particular, $\omega(I) = \sum_{\mathfrak{p} \in \text{Ass } I} \ell(A_{\mathfrak{p}}) \cdot \omega(\mathfrak{p})$.*

Proof. The assertion follows from the multiplicity formula

$$e(Q; R) = \sum_{\mathfrak{p} \in \text{Ass } R} \ell(R_{\mathfrak{p}}) \cdot e(Q; R/\mathfrak{p})$$

where $(R, \mathfrak{m})$ is a local ring, and Q is an $\mathfrak{m}$ -primary ideal (see e.g. [18], Corollary 23.4). $\square$

Lemma 3.3. *Let A be an irreducible algebroid curve over k . Let $B = k'[[s]]$ be the integral closure of A with the coefficient field k' . For $\eta \in A$, $\dim_k A/\eta A = [k' : k] \cdot \text{ord}_s(\eta)$.*

Proof. As $0 \rightarrow A \rightarrow B \rightarrow B/A \rightarrow 0$ is an exact sequence, and B/A is an A -module of finite length, $\dim_k A/\eta A = e_A(\eta; A) = e_A(\eta; B) = \dim_k B/\eta B = [k' : k] \cdot \text{ord}_s(\eta)$. $\square$

Corollary 3.4. *Let $I \subset k[[\mathbf{x}]]$ be an equidimensional ideal of dimension one, and $f, g \in k[[\mathbf{x}]]$. Then $\text{int}(fg; I) = \text{int}(f; I) + \text{int}(g; I)$. In particular, $\text{int}(\mathbf{x}^a; I) = \mathbf{a} \cdot \boldsymbol{\omega}(I)$.*

Corollary 3.4 is useful for computing intersection numbers by hand.

Example 3.5. Let $I = \langle (y^2 - xz)^d - x^a y^b z^c, x^3 - z^2 - y^3 \rangle \subset K[[x, y, z]]$ where $d \in \mathbb{N}_+$, and $a, b, c \in \mathbb{N}$. Then $\text{int}(y^2 - xz; I) = \dim_K K[[x, y, z]] / \langle y^2 - xz, x^a y^b z^c, x^3 - z^2 - y^3 \rangle = \text{int}(x^a y^b z^c; \langle y^2 - xz, x^3 - z^2 - y^3 \rangle) = \boldsymbol{\omega}(\langle y^2 - xz, x^3 - z^2 - y^3 \rangle) \cdot (a, b, c) = (4, 5, 6) \cdot (a, b, c) = 4a + 5b + 6c$.

In the tropical algebraic geometry, a tropical variety defined by a one-dimensional prime ideal with constant coefficient is a finite union of rays by Bieri–Groves Theorem ([3] Theorem A, [22] Theorem 9.6). We will prove that a stronger result holds for local tropical varieties; the local tropical variety defined by a one-dimensional prime ideal consists of a single ray.

Theorem 3.6. *Let $\mathfrak{p} \subset k[[\mathbf{x}]]$ be a one-dimensional prime ideal. Then*

$$\mathcal{T}_{\text{loc}}(\mathfrak{p}) = \text{ord}_t(V_{\mathcal{M}}(\mathfrak{p})) = \widetilde{\mathbb{R}}_+ \cdot \boldsymbol{\omega}(\mathfrak{p}).$$

Proof. As $\mathcal{T}_{\text{loc}}(\mathfrak{p}) = \mathcal{T}_{\text{loc}}(\mathfrak{p}K[[\mathbf{x}]])$ and by Theorem 2.13, $\mathcal{T}_{\text{loc}}(\mathfrak{p}) = \text{ord}_t(V_{\mathcal{M}}(\mathfrak{p}))$ is satisfied.

We will prove that $\boldsymbol{\omega}(\mathfrak{p}) \in \text{ord}_t(V_{\mathcal{M}}(\mathfrak{p}))$. Let $A = k[[\mathbf{x}]]/\mathfrak{p}$, and B the integral closure of A . Then $B = k'[[s]]$ where $k' \subset K$ is the coefficient field of B . Since k is perfect, we may assume that the inclusion $k \subset k'$ holds under the inclusion $A \subset B$. Let $\xi_i(s) \in k'[[s]]$ be the image of x_i in $A \subset B$, and set $\boldsymbol{\xi} = (\xi_1(t), \dots, \xi_r(t))$. Since $\mathfrak{p}$ is the kernel of the k -algebra homomorphism $k[[\mathbf{x}]] \rightarrow k[[\xi_1(t), \dots, \xi_r(t)]] \cong A$, $f(\mathbf{x}) \mapsto f(\boldsymbol{\xi})$, we have $\boldsymbol{\xi} \in V_{\mathcal{M}}(\mathfrak{p})$. By Lemma 3.3, $\boldsymbol{\omega}(\mathfrak{p}) = [k' : k] \cdot \text{ord}_t(\boldsymbol{\xi}) \in \text{ord}_t(V_{\mathcal{M}}(\mathfrak{p}))$.

To complete the proof, it is enough to show that $\text{ord}_t(V_{\mathcal{M}}(\mathfrak{p})) \subset \widetilde{\mathbb{R}}_+ \cdot \boldsymbol{\omega}(\mathfrak{p})$. Take $(0, \dots, 0) \neq \boldsymbol{\xi} = (\xi_1, \dots, \xi_r) \in V_{\mathcal{M}}(\mathfrak{p})$. Let $A = k[[\xi_1, \dots, \xi_r]] \subset \mathbb{K}^\circ$ and $\phi : k[[\mathbf{x}]] \rightarrow A$, $f(\mathbf{x}) \mapsto f(\boldsymbol{\xi})$. As $\boldsymbol{\xi} \in V_{\mathcal{M}}(\mathfrak{p})$, we have $\mathfrak{p} \subset \text{Ker } \phi$. Since $\mathfrak{p}$ is a one-dimensional prime ideal and A is a domain, $\text{Ker } \phi$ cannot be larger than $\mathfrak{p}$. Thus $\text{Ker } \phi = \mathfrak{p}$ and $A \cong k[[\mathbf{x}]]/\mathfrak{p}$. In particular, A is of dimension one. Let $B = k'[[s]]$ be the integral closure of A . Then $\boldsymbol{\omega}(\mathfrak{p}) = [k' : k] \cdot \text{ord}_s(\boldsymbol{\xi})$ by Lemma 3.3. As $\mathbb{K}^\circ$ is integrally closed, we may assume that $B \subset \mathbb{K}^\circ$. Since $0 \neq \beta \in k'$ is invertible also in $\mathbb{K}^\circ$, we have $\text{ord}_t(\xi) = \text{ord}_t(s) \cdot \text{ord}_s(\xi)$ for $\xi \in B$. Therefore $\text{ord}_t(\boldsymbol{\xi}) = \text{ord}_t(s) \cdot \text{ord}_s(\boldsymbol{\xi}) = \text{ord}_t(s) \cdot [k' : k]^{-1} \cdot \boldsymbol{\omega}(\mathfrak{p}) \in \mathbb{R}_+ \cdot \boldsymbol{\omega}(\mathfrak{p})$. $\square$

Using Theorem 3.6, we obtain an irreducibility criterion for algebroid curves in terms of local tropical variety.

Theorem 3.7. *Let $I \subset k[[x]]$ be an equidimensional ideal of dimension one. Then the following hold.*

- (1) *If $\mathcal{T}_{\text{loc}}(I) \neq \widetilde{\mathbb{R}}_+ \cdot \omega(I)$ (e.g. $\text{in}_{\omega(I)}(I)$ contains monomials, or I has at least two tropisms), then I is not prime.*
- (2) *If $\omega(I)$ is a tropism of I , then I is prime.*
- (3) *Let $\{\mathbf{w}_1, \dots, \mathbf{w}_l\} \subset \widetilde{\mathbb{N}}_+^r$ be the set of the tropisms of I . If $\omega(I) \neq (\infty, \dots, \infty)$ and $\omega(I) = \mathbf{w}_1 + \dots + \mathbf{w}_l$, then there exist prime ideals $\mathfrak{p}_j \subset k[[x]]$ for $1 \leq j \leq l$ such that $\omega(\mathfrak{p}_j) = \mathbf{w}_j$ and $I = \bigcap_{j=1}^l \mathfrak{p}_j$.*

Proof. Recall that, by Lemma 3.2, $\omega(I) = \sum_{\mathfrak{p} \in \text{Ass } I} \ell(A_{\mathfrak{p}}) \cdot \omega(\mathfrak{p})$.

(1) The assertion follows immediately from Theorem 3.6.

(2) Since $\omega(I) \in \mathcal{T}_{\text{loc}}(I) = \bigcup_{\mathfrak{p} \in \text{Ass } I} \mathbb{R}_+ \cdot \omega(\mathfrak{p})$, there exist $\mathfrak{p}_0 \in \text{Ass } I$ such that $\omega(I) = \gcd(\omega(\mathfrak{p}_0))^{-1} \cdot \omega(\mathfrak{p}_0)$. By Lemma 3.2, $\gcd(\omega(\mathfrak{p}_0))^{-1} \cdot \omega(\mathfrak{p}_0) = \sum_{\mathfrak{p} \in \text{Ass } I} \ell(A_{\mathfrak{p}}) \cdot \omega(\mathfrak{p})$. Since $\gcd(\omega(\mathfrak{p}_0))^{-1} \leq 1$ and $\ell(A_{\mathfrak{p}_0}) \geq 1$, this equality is possible only if $\text{Ass } I = \{\mathfrak{p}_0\}$ and $\ell(A_{\mathfrak{p}_0}) = 1$. This shows that $I = \mathfrak{p}_0$.

(3) Since $\mathbf{w}_j \in \mathcal{T}_{\text{loc}}(I) = \bigcup_{\mathfrak{p} \in \text{Ass } I} \widetilde{\mathbb{R}}_+ \cdot \omega(\mathfrak{p}_j)$ and $\gcd(\mathbf{w}_j) = 1$, there exists $\mathfrak{p}_j \in \text{Ass } I$ such that $\mathbf{w}_j = \gcd(\omega(\mathfrak{p}_j))^{-1} \cdot \omega(\mathfrak{p}_j)$. Hence the equality $\sum_{\mathfrak{p} \in \text{Ass } I} \ell(A_{\mathfrak{p}}) \cdot \omega(\mathfrak{p}) = \omega(I) = \sum_{j=1}^l \gcd(\omega(\mathfrak{p}_j))^{-1} \cdot \omega(\mathfrak{p}_j)$ is possible only if $\text{Ass } I = \{\mathfrak{p}_1, \dots, \mathfrak{p}_l\}$ and $\ell(A_{\mathfrak{p}_j}) = \gcd(\omega(\mathfrak{p}_j)) = 1$ for all j . Thus $I = \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_l$, and $\mathbf{w}_j = \omega(\mathfrak{p}_j)$ for all j . This proves the assertion. $\square$

The next well-known fact and Corollary 3.4 is useful for computing examples by hand.

Lemma 3.8. *Let $\mathbf{w} \in \mathbb{R}_+^r$ and $f_1, \dots, f_n \in k[[x]]$. If $\text{in}_{\mathbf{w}}(f_1), \dots, \text{in}_{\mathbf{w}}(f_n)$ is a regular sequence, then $\text{in}_{\mathbf{w}}(\langle f_1, \dots, f_n \rangle) = \langle \text{in}_{\mathbf{w}}(f_1), \dots, \text{in}_{\mathbf{w}}(f_n) \rangle$.*

We will present some examples of Theorem 3.7.

Example 3.9 ([1] Kuo's Example). Let $F(x, y) = (y^2 - x^3)^2 - x^7 \in k[[x, y]]$. Let $J = \langle F, z - (y^2 - x^3 - x^2y) \rangle \subset k[[x, y, z]]$. Then $\omega(J) = (4, 6, 15)$. Note that F is irreducible if and only if J is prime. If $\text{char}(k) \neq 2$, then $G = 2x^2yz + z^2 + x^6y + x^4z \in J$ and $\text{in}_{(4,6,15)}(G) = 2x^2yz$ is a monomial. Hence J is not prime, and thus F is reducible. If $\text{char}(k) = 2$, then $\omega(J)$ is a tropism of J as $\text{in}_{(4,6,15)}(J) = \langle x^3 - y^2, y^5 - z^2 \rangle$ contains no monomial. Hence J is prime, and thus F is irreducible.

Example 3.10. Let $F(x, y) = y^2 + x^3 + xy \in k[[x, y]]$. Then $\mathcal{T}_{\text{loc}}(F) = \widetilde{\mathbb{R}}_+ \cdot (1, 1) \cup \widetilde{\mathbb{R}}_+ \cdot (1, 2)$. As $\omega(F) = (2, 3) = (1, 1) + (1, 2)$, F is factored as $F = F_1 F_2$ with $\omega(F_1) = (1, 1)$ and $\omega(F_2) = (1, 2)$. For example, if $k = \mathbb{Q}$, $F(x, y) = (y + \frac{x}{2} + \frac{x}{2}\sqrt{1-4x})(y + \frac{x}{2} - \frac{x}{2}\sqrt{1-4x})$, and if $k = \mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$, $F(x, y) = (y + x + \sum_{i=0}^{\infty} x^{2^i+1})(y + \sum_{i=0}^{\infty} x^{2^i+1})$.

Example 3.11. Let $I = I_2 \begin{pmatrix} x^3 + y^2 & y & z \\ z^2 & x & y \end{pmatrix} = \langle (x^3 + y^2)x - yz^2, y^2 - xz, z^3 - (x^3 + y^2)y \rangle \subset k[[x, y, z]]$. Then $\omega(I) = (5, 6, 7)$. Since $\text{in}_{(5,6,7)}((x^3 + y^2)x - yz^2) = xy^2$ is a monomial, $\omega(I) \notin \mathcal{T}_{\text{loc}}(I)$, and thus I is not prime.

Example 3.12. Let $I = \langle x^3 - y^2 - z^2, (z^2 - x^2y)^2 - x^3y^2z \rangle \subset k[[x, y, z]]$. Then $\omega(I) = (8, 12, 14)$ and $\text{in}_{\omega(I)}(I) = \langle x^3 - y^2, (z^2 - x^2y)^2 \rangle$. Let $J = \langle I, u - (z^2 - x^2y) \rangle \subset k[[x, y, z, u]]$. Then $\omega(J) = (8, 12, 10, 31)$, and $\gcd(\omega(J)) = 1$. As $\text{in}_{\omega(J)}(J) = \langle x^3 - y^2, u^2 - x^3y^2z, z^2 - xy^2 \rangle$ contains no monomial, $\omega(J)$ is a tropism of J . Hence J is prime, and thus I is also prime.

Example 3.13. Let $a, b, c \in \mathbb{N}_+$ such that $a < b$, $a < c$, and $\gcd(a, b) = \gcd(a, c) = 1$. Let $F_1 = x^a + y^b + z^c$, $F_2 = xy + yz + zx$, and $I = \langle F_1, F_2 \rangle \subset k[[x, y, z]]$. Then $\omega(I) = (b + c, c + a, a + b)$, and $\mathcal{T}_{\text{loc}}(I) = \widetilde{\mathbb{R}}_+ \cdot (c, c, a) \cup \mathbb{R}_+ \cdot (b, a, b)$. Since $\omega(I) = (c, c, a) + (b, a, b)$, there exist one-dimensional prime ideals $\mathfrak{p}_1$ and $\mathfrak{p}_2$ such that $\omega(\mathfrak{p}_1) = (c, c, a)$, $\omega(\mathfrak{p}) = (b, a, b)$, and $I = \mathfrak{p}_1 \cap \mathfrak{p}_2$.

Let $I \subset K[[\mathbf{x}]]$ be a one-dimensional radical ideal. In the last section, we will give an algorithm for computing $J \subset K[[x_1, \dots, x_{r'}]]$ such that $K[[\mathbf{x}]]/I \cong K[[x_1, \dots, x_{r'}]]/J$ and the irreducibility of $K[[x_1, \dots, x_{r'}]]/J$ can be determined by Theorem 3.7 (1) or (2). In case where I is a prime ideal, this can be done by computing the value-semigroup of $K[[\mathbf{x}]]/I$ (Definition 4.1). If I is prime, there exists $f \in K[[\mathbf{x}]]$ such that $\gcd(\omega(I), \text{int}(f; I)) = 1$. Then $J := \langle I, x_{r+1} - f \rangle \subset K[[x_1, \dots, x_{r+1}]]$ satisfies $K[[\mathbf{x}]]/I \cong K[[x_1, \dots, x_{r+1}]]/J$ and $\gcd(\omega(J)) = 1$. As J is prime, $\omega(J)$ should be a tropism of J .

4. VALUE-SEMGROUPS OF IRREDUCIBLE ALGEBROID CURVES

In the rest of this paper, we consider only algebroid curves over the algebraically closed field K .

In [13], Hefez and Hernandes gave an algorithm for computing the value-semigroup of an irreducible algebroid curve A with expression $A = K[[\xi_1, \dots, \xi_r]] \subset K[[t]]$ as a subring of $K[[t]]$ ([13], Algorithm 3.2). They also gave an algorithm for computing the value-semigroup of A with expression $A = K[[\mathbf{x}]]/\mathfrak{p}$ as a quotient ring of $K[[\mathbf{x}]]$ ([13] before Example 3.4). In this section, we will present an improved version of their algorithm for computing the value-semigroup of $K[[\mathbf{x}]]/\mathfrak{p}$.

4.1. Numerical semigroups. We call a subsemigroup of $\mathbb{N}$ a *numerical semigroup*.

Definition 4.1. (1) Let A be an irreducible algebroid curve over K , and $K[[t]]$ the integral closure of A . We set

$$\text{Semi}(A) = \{\text{ord}_t(\eta) \mid 0 \neq \eta \in A\} = \{\ell_A(A/\eta A) \mid 0 \neq \eta \in A\},$$

and call it the *value-semigroup* of A .

(2) For $\mathbf{w} = (w_1, \dots, w_r) \in \mathbb{N}_+^r$, we set $\text{Span}_{\mathbb{N}}(\mathbf{w}) = \sum_{i=1}^r \mathbb{N} \cdot w_i$.

Note that

$$\text{Span}_{\mathbb{N}}(\omega(\mathfrak{p})) \subset \text{Semi}(K[[\mathbf{x}]]/\mathfrak{p}) = \{\text{int}(f; \mathfrak{p}) \mid f \in K[[\mathbf{x}]], f \notin \mathfrak{p}\}$$

for a one-dimensional prime ideal $\mathfrak{p} \subset K[[\mathbf{x}]]$. By Corollary 3.4, $\text{Semi}(A)$ is actually a semigroup. As t is in the fractional field of A , it follows that $\gcd(\text{Semi}(A)) = 1$. The value-semigroup of an irreducible algebroid curve A is deeply related to the singularity of A . It is known that A is Gorenstein if and only if $\text{Semi}(A)$ is symmetric ([15]). In the case where $\text{char}(k) = 0$, for an irreducible bivariate power series $F \in K[[x, y]]$, the conductor of $\text{Semi}(K[[x, y]]/\langle F \rangle)$ coincides with the Milnor number of F . The conductor of a numerical semigroup H is the smallest integer c such that all integers n satisfying $n \geq c$ belong to H .

Definition 4.2. For a positive integer vector $\mathbf{w} = (w_1, \dots, w_r) \in \mathbb{N}_+^r$, we denote by

$$\text{Prim}(\mathbf{w}) = \langle \mathbf{x}^{\mathbf{a}} - \mathbf{x}^{\mathbf{b}} \mid \mathbf{a}, \mathbf{b} \in \mathbb{N}^r, \mathbf{a} \cdot \mathbf{w} = \mathbf{b} \cdot \mathbf{w} \rangle \subset K[\mathbf{x}],$$

the kernel of the K -algebra homomorphism $K[\mathbf{x}] \rightarrow K[t]$, $x_i \mapsto t^{w_i}$.

The ideal $\text{Prim}(\mathbf{w})$ is a $\mathbf{w}$ -homogeneous prime ideal. If $\text{Span}_{\mathbb{N}}(\mathbf{w})$ is a complete intersection, a system of generators of $\text{Prim}(\mathbf{w})$ is given in [7].

Lemma 4.3. For a $\mathbf{w}$ -homogeneous ideal $J \subset \text{Prim}(\mathbf{w})$, J coincides with $\text{Prim}(\mathbf{w})$ if and only if $\dim_K K[\mathbf{x}]/\langle x_i, J \rangle = w_i$ for some i .

Proof. Similarly to Lemma 3.3, $\dim_K K[\mathbf{x}]/\langle x_i, \text{Prim}(\mathbf{w}) \rangle = w_i$. Consider the natural surjection $K[\mathbf{x}]/J \rightarrow K[\mathbf{x}]/\text{Prim}(\mathbf{w})$. If $\dim_K K[\mathbf{x}]/\langle x_i, J \rangle = w_i$, then $K[\mathbf{x}]/\langle x_i, J \rangle \rightarrow K[\mathbf{x}]/\langle x_i, \text{Prim}(\mathbf{w}) \rangle$ is an isomorphism. Since x_i is a regular element on $K[\mathbf{x}]/\text{Prim}(\mathbf{w})$, $K[\mathbf{x}]/J \rightarrow K[\mathbf{x}]/\text{Prim}(\mathbf{w})$ is also an isomorphism by Nakayama's Lemma. $\square$

Example 4.4. Let $a, b, d \in \mathbb{N}_+$, $c_1, c_2 \in \mathbb{N}$ such that $\gcd(a, b) = 1$ and $\gcd(d, c_1a + c_2b) = 1$. Let $J = \langle x^b - y^a, z^d - x^{c_1}y^{c_2} \rangle \subset K[x, y, z]$. Since $J \subset \text{Prim}(da, db, c_1a + c_2b)$ and $\dim_K K[x, y, z]/\langle x, J \rangle = \dim_K K[x, y, z]/\langle x, y^a, z^d \rangle = da$, then $J = \text{Prim}(da, db, c_1a + c_2b)$.

The next lemma follows from the elimination theory using Gröbner basis (see [5], [6]).

Lemma 4.5. Let $\mathbf{w} = (w_1, \dots, w_r) \in \mathbb{N}^r$, and $N \in \mathbb{N}$. Let $\prec$ be a global ordering on $K[\mathbf{x}, t]$ such that $u \prec v$ for any two monomials $u \in K[\mathbf{x}]$ and $v \in K[\mathbf{x}, t] \setminus K[\mathbf{x}]$. Let G be a Gröbner basis of $\langle x_i - t^{w_i} \mid 1 \leq i \leq r \rangle \subset K[t, \mathbf{x}]$ with respect to $\prec$. Then the following hold.

- (1) $G \cap K[\mathbf{x}]$ is a Gröbner basis of $\text{Prim}(\mathbf{w})$ with respect to $\prec$.
- (2) Let u be the remainder of t^N on the division by G . Then $N \in \text{Span}_{\mathbb{N}}(\mathbf{w})$ if and only if $u \in K[\mathbf{x}]$. Furthermore, if this is the case, $u = \mathbf{x}^{\mathbf{a}}$ where $\mathbf{a} \in \mathbb{N}^r$ with $N = \mathbf{a} \cdot \mathbf{w}$.

The ascending chain condition holds for numerical semigroups.

Lemma 4.6. Let $H_1 \subset H_2 \subset \dots$ be a sequence of numerical semigroups. Then there exists $i_0 \in \mathbb{N}$ such that $H_i = H_{i_0}$ for all $i \geq i_0$.

Proof. Let $\preceq_{lex}$ be the lexicographic order on $\mathbb{N}^2$, that is, $(a_1, a_2) \prec_{lex} (b_1, b_2)$ if and only if $a_1 < b_1$, or $a_1 = b_1$ and $a_2 < b_2$. Set $m_i = \gcd(H_i)$, and $n_i = \#((\mathbb{N} \cdot m_i) \setminus H_i) \in \mathbb{N}$. It is easy to see that $(m_{i+1}, n_{i+1}) \prec_{lex} (m_i, n_i)$ for all i . Since $\preceq_{lex}$ is a well-ordering, there exists $i_0 \in \mathbb{N}$ such that $(m_i, n_i) = (m_{i_0}, n_{i_0})$ for all $i \geq i_0$. Therefore $H_i = H_{i_0}$ for all $i \geq i_0$. $\square$

4.2. Local SAGBI bases. In this section, we define local SAGBI basis which is a variant of SAGBI basis [19] in local rings. This notion is introduced by Hefez and Hernandez [13], and they call it standard basis. See also [21], Chapter 11 for SAGBI bases.

Let A be an algebroid curve over K . Since K is algebraically closed, the integral closure of A is isomorphic to $K[[t]]$. Let $\boldsymbol{\xi} = (\xi_1, \dots, \xi_r)$ such that $0 \neq \xi_i \in tK[[t]]$ for all i , and $A = K[[\boldsymbol{\xi}]]$. We set

$$\begin{aligned} \phi_{\boldsymbol{\xi}} : K[\mathbf{x}] &\rightarrow K[[t]], & f(\mathbf{x}) &\mapsto f(\boldsymbol{\xi}), \\ \phi_{\text{in}_t(\boldsymbol{\xi})} : K[\mathbf{x}] &\rightarrow K[t], & g(\mathbf{x}) &\mapsto g(\text{in}_t(\boldsymbol{\xi})). \end{aligned}$$

Note that $K[[\mathbf{x}]]$ and $K[[t]]$ are equipped with topologies defined by the filtrations given by powers of their maximal ideals, and ϕ_{ξ} is continuous with respect to these topologies.

Definition 4.7. We call $K[\text{in}_t(A)] := K[\text{in}_t(\eta) \mid \eta \in A] = K[t^i \mid i \in \text{Semi}(A)]$ the *initial algebra* of A . We say that ξ is a *local SAGBI basis* of A if $K[\text{in}_t(A)] = K[\text{in}_t(\xi)]$, in other words, $\text{Semi}(A) = \text{Span}_{\mathbb{N}}(\text{ord}_t(\xi))$.

An algorithm for computing local SAGBI bases is given in [13], Algorithm 3.2.

Remark 4.8. It holds that $K[\text{in}_t(\xi)] \subset K[\text{in}_t(A)]$, but the equality does not hold in general. Since $\text{Semi}(A)$ is finitely generated as a semigroup, there exists a finite local SAGBI basis of A . The assumption that A is of dimension one is essential for the existence of finite local SAGBI bases.

Proposition-Definition 4.9 (local reduction, see [13] Section 2). *Let $\eta \in K[[t]]$. We set $\mathbf{w} = \text{ord}_t(\xi)$. Then there exist $q \in K[[\mathbf{x}]]$ and $\zeta \in K[[\xi]]$ satisfying the following:*

- (1) $\eta = q(\xi) + \zeta$.
- (2) $\text{in}_t(\zeta) \notin K[\text{in}_t(\xi)]$ if $\zeta \neq 0$.
- (3) $\text{ord}_t(\eta) = \text{ord}_{\mathbf{w}}(q)$ if $q \neq 0$.

We call q a *quotient*, and ζ a *remainder* of η on local reduction by ξ .

Proof. We define $\eta_i \in K[[\xi]]$, $\alpha_i \in K$, and $\mathbf{c}_i \in \mathbb{N}^r$ inductively on i in the following manner: Set $\eta_0 = \eta$. If $\text{in}_t(\eta_i) \in K[\text{in}_t(\xi)]$, take $\mathbf{c}_i \in \mathbb{N}^r$ and $\beta_i \in K$ such that $\text{ord}_t(\eta_i) = \text{ord}_t(\xi^{\mathbf{c}_i})$ and $\text{in}_t(\eta_i) = \beta_i \text{in}_t(\xi^{\mathbf{c}_i})$. We set $\eta_{i+1} = \eta_i - \beta_i \xi^{\mathbf{c}_i}$.

If $\eta_m \notin K[\text{in}_t(\xi)]$ or $\eta_m = 0$ for some $m \in \mathbb{N}$, then $q := \sum_{i=0}^{m-1} \beta_i \mathbf{x}^{\mathbf{c}_i}$ and $\zeta := \eta_m$ satisfy the desired conditions. If $0 \neq \eta_i \in K[\text{in}_t(\xi)]$ for all i , then $q := \sum_{i=0}^{\infty} \beta_i \mathbf{x}^{\mathbf{c}_i}$ and $\zeta := 0$ satisfy the desired conditions. $\square$

If $q \neq 0$, then $\text{in}_t(\eta) = \text{in}_{\mathbf{w}}(q)(\text{in}_t(\xi))$ by Proposition-Definition 4.9 (3).

Lemma 4.10. *Let $\mathbf{w} =: \text{ord}_t(\xi)$. Then $\text{in}_{\mathbf{w}}(\text{Ker } \phi_{\xi}) \subset \text{Ker } \phi_{\text{in}_t(\xi)}$.*

Proof. Let $f \in \text{Ker } \phi_{\xi}$, and $f_0 = \text{in}_{\mathbf{w}}(f) \in K[\mathbf{x}]$. Since the lowest order terms appearing in the expansion of $f(\xi)$ is $f_0(\text{in}_t(\xi))$ which should also be zero. Hence $f_0 \in \text{Ker } \phi_{\text{in}_t(\xi)}$. Thus $\text{in}_{\mathbf{w}}(\text{Ker } \phi_{\xi}) \subset \text{Ker } \phi_{\text{in}_t(\xi)}$. $\square$

Furthermore, the following holds.

Theorem 4.11. *Let $\mathbf{w} = (w_1, \dots, w_r) =: \text{ord}_t(\xi)$. Then $\sqrt{\text{in}_{\mathbf{w}}(\text{Ker } \phi_{\xi})} = \text{Ker } \phi_{\text{in}_t(\xi)}$.*

Proof. Set $\mathfrak{p} = \text{Ker } \phi_{\xi}$. Note that $\mathbf{w} = \omega(\mathfrak{p})$ since $K[[t]]$ is the integral closure of $K[[\xi]]$. By Hilbert's Nullstellensatz, it is enough to show that $V_K(\text{in}_{\mathbf{w}}(\mathfrak{p})) \subset V_K(\text{Ker } \phi_{\text{in}_t(\xi)})$. Let $0 \neq (\alpha_1, \dots, \alpha_r) \in V_K(\text{in}_{\mathbf{w}}(\mathfrak{p}))$. Then there exists $\eta_0 = (\eta_1, \dots, \eta_r) \in V_{\mathcal{M}}(\mathfrak{p})$ such that $\text{in}_t(\eta_i) = \alpha_i t^{w_i}$ if $\alpha_i \neq 0$ and $\text{ord}_t(\eta_i) \succ w_i$ otherwise by Theorem 2.11. Since $\text{ord}_t(\eta_0) \in \text{ord}_t(V_{\mathcal{M}}(\mathfrak{p})) = \mathcal{T}_{\text{loc}}(\mathfrak{p}) = \mathbb{R}_+ \cdot \mathbf{w}$ by Theorem 3.6, it should hold that $\alpha_i \neq 0$ for all $1 \leq i \leq r$. For $0 \neq \eta \in V_{\mathcal{M}}(\mathfrak{p})$ we set $P_{\text{in}_t(\eta)} = \{g \in K[\mathbf{x}] \mid g(\text{in}_t(\eta)) = 0\}$. Then $(\alpha_1, \dots, \alpha_r) \in V_K(P_{\text{in}_t(\eta_0)})$ as $\text{in}_t(\eta_0) = (\alpha_1 t^{w_1}, \dots, \alpha_r t^{w_r})$. We will prove that $P_{\text{in}_t(\eta_0)} = \text{Ker } \phi_{\text{in}_t(\xi)}$. Since $\xi \in V_{\mathcal{M}}(\mathfrak{p})$ and $P_{\text{in}_t(\xi)} = \text{Ker } \phi_{\text{in}_t(\xi)}$, it is enough to show that $P_{\text{in}_t(\eta)}$ is independent from the choice of η . Let $K[[s]]$ be the integral closure of $K[[\eta]]$, and $f \in K[[\mathbf{x}]]$. Then we may assume that $K[[s]] \subset \mathbb{K}^\circ$, and it holds that $\text{ord}_t(f(\eta)) = \text{ord}_t(s) \cdot \text{ord}_s(f(\eta)) = \text{ord}_t(s) \cdot \text{int}(f; \mathfrak{p})$, and $\text{ord}_t(\eta) = \text{ord}_t(s) \cdot \text{ord}_s(\eta) = \text{ord}_t(s) \cdot \mathbf{w}$ as in

the proof of Theorem 3.6. The lowest order terms in t appearing in the expansion of $f(\boldsymbol{\eta})$ is $\text{in}_{\mathbf{w}}(f)(\text{in}_t(\boldsymbol{\eta}))$ which is a sum of monomials in t of order $\text{ord}_t(s) \cdot \text{ord}_{\mathbf{w}}(f)$. Thus $\text{in}_{\mathbf{w}}(f)(\text{in}_t(\boldsymbol{\eta})) = 0$ if and only if $\text{ord}_t(f(\boldsymbol{\eta})) > \text{ord}_t(s) \cdot \text{ord}_{\mathbf{w}}(f)$ which is equivalent to saying that $\text{int}(f; \mathfrak{p}) > \text{ord}_{\mathbf{w}}(f)$. Therefore $P_{\text{in}_t(\boldsymbol{\eta})} = \langle g \in K[\mathbf{x}] \mid g \text{ is } \mathbf{w}\text{-homogeneous, } \text{int}(g; \mathfrak{p}) > \deg_{\mathbf{w}}(g) \rangle$. Hence $P_{\text{in}_t(\boldsymbol{\eta})}$ is independent from the choice of $\boldsymbol{\eta} \in V_{\mathcal{M}}(\mathfrak{p})$. Thus we conclude that $(\alpha_1, \dots, \alpha_r) \in V_K(P_{\text{in}_t(\boldsymbol{\eta}_0)}) = V_K(\text{Ker } \phi_{\text{in}_t(\boldsymbol{\xi})})$. This proves $V_K(\text{in}_{\mathbf{w}}(\mathfrak{p})) \subset V_K(\text{Ker } \phi_{\text{in}_t(\boldsymbol{\xi})})$. $\square$

Corollary 4.12. *Let $\mathfrak{p} \subset K[\mathbf{x}]$ be a one-dimensional prime ideal such that $x_i \notin \mathfrak{p}$ for all i , and $f \in K[\mathbf{x}]$. Set $\mathbf{w} = \boldsymbol{\omega}(\mathfrak{p})$. Then $\text{int}(f; I) > \text{ord}_{\mathbf{w}}(f)$ if $\text{in}_{\mathbf{w}}(f) \in \sqrt{\text{in}_{\mathbf{w}}(\mathfrak{p})}$, and $\text{int}(f; I) = \text{ord}_{\mathbf{w}}(f)$ otherwise.*

Proof. Let $K[[t]]$ be the integral closure of $A := K[\mathbf{x}]/\mathfrak{p}$, and write $A = K[[\boldsymbol{\xi}]] \subset K[[t]]$. Then $\mathfrak{p} = \text{Ker } \phi_{\boldsymbol{\xi}}$, and by Theorem 4.11, $\sqrt{\text{in}_{\mathbf{w}}(\mathfrak{p})} = \text{Ker } \phi_{\text{in}_t(\boldsymbol{\xi})}$. Since $\text{int}(f; \mathfrak{p}) = \text{ord}_t(f(\boldsymbol{\xi}))$ by Lemma 3.3, and the lowest order part in t appearing in the expansion of $f(\boldsymbol{\xi})$ is $\text{in}_{\mathbf{w}}(f)(\text{in}_t(\boldsymbol{\xi}))$ which is a sum of monomials in t of order $\text{ord}_{\mathbf{w}}(f)$, we conclude the assertion. $\square$

Proposition 4.13. *Assume that $\boldsymbol{\xi} = (\xi_1, \dots, \xi_r)$ is a local SAGBI basis of A . For $\eta \in K[[t]]$, $\eta \in A$ if and only if some (any) remainder of η on local reduction by $\boldsymbol{\xi}$ is zero. In particular, if $\boldsymbol{\xi}$ is a local SAGBI basis of A , then $K[[\boldsymbol{\xi}]] = A$.*

Proof. The assertion follows obviously from the definition of local SAGBI bases and remainders of local reduction. $\square$

Lemma 4.14. *Let $\mathbf{w} = \text{ord}_t(\boldsymbol{\xi})$, and $f \in \text{Ker}(\phi_{\text{in}_t(\boldsymbol{\xi})})$ a $\mathbf{w}$ -homogeneous polynomial. Assume that one can take zero as a remainder of $f(\boldsymbol{\xi})$ with the quotient g on local reduction by $\boldsymbol{\xi}$. Then $f - g \in \text{Ker } \phi_{\boldsymbol{\xi}}$ and $f \in \text{in}_{\mathbf{w}}(\text{Ker } \phi_{\boldsymbol{\xi}})$.*

Proof. By Proposition-Definition 4.9 (1), $f(\boldsymbol{\xi}) = g(\boldsymbol{\xi})$ and thus $f - g \in \text{Ker } \phi_{\boldsymbol{\xi}}$. By Proposition-Definition 4.9 (3) and $f \in \text{Ker}(\phi_{\text{in}_t(\boldsymbol{\xi})})$, we have $\text{ord}_{\mathbf{w}}(g) = \text{ord}_t(f(\boldsymbol{\xi})) > \text{ord}_{\mathbf{w}}(f)$. Hence $f = \text{in}_{\mathbf{w}}(f - g) \in \text{in}_{\mathbf{w}}(\text{Ker } \phi_{\boldsymbol{\xi}})$. $\square$

Similarly to SAGBI bases ([21], Theorem 11.4), the following holds.

Theorem 4.15. *Let $A = K[[\boldsymbol{\xi}]] \subset K[[t]]$ be an irreducible algebroid curve over K , and set $\mathbf{w} = \text{ord}_t(\boldsymbol{\xi})$. Let G be a system of binomial generators of $\text{Ker}(\phi_{\text{in}_t(\boldsymbol{\xi})})$. Then the following are equivalent:*

- (1) $\boldsymbol{\xi}$ is a local SAGBI basis of A .
- (2) Any remainder of $f(\boldsymbol{\xi})$ on local reduction by $\boldsymbol{\xi}$ is zero for all $f \in K[[\mathbf{x}]]$.
- (3) One can take zero as a remainder of $f(\boldsymbol{\xi})$ on local reduction by $\boldsymbol{\xi}$ for all $f \in G$.
- (4) $\text{in}_{\mathbf{w}}(\text{Ker } \phi_{\boldsymbol{\xi}}) = \text{Ker}(\phi_{\text{in}_t(\boldsymbol{\xi})})$.

Proof. (1) $\Rightarrow$ (2) immediately follows from the definition of local SAGBI bases and local reduction. (2) $\Rightarrow$ (3) is obvious. (3) $\Rightarrow$ (4) follows from Lemma 4.10 and Lemma 4.14.

(4) $\Rightarrow$ (1): Let $\eta \in A$, and write $f(\boldsymbol{\xi}) = \eta$ for $f \in K[[\mathbf{x}]]$. Then $\text{ord}_{\mathbf{w}}(f) \leq \text{ord}_t(\eta)$. Hence we may take $f \in K[[\mathbf{x}]]$ so that $\text{ord}_{\mathbf{w}}(f)$ is maximal among the power series satisfying $f(\boldsymbol{\xi}) = \eta$. We claim that $\text{in}_{\mathbf{w}}(f) \notin \text{Ker}(\phi_{\text{in}_t(\boldsymbol{\xi})})$. Assume, to the contrary, that $\text{in}_{\mathbf{w}}(f) \in \text{Ker}(\phi_{\text{in}_t(\boldsymbol{\xi})})$. By the assumption $\text{Ker}(\phi_{\text{in}_t(\boldsymbol{\xi})}) = \text{in}_{\mathbf{w}}(\text{Ker } \phi_{\boldsymbol{\xi}})$, there exists $g \in \text{Ker } \phi_{\boldsymbol{\xi}}$ such that $\text{in}_{\mathbf{w}}(f) = \text{in}_{\mathbf{w}}(g)$. Then $\text{ord}_{\mathbf{w}}(f - g) > \text{ord}_{\mathbf{w}}(f)$ and $(f - g)(\boldsymbol{\xi}) = f(\boldsymbol{\xi}) - g(\boldsymbol{\xi}) = f(\boldsymbol{\xi}) = \eta$. This is a contradiction. Thus $\text{in}_{\mathbf{w}}(f) \notin$

$\text{Ker}(\phi_{\text{in}_t(\xi)})$. Therefore $\text{in}_t(\eta) = \text{in}_t(f(\xi)) = \text{in}_{\mathbf{w}}(f)(\text{in}_t(\xi)) \in K[\text{in}_t(\xi)]$. This proves that ξ is a local SAGBI basis of A . $\square$

Corollary 4.16. *Let $\mathfrak{p} \subset K[[\mathbf{x}]]$ be a one-dimensional prime ideal such that $x_i \notin \mathfrak{p}$ for all i , and set $\mathbf{w} = \omega(\mathfrak{p})$. Then $\text{Semi}(K[[\mathbf{x}]]/\mathfrak{p}) = \text{Span}_{\mathbb{N}}(\mathbf{w})$ if and only if $\text{in}_{\mathbf{w}}(\mathfrak{p}) = \sqrt{\text{in}_{\mathbf{w}}(\mathfrak{p})}$.*

Proof. Note that $K[[\mathbf{x}]]/\mathfrak{p} \cong K[[\xi]] \subset K[[t]]$ and $\mathbf{w} = \text{ord}_t(\xi)$ for some ξ . As $\mathfrak{p} = \text{Ker } \phi_{\xi}$ and $\omega(\mathfrak{p}) = \text{ord}_t(\xi)$, the assertion follows from Theorem 4.11, Theorem 4.15 and the definition of local SAGBI basis. $\square$

4.3. Algorithm for computing value-semigroups of irreducible algebroid curves. We give an algorithm for computing $\text{Semi}(K[[\mathbf{x}]]/\mathfrak{p})$. By Lemma 3.3, we have the following.

Lemma 4.17. *Let $\mathfrak{p} \subset K[[\mathbf{x}]]$ be a one-dimensional prime ideal, and $f, g \in K[[\mathbf{x}]]$ with $\text{int}(f; \mathfrak{p}) = \text{int}(g; \mathfrak{p}) < \infty$. Then there exists $\alpha \in K^{\times}$ such that $\text{int}(f - \alpha g; \mathfrak{p}) > \text{int}(f; \mathfrak{p})$ and $\text{int}(f - \beta g; \mathfrak{p}) = \text{int}(f; \mathfrak{p})$ for $\beta \neq \alpha$.*

Algorithm 4.18 (Algorithm for computing value-semigroups).

Input: a one-dimensional prime ideal $\mathfrak{p} \subset K[[\mathbf{x}]] = K[[x_1, \dots, x_r]]$ with $x_i \notin \mathfrak{p}$ for all i .

Output: $(\mathfrak{p}', \mathbf{w})$ where $\mathfrak{p}' \subset K[[x_1, \dots, x_{r'}]]$, $r' \geq r$, and $\mathbf{w} = \omega(\mathfrak{p}')$ such that $K[[\mathbf{x}]]/\mathfrak{p} \cong K[[x_1, \dots, x_{r'}]]/\mathfrak{p}'$ and $\text{Semi}(K[[\mathbf{x}]]/\mathfrak{p}) = \text{Span}_{\mathbb{N}}(\mathbf{w})$.

```

1:  $\mathbf{x} := (x_1, \dots, x_r)$ ,  $\mathbf{w} := \omega(\mathfrak{p})$ .
2: while  $(\text{in}_{\mathbf{w}}(\mathfrak{p}) \neq \sqrt{\text{in}_{\mathbf{w}}(\mathfrak{p})})$  do
3:    $G :=$  a system of binomial generators of  $\sqrt{\text{in}_{\mathbf{w}}(\mathfrak{p})}$ .
4:   Take  $f = \mathbf{x}^{\mathbf{a}} - \alpha \mathbf{x}^{\mathbf{b}} \in G \setminus \text{in}_{\mathbf{w}}(\mathfrak{p})$ , and let  $N := \text{int}(f; \mathfrak{p})$ .
5:   while  $(N \in \text{Span}_{\mathbb{N}}(\mathbf{w}))$  do
6:     Take  $\mathbf{c} \in \mathbb{N}^r$  such that  $N = \mathbf{w} \cdot \mathbf{c}$ .
7:     Take the unique  $\beta \in K^{\times}$  such that  $\text{int}(f - \beta \mathbf{x}^{\mathbf{c}}; \mathfrak{p}) > \text{int}(f; \mathfrak{p})$ .
8:      $N := \text{int}(f - \beta \mathbf{x}^{\mathbf{c}}; \mathfrak{p})$ ,  $f := f - \beta \mathbf{x}^{\mathbf{c}}$ .
9:   end while
10:   $\mathfrak{p} := \langle \mathfrak{p}, x_{r+1} - f \rangle$ ,  $\mathbf{x} := (\mathbf{x}, x_{r+1})$ ,  $\mathbf{w} := (\mathbf{w}, N)$ ,  $r := r + 1$ .
11: end while
12: return  $(\mathfrak{p}, \mathbf{w})$ .
```

Proof. We give a proof of the correctness and of this algorithm.

There exists $\xi = (\xi_1, \dots, \xi_r)$, $\xi_i \in tK[[t]]$, such that $K[[\mathbf{x}]]/\mathfrak{p} \cong K[[\xi]] \subset K[[t]]$ where $K[[t]]$ is the integral closure of A . Then $\mathfrak{p} = \text{Ker } \phi_{\xi}$, and $\sqrt{\text{in}_{\mathbf{w}}(\mathfrak{p})} = \text{in}_{\mathbf{w}}(\text{Ker } \phi_{\xi})$ by Theorem 4.11. The ideal $\mathfrak{p}$ changes only at Line 10, and the residue class ring of $\mathfrak{p}$ does not change. Note that $\text{Ker } \phi_{\xi'} = \langle \mathfrak{p}, x_{r+1} - f \rangle$ where $\xi' = (\xi, f(\xi))$.

We will prove the termination of this algorithm. First, we show that the while loop from Line 5 to 9 terminates in finite time. Assume, to the contrary, that this while loop causes infinite steps. This means that there exist $\mathbf{x}^{\mathbf{a}} - \alpha \mathbf{x}^{\mathbf{b}} \in \sqrt{\text{in}_{\mathbf{w}}(\mathfrak{p})} \setminus \text{in}_{\mathbf{w}}(\mathfrak{p})$, and $\beta_i \in K^{\times}$, $\mathbf{c}_i \in \mathbb{N}^r$ for $i = 1, 2, \dots$ satisfying $\text{int}(f_{i+1}; \mathfrak{p}) > \text{int}(f_i; \mathfrak{p}) = \omega(\mathfrak{p}) \cdot \mathbf{c}_{i+1}$ where $f_0 = \mathbf{x}^{\mathbf{a}} - \alpha \mathbf{x}^{\mathbf{b}}$ and $f_i = f_{i-1} - \beta_i \mathbf{x}^{\mathbf{c}_i} = (\mathbf{x}^{\mathbf{a}} - \alpha \mathbf{x}^{\mathbf{b}}) - \beta_1 \mathbf{x}^{\mathbf{c}_1} - \beta_2 \mathbf{x}^{\mathbf{c}_2} - \dots - \beta_i \mathbf{x}^{\mathbf{c}_i}$ for $i \geq 1$. Since $|\mathbf{c}_i| \rightarrow \infty$ as $i \rightarrow \infty$, the limit $F := \lim_{i \rightarrow \infty} f_i$ exists. Since $\text{ord}_t(\phi_{\xi}(f_i)) = \text{int}(f_i; \mathfrak{p}) \rightarrow \infty$ as $i \rightarrow \infty$, $\lim_{i \rightarrow \infty} \phi_{\xi}(f_i) = 0$. As ϕ_{ξ} is continuous, this shows $F \in \text{Ker } \phi_{\xi} = \mathfrak{p}$. On the other hand, $\text{in}_{\mathbf{w}}(F) = f_0 = \mathbf{x}^{\mathbf{a}} - \alpha \mathbf{x}^{\mathbf{b}}$ as $\text{ord}_{\mathbf{w}}(f_0) < \text{int}(f_0; \mathfrak{p}) = \mathbf{w} \cdot \mathbf{c}_1 < \text{int}(f_1; \mathfrak{p}) =$

$\mathbf{w} \cdot \mathbf{c}_2 < \dots$, which contradicts to $f_0 \notin \text{in}_{\mathbf{w}}(\mathbf{p})$. Thus the while loop from Line 5 to 9 terminates in finite time. By the same reason, N cannot be ∞ in this while loop.

At Line 10, the semigroup $\text{Span}_{\mathbb{N}}(\mathbf{w})$ become strictly larger. Thus the while loop from Line 2 to 11 terminates in finite time by Lemma 4.6.

By Corollary 4.16, $\text{Semi}(K[[\mathbf{x}]]/\mathbf{p}) = \text{Semi}(K[[\mathbf{x}]]/\mathbf{p}') = \text{Span}_{\mathbb{N}}(\omega(\mathbf{p}'))$ is satisfied for the output $\mathbf{p}'$. $\square$

Observation 4.19. In some cases, it is not easy to compute $\sqrt{\text{in}_{\mathbf{w}}(\mathbf{p})}$ directly. One can compute $\sqrt{\text{in}_{\mathbf{w}}(\mathbf{p})}$ as follows. Let $\{\mathbf{x}^{a_i} - \mathbf{x}^{b_i} \mid i = 1, \dots, n\}$ be a system of binomial generators of $\text{Prim}(\omega(\mathbf{p}))$. For each i , there exists the unique $\alpha_i \in K^\times$ satisfying the following equivalent conditions:

- (1) $\mathbf{x}^{a_i} - \alpha_i \mathbf{x}^{b_i} \in \sqrt{\text{in}_{\mathbf{w}}(\mathbf{p})}$.
- (2) $x_j \notin \sqrt{\langle \mathbf{x}^{a_i} - \alpha_i \mathbf{x}^{b_i}, \text{in}_{\mathbf{w}}(\mathbf{p}) \rangle}$ for some (any) j .
- (3) $\text{int}(\mathbf{x}^{a_i} - \alpha_i \mathbf{x}^{b_i}; \mathbf{p}) > \text{int}(\mathbf{x}^{a_i}; \mathbf{p})$ ([13] before Example 3.4).

Then $\{\mathbf{x}^{a_i} - \alpha_i \mathbf{x}^{b_i} \mid i = 1, \dots, n\}$ is a system of generators of $\sqrt{\text{in}_{\mathbf{w}}(\mathbf{p})}$. For an ideal $I \subset K[\mathbf{x}]$ and a polynomial $h \in K[\mathbf{x}]$, $h \in \sqrt{I}$ if and only if $\langle hu - 1, I \rangle = K[\mathbf{x}, u]$. Thus one can find α_i satisfying (2) by computing comprehensive Gröbner basis [25] of $\langle ux_1 - 1, \mathbf{x}^{a_i} - \alpha \mathbf{x}^{b_i}, \text{in}_{\mathbf{w}}(\mathbf{p}) \rangle \subset K[\mathbf{x}, u]$ with $\alpha \in K$ a parameter. One can find α_i using the condition (3) by computing a standard basis of $\langle \mathbf{x}^{a_i} - \alpha \mathbf{x}^{b_i}, \mathbf{p} \rangle$ with $\alpha \in K$ a parameter in a similar way to comprehensive Gröbner basis.

For $f \in K[[\mathbf{x}]]$ and $\mathbf{c} \in \mathbb{N}^r$ with $\text{int}(f; \mathbf{p}) = \omega(\mathbf{p}) \cdot \mathbf{c} \in \text{Span}_{\mathbb{N}}(\omega(\mathbf{p}))$, one can find $\beta \in K$ such that $\text{int}(f - \beta \mathbf{x}^{\mathbf{c}}; \mathbf{p}) > \text{int}(f; \mathbf{p})$ in a similar way. This β is the unique element satisfying $x_{r+1} - \beta \mathbf{x}^{\mathbf{c}} \in \sqrt{\text{in}_{\mathbf{w}}(\langle \mathbf{p}, x_{r+1} - f \rangle)} \subset K[\mathbf{x}, x_{r+1}]$ where $\mathbf{w} = \omega(\langle \mathbf{p}, x_{r+1} - f \rangle) = (\omega(\mathbf{p}), \text{int}(f; \mathbf{p}))$.

Remark 4.20. Algorithm 4.18 is an improved version of the Hefez–Hernandes algorithm used in [13] Example 3.4. The difference is the terminal condition $\text{in}_{\mathbf{w}}(\mathbf{p}) = \sqrt{\text{in}_{\mathbf{w}}(\mathbf{p})}$, and this reduces the computation. Hefez and Hernandez define *minimal S-process*, and their algorithm computes the while loop from Line 5 to 9 of Algorithm 4.18 for all minimal S-processes f until $N \notin \text{Span}_{\mathbb{N}}(\mathbf{w})$ or $N \geq c'$ where c' is a number not less than the conductor of $\text{Semi}(K[[\mathbf{x}]]/\mathbf{p})$. In many cases, the set $G \setminus \text{in}_{\mathbf{w}}(\mathbf{p})$, where G a minimal system of binomial generators of $\sqrt{\text{in}_{\mathbf{w}}(\mathbf{p})}$, is a proper subset of the set of minimal S-processes, and one need not to estimate the conductor of $\text{Semi}(K[[\mathbf{x}]]/\mathbf{p})$ in our algorithm.

For the output $\mathbf{p}'$ of Algorithm 4.18, $\omega(\mathbf{p}')$ is a tropism of $\mathbf{p}'$. Thus one can use Algorithm 4.18 to obtain an evidence of the primeness of $\mathbf{p}$. Moreover, $\text{in}_{\omega(\mathbf{p}')}(\mathbf{p}')$ is a prime ideal, and it is not hard to show that an ideal $I \subset K[[\mathbf{x}]]$ is prime if $\text{in}_{\mathbf{w}}(I) \subset K[\mathbf{x}]$ is prime for some $\mathbf{w} \in \mathbb{N}_+^r$.

Example 4.21. Let $F(x, y) = (y^{a_1} - x^{a_2})^d - x^{c_1} y^{c_2} \in K[[x, y]]$ where $a_1, a_2, d \in \mathbb{N}_+$ and $c_1, c_2 \in \mathbb{N}$ such that $d \geq 2$, $\gcd(a_1, a_2) = 1$, $\gcd(c_1 a_1 + c_2 a_2, d) = 1$ and $c_1 a_1 + c_2 a_2 > d a_1 a_2$. Then $\omega(F) = (da_1, da_2)$, and $\text{in}_{\omega(F)}(F) = (y^{a_1} - x^{a_2})^d$. Let $g = y^{a_1} - x^{a_2}$. Then $\text{int}(g; F) = \dim_K K[[x, y]] / \langle x^{c_1} y^{c_2}, g \rangle = \text{int}(x^{c_1} y^{c_2}; g) = \omega(g) \cdot (c_1, c_2) = c_1 a_1 + c_2 a_2 \notin \text{Span}_{\mathbb{N}}(da_1, da_2)$ as $\gcd(c_1 a_1 + c_2 a_2, d) = 1$. Let $J = \langle F, z - g \rangle = \langle (y^{a_1} - x^{a_2})^d - x^{c_1} y^{c_2}, z - (y^{a_1} - x^{a_2}) \rangle = \langle z^d - x^{c_1} y^{c_2}, (y^{a_1} - x^{a_2}) - z \rangle \subset K[[x, y, z]]$. Then $\omega(J) = (da_1, da_2, b)$, and $\text{in}_{\omega(J)}(J) = \langle z^d - x^{c_1} y^{c_2}, y^{a_1} - x^{a_2} \rangle = \text{Prim}(da_1, da_2, b)$. Thus F is irreducible, and $\text{Semi}(K[[x, y]]/\langle F \rangle) = \text{Span}_{\mathbb{N}}(da_1, da_2, b)$.

Example 4.22. Let a, b, c, d, e be positive integers such that $\gcd(a, b) = 1$, and $a + b$ is odd. Let $\mathbf{p} = \langle x^b - z^a + x^{b+1} y z, (y^2 - x z)^2 - x^c y^{2d+1} z^e \rangle \subset K[[x, y, z]]$. Then

$\omega(\mathfrak{p}) = (4a, 2a + 2b, 4b)$, and $\text{in}_{\omega(\mathfrak{p})}(\mathfrak{p}) = \langle x^b - z^a, (y^2 - xz)^2 \rangle$. Take $y^2 - xz \in \sqrt{\text{in}_{\omega(\mathfrak{p})}(\mathfrak{p})} \setminus \text{in}_{\omega(\mathfrak{p})}(\mathfrak{p})$. Then $\text{int}(y^2 - xz; \mathfrak{p}) = \text{int}(x^c y^{2d+1} z^e; \langle x^b - z^a + x^{b+1} yz, y^2 - xz \rangle) = (2a, a+b, 2b) \cdot (c, 2d+1, e) = 2ac + (a+b)(2d+1) + 2be \notin \text{Span}_{\mathbb{N}}(4a, 4b, 2a+2b)$. Let $\mathfrak{p}' = \langle \mathfrak{p}, u - (y^2 - xz) \rangle \subset K[[x, y, z, u]]$. Then $\omega(\mathfrak{p}') = (4a, 2a + 2b, 4b, 2ac + (a+b)(2d+1) + 2be)$, and $\text{in}_{\omega(\mathfrak{p}')}(\mathfrak{p}') = \langle x^b - z^a, u^2 - x^c y^{2d+1} z^e, y^2 - xz \rangle = \text{Prim}(\omega(\mathfrak{p}'))$. Thus $\mathfrak{p}$ is prime, and $\text{Semi}(K[[x, y, z]]/\mathfrak{p}) = \text{Span}_{\mathbb{N}}(4a, 2a + 2b, 4b, 2ac + 2bd + (a+b)(2e+1))$.

Example 4.23. Assume $\text{char } K = 2$. Let $F = (y^2 - x^3)^2 - x^5 y^2$. Then $\omega(F) = (4, 6)$ and $\text{in}_{\omega(F)}(F) = (y^2 - x^3)^2$. Let $g = y^2 - x^3$. Then $\text{int}(g; F) = 16 = \text{int}(xy^2; F)$. Note that $\langle g - \beta xy^2, F \rangle = \langle x^2 y^2 (\beta^2 y^2 - x^3), g - \beta xy^2 \rangle$, and $g - \beta xy^2$ is an irreducible power series with $\omega(g - \beta xy^2) = (2, 3)$. Thus $\text{int}(g - \beta xy^2; F) = \text{int}(x^2 y^2 (\beta^2 y^2 - x^3); g - \beta xy^2) = \text{ord}_{(2,3)}(x^2 y^2 (\beta^2 y^2 - x^3)) = 16$ for $\beta \neq 1$ by Corollary 4.12, and $\text{int}(g - xy^2; F) = \text{int}(x^3 y^4; g - xy^2) = (2, 3) \cdot (3, 4) = 18 = \text{int}(x^3 y; F)$. In a similar way as above, we have $\text{int}(g - xy^2 - \beta x^3 y; F) = 18$ for $\beta \neq 1$, and $\text{int}(g - xy^2 - x^3 y; F) = \text{int}(x^5 y^3 + x^3 y^2 (xy^2 - x^3 y); g - xy^2 - x^3 y) = 19$. Let $J = \langle F, z - (y^2 - x^3 - xy^2 - x^3 y) \rangle \subset K[[x, y, z]]$. Then $\omega(J) = (4, 6, 19)$, and $\text{in}_{\omega(J)}(J) = \langle y^2 - x^3, z^2 - x^2 y^5 \rangle = \text{Prim}(4, 6, 19)$. Thus F is irreducible, and $\text{Semi}(K[[x, y]]/\langle F \rangle) = \text{Span}_{\mathbb{N}}(4, 6, 19)$.

Example 4.24. Let $\mathfrak{p} = \langle y^2 - xz, x^5 + z^3 - 2x^2 yz - y^4 \rangle \subset K[[x, y, z]]$. Then $\omega(\mathfrak{p}) = (6, 8, 10)$, $\text{in}_{\omega(\mathfrak{p})}(\mathfrak{p}) = \langle y^2 - xz, x^5 + z^3 - 2x^2 yz \rangle$, and $\sqrt{\text{in}_{\omega(\mathfrak{p})}(\mathfrak{p})} = \langle x^3 - yz, y^2 - xz, z^2 - x^2 y \rangle$. Let $g_1 = x^3 - yz$ and $g_2 = x^2 y - z^2$. Then $\text{int}(g_1; \mathfrak{p}) = 19$, and $\text{int}(g_2; \mathfrak{p}) = 21$. Let $\mathfrak{p}' = \langle \mathfrak{p}, u_1 - g_1, u_2 - g_2 \rangle \subset K[[x, y, z, u_1, u_2]]$. Then $\omega(\mathfrak{p}') = (6, 8, 10, 19, 21)$, and $\text{in}_{\omega(\mathfrak{p}')}(\mathfrak{p}') = \text{Prim}(6, 8, 10, 19, 21)$. Thus $\mathfrak{p}$ is prime, and $\text{Semi}(K[[x, y, z]]/\mathfrak{p}) = \text{Span}_{\mathbb{N}}(6, 8, 10, 19, 21)$.

Example 4.25. Assume that $\text{char } K = p > 0$. Let a, b, c_1, c_2, c_3 be non-negative integers such that $\gcd(a, b) = 1$, $2ab < 2ac_1 + (a+b)c_2 + 2bc_3$, $2ac_1 + (a+b)c_2 + 2bc_3 - 2ab$ is not divisible by p , and $2p > b$. Let

$$\mathfrak{p} = \langle x^b - z^a - x^{c_1} y^{c_2} z^{c_3}, (y^2 - xz)^p - x^{2p-b} z^{p+a} \rangle \subset K[[x, y, z]].$$

Then $\omega(\mathfrak{p}) = (2ap, (a+b)p, 2bp)$, and $\text{in}_{\omega(\mathfrak{p})}(\mathfrak{p}) = \langle x^b - z^a, (y^2 - xz)^p \rangle$. Take $y^2 - xz \in \sqrt{\text{in}_{\omega(\mathfrak{p})}(\mathfrak{p})} \setminus \text{in}_{\omega(\mathfrak{p})}(\mathfrak{p})$. Then $\text{int}(y^2 - xz; \mathfrak{p}) = \text{int}(x^{2p-b} z^{p+a}; \langle x^b - z^a - x^{c_1} y^{c_2} z^{c_3}, y^2 - xz \rangle) = (2a, a+b, 2b) \cdot (2p-b, 0, p+a) = 4ap + 2bp = \text{int}(x^2 z; \mathfrak{p})$.

We will compute $\text{int}(y^2 - xz - \beta x^2 z; \mathfrak{p})$ for $\beta \in K^\times$. Since $\langle y^2 - xz - \beta x^2 z, \mathfrak{p} \rangle = \langle y^2 - xz - \beta x^2 z, x^b - z^a - x^{c_1} y^{c_2} z^{c_3}, x^{2p-b} z^p (\beta^p x^b - z^a) \rangle$, it follows that

$$\begin{aligned} \text{int}(y^2 - xz - \beta x^2 z; \mathfrak{p}) &= \text{int}(x^{2p-b} z^p (\beta^p x^3 - z^2); \langle y^2 - xz - \beta x^2 z, x^b - z^a - x^{c_1} y^{c_2} z^{c_3} \rangle) \\ &= 4ap + 2bp \end{aligned}$$

for $\beta \neq 1$, and $\text{int}(y^2 - xz - x^2 z; \mathfrak{p}) = \text{int}(x^{c_1+2p-b} y^{c_2} z^{c_3+p}; \langle y^2 - xz - x^2 z, x^b - z^a - x^{c_1} y^{c_2} z^{c_3} \rangle) = 4ap + 2bp + (2ac_1 + (a+b)c_2 + 2bc_3 - 2ab) > 4ap + 2bp$. As $2ac_1 + (a+b)c_2 + 2bc_3 - 2ab$ is not divisible by p , $\text{int}(y^2 - xz - x^2 z; \mathfrak{p}) \notin \text{Span}_{\mathbb{N}}(\omega(\mathfrak{p}))$. Let $\mathfrak{p}' = \langle \mathfrak{p}, u - (y^2 - xz - x^2 z) \rangle \subset K[[x, y, z]]$. Then $\omega(\mathfrak{p}') = (2ap, (a+b)p, 2bp, (4a+2b)p + (2ac_1 + (a+b)c_2 + 2bc_3 - 2ab))$, and $\text{in}_{\omega(\mathfrak{p}')}(\mathfrak{p}') = \langle x^b - z^a, y^2 - xz, u^p - x^{c_1+2p-b} y^{c_2} z^{c_3+p} \rangle = \text{Prim}(\omega(\mathfrak{p}'))$. Thus $\mathfrak{p}$ is prime, and $\text{Semi}(K[[x, y, z]]/\mathfrak{p}) = \text{Span}_{\mathbb{N}}(2ap, (a+b)p, 2bp, (4a+2b)p + (2ac_1 + (a+b)c_2 + 2bc_3 - 2ab))$.

If one applies Algorithm 4.18 to a non-prime ideal I , then some errors would occur. For example, the uniqueness of β in Line 7 fails. One may regard it as

evidence of the non-primeness of I , and this lead us to an algorithm for deciding irreducibility of algebroid curves.

5. ALGORITHM FOR DECIDING IRREDUCIBILITY OF REDUCED ALGEBROID CURVES

We will present an algorithm for proving irreducibility of algebroid curves over an algebraically closed field.

5.1. Parametric intersection numbers. Let $I \subset K[[\mathbf{x}]] = K[[x_1, \dots, x_r]]$ be an equidimensional ideal of dimension one, and take $f, g \in K[[\mathbf{x}]]$ such that $\text{int}(f; I) = \text{int}(g; I)$. We will investigate how the value $\text{int}(f - \alpha g; I)$ varies as $\alpha \in K^\times$ changes. One can solve this problem this by computing a standard basis of $\langle I, f - \alpha g \rangle$ with a parameter α in a similar way as comprehensive Gröbner basis [25]. In case where I is prime, we already have the answer in Lemma 4.17. One can use Lemma 4.17 for testing the primeness of a one-dimensional ideal. If I is decided to be not prime by Lemma 4.17, we can construct J such that its residue class ring is isomorphic to that of I , and J has at least two tropisms.

Theorem 5.1. *Let $I \subset K[[\mathbf{x}]]$ be an equidimensional ideal of dimension one, and $f, g \in K[[\mathbf{x}]]$ with $\text{int}(f; I) = \text{int}(g; I) < \infty$. Then the following holds.*

- (1) $\text{int}(f - \alpha g; I) \leq \text{int}(f; I)$ for generic $\alpha \in K^\times$.
- (2) If $\text{int}(f - \alpha g; I) < \text{int}(f; I)$ for generic $\alpha \in K^\times$, then $J = \langle I, x_{r+1} - f, x_{r+2} - g \rangle \subset K[[\mathbf{x}, x_{r+1}, x_{r+2}]]$ has at least two tropisms.
- (3) Assume that $\text{int}(f - \alpha g; I) = \text{int}(f; I)$ for generic $\alpha \in K^\times$, that is, there exists a non-empty finite set $\Gamma \in K^\times$ such that $\text{int}(f - \alpha g; I) > \text{int}(f; I)$ if $\alpha \in \Gamma$, and $\text{int}(f - \alpha g; I) = \text{int}(f; I)$ otherwise. If $\#\Gamma \geq 2$, then, for $\beta_1, \beta_2 \in \Gamma$ with $\beta_1 \neq \beta_2$,

$$J = \langle I, x_{r+1} - (f - \beta_1 g), x_{r+1} - (f - \beta_2 g) \rangle \subset K[[\mathbf{x}, x_{r+1}, x_{r+2}]]$$

has at least two tropisms.

- (4) If $h \notin \sqrt{I}$ and $\text{int}(h; I) = \infty$, then $J = \langle I, x_{r+1} - h \rangle \subset K[[\mathbf{x}, x_{r+1}]]$ has at least two tropisms.

Proof. Let $I = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_l$ be the irredundant primary decomposition, and $\mathfrak{p}_j = \sqrt{\mathfrak{q}_j}$. Note that $\{(\gcd \omega(\mathfrak{p}_j))^{-1} \omega(\mathfrak{p}_j) \mid 1 \leq j \leq l\}$ is the set of the tropism of I by Theorem 3.6 and Lemma 2.4. Set $\mathbf{w} = \omega(I)$. Let $A = K[[\mathbf{x}]]/I$ and $A_j = K[[\mathbf{x}]]/\mathfrak{p}_j$. We identify the integral closure of A_j with $K[[t]]$ for all $1 \leq j \leq l$. Let $\phi_j : K[[\mathbf{x}]] \rightarrow K[[t]]$ be the composition of the natural surjection $K[[\mathbf{x}]] \rightarrow A_j$ and the inclusion $A_j \hookrightarrow K[[t]]$. Recall that $\text{int}(h; I) = \sum_{j=1}^l \ell(A_{\mathfrak{p}_j}) \cdot \text{ord}_t(\phi_j(h))$ for $h \in K[[\mathbf{x}]]$ by Lemma 3.2 and Lemma 3.3. We set $u_j = \text{ord}_t(\phi_j(f))$ and $v_j = \text{ord}_t(\phi_j(g))$. Then $\sum_{j=1}^l \ell(A_{\mathfrak{p}_j}) \cdot u_j = \text{int}(f; I) = \text{int}(g; I) = \sum_{j=1}^l \ell(A_{\mathfrak{p}_j}) \cdot v_j$. Hence, for generic $\alpha \in K^\times$, it holds that

$$\text{int}(f - \alpha g; I) = \sum_{j=1}^l \ell(A_{\mathfrak{p}_j}) \cdot \text{ord}_t(\phi_j(f) - \alpha \phi_j(g)) = \sum_{j=1}^l \ell(A_{\mathfrak{p}_j}) \cdot \min\{u_j, v_j\}.$$

Therefore $\text{int}(f - \alpha g; I) = \text{int}(f; I)$ holds for generic $\alpha \in K^\times$ if and only if $u_j = v_j$ for all $1 \leq j \leq l$.

- (1) As $\text{int}(f - \alpha g; I) = \sum_{j=1}^l \ell(A_{\mathfrak{p}_j}) \cdot \min\{u_j, v_j\}$, it holds that $\text{int}(f - \alpha g; I) \leq \text{int}(f; I)$ for generic $\alpha \in K^\times$.

(2) Since $\text{int}(f - \alpha g; I) < \text{int}(f; I)$ for generic $\alpha \in K^\times$, there exists $1 \leq j \leq l$ such that $u_j \neq v_j$. Moreover, since $\sum_{j=1}^l \ell(A_{\mathfrak{p}_j}) \cdot u_j = \sum_{j=1}^l \ell(A_{\mathfrak{p}_j}) \cdot v_j$, there exist $1 \leq j_1, j_2 \leq l$ such that $u_{j_1} > v_{j_1}$ and $u_{j_2} < v_{j_2}$. For $1 \leq j \leq l$, set $\mathfrak{p}'_j = \langle \mathfrak{p}_j, x_{r+1} - f, x_{r+2} - g \rangle \subset K[[\mathbf{x}, x_{r+1}, x_{r+2}]]$. Then $\mathfrak{p}'_1, \dots, \mathfrak{p}'_l$ are the associated prime ideals of J , and $\omega(\mathfrak{p}'_j) = (\omega(\mathfrak{p}_j), u_j, v_j)$. Since $u_{j_1} > v_{j_1}$ and $u_{j_2} < v_{j_2}$, we have $\mathbb{R}_+ \cdot \omega(\mathfrak{p}'_{j_1}) \neq \mathbb{R}_+ \cdot \omega(\mathfrak{p}'_{j_2})$.

(3) Since $\text{int}(f - \alpha g; I) = \text{int}(f; I)$ for generic $\alpha \in K^\times$, it holds that $u_j = v_j$ for all $1 \leq j \leq l$. Thus, for each $1 \leq j \leq l$, there exists $\alpha_j \in K^\times$ such that $\phi_j(f) = \alpha_j \phi_j(g)$. Then $\text{ord}_t(\phi_j(f) - \alpha_j \phi_j(g)) > u_j$, and $\text{ord}_t(\phi_j(f) - \beta \phi_j(g)) = u_j$ for $\beta \neq \alpha_j$. Therefore $\Gamma = \{\alpha_1, \dots, \alpha_l\}$ satisfies the desired condition. Assume that $\#\Gamma \geq 2$. Then there exist $1 \leq j_1 < j_2 \leq l$ such that $\alpha_{j_1} \neq \alpha_{j_2}$. Set $\mathfrak{p}'_j = \langle \mathfrak{p}_j, x_{r+1} - (f - \alpha_{j_1}g), x_{r+1} - (f - \alpha_{j_2}g) \rangle \subset K[[\mathbf{x}, x_r, x_{r+1}]]$ for $1 \leq j \leq l$. Then $\mathfrak{p}'_1, \dots, \mathfrak{p}'_l$ are the associated prime ideals of J , and

$$\begin{aligned}\omega(\mathfrak{p}'_{j_1}) &= (\omega(\mathfrak{p}_{j_1}), \text{ord}_t(\phi_{j_1}(f) - \alpha_{j_1}\phi_{j_1}(g)), \text{ord}_t(\phi_{j_1}(f))), \\ \omega(\mathfrak{p}'_{j_2}) &= (\omega(\mathfrak{p}_{j_2}), \text{ord}_t(\phi_{j_2}(f)), \text{ord}_t(\phi_{j_2}(f) - \alpha_{j_2}\phi_{j_2}(g))).\end{aligned}$$

As $\text{ord}_t(\phi_{j_1}(f) - \alpha_{j_1}\phi_{j_1}(g)) > \text{ord}_t(\phi_{j_1}(f))$ and $\text{ord}_t(\phi_{j_2}(f)) < \text{ord}_t(\phi_{j_2}(f) - \alpha_{j_2}\phi_{j_2}(g))$, we have $\mathbb{R}_+ \cdot \omega(\mathfrak{p}'_{j_1}) \neq \mathbb{R}_+ \cdot \omega(\mathfrak{p}'_{j_2})$.

(4) Since $h \notin \sqrt{I} = \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_\ell$, there exists $1 \leq j_1 \leq \ell$ such that $h \notin \mathfrak{p}_{j_1}$. As $\text{int}(h; I) = \infty$, there exists $1 \leq j_2 \leq \ell$ such that $h \in \mathfrak{p}_{j_2}$. For $1 \leq j \leq l$, we set $\mathfrak{p}'_j = \langle \mathfrak{p}_j, x_{r+1} - h \rangle \subset K[[\mathbf{x}, x_{r+1}]]$. Then $\mathfrak{p}'_1, \dots, \mathfrak{p}'_l$ are the associated prime ideals of J , and $\omega(\mathfrak{p}'_j) = (\omega(\mathfrak{p}_j), \text{ord}_t(\phi_j(h)))$. Since $h \notin \mathfrak{p}_{j_1}$ and $h \in \mathfrak{p}_{j_2}$, $\text{ord}_t(\phi_{j_1}(h)) < \infty$ and $\text{ord}_t(\phi_{j_2}(h)) = \infty$. Thus $\mathbb{R}_+ \cdot \omega(\mathfrak{p}'_{j_1}) \neq \mathbb{R}_+ \cdot \omega(\mathfrak{p}'_{j_2})$. $\square$

5.2. Main algorithm. Combining Algorithm 4.18, Theorem 3.7 (1) and Theorem 5.1, we obtain an algorithm for deciding irreducibility of reduced algebroid curves. Concerning Theorem 5.1, we define functions which we use in the main algorithm.

Algorithm 5.2. `find_coeff( $f, g, I$ )`: Let I be an equidimensional ideal of dimension one, and $f, g \in K[[\mathbf{x}]]$ with $\text{int}(f; I) = \text{int}(g; I)$. We define the function `find_coeff( $f, g, I$ )` which returns $\Gamma \subset K^\times$ as follows: If $\text{int}(f - \alpha g; I) < \text{int}(f; I)$ for generic $\alpha \in K^\times$, then $\Gamma = \emptyset$. If $\text{int}(f - \alpha g; I) = \text{int}(f; I)$ for generic $\alpha \in K^\times$, then $\Gamma = \{\beta \in K \mid \text{int}(f - \beta g; I) > \text{int}(f; I)\}$.

Algorithm 5.3. `distinguish_branches( $\Gamma, f, g, I$ )`: Let I be an equidimensional ideal of dimension one, $f, g \in K[[\mathbf{x}]]$ with $\text{int}(f; I) = \text{int}(g; I)$, and Γ a finite subset of $K^\times$ such that $\#\Gamma \neq 1$. We define the function `distinguish_branches( $\Gamma, f, g, I$ )` which returns $J \subset K[[\mathbf{x}, x_{r+1}, x_{r+2}]]$ as follows: If $\Gamma = \emptyset$, then $J = \langle I, x_{r+1} - f, x_{r+1} - g \rangle$. If $\#\Gamma \geq 2$, then $J = \langle I, x_{r+1} - (f - \beta_1 g), x_{r+1} - (f - \beta_2 g) \rangle$ for $\beta_1, \beta_2 \in \Gamma$, $\beta_1 \neq \beta_2$.

Lemma 5.4. Let $I \subset K[[\mathbf{x}]]$ be an equidimensional ideal of dimension one, and $\mathbf{w} = (w_1, \dots, w_r) \in \mathcal{T}_{\text{loc}}(I) \cap \mathbb{N}_+^r$. If $\mathcal{T}_{\text{loc}}(I) = \mathbb{R}_+ \cdot \mathbf{w}$, then $\sqrt{\text{in}_{\mathbf{w}}(I)} = \bigcap_{\mathfrak{p} \in \text{Ass } I} \sqrt{\text{in}_{\mathbf{w}}(\mathfrak{p})}$.

Proof. The inclusion $\sqrt{\text{in}_{\mathbf{w}}(I)} \subset \bigcap_{\mathfrak{p} \in \text{Ass } I} \sqrt{\text{in}_{\mathbf{w}}(\mathfrak{p})}$ is trivial. We will prove the converse inclusion similarly to Theorem 4.11. Let $0 \neq (\alpha_1, \dots, \alpha_r) \in V_K(\text{in}_{\mathbf{w}}(I))$. Then $\alpha_i \neq 0$ for all i , and there exists $\boldsymbol{\eta} \in V_{\mathcal{M}}(I)$ such that $\text{in}_t(\boldsymbol{\eta}) = (\alpha_1 t^{w_1}, \dots, \alpha_r t^{w_r})$. Let $\mathfrak{p}_0 = \{f \in K[[\mathbf{x}]] \mid f(\boldsymbol{\eta}) = 0\}$. Since $I \subset \mathfrak{p}_0$ and $\mathfrak{p}_0 \neq \langle \mathbf{x} \rangle$, $\mathfrak{p}_0 \in \text{Ass } I$. As we saw in the proof of Theorem 4.11, $\sqrt{\text{in}_{\mathbf{w}}(\mathfrak{p}_0)} = \{g \in K[[\mathbf{x}]] \mid g(\text{in}_t(\boldsymbol{\eta})) = 0\}$. Thus $(\alpha_1, \dots, \alpha_r) \in V_K(\text{in}_{\mathbf{w}}(\mathfrak{p}_0)) \subset V_K(\bigcap_{\mathfrak{p} \in \text{Ass } I} \sqrt{\text{in}_{\mathbf{w}}(\mathfrak{p})})$. This proves $\sqrt{\text{in}_{\mathbf{w}}(I)} = \bigcap_{\mathfrak{p} \in \text{Ass } I} \sqrt{\text{in}_{\mathbf{w}}(\mathfrak{p})}$. $\square$

The following is the main algorithm in this paper.

Algorithm 5.5 (Algorithm for deciding irreducibility of reduced algebroid curves).

Input: a one-dimensional radical ideal $I \subset K[[x_1, \dots, x_r]]$ with $x_i \notin I$ for all i .

Output: (“*result*”, J) such that J is a one-dimensional ideal of $K[[x_1, \dots, x_{r'}]]$, $r' \geq r$, and

- (1) $K[[x_1, \dots, x_r]]/I \cong K[[x_1, \dots, x_{r'}]]/J$,
  - (2) if I is prime, then “*result*” = “*true*” and $\omega(J)$ is a tropism of J ,
  - (3) if I is not prime, then “*result*” = “*false*” and J has at least two tropisms.
- 1: $\mathbf{x} := (x_1, \dots, x_r)$, $\mathbf{w} = (w_1, \dots, w_r) := \omega(I)$.
  - 2: **if** ($\text{in}_{\mathbf{w}}(I)$ contains monomials, or $w_i = \infty$ for some i) **return** (“*false*”, I).
  - 3: **while** ($\text{gcd}(\mathbf{w}) \neq 1$) **do**
  - 4: $G :=$ a system of binomial generators of $\text{Prim}(\mathbf{w})$, $G' := \emptyset$.
  - 5:   **for all** ($\mathbf{x}^a - \mathbf{x}^b \in G$) **do**
  - 6: $\Gamma := \text{find\_coeff}(\mathbf{x}^a, \mathbf{x}^b, I)$.
  - 7:   **if** ($\#\Gamma \neq 1$) **return** (“*false*”, $\text{distinguish\_branches}(\Gamma, \mathbf{x}^a, \mathbf{x}^b, I)$).
  - 8: $G' := G' \cup \{\mathbf{x}^a - \alpha \mathbf{x}^b\}$ where $\Gamma = \{\alpha\}$.
  - 9:   **end for**
  - 10:   Take $f = \mathbf{x}^a - \alpha \mathbf{x}^b \in G' \setminus \text{in}_{\mathbf{w}}(I)$, and let $N := \text{int}(f; I)$.
  - 11:   **if** ($N = \infty$) **return** (“*false*”, $\langle I, x_{r+1} - f \rangle$).
  - 12:   **while** ($N \in \text{Span}_{\mathbb{N}}(\mathbf{w})$) **do**
  - 13:   Take $\mathbf{c} \in \mathbb{N}^r$ such that $N = \mathbf{c} \cdot \mathbf{w}$.
  - 14: $\Gamma := \text{find\_coeff}(f, \mathbf{x}^{\mathbf{c}}, I)$.
  - 15:   **if** ($\#\Gamma \neq 1$) **return** (“*false*”, $\text{distinguish\_branches}(\Gamma, \mathbf{x}^a, \mathbf{x}^b, I)$).
  - 16: $N := \text{int}(f - \beta \mathbf{x}^{\mathbf{c}}; I)$, $f := f - \beta \mathbf{x}^{\mathbf{c}}$, where $\Gamma = \{\beta\}$.
  - 17:   **if** ($N = \infty$) **return** (“*false*”, $\langle I, x_{r+1} - f \rangle$).
  - 18:   **end while**
  - 19: $I := \langle I, x_{r+1} - f \rangle$, $\mathbf{x} := (\mathbf{x}, x_{r+1})$, $\mathbf{w} := (\mathbf{w}, N)$, $r := r + 1$.
  - 20:   **if** ($\text{in}_{\mathbf{w}}(I)$ contains monomials) **return** (“*false*”, I).
  - 21: **end while**
  - 22: **return** (“*true*”, I).

Proof. We will give a proof of the correctness of this algorithm.

Let $I = \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_l$ be the prime decomposition of I (at each step of this algorithm), and set $A_j = K[[\mathbf{x}]]/\mathfrak{p}_j$ and $\phi_j : K[[\mathbf{x}]] \rightarrow K[[t]]$ as in the proof of Theorem 5.1. Let $\boldsymbol{\xi}^{(j)} = (\phi_j(x_1), \dots, \phi_j(x_r))$. Then $\phi_j = \phi_{\boldsymbol{\xi}^{(j)}}$. Recall that $\text{Ker } \phi_{\text{int}(\boldsymbol{\xi}^{(j)})} = \sqrt{\text{in}_{\omega(\mathfrak{p}_j)}(\mathfrak{p}_j)}$ by Theorem 4.11.

At Line 2: If $\text{in}_{\mathbf{w}}(I)$ contains monomials, then I has at least two tropisms. If $\text{int}(x_i; I) = \infty$, there exists $1 \leq j_0 \leq l$ such that $x_i \in \mathfrak{p}_{j_0}$. On the other hand, since $x_i \notin I$, there exists $1 \leq j_1 \leq l$ such that $x_i \notin \mathfrak{p}_{j_1}$. Hence $\tilde{\mathbb{R}}_+ \cdot \omega(\mathfrak{p}_{j_0}) \neq \tilde{\mathbb{R}}_+ \cdot \omega(\mathfrak{p}_{j_1})$, and thus I has at least two tropisms.

The for loop from Line 5 to 9: If $\#\Gamma \neq 1$, then $\text{distinguish\_branches}(\Gamma, \mathbf{x}^a, \mathbf{x}^b, I)$ has at least two tropisms by Theorem 5.1 (2), (3). Assume that $\Gamma = \{\alpha\}$, that is, $\text{int}(\mathbf{x}^a - \alpha \mathbf{x}^b; I) > \mathbf{w} \cdot \mathbf{a}$ and $\text{int}(\mathbf{x}^a - \beta \mathbf{x}^b; I) = \mathbf{w} \cdot \mathbf{a}$ for $\alpha \neq \beta \in K$. Then $\text{int}(\mathbf{x}^a; \mathfrak{p}_j) = \text{int}(\mathbf{x}^b; \mathfrak{p}_j)$, and $\mathbf{x}^a - \alpha \mathbf{x}^b \in \text{Ker } \phi_{\text{int}(\boldsymbol{\xi}^{(j)})} = \sqrt{\text{in}_{\omega(\mathfrak{p}_j)}(\mathfrak{p}_j)}$ for all $1 \leq j \leq l$ as we saw in the first part of the proof of Theorem 5.1. In particular, $\mathbf{x}^a - \mathbf{x}^b$ is $\omega(\mathfrak{p}_j)$ -homogeneous for all $1 \leq j \leq l$. Therefore, when this for loop terminates, any binomial in G is $\omega(\mathfrak{p}_j)$ -homogeneous, and $G' \subset \sqrt{\text{in}_{\omega(\mathfrak{p}_j)}(\mathfrak{p}_j)}$ for

all $1 \leq j \leq l$. This shows that $\mathbb{R}_+ \cdot \mathbf{w} = \mathbb{R}_+ \cdot \boldsymbol{\omega}(\mathfrak{p}_j)$, and $\sqrt{\text{in}_{\mathbf{w}}(\mathfrak{p}_j)} = \langle G' \rangle$ for all $1 \leq j \leq l$. Since $\mathcal{T}_{\text{loc}}(I) = \bigcup_{j=1}^l \widetilde{\mathbb{R}}_+ \cdot \boldsymbol{\omega}(\mathfrak{p}_j) = \mathbb{R}_+ \cdot \mathbf{w}$, and by Lemma 5.4, G' is also a system of generators of $\sqrt{\text{in}_{\mathbf{w}}(I)}$.

At Line 10: Here, $\gcd(\mathbf{w}) \neq 1$, $\mathcal{T}_{\text{loc}}(I) = \mathbb{R}_+ \cdot \mathbf{w}$, and $\sqrt{\text{in}_{\mathbf{w}}(I)} = \sqrt{\text{in}_{\mathbf{w}}(\mathfrak{p}_j)} = \langle G' \rangle$ for all $1 \leq j \leq l$ are satisfied. We will prove that $\text{in}_{\mathbf{w}}(I) \neq \sqrt{\text{in}_{\mathbf{w}}(I)}$. Assume, to the contrary, that $\text{in}_{\mathbf{w}}(I) = \sqrt{\text{in}_{\mathbf{w}}(I)}$. Then $\text{in}_{\mathbf{w}}(I) = \sqrt{\text{in}_{\mathbf{w}}(\mathfrak{p}_j)} \supset \text{in}_{\mathbf{w}}(\mathfrak{p}_j)$ and $\text{in}_{\mathbf{w}}(I) \subset \text{in}_{\mathbf{w}}(\mathfrak{p}_j)$ for all $1 \leq j \leq l$. Thus $\text{in}_{\mathbf{w}}(I) = \text{in}_{\mathbf{w}}(\mathfrak{p}_j)$ which implies $I = \mathfrak{p}_j$ for all $1 \leq j \leq l$, that is, $l = 1$ and I is prime. Thus $\text{in}_{\mathbf{w}}(I) = \sqrt{\text{in}_{\mathbf{w}}(I)}$ implies that $\text{Semi}(K[[\mathbf{x}]]/I) = \text{Span}_{\mathbb{N}}(\mathbf{w})$ by Corollary 4.16 which contradicts to $\gcd(\mathbf{w}) \neq 1$. Thus $\text{in}_{\mathbf{w}}(I) \neq \sqrt{\text{in}_{\mathbf{w}}(I)}$ is satisfied, and one can take $f = \mathbf{x}^{\mathbf{a}} - \alpha \mathbf{x}^{\mathbf{b}} \in G' \setminus \text{in}_{\mathbf{w}}(I)$.

At Line 11: Since $\text{in}_{\mathbf{w}}(f) = f \notin \text{in}_{\mathbf{w}}(I)$ in any step, $f \notin I = \sqrt{I}$. Thus if $\text{int}(f; I) = \infty$, then $\langle I, x_{r+1} - f \rangle$ has at least two tropisms by Theorem 5.1 (4).

The while loop from Line 12 to 18: We will prove that this while loop terminates in finite time. Assume, to the contrary, that this while loop causes an infinite loop. This means that there exist $\mathbf{x}^{\mathbf{a}} - \alpha \mathbf{x}^{\mathbf{b}} \in \sqrt{\text{in}_{\mathbf{w}}(I)} \setminus \text{in}_{\mathbf{w}}(I)$, and $\beta_i \in K^\times$, $\mathbf{c}_i \in \mathbb{N}^r$ for $i \geq 1$ satisfying $\text{int}(f_{i+1}; I) > \text{int}(f_i; I) = \mathbf{w} \cdot \mathbf{c}_i$, and β_i is the unique element such that $\text{int}(f_i - \beta_i \mathbf{x}^{\mathbf{c}_i}; I) > \text{int}(f_i; I)$ where $f_0 = \mathbf{x}^{\mathbf{a}} - \alpha \mathbf{x}^{\mathbf{b}}$ and $f_i = f_{i-1} - \beta_i \mathbf{x}^{\mathbf{c}_i} = (\mathbf{x}^{\mathbf{a}} - \alpha \mathbf{x}^{\mathbf{b}}) - \beta_1 \mathbf{x}^{\mathbf{c}_1} - \beta_2 \mathbf{x}^{\mathbf{c}_2} - \dots - \beta_i \mathbf{x}^{\mathbf{c}_i}$ for $i \geq 1$. Since $|\mathbf{c}_i| \rightarrow \infty$ as $i \rightarrow \infty$, the limit $F := \lim_{i \rightarrow \infty} f_i$ exists. As we observed in the proof of Theorem 5.1, the uniqueness of β_i shows that $\text{int}(\phi_j(f_{i-1})) = \beta_i \cdot \text{int}(\phi_j(\mathbf{x}^{\mathbf{c}_i}))$ for all $i \geq 1$ and $1 \leq j \leq l$. Hence $\lim_{i \rightarrow \infty} \text{ord}_t(\phi_j(f_i)) = \infty$, and thus $\lim_{i \rightarrow \infty} \phi_j(f_j) = 0$ for all j . As ϕ_j is continuous, this implies $F \in \text{Ker } \phi_j = \mathfrak{p}_j$ for all j . Hence $F \in I = \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_l$. On the other hand, $\text{in}_{\mathbf{w}}(F) = f_0 = \mathbf{x}^{\mathbf{a}} - \alpha \mathbf{x}^{\mathbf{b}}$ since $\text{ord}_{\mathbf{w}}(f_0) < \text{int}(f_0; I) = \mathbf{w} \cdot \mathbf{c}_1 < \text{int}(f_1; I) = \mathbf{w} \cdot \mathbf{c}_2 < \dots$, which contradicts to $f_0 \notin \text{in}_{\mathbf{w}}(I)$. At Line 17: As $\text{in}_{\mathbf{w}}(f) = \mathbf{x}^{\mathbf{a}} - \alpha \mathbf{x}^{\mathbf{b}} \notin \text{in}_{\mathbf{w}}(I)$, $f \notin I$. Thus if $\text{int}(f; I) = \infty$, then $\langle I, x_{r+1} - f \rangle$ has at least two tropisms by Theorem 5.1 (4).

At Line 20: If $\text{in}_{\mathbf{w}}(I)$ contains monomials, then I has at least two tropisms.

The while loop from Line 3 to 21: At Line 19, the semigroup $\text{Span}_{\mathbb{N}}(\mathbf{w})$ become strictly larger. Hence this while loop does not cause infinite loop by Lemma 4.6. When this while loop terminates, it holds that $\mathbf{w} = \boldsymbol{\omega}(I) \in \mathcal{T}_{\text{loc}}(I)$ and $\gcd(\mathbf{w}) = 1$, and thus $\boldsymbol{\omega}(I)$ is a tropism of I .

Therefore, this algorithm terminates in finite steps, and the output satisfies the desired conditions. $\square$

Remark 5.6. If one apply Algorithm 5.5 to a non-radical ideal, it takes infinite time to complete the while loop from Line 19 to 26 in general. Note that the question of determining whether or not a given ideal is a radical ideal is uncomputable in general ([23] Lemma 1). When we consider the implementation in computer algebra systems, we assume that k is a computable perfect field (e.g. $\mathbb{Q}$ or $\mathbb{F}_p$), and $I = I_0 K[[\mathbf{x}]]$ for some $I_0 \subset k[\mathbf{x}]$. Since $K[\mathbf{x}]$ is an excellent ring, one can compute $\sqrt{I} = \sqrt{I_0} K[[\mathbf{x}]]$ in this situation. If $\alpha, \beta \in K$ are conjugate over k and $f, g \in k[\mathbf{x}]$, then $\text{int}(f - \alpha g; I) = \text{int}(f - \beta g; I)$. Hence if the element $\alpha \in K$ satisfying $\text{int}(f - \alpha g; I) > \text{int}(f; I)$ is unique, then $\alpha \in k$. Thus all power series appearing in Algorithm 5.5 are polynomials over k except for the output.

Example 5.7. Let $a_1, a_2 \in \mathbb{N}_+$ with $\gcd(a_1, a_2) = 1$. Take $2 \leq d \in \mathbb{N}_+$ and $c_1, c_2 \in \mathbb{N}$ such that $c_1 a_1 + c_2 a_2 > d a_1 a_2$. Let $d_1 = \gcd(c_1 a_1 + c_2 a_2, d)$, and assume that $d_1 \neq 1$ and d_1 is not a pure power of $p = \text{char } K \geq 0$. Let $F(x, y) = (y^{a_1} -$

$x^{a_2} \rangle^d - x^{c_1} y^{c_2} \in K[x, y]$. Then $\omega(F) = (da_1, da_2)$, and $\text{in}_{\omega(F)}(F) = (y^{a_1} - x^{a_2})^d$. Take $y^{a_1} - x^{a_2} \in \sqrt{\langle \text{in}_{\omega(F)}(F) \rangle} \setminus \langle \text{in}_{\omega(F)}(F) \rangle$. Then $\text{int}(y^{a_1} - x^{a_2}; F) = c_1 a_1 + c_2 a_2$.

(1) Suppose that $d_1 < d$. Then $\text{int}(y^{a_1} - x^{a_2}; F) = c_1 a_1 + c_2 a_2 \notin \text{Span}_{\mathbb{N}}(da_1, da_2)$. Let $I = \langle F, z - (y^{a_1} - x^{a_2}) \rangle = \langle z^d - x^{c_1} y^{c_2}, y^{a_1} - x^{a_2} - z \rangle \subset K[x, y, z]$. Then $\omega(I) = (da_1, da_2, c_1 a_1 + c_2 a_2)$ and $\gcd(\omega(I)) = d_1$. Write $c_1 a_1 + c_2 a_2 = d_1 b$ and $d = d_1 d_2$ for $2 \leq b, d_2 \in \mathbb{N}$. Since $b > d_2 a_1 a_2 > (a_1 - 1)(a_2 - 1)$, and the conductor of $\text{Span}_{\mathbb{N}}(a_1, a_2)$ is $(a_1 - 1)(a_2 - 1)$, $b \in \text{Span}_{\mathbb{N}}(a_1, a_2)$. Write $b = e_1 a_1 + e_2 a_2$, $e_1, e_2 \in \mathbb{N}$. Then

$$\text{Prim}(\omega(I)) = \text{Prim}(da_1, da_2, d_1 b) = \text{Prim}(d_2 a_1, d_2 a_2, b) = \langle z^{d_2} - x^{e_1} y^{e_2}, y^{a_1} - x^{a_2} \rangle.$$

We will compute $\text{int}(z^{d_2} - \alpha x^{e_1} y^{e_2}; I)$ for $\alpha \in K^\times$. As $\langle z^{d_2} - \alpha x^{e_1} y^{e_2}, I \rangle = \langle \mathfrak{p}_\alpha, h_\alpha \rangle$, where $\mathfrak{p}_\alpha := \langle z^{d_2} - \alpha x^{e_1} y^{e_2}, y^{a_1} - x^{a_2} - z \rangle$ and $h_\alpha := \alpha^{d_1} x^{d_1 e_1} y^{d_1 e_2} - x^{c_1} y^{c_2}$, we have $\text{int}(z^{d_2} - \alpha x^{e_1} y^{e_2}; I) = \text{int}(h_\alpha, \mathfrak{p}_\alpha)$. Note that $\mathfrak{p}_\alpha$ is a prime ideal, $\omega(\mathfrak{p}) = (d_2 a_1, d_2 a_2, e_1 a_1 + e_2 a_2)$, and $\text{in}_{\omega(\mathfrak{p}_\alpha)}(\mathfrak{p}_\alpha) = \sqrt{\text{in}_{\omega(\mathfrak{p}_\alpha)}(\mathfrak{p}_\alpha)} = \langle z^{d_2} - \alpha x^{e_1} y^{e_2}, y^{a_1} - x^{a_2} \rangle = \text{Prim}(\omega(\mathfrak{p}_\alpha))$. Since $\text{in}_{\omega(\mathfrak{p}_\alpha)}(h_\alpha) = h_\alpha \in \sqrt{\text{in}_{\omega(\mathfrak{p}_\alpha)}(\mathfrak{p}_\alpha)}$ if and only if $\alpha^{d_1} = 1$, we have $\text{int}(h_\alpha; \mathfrak{p}_\alpha) > db$ if $\alpha^{d_1} = 1$, and $\text{int}(h_\alpha; \mathfrak{p}_\alpha) = \text{ord}_{\omega(\mathfrak{p}_\alpha)}(h) = db$ otherwise by Corollary 4.12. Hence F is reducible, and $\langle I, u_1 - (z^{d_2} - x^{e_1} y^{e_2}), u_2 - (z^{d_2} - \beta x^{e_1} y^{e_2}) \rangle \subset K[x, y, u_1, u_2]$ has at least two tropisms for $\beta \in K$ such that $\beta^{d_1} = 1$ and $\beta \neq 1$.

(2) Suppose that $d_1 = d$. Since $\text{Span}_{\mathbb{N}}(da_1, da_2)$ contains all multiples of d not less than $d(a_1 - 1)(a_2 - 1)$, and $c_1 a_1 + c_2 a_2 > da_1 a_2 > d(a_1 - 1)(a_2 - 1)$, $\text{int}(y^{a_1} - x^{a_2}; F) = c_1 a_1 + c_2 a_2 \in \text{Span}_{\mathbb{N}}(da_1, da_2)$. Write $c_1 a_1 + c_2 a_2 = e_1 da_1 + e_2 da_2$. Then $\text{int}(y^{a_1} - x^{a_2}; F) = \text{int}(x^{e_1} y^{e_2}; F)$. We will compute $\text{int}(y^{a_1} - x^{a_2} - \alpha x^{e_1} y^{e_2}; F)$ for $\alpha \in K$. In the same way as in the case (1), one concludes that $\text{int}(y^{a_1} - x^{a_2} - \alpha x^{e_1} y^{e_2}; F) > c_1 a_1 + c_2 a_2$ if $\alpha^d = 1$, and $\text{int}(y^{a_1} - x^{a_2} - \alpha x^{e_1} y^{e_2}; F) = c_1 a_1 + c_2 a_2$ otherwise. Therefore, F is reducible, and $\langle I, u_1 - (y^{a_1} - x^{a_2} - \alpha x^{e_1} y^{e_2}), u_2 - (y^{a_1} - x^{a_2} - \beta x^{e_1} y^{e_2}) \rangle \subset K[x, y, u_1, u_2]$ has at least two tropisms for $\beta \in K$ such that $\beta^d = 1$ and $\beta \neq 1$.

By Example 4.21 and Example 5.7, we have the following:

Proposition 5.8. *Let*

$$F(x, y) = (y^{a_1} - x^{a_2})^d - x^{c_1} y^{c_2} \in K[x, y]$$

where $d, a_1, a_2 \geq 1$, c_1, c_2 are non-negative integers such that $c_1 a_1 + c_2 a_2 > da_1 a_2$, $\gcd(a_1, a_2) = 1$, and $\gcd(d, c_1 a_1 + c_2 a_2)$ is not a pure power of $p = \text{char } K \geq 0$ (e.g. $\text{char } K = 0$, or $\text{char } K = p > 0$ and $\gcd(p, d, c_1 a_1 + c_2 a_2) = 1$). Then F is irreducible if and only if $\gcd(d, c_1 a_1 + c_2 a_2) = 1$.

Example 5.9. Assume that $\text{char}(K) = 0$. Let $I = \langle x^3 - y^2 - z^d, (z^2 - xy)^2 - x^a y^a z^b \rangle \subset K[x, y, z]$ where $d, a, b \in \mathbb{N}$ such that $d \geq 3$ and $a \geq 3$. Then $\omega(I) = (8, 12, 10)$, $\text{in}_{\omega(I)}(I) = \langle x^3 - y^2, (z^2 - xy)^2 \rangle$. Take $z^2 - xy \in \sqrt{\text{in}_{\omega(I)}(I)} \setminus \text{in}_{\omega(I)}(I)$. Then $\text{int}(z^2 - xy; I) = 10a + 5b$. We will show that I is prime if and only if b is odd.

Assume that b is odd. Then $\text{int}(z^2 - xy; I) \notin \text{Span}_{\mathbb{N}}(8, 12, 10)$. Let $J = \langle I, u - (z^2 - xy) \rangle \subset K[x, y, z, u]$. Then $\omega(J) = (8, 12, 10, 10a + 5b)$ and $\text{in}_{\omega(J)}(J) = \langle x^3 - y^2, z^2 - xy, u^2 - x^a y^a z^b \rangle = \text{Prim}(\omega(J))$. Hence $\omega(J)$ is a tropism of J , and thus I is prime.

Assume that b is even, and write $b = 2c$ for $c \in \mathbb{N}$. Then $\text{int}(z^2 - xy; I) = 10(a + c) = \text{int}(z^{a+c}; I)$. We will compute $\text{int}((z^2 - xy) - \alpha z^{a+c}; I)$ for $\alpha \in K$. Let $\mathfrak{p} = \langle (z^2 - xy) - \alpha z^{a+c}, x^3 - y^2 - z^d \rangle$ and $h = z^{2c}(\alpha^2 z^{2a} - x^a y^a)$. Since

$\langle (z^2 - xy) - \alpha z^{a+c}, I \rangle = \langle \mathbf{p}, h \rangle$, $\text{int}((z^2 - xy) - \alpha z^{a+c}; I) = \text{int}(h; \mathbf{p})$. In the same way as in Example 5.7, one can show that $\text{int}((z^2 - xy) - \alpha z^{a+c}; I) = \text{int}(h; \mathbf{p}) > 10(a+c)$ if $\alpha^2 = 1$, and $\text{int}((z^2 - xy) - \alpha z^{a+c}; I) = \text{int}(h; \mathbf{p}) = 10(a+c)$ otherwise. Thus I is not prime, and $J := \langle I, u_1 - (z^2 - xy - z^{a+c}), u_2 - (z^2 - xy + z^{a+c}) \rangle \subset K[[x, y, z, u_1, u_2]]$ has two tropisms.

For $\mathbf{w} \in \mathbb{R}_+^r$, we define the local ordering $\prec_{ds}^{(\mathbf{w})}$ with $x_1 \prec_{ds}^{(\mathbf{w})} \cdots \prec_{ds}^{(\mathbf{w})} x_r$ as follows; for $\mathbf{a}, \mathbf{b} \in \mathbb{N}^r$, $\mathbf{x}^{\mathbf{a}} \prec_{ds}^{(\mathbf{w})} \mathbf{x}^{\mathbf{b}}$ if (i) $\mathbf{w} \cdot \mathbf{a} > \mathbf{w} \cdot \mathbf{b}$, or (ii) $\mathbf{w} \cdot \mathbf{a} = \mathbf{w} \cdot \mathbf{b}$ and $a_n > b_n$ for $n = \min\{i \mid a_i \neq b_i\}$. Note that a standard basis of a ideal $I \subset K[[\mathbf{x}]]$ with respect to $\prec_{ds}^{(\mathbf{w})}$ is also a standard basis of I with respect to $\mathbf{w}$.

Example 5.10. Let $I = \langle y^2 - xz, x^5 + z^3 - 2x^2yz - y^c \rangle \subset K[[x, y, z]]$, $c \geq 4$. Then $\omega(I) = (6, 8, 10)$, $\text{in}_{\omega(I)}(I) = \langle y^2 - xz, x^5 + z^3 - 2x^2yz \rangle$, and $\sqrt{\text{in}_{\omega(I)}(I)} = \langle x^3 - yz, y^2 - xz, z^2 - x^2y \rangle$. We will compute $\text{int}(x^3 - yz; I)$. The set

$$\{xz - y^2, x^3 - yz, yz^2 - x^2y^2, z^3 - xy^3 - y^c, xy^c, y^{c+1}\}$$

is a standard basis of $\langle I, x^3 - yz \rangle$ with respect to $\prec_{ds}^{(\omega(I))}$ with $y \prec_{ds}^{(\omega(I))} z \prec_{ds}^{(\omega(I))} x$. Thus

$$\begin{aligned} \text{int}(x^3 - yz; I) &= \dim_K K[x, y, z] / \langle xz, x^3, yz^2, z^3, xy^c, y^{c+1} \rangle \\ &= 4c + 3 \notin \text{Span}_{\mathbb{N}}(6, 8, 10). \end{aligned}$$

Let $J = \langle I, x^3 - yz \rangle \subset K[[x, y, z, u]]$. Then $\omega(J) = (6, 8, 10, 4c + 3)$, and

$$\{xz - y^2, x^3 - yz - u, yz^2 - x^2y^2 + zu, z^3 - xy^3 + x^2u - y^c, xy^c - u^2, x^2yu - z^2u - y^{c+1}\}$$

is a standard basis of J with respect to $\prec_{ds}^{(\omega(J))}$ with $y \prec_{ds}^{(\omega(J))} z \prec_{ds}^{(\omega(J))} x \prec_{ds}^{(\omega(J))} u$. Thus

$$\text{in}_{\omega(J)}(J) = \langle xz - y^2, xy^c - u^2, x^3 - yz, x^2y^2 - yz^2, x^2yu - z^2u, xy^3 - z^3 \rangle.$$

Hence $\omega(J)$ is a tropism of J , and thus I is prime.

Remark 5.11. One can decide irreducibility of an reduced algebroid curve using resolution of singularities; a reduced algebroid curve is irreducible if and only if the strict transform does not bifurcate. Our method determines the irreducibility of algebroid curves more directly, and it is easier to follow the computation in some cases. For example, consider $F = (x^a - y^{2b+1})^2 - x^{2a+1} \in K[[x, y]]$, $\gcd(a, 2b+1) = 1$. We regard a and b as parameters. To construct the resolution of singularity of F by a succession of blowing-ups at the singular point, it needs infinitely many steps, and one should consider infinitely many cases: we first compute the blowing-up at the origin. We should consider three cases $a < 2b + 1$, $a > 2b + 1$, and $a = 2b + 1 = 1$. In the case of $a < 2b + 1$, the strict transform of F is $(x^a - y^{2b+1-a})^2 - xy^{2a+1}$, and then we should consider three cases $a < 2b + 1 - a$, $a > 2b + 1 - a$, and $a = 2b + 1 - a = 1$, and so on. This procedure does not terminate in finite steps. On the other hand, our method decides the irreducibility in a few steps: the output is $J = \langle F, z - (x^a - y^{2b+1}) \rangle \subset K[[x, y, z]]$, and $\omega(J) = (4b + 2, 2a, (2a + 1)(2b + 1))$ is a tropism of J since $\gcd(\omega(J)) = \gcd(2, (2a + 1)(2b + 1)) = 1$ and $\text{in}_{\omega(J)}(J) = \langle z^2 - x^{2a+1}, x^a - y^{2b+1} \rangle \subset K[[x, y, z]]$ contains no monomial. If some positive integers are substituted to the parameters, the numbers of steps of blowing-ups can be arbitrary large. For curves of higher codimension (e.g. Examples 4.22, 4.25, 5.9), the resolution of singularity can be more complicated. Hence our method is easier to follow the computation in these cases.

ACKNOWLEDGMENTS

This research was supported by JST, CREST. The author wishes to thank Thomas Markwig and Anders Nedergaard Jensen for their useful comments and discussions.

REFERENCES

- [1] S. Abhyankar, Irreducibility criterion for germs of analytic functions of two complex variables, *Adv. Math.* **74** (1989), 190–257. MR997097 (90h:32018)
- [2] R. Bahloul, N. Takayama, Local Gröbner fans: polyhedral and computational approach, preprint (2004), arXiv:math/0412044v2.
- [3] R. Bieri, J.R.J. Groves, The geometry of the set of characters induced by valuations, *J. reine und angewandte Mathematik* **347** (1984), 168–195. MR733052 (86c:14001)
- [4] T. Bogart, A. Jensen, D. Speyer, B. Sturmfels, and R. Thomas: Computing tropical varieties, *J. Symbolic Comput.* **42** (2007), 54–73. MR2284285 (2007j:14103)
- [5] D. Cox, J. Little, D. O’Shea, *Ideals, Varieties and Algorithms*, Springer-Verlag, 1992. MR1189133 (93j:13031)
- [6] D. Cox, J. Little, D. O’Shea, *Using Algebraic Geometry*, Springer-Verlag, 1998. MR1639811 (99h:13033)
- [7] C. Delorme, Sous-monoides d’intersection complète de $\mathbf{N}$. *Ann. Scient. Ec. Norm. Sup.* 4^e Série, t. 9: 145–154, 1976. MR0407038 (53:10821)
- [8] J. Draisma, A tropical approach to secant dimensions, *Journal of Pure and Applied Algebra* **212** (2008), 349–363. MR2357337 (2008j:14102)
- [9] M. Einsiedler, M. Kapranov, and D. Lind, Non-Archimedean amoebas and tropical varieties, *J. Reine Angew. Math.*, **601** (2006), 139–157. MR2289207 (2007k:14038)
- [10] J. Fresnel, M. van der Put, *Rigid Analytic Geometry and its Applications*, Progress in Mathematics 218, Birkhäuser, 2003. MR2014891 (2004i:14023)
- [11] G.-M. Greuel, G. Pfister, *A Singular Introduction to Commutative Algebra*, Springer-Verlag, Berlin, 2002, With contributions by Olaf Bachmann, Christoph Lossen and Hans Schöneemann, With 1 CD-ROM (Windows, Macintosh, and UNIX). MR2363237 (2008j:13001)
- [12] A. Hefez, M.E. Hernandez, *Computational Methods in the Local Theory of Curves*, 23^o Colóquio Brasileiro de Matemática, IMPA, 2001. MR1849596 (2002g:14043)
- [13] A. Hefez, M.E. Hernandez, Standard bases for local rings of branches and their modules of differentials, *J. Symb. Comp.* **42** (2007), 178–191. MR2284291 (2008g:13035)
- [14] A. Jensen, H. Markwig, T. Markwig, An algorithm for lifting points in a tropical variety, *Collect. Math.* **59** (2008), no. 2, 129–165. MR2414142 (2009a:14077)
- [15] E. Kunz, The value-semigroup of a one dimensional Gorenstein ring, *Proc. Amer. Math. Soc.* **25**, (1970), 748–751. MR0265353 (42:263)
- [16] T. C. Kuo, A simple algorithm for deciding primes in $K[[x, y]]$. *Canad. J. Math.* **47**, no. 4 (1995), 801–816. MR1346164 (96k:13031)
- [17] J. Maurer, Puiseux expansion for space curves, *Manuscripta Math.*, **32** (1980), 91–100. MR592712 (81m:32009)
- [18] M. Nagata, *Local Rings*, Intersciences Publishers, New York, London, 1962. MR0155856 (27:5790)
- [19] L. Robbiano, M. Sweedler, Subalgebra Bases, *Lecture Notes in Math.*, vol. **1430**, Springer, 1990, 67–87. MR1068324 (91f:13027)
- [20] D. Speyer, B. Sturmfels, The tropical Grassmannian, *Advances in Geometry*, **4** (2004), 389–411. MR2071813 (2005d:14089)
- [21] B. Sturmfels, *Gröbner Bases and Convex Polytopes*, Univ. Lecture Ser., vol. 8, Amer.Math. Soc., Providence, 1996. MR1363949 (97b:13034)
- [22] B. Sturmfels, *Solving Systems of Polynomial Equations*, CBMS Regional Conference Series in Math., **97**, American Mathematical Society, 2002. MR1925796 (2003i:13037)
- [23] J. Teitelbaum, The computational complexity of the resolution of plane curve singularities, *Math. Comput.* **54** 190 (1990), 797–837. MR1010602 (90j:14007)

- [24] N. Touda, Local Tropical Variety, preprint (2005), arXiv:math.AG/0511486.
- [25] V. Weispfenning, Comprehensive Gröbner bases, J. Symbolic Computation, **14** (1991), 1–29.
MR1177987 (93h:13029)

DEPARTMENT OF MATHEMATICS, RIKKYO UNIVERSITY, NISHI-IKEBUKURO, TOKYO 171-8501,
JAPAN

E-mail address: `shibuta@rikkyo.ac.jp`