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N-TUPLES OF POSITIVE INTEGERS WITH THE SAME SUM AND THE SAME PRODUCT

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Abstract. In this paper, by using the theory of elliptic curves, we prove that for every k, there exists infinitely many primitive sets of k n-tuples of positive integers with the same sum and the same product.

1. Introduction

In 1981, J. G. Mauldon [3] proposed the Problem E2872: Find five different triples of positive integers with the same sum and the same product. In 1982, L. L. Foster and G. Robins [1] gave ten triples with the sum 132600 and the product $2^7 3^6 5^4 7^2 13^3 17^3$. In 1996, A. Schinzel [4] proved that for every k, there exist infinitely many primitive sets of k triples of positive integers with the same sum and the same product, i.e., the system of equations

$$\begin{cases} x_{i1}+x_{i2}+x_{i3}=A, \\ x_{i1}x_{i2}x_{i3}=B, \\ x_{ij}>0,\ A>0,\ B>0, \\ i=1,\cdots,k,\ j=1,2,3, \end{cases} \tag{1.1}$$

has k solutions for every $k \in \mathbb{N}-\{0\}$. A set S of triples is called primitive if the greatest common divisor of all elements of all triples of S is 1.

More information on this problem can be found in [2]: *D16 Triples with the same sum and the same product* and *D24 Sum equals product*.

In this paper, we consider the generalized system of equations

$$\begin{cases} x_{i1}+\cdots+x_{in}=A, \\ x_{i1}\cdots x_{in}=B, \\ x_{ij}>0,\ A>0,\ B>0, \\ i=1,\cdots,k,\ j=1,\cdots,n,\ n\geq 4. \end{cases} \tag{1.2}$$

By using the theory of elliptic curves, we prove the following theorem.

Theorem 1.1. *For every k, there exist infinitely many primitive sets of k n-tuples of positive integers with the same sum and the same product.*

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2. An important lemma

In [4], A. Schinzel proved that:

The system of equations

$$x_1 + x_2 + x_3 = x_1x_2x_3 = 6$$

has infinitely many solutions in rational numbers $x_j > 0$.

The equation

$$x_1 + \cdots + x_n = x_1 \cdots x_n,\ n \geq 3$$

always has at least one solution $(1, 1, \cdots, 1, 2, n)$ such that it has the same sum and product $2n$.

To prove the theorem, we consider an analogous system of equations and get the following lemma.

Lemma 2.1. *The system of equations*

$$x_1 + \cdots + x_n = x_1 \cdots x_n = 2n \tag{2.1}$$

has infinitely many solutions in rational numbers $x_j > 0$ *for* $n \geq 3$.

Proof. Because of Schinzel's result, we can assume that $n \geq 4$. Taking $x_1 = 1, \cdots, x_{n-3} = 1$, we have

$$\begin{cases} x_{n-2} + x_{n-1} + x_n = n + 3, \\ x_{n-2}x_{n-1}x_n = 2n. \end{cases} \tag{2.2}$$

Eliminating x_n of (2.2), we get

$$x_{n-2}^2 x_{n-1} + x_{n-2}x_{n-1}^2 - (n+3)x_{n-2}x_{n-1} + 2n = 0,$$

leading to

$$(\frac{x_{n-2}}{x_{n-1}})^2 + \frac{x_{n-2}}{x_{n-1}} - (n+3)\frac{x_{n-2}}{x_{n-1}}\frac{1}{x_{n-1}} + 2n(\frac{1}{x_{n-1}})^3 = 0.$$

Taking

$$u = \frac{x_{n-2}}{x_{n-1}},\ v = \frac{1}{x_{n-1}},$$

we have

$$u^2 + u - (n+3)uv + 2nv^3 = 0.$$

Let

$$y = 216n(2u + 1 - (n+3)v),\ x = -72nv + 3(n+3)^2, \tag{2.3}$$

and we get

$$\begin{aligned} E_n : y^2 =& x^3 - 27(n+3)(n^3 + 9n^2 - 21n + 27)x + 54n^6 + 972n^5 + 3402n^4 \\ & - 5832n^3 + 7290n^2 - 26244n + 39366. \end{aligned}$$

This is a family of elliptic curves, which is defined over $\mathbb{Q}$. We study the rational points on E_n.

The discriminant of E_n is $\Delta = 2^{15}3^{12}n^3(n^3 + 9n^2 - 27n + 27)$, and when $n \geq 4$, we have $\Delta > 0$, this means that E_n is nonsingular. Meanwhile, the cubic equation

$$\begin{aligned} & x^3 - 27(n+3)(n^3 + 9n^2 - 21n + 27)x \\ & + 54n^6 + 972n^5 + 3402n^4 - 5832n^3 + 7290n^2 - 26244n + 39366 = 0 \end{aligned}$$

has three different real roots $x_1(n),\ x_2(n),\ x_3(n)$. Suppose $x_1(n) < x_2(n) < x_3(n)$, then from the relationship of roots and coefficients, we have

$$\begin{cases} x_1(n)+x_2(n)+x_3(n)=0, \\ x_1(n)x_2(n)x_3(n)=-f(n), \end{cases}$$

where $f(n)=54n^6+972n^5+3402n^4-5832n^3+7290n^2-26244n+39366$. An easy calculation shows that $f(n)>0$ when $n \geq 4$, then

$$x_1(n) < 0 < x_2(n) < x_3(n),\ -x_1(n) = x_2(n)+x_3(n).$$

It is easy to check that the points $P=(3(n+3)^2, 216n)$, $Q=(3(n-3)^2, 108n(n-1))$ and $R=(3(n+3)^2-72n, 216n(n-2))$ lie on E_n. Using the Group Law on elliptic curves, we obtain the points

$$[2]P=O,\ P+Q+R=O$$

and

$$[2]R=\left(\frac{3(n^4+2n^3+13n^2-36n+36)}{(n-2)^2}, -\frac{216(2n^3-6n^2+7n-2)}{(n-2)^3}\right),$$

$$[3]R=\Big(3(n^6-36n^4+126n^3-180n^2+108n-15)/(n^2-3n+3)^2,$$

$$108(n-1)(n-2)(7n^4-33n^3+67n^2-66n+28)/(n^2-3n+3)^3\Big),$$

where O denotes the point at infinity on E_n and $[m]$ is the isogeny multiplication by m, which is defined by $[m](P)=P+\cdots+P(m \text{ terms})$. This means that P is a point of order 2 and P, Q, R lie on a line.

To prove that there are infinitely many rational points on E_n, it is enough to find a point on E_n with x-coordinate not in $\mathbb{Z}$. When the numerator of the x-coordinate of $[3]R$ is divided by $(n^2-3n+3)^2$, the remainder equals $r=-36(3n^3-12n^2+18n-10)$ and $r \neq 0$ when $n \geq 4$, so the x-coordinate of $[3]R$ is not a polynomial. For $4 \leq n \leq 109$ one can check that $r/(n^2-3n+3)^2$ is not an integer, and that it is nonzero and less than 1 in modulus for $n > 109$. Hence for all $n \geq 4$ the point $[3]R$ has nonintegral x-coordinate and hence, by the Nagell-Lutz Theorem (see p. 56 of [5]), is of infinite order. Then there are infinitely many rational points on E_n.

From the transformation (2.3), we have

$$u=\frac{y-3xn-9x+9n^3+81n^2+27n+243}{432n},\ v=\frac{3(n+3)^2-x}{72n},$$

leading to

$$x_{n-2}=\frac{y-3xn-9x+9n^3+81n^2+27n+243}{6(-x+3n^2+18n+27)},\ x_{n-1}=\frac{72n}{3(n+3)^2-x};$$

then

$$x_n=n+3-x_{n-2}-x_{n-1}=\frac{-y-3xn-9x+9n^3+81n^2+27n+243}{6(-x+3n^2+18n+27)}.$$

Therefore,

$$\begin{aligned}&(x_1,\cdots,x_{n-3},x_{n-2},x_{n-1},x_n)\\&=\Bigg(1,\cdots,1,\frac{y-3xn-9x+9n^3+81n^2+27n+243}{6(-x+3n^2+18n+27)},\\&\frac{72n}{3(n+3)^2-x},\frac{-y-3xn-9x+9n^3+81n^2+27n+243}{6(-x+3n^2+18n+27)}\Bigg)\end{aligned}$$

is a solution of (2.1).

In view of $x_j>0,\ j=1,\cdots,n$, we have the condition

$$x<3(n+3)^2,\ |y|<-3xn-9x+9n^3+81n^2+27n+243.$$

From the graph of $|y|=-3xn-9x+9n^3+81n^2+27n+243$, it is easy to see that when

$$x<\frac{3(n^3+9n^2+3n+27)}{n+3},$$

the above condition is satisfied, because

$$\frac{3(n^3+9n^2+3n+27)}{(n+3)}-3(n+3)^2=-\frac{72n}{(n+3)}<0$$

for $n\geq 4$, and

$$|y|=g\bigg(\frac{3(n^3+9n^2+3n+27)}{n+3}\bigg)=0,$$

where $g(x)=-3xn-9x+9n^3+81n^2+27n+243$.

Since $|y|=-3xn-9x+9n^3+81n^2+27n+243$ is the equation of the pair of tangent at $P=(3(n+3)^2,216n)$ and $-P=(3(n+3)^2,-216n)$, they intersect at the point

$$\bigg(\frac{3(n^3+9n^2+3n+27)}{n+3},0\bigg).$$

It is easy to see that

$$x_2(n)<\frac{3(n^3+9n^2+3n+27)}{n+3}<x_3(n).$$

In virtue of the theorem of Poincaré and Hurwitz (see [6], Chap. V, p. 78, Satz 11), E_n has infinitely many rational points in every neighborhood of any one of them. The point $[3]R$ satisfies the inequality $|y|<-3xn-9x+9n^3+81n^2+27n+243$, since

$$\begin{aligned}&\frac{3(n^6-36n^4+126n^3-180n^2+108n-15)}{(n^2-3n+3)^2}-\frac{3(n^3+9n^2+3n+27)}{n+3}\\&=\frac{-36(3n^2-7n+6)(3n^2-6n+4)}{(n^2-3n+3)^2(n+3)}<0,\end{aligned}$$

when $n\geq 4$. Hence, there are infinitely many rational points of E_n satisfying $|y|<-3xn-9x+9n^3+81n^2+27n+243$. Therefore, we can find infinitely many solutions in rational numbers $x_j>0,\ j=1,\cdots,n$ satisfying (2.1). □

As an example, when $n=4$, we have $R=(-141,1728)$, $[2]R=(363,-6264)$, $[3]R=(1443/49,334368/343)$.

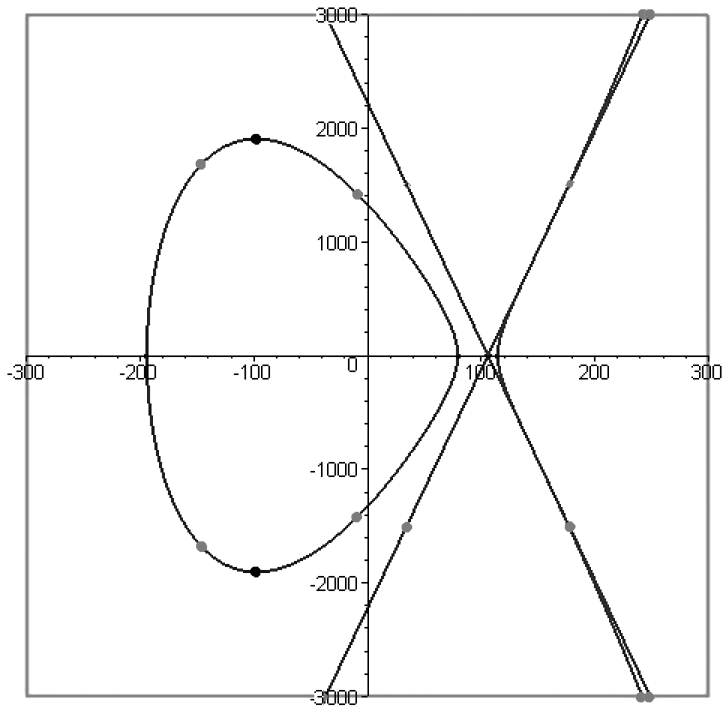

FIGURE 1. $E_4 : y^2 = x^3 - 28539x + 1765206$ *and* $|y| = 2223 - 21x$

From the transformation (2.3), we have

$$u = \frac{y + 6048x - 864}{1728}, \ v = \frac{147 - x}{288},$$

leading to

$$x_2 = \frac{21x - 2223 - y}{6(x - 147)}, \ x_3 = \frac{288}{147 - x};$$

then

$$x_4 = 8 - 1 - x_2 - x_3 = \frac{21x - 2223 + y}{6(x - 147)}.$$

Therefore,

$$(x_1, x_2, x_3, x_4) = \left(1, \frac{21x - 2223 - y}{6(x - 147)}, \frac{288}{147 - x}, \frac{21x - 2223 + y}{6(x - 147)}\right)$$

is a solution of (2.1) for $n = 4$.

In Figure 1, we display the elliptic curve E_4 and the lines $|y| = 2223 - 21x$. It is easy to see that if (x, y) on E_4 and $x \leq x_2(4)$, we have $|y| < 2223 - 21x$, then there are infinitely many solutions in rational numbers $x_j > 0,\ j = 1, \cdots, 4$ for (2.1). By an easy calculation, we have the following examples. The points

$$(x, y) = (3, 1296), \left(\frac{1443}{49}, \frac{334368}{343}\right), \left(\frac{-3423813}{34969}, \frac{12443156928}{6539203}\right),$$

lead to

$$(x_1, x_2, x_3, x_4) = (1, 4, 2, 1), \left(1, \frac{49}{20}, \frac{128}{35}, \frac{25}{28}\right), \left(1, \frac{103058}{24497}, \frac{34969}{29737}, \frac{68644}{42449}\right),$$

respectively.

3. Proof of the theorem

Proof of Theorem 1.1. The method is as used in [4]. Take any k positive rational solutions $(x_{i1}, \cdots, x_{in})$, where $x_{i1} = 1, \cdots, x_{i,n-3} = 1$, of (2.1). Let $d = lcm_{i,j}\ (x_{ij},\ j = 1, \cdots, n,\ i \le k)$, we set

$$x_{ij} = \frac{a_{ij}}{d},\ a_{ij} \in \mathbb{N} - \{0\},\ \big(gcd_{i,j}\ (a_{ij}), d\big) = 1,$$

where $a_{i1} = d, \cdots, a_{i,n-3} = d$. Then

$$\sum_{i=1}^{n} a_{ij} = 2nd,\ \prod_{i=1}^{n} a_{ij} = 2nd^n\ (i \le k), \tag{3.1}$$

hence

$$gcd_{i,j}\ (a_{ij}) = 1.$$

For two sets of solutions $\{(x_{i1}, \cdots, x_{in}), i \le k\}$ and $\{(x'_{i1}, \cdots, x'_{in}), i \le k\}$, if the sets of n-tuples $\{(a_{i1}, \cdots, a_{in}), i \le k\}$ and $\{(a'_{i1}, \cdots, a'_{in}), i \le k\}$ coincides, then we have $d = d'$ by (3.1). Hence, the sets of solutions themselves coincide. Since there are infinitely many choices of k elements from an infinite set, then for every k there exist infinitely many primitive sets of k n-tuples of positive integers with the same sum and the same product. $\square$

Example 1. For $n = 4$, we have three rational quadruples

$$(1, 4, 2, 1), \left(1, \frac{49}{20}, \frac{128}{35}, \frac{25}{28}\right), \left(1, \frac{103058}{24497}, \frac{34969}{29737}, \frac{68644}{42449}\right).$$

Then $d = 778514660$, leading to three integral quadruples

$$(778514660, 3114058640, 1557029320, 778514660),$$

$$(778514660, 1907360917, 2847139328, 695102375),$$

$$(778514660, 3275183240, 915488420, 1258930960),$$

with the sum 6228117280 and product $2938712953198523150291392472986880000 = 2^{11}5^4 7^4 11^4 17^4 131^4 227^4$.

Example 2. For $n = 5$, we have two rational quintuples

$$(1, 1, 1, 2, 5), \left(1, 1, \frac{841}{221}, \frac{1690}{493}, \frac{289}{377}\right).$$

Then $d = 6409$, leading to two integral quintuples

$$(6409, 6409, 6409, 12818, 32045),\ (6409, 6409, 24389, 21970, 4913),$$

with the sum 64090 and the product 10813128347448 4110490.

Example 3. For $n = 6$, we have two rational sextuples

$$(1, 1, 1, 1, 2, 6), \left(1, 1, 1, \frac{1058}{273}, \frac{1323}{299}, \frac{388}{483}\right).$$

Then $d = 6279$, leading to two integral sextuples

$$(6279, 6279, 6279, 6279, 12558, 37674), \ (6279, 6279, 6279, 24334, 27783, 4394),$$

with the sum 75348 and the product 7354008786053535611798552.

4. Further consideration

The referee asks whether there exists infinitely many n-tuples of positive integers with the same sum, the same product, and the same second elementary symmetric function $\sum_{i<j} x_i x_j$. It's an interesting problem. However, we think it needs a new method even for $n = 3$.

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