TAILORED FINITE POINT METHOD BASED ON EXPONENTIAL BASES FOR CONVECTION-DIFFUSION-REACTION EQUATION

HOUDE HAN AND ZHONGYI HUANG

ABSTRACT. In this paper, we propose a class of new tailored finite point methods (TFPM) for the numerical solution of a type of convection-diffusion-reaction problems in two dimensions. Our finite point method has been tailored based on the local exponential basis functions. Furthermore, our TFPM satisfies the discrete maximum principle automatically. We also study the error estimates of our TFPM. We prove that our TFPM can achieve good accuracy even when the mesh size $h \gg \varepsilon$ for some cases without any prior knowledge of the boundary layers. Our numerical examples show the efficiency and reliability of our method.

1. INTRODUCTION

We consider the following convection-diffusion-reaction problem:

$$(1.1) \quad \mathbb{L}u \equiv -\varepsilon^2 \Delta u + p(\mathbf{x})u_x + q(\mathbf{x})u_y + b(\mathbf{x})u = f(\mathbf{x}), \quad \forall \mathbf{x} = (x, y) \in \Omega,$$

$$(1.2) \quad u|_{\partial\Omega} = 0,$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain, $p(\mathbf{x})$, $q(\mathbf{x})$, $b(\mathbf{x}) \in L^\infty(\Omega)$ and $f(\mathbf{x}) \in L^2(\Omega)$ are four given functions on Ω and

$$(1.3) \quad b(\mathbf{x}) \geq 0, \quad \forall \mathbf{x} \in \overline{\Omega}.$$

When $\varepsilon \ll 1$, the problem (1.1)–(1.2) is a so-called singular perturbation problem, and is devised to represent some difficulties in the numerical simulation. When $\varepsilon \ll 1$, the solution of problem (1.1)–(1.2) has some *boundary layers* on a portion of $\partial\Omega$, and may also have some *interior layers* in Ω .

- These layers are characterized by *rapid transitions* in the solution, and are thus *difficult* to capture in a numerical approximation *without using a large number of unknowns*.
- Also, such layers tend to cause *spurious oscillations* in a numerical solution to the problem.

The numerical solution of similar singular perturbation problems have been studied by many mathematicians, for example, Il'in [12], Miller [18], Berger, Han and

---

Received by the editor November 29, 2009 and, in revised form, May 2, 2010.

2010 *Mathematics Subject Classification.* Primary 65-xx; Secondary 35-xx.

Key words and phrases. Tailored finite point method, singular perturbation problem, boundary layer, discrete maximum principle.

H. Han was supported by the NSFC Project No. 10971116.

Z. Huang was supported by the NSFC Project No. 11071139, the National Basic Research Program of China under the grant 2011CB309705, Tsinghua University Initiative Scientific Research Program.

Kellogg [2] studied some one-dimensional problems, they used the method of “exponential fitting” to get the uniformly convergent approximation. But for high dimensional problems, people did not get the uniform convergence results by such technique. Recently, the typical methods for high dimensional problems are proposed to solve such singular perturbation problems based on the refined mesh technique, such as Shishkin meshes [3, 14, 15, 16, 20, 22], in which people can get the uniform convergence on nonuniform meshes. Melenk et al. also used so-called *hp*-version finite element method to achieve uniform convergence (cf. [17] and the references in it). For more related work, please refer to the books by Morton [19], Roos, Stynes and Tobiska [21], and the review paper by Stynes [23]. Generally speaking, in those methods for high dimensional problems, one usually needs the finest mesh size $h \sim \mathcal{O}(\varepsilon)$ to achieve a satisfied numerical result. Therefore, most of those methods need many degrees of freedom when $\varepsilon \ll 1$, although they are uniformly convergent, especially if they want to have small pointwise errors on all mesh points. For example, by Shishkin meshes, one can get the uniform convergence rate, but they need a prior knowledge of the position of boundary layer (cf. [3, 14, 15, 16, 20, 22]).

Recently, we develop an efficient method, so-called “*tailored finite point method*”, for the numerical simulation of singular perturbation problems and high frequency wave propagation problems [6, 7, 8, 9, 11]. In our work, we proposed a kind of finite point methods which used the exact solutions of the local approximation problems to construct the global approximate solution. Therefore, we can have high accuracy even on the uniform coarse mesh $h \gg \varepsilon$ without any prior knowledge of the boundary layers. In [7], we have proposed some finite point schemes for convection-diffusion problems using Bessel functions. In that paper, our schemes also obtained very good approximate solutions on coarse mesh, but those schemes did not satisfy the *discrete maximum principle* in some cases.

In this paper, we propose a class of new tailored finite point methods to solve the problem (1.1)–(1.2) numerically, in which we use the proper exponential functions to construct the approximate solution. These exponential functions are the exact solution of the proper local approximate problems. We prove that these new schemes can always satisfy the *discrete maximum principle*, we also prove the error analysis of our method on coarse mesh $h \gg \varepsilon$ in some cases.

2. THE TAILORED FINITE POINT SCHEME

For the sake of simplicity, we assume that $\Omega = [0, 1] \times [0, 1]$ and we have a uniform mesh, i.e., let $h = N^{-1}$ be the mesh size and

$$(2.1) \quad x_i = ih, \quad y_j = jh, \quad 0 \leq i, j \leq N.$$

Then $\{P_{i,j} = (x_i, y_j), \quad 0 \leq i, j \leq N\}$ are the mesh points.

We now construct our tailored finite point scheme for (1.1) on a cell Ω_0 (see Figure 1).

First we approximate the model problem (1.1) on the cell Ω_0 by

$$(2.2) \quad -\varepsilon^2 \Delta u + p_0 u_x + q_0 u_y + b_0 u = f_0,$$

with $p_0 = p(\mathbf{x}^0)$, $q_0 = q(\mathbf{x}^0)$, $b_0 = b(\mathbf{x}^0)$, $f_0 = f(\mathbf{x}^0)$.

We consider two cases, respectively:

Case 1: $b_0 > 0$.

FIGURE 1. The local mesh.

Let

$$(2.3) \quad u(x, y) = \frac{f_0}{b_0} + v(x, y) \exp\left(\frac{p_0 x + q_0 y}{2\varepsilon^2}\right).$$

Then v satisfies

$$(2.4) \quad -\varepsilon^2 \Delta v + d_0^2 v = 0,$$

with $d_0^2 = b_0 + \frac{p_0^2 + q_0^2}{4\varepsilon^2}$.

Let $\mu_0 = d_0/\varepsilon$, and

$$H_4 = \left\{ v(x, y) \mid v = c_1 e^{-\mu_0 x} + c_2 e^{\mu_0 x} + c_3 e^{-\mu_0 y} + c_4 e^{\mu_0 y}, \quad \forall c_i \in \mathbb{R} \right\}.$$

Then we take a scheme as

$$(2.5) \quad \alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 + \alpha_4 V_4 + \alpha_0 V_0 = 0,$$

with $V_j = v(x^j)$, such that (2.5) holds for all $v \in H_4$. Thus we obtain

$$(2.6) \quad \alpha_1 e^{-\mu_0 h} + \alpha_2 + \alpha_3 e^{\mu_0 h} + \alpha_4 + \alpha_0 = 0,$$

$$(2.7) \quad \alpha_1 e^{\mu_0 h} + \alpha_2 + \alpha_3 e^{-\mu_0 h} + \alpha_4 + \alpha_0 = 0,$$

$$(2.8) \quad \alpha_1 + \alpha_2 e^{-\mu_0 h} + \alpha_3 + \alpha_4 e^{\mu_0 h} + \alpha_0 = 0,$$

$$(2.9) \quad \alpha_1 + \alpha_2 e^{\mu_0 h} + \alpha_3 + \alpha_4 e^{-\mu_0 h} + \alpha_0 = 0.$$

That means, for any given $\alpha_0 \in \mathbb{R}$, the system (2.6)–(2.9) has the unique solution

$$(2.10) \quad \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \frac{-\alpha_0}{e^{\mu_0 h} + e^{-\mu_0 h} + 2} \equiv \frac{-\alpha_0}{4 \cosh^2(\frac{\mu_0 h}{2})}.$$

If we take

$$(2.11) \quad \alpha_0 = \frac{e^{\mu_0 h} + e^{-\mu_0 h} + 2}{e^{\mu_0 h} + e^{-\mu_0 h} - 2} \equiv \frac{\cosh^2(\frac{\mu_0 h}{2})}{\sinh^2(\frac{\mu_0 h}{2})},$$

we get

$$(2.12) \quad \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = -\frac{1}{e^{\mu_0 h} + e^{-\mu_0 h} - 2} \equiv -\frac{1}{4 \sinh^2(\frac{\mu_0 h}{2})}.$$

We finally have the following five-point scheme for (1.1):

$$(2.13) \quad U_0 - \frac{e^{-\frac{p_0 h}{2\varepsilon^2}} U_1 + e^{-\frac{q_0 h}{2\varepsilon^2}} U_2 + e^{\frac{p_0 h}{2\varepsilon^2}} U_3 + e^{\frac{q_0 h}{2\varepsilon^2}} U_4}{4 \cosh^2\left(\frac{\mu_0 h}{2}\right)} \\ = \frac{f_0}{b_0} \left(1 - \frac{e^{-\frac{p_0 h}{2\varepsilon^2}} + e^{-\frac{q_0 h}{2\varepsilon^2}} + e^{\frac{p_0 h}{2\varepsilon^2}} + e^{\frac{q_0 h}{2\varepsilon^2}}}{4 \cosh^2\left(\frac{\mu_0 h}{2}\right)} \right),$$

with

$$U_j = u(\mathbf{x}^j) = \frac{f_0}{b_0} + V_j \exp\left(\frac{p_0 x_j + q_0 y_j}{2\varepsilon^2}\right).$$

Case 2: $b_0 = 0$.

If $p_0 = q_0 = 0$, we can just use the standard five points difference scheme:

$$(2.14) \quad -\varepsilon^2 \frac{U_1 + U_2 + U_3 + U_4 - 4U_0}{h^2} = f_0.$$

Otherwise, if $p_0^2 + q_0^2 > 0$, let

$$(2.15) \quad u(x, y) = \frac{p_0 x + q_0 y}{p_0^2 + q_0^2} f_0 + v(x, y) \exp\left(\frac{p_0 x + q_0 y}{2\varepsilon^2}\right).$$

Then v satisfies:

$$(2.16) \quad -\varepsilon^2 \Delta v + r_0^2 v = 0,$$

with $r_0^2 = \frac{p_0^2 + q_0^2}{4\varepsilon^2}$. From the results before (cf. (2.5)–(2.12)), we finally have the following five-point scheme in this case:

$$(2.17) \quad U_0 - \frac{e^{-\frac{p_0 h}{2\varepsilon^2}} U_1 + e^{-\frac{q_0 h}{2\varepsilon^2}} U_2 + e^{\frac{p_0 h}{2\varepsilon^2}} U_3 + e^{\frac{q_0 h}{2\varepsilon^2}} U_4}{4 \cosh^2\left(\frac{\nu_0 h}{2}\right)} \\ = \frac{p_0(e^{\frac{p_0 h}{2\varepsilon^2}} - e^{-\frac{p_0 h}{2\varepsilon^2}}) + q_0(e^{\frac{q_0 h}{2\varepsilon^2}} - e^{-\frac{q_0 h}{2\varepsilon^2}})}{4(p_0^2 + q_0^2) \cosh^2\left(\frac{\nu_0 h}{2}\right)} h f_0,$$

with

$$\nu_0 = r_0/\varepsilon, \quad U_j = u(\mathbf{x}^j) = \frac{p_0 x_j + q_0 y_j}{p_0^2 + q_0^2} f_0 + V_j \exp\left(\frac{p_0 x_j + q_0 y_j}{2\varepsilon^2}\right).$$

Lemma 2.1. *It is easy to check that our discrete operator (defined by schemes (2.13), (2.14) and (2.17)) satisfies the **discrete maximum principle**.*

Proof. It is straightforward to check that (2.14) is diagonally dominant. To prove that schemes (2.13) and (2.16) are also diagonally dominant, we need to prove

$$(2.18) \quad 4 \cosh^2\left(\frac{\mu_0 h}{2}\right) \geq e^{-\frac{p_0 h}{2\varepsilon^2}} + e^{\frac{p_0 h}{2\varepsilon^2}} + e^{-\frac{q_0 h}{2\varepsilon^2}} + e^{\frac{q_0 h}{2\varepsilon^2}} \equiv 2 \cosh\left(\frac{p_0 h}{2\varepsilon^2}\right) + 2 \cosh\left(\frac{q_0 h}{2\varepsilon^2}\right),$$

$$(2.19) \quad 4 \cosh^2\left(\frac{\nu_0 h}{2}\right) \geq e^{-\frac{p_0 h}{2\varepsilon^2}} + e^{\frac{p_0 h}{2\varepsilon^2}} + e^{-\frac{q_0 h}{2\varepsilon^2}} + e^{\frac{q_0 h}{2\varepsilon^2}} \equiv 2 \cosh\left(\frac{p_0 h}{2\varepsilon^2}\right) + 2 \cosh\left(\frac{q_0 h}{2\varepsilon^2}\right).$$

Since $\mu_0 = \sqrt{\frac{b_0}{\varepsilon^2} + \frac{p_0^2 + q_0^2}{4\varepsilon^4}} \geq \sqrt{\frac{p_0^2 + q_0^2}{4\varepsilon^4}} \equiv \nu_0$, we only need to prove (2.19). If we let $\xi = \frac{p_0 h}{2\varepsilon^2}$ and $\eta = \frac{q_0 h}{2\varepsilon^2}$, we have

$$4 \cosh^2 \left(\frac{\nu_0 h}{2} \right) = 2 \cosh \left(\sqrt{\frac{p_0^2 + q_0^2}{4\varepsilon^4}} \right) + 2 = 2 \cosh(\sqrt{\xi^2 + \eta^2}) + 2.$$

That means we only need to prove

$$\phi(x, y) \equiv \cosh(\sqrt{x^2 + y^2}) + 1 - \cosh(x) - \cosh(y) \geq 0, \quad \forall x, y \in \mathbb{R}.$$

It is easy to check that

$$\phi(0, y) = \phi(x, 0) = 0, \quad \forall x, y \in \mathbb{R}.$$

Furthermore, because $\sinh(z)$ is convex for $z > 0$, we have

$$\frac{\partial \phi}{\partial x}(x, y) \geq 0, \quad \frac{\partial \phi}{\partial y}(x, y) \geq 0, \quad \forall x, y \in \mathbb{R}_+.$$

Because $\cosh(z)$ is an even function, then it is easy to get

$$\phi(x, y) \geq 0, \quad \forall x, y \in \mathbb{R}.$$

That means our discrete operator (defined by schemes (2.13), (2.14) and (2.17)) satisfies the discrete maximum principle. $\square$

Remark 2.1. If $h \ll \varepsilon^2$, i.e., $\mu_0 h \ll 1$ and $\nu_0 h \ll 1$, we have

$$\begin{aligned} \cosh^2 \left(\frac{\mu_0 h}{2} \right) &= 1 + \frac{(\mu_0 h)^2}{4} + \mathcal{O} \left((\mu_0 h)^4 \right), \\ \cosh^2 \left(\frac{\nu_0 h}{2} \right) &= 1 + \frac{(\nu_0 h)^2}{4} + \mathcal{O} \left((\nu_0 h)^4 \right), \\ e^{\pm \frac{p_0 h}{2\varepsilon^2}} &= 1 \pm \frac{p_0 h}{2\varepsilon^2} + \frac{\left(\frac{p_0 h}{2\varepsilon^2} \right)^2}{2} \pm \frac{\left(\frac{p_0 h}{2\varepsilon^2} \right)^3}{6} + \mathcal{O} \left(\left(\frac{p_0 h}{2\varepsilon^2} \right)^4 \right), \\ e^{\pm \frac{q_0 h}{2\varepsilon^2}} &= 1 \pm \frac{q_0 h}{2\varepsilon^2} + \frac{\left(\frac{q_0 h}{2\varepsilon^2} \right)^2}{2} \pm \frac{\left(\frac{q_0 h}{2\varepsilon^2} \right)^3}{6} + \mathcal{O} \left(\left(\frac{q_0 h}{2\varepsilon^2} \right)^4 \right). \end{aligned}$$

If we omit the high order term, the tailored finite point schemes (2.13) and (2.17) are reduced to the *standard second-order finite difference scheme* of equation (1.1) at $\mathbf{x}^0$, i.e., scheme (2.13) is reduced to

$$-\varepsilon^2 \frac{U_1 + U_2 + U_3 + U_4 - 4U_0}{h^2} + p_0 \frac{U_1 - U_3}{2h} + q_0 \frac{U_2 - U_4}{2h} + b_0 U_0 = f_0,$$

scheme (2.17) is reduced to

$$-\varepsilon^2 \frac{U_1 + U_2 + U_3 + U_4 - 4U_0}{h^2} + p_0 \frac{U_1 - U_3}{2h} + q_0 \frac{U_2 - U_4}{2h} = f_0.$$

3. ERROR ANALYSIS

In this paper, we only consider the error analysis for the case of $p \equiv p_0 > 0$, $q \equiv 0$, $b \equiv b_0 > 0$. A more general case will be studied in the future.

Therefore, in this section, we assume that

$$(3.1) \quad \Omega = (0, 1)^2, \quad p(\mathbf{x}) \equiv p_0 > 0, \quad q(\mathbf{x}) \equiv 0, \quad b(\mathbf{x}) \equiv b_0 > 0, \quad f(\mathbf{x}) \in \mathbf{C}^{2l, \alpha}(\bar{\Omega}),$$

for some integers $3 \leq l \in \mathbb{N}$ and some real numbers $\alpha \in (0, 1)$.

3.1. Derivative bounds for the model problem (1.1)–(1.2). Let us first recall some derivative bounds for the solution to our problem (1.1). (cf. the corollary of Theorem 4.2 in [10] and Theorem 1 in [13]).

Theorem 3.1. *Let m, n be nonnegative integers satisfying $2m+n \leq 2l$ and $m+n \leq 2l-4$. Then for $(x, y) \in \Omega$, the solution of (1.1)–(1.2) satisfies*

$$(3.2) \quad |\partial_x^m \partial_y^n u(x, y)| \leq C(1 + T_{00} + T_{01} + T_{10} + T_{11} + T_E)$$

with

$$T_E = \varepsilon^{-2m} e^{-p_0(1-x)/\varepsilon^2},$$

where for $j = 0, 1$, one has

$$\begin{aligned} T_{0j} &= \begin{cases} \varepsilon^{-n} + \varepsilon^{2-2m-2n}, & \text{for } m+n < 2 \text{ and } r_{0j} < \varepsilon^2, \\ \varepsilon^{-n} + \varepsilon^{-2} |\ln r_{0j}|, & \text{for } m+n = 2 \text{ and } r_{0j} < \varepsilon^2, \\ \varepsilon^{-n} + \varepsilon^{-2} r_{0j}^{2-m-n}, & \text{for } m+n > 2 \text{ and } r_{0j} < \varepsilon^2, \end{cases} \\ T_{00} &= \varepsilon^{-n} [1 + r_{00}^{1-m-n/2}] e^{-\beta y/(2\varepsilon)}, \quad \text{for } \varepsilon^2 \leq r_{00} \leq 1, \\ T_{01} &= \varepsilon^{-n} [1 + r_{01}^{1-m-n/2}] e^{-\beta(1-y)/(2\varepsilon)}, \quad \text{for } \varepsilon^2 \leq r_{01} \leq 1, \\ T_{1j} &= \begin{cases} \varepsilon^{-2m-n} + \varepsilon^{2-2m-2n}, & \text{for } m+n < 2 \text{ and } r_{1j} < \varepsilon^2, \\ \varepsilon^{-2m-n} + \varepsilon^{-2} |\ln r_{1j}|, & \text{for } m+n = 2 \text{ and } r_{1j} < \varepsilon^2, \\ \varepsilon^{-2m-n} + \varepsilon^{-2} r_{1j}^{2-m-n}, & \text{for } m+n > 2 \text{ and } r_{1j} < \varepsilon^2, \end{cases} \\ T_{10} &= \varepsilon^{-2m-n} [1 + r_{10}^{1-n/2}] e^{-\beta y/(2\varepsilon) - p_0(1-x)/\varepsilon^2}, \quad \text{for } \varepsilon^2 \leq r_{10} \leq 1, \\ T_{11} &= \varepsilon^{-2m-n} [1 + r_{11}^{1-n/2}] e^{-\beta(1-y)/(2\varepsilon) - p_0(1-x)/\varepsilon^2}, \quad \text{for } \varepsilon^2 \leq r_{11} \leq 1. \end{aligned}$$

Here $r_{ij} = \sqrt{(x-i)^2 + (y-j)^2}$ denotes the distance between (x, y) and the vertex (i, j) of Ω , $\beta \in (0, \sqrt{b_0}]$ is a positive constant, constant C depends on m, n and l .

3.2. Truncation error. We now consider the truncation error of our scheme at mesh point $\mathbf{x}^0 \in \Omega$. Because of our assumption (3.1), we need only consider the truncation error of scheme (2.13).

For function u , the solution of problem (1.1)–(1.2), we have:

$$\begin{aligned} (\mathbb{L}_h u)(\mathbf{x}^0) &\equiv \frac{4b_0 \cosh^2(\frac{\mu_0 h}{2})}{4 \cosh^2(\frac{\mu_0 h}{2}) - \eta_0} \left\{ u_0 - \frac{e^{-\frac{p_0 h}{2\varepsilon^2}} u_1 + u_2 + e^{\frac{p_0 h}{2\varepsilon^2}} u_3 + u_4}{4 \cosh^2(\frac{\mu_0 h}{2})} \right\} \\ (3.3) \quad &= b_0 u_0 - \frac{b_0 (e^{-\frac{p_0 h}{2\varepsilon^2}} u_1 + u_2 + e^{\frac{p_0 h}{2\varepsilon^2}} u_3 + u_4 - \eta_0 u_0)}{4 \cosh^2(\frac{\mu_0 h}{2}) - \eta_0}, \end{aligned}$$

$$(3.4) \quad (\mathbb{L} u)(\mathbf{x}^0) = f_0,$$

with

$$\begin{aligned} u_j &= u(\mathbf{x}^j), \quad (j = 0, 1, \dots, 4); \\ \mu_0 &= \sqrt{\frac{b_0}{\varepsilon^2} + \frac{p_0^2}{4\varepsilon^4}}; \\ \eta_0 &= e^{-\frac{p_0 h}{2\varepsilon^2}} + e^{\frac{p_0 h}{2\varepsilon^2}} + 2 \equiv 4 \cosh^2\left(\frac{p_0 h}{4\varepsilon^2}\right). \end{aligned}$$

The truncation error of scheme (2.13) at mesh point $\mathbf{x}^0$ is given by

$$(3.5) \quad \begin{aligned} (\mathbb{T}_h u)(\mathbf{x}^0) &= (\mathbb{L}_h u)(\mathbf{x}^0) - (\mathbb{L}u)(\mathbf{x}^0) \\ &= b_0 u_0 - \frac{b_0(e^{-\frac{p_0 h}{2\varepsilon^2}} u_1 + u_2 + e^{\frac{p_0 h}{2\varepsilon^2}} u_3 + u_4 - \eta_0 u_0)}{4 \cosh^2(\frac{\mu_0 h}{2}) - \eta_0} - f_0. \end{aligned}$$

In this paper, we hope that the truncation error $(\mathbb{T}_h u)(\mathbf{x}^0)$ is *small even when* $h \gg \varepsilon^2$. Precisely, we assume that

$$(3.6) \quad \frac{\varepsilon}{h^2} \leq h^{\gamma_0},$$

where $\gamma_0 > 0$ is a constant. It means that we used a coarse mesh comparing with the small parameter ε . We now estimate the truncation error at point $\mathbf{x}^0 = (x_0, y_0)$.

Since $\mathbf{x}^0$ is a mesh point in Ω , we know that

$$(3.7) \quad h \leq x_0 \leq 1 - h, \quad h \leq y_0 \leq 1 - h.$$

By Theorem 3.1, we know that

$$\begin{aligned} |\varepsilon^2 \Delta u(\mathbf{x}_0)| &\leq C(\varepsilon^2 + e^{-\frac{\beta h}{2\varepsilon}} + \varepsilon^{-2} e^{-\frac{p_0 h}{\varepsilon^2}}), \\ \frac{u(\mathbf{x}_0) - u(\mathbf{x}_3)}{h} &= u_x(\mathbf{x}_0) + \mathcal{O}(h + e^{-\frac{\beta h}{2\varepsilon}} + \varepsilon^{-2} e^{-\frac{p_0 h}{\varepsilon^2}}), \\ \eta_0 &= e^{\frac{p_0 h}{2\varepsilon^2}} \left(1 + 2e^{-\frac{p_0 h}{2\varepsilon^2}} + e^{-\frac{p_0 h}{\varepsilon^2}}\right), \\ 4 \cosh^2\left(\frac{\mu_0 h}{2}\right) - \eta_0 &= e^{\frac{p_0 h}{2\varepsilon^2}} \left(e^{\frac{b_0 h}{p_0}} - 1 + \mathcal{O}(\varepsilon^2)\right). \end{aligned}$$

Then solution $u(x, y)$ of problem (1.1)–(1.2) satisfies

$$\begin{aligned} &\left| b_0 u(\mathbf{x}^0) - f_0 - \frac{b_0(e^{-\frac{p_0 h}{2\varepsilon^2}} u_1 + u_2 + e^{\frac{p_0 h}{2\varepsilon^2}} u_3 + u_4 - \eta_0 u_0)}{4 \cosh^2(\frac{\mu_0 h}{2}) - \eta_0} \right| \\ &= \left| \varepsilon^2 \Delta u(\mathbf{x}^0) - p_0 u_x(\mathbf{x}^0) - \frac{b_0(u_3 - u_0)}{e^{\frac{b_0 h}{p_0}} - 1} + \mathcal{O}(\varepsilon^2 + e^{-\frac{p_0 h}{2\varepsilon^2}}) \right| \\ &= \left| -p_0 u_x(\mathbf{x}^0) - \frac{b_0(u_3 - u_0)}{\frac{b_0 h}{p_0} + \mathcal{O}(h^2)} + \mathcal{O}(\varepsilon^2 + e^{-\frac{p_0 h}{2\varepsilon^2}} + e^{-\frac{\beta h}{2\varepsilon}} + \varepsilon^{-2} e^{-\frac{p_0 h}{\varepsilon^2}}) \right| \\ (3.8) \quad &\leq C \left\{ h + \varepsilon^{-2} e^{-\frac{p_0 h}{\varepsilon^2}} + e^{-\frac{\beta h}{2\varepsilon}} \right\}, \end{aligned}$$

with constants C , β independent of ε and $\beta \in (0, \sqrt{b_0})$.

3.3. Error estimate. We now consider the discrete scheme of problem (1.1)–(1.2) on the mesh (2.1) and h satisfies condition (3.6). Also because of our assumption (3.1), we need only consider the error estimate of scheme (2.13).

For scheme (2.13), we have:

$$(3.9) \quad \mathbb{L}_h U_{i,j} \equiv \frac{4b_0 \cosh^2(\frac{\mu_0 h}{2})}{4 \cosh^2(\frac{\mu_0 h}{2}) - \eta_0} \left\{ U_{ij} - \frac{e^{-\frac{p_0 h}{2\varepsilon^2}} U_{i+1,j} + U_{i,j+1} + e^{\frac{p_0 h}{2\varepsilon^2}} U_{i-1,j} + U_{i,j-1}}{4 \cosh^2(\frac{\mu_0 h}{2})} \right\}$$

$$(3.10) \quad = f_{i,j}, \quad 1 \leq i, j \leq N-1,$$

$$(3.11) \quad U_{i,0} = U_{i,N} = 0, \quad i = 0, 1, \dots, N,$$

$$U_{0,j} = U_{N,j} = 0, \quad j = 1, \dots, N-1.$$

For the discrete problem (3.9)–(3.11), we have the following result:

Theorem 3.2. *Suppose that (3.1) holds, $\{U_{i,j}, 0 \leq i, j \leq N\}$ is a solution of problem (3.9)–(3.11), then the following estimate holds:*

$$(3.12) \quad |U_{i,j}| \leq \max_{0 \leq i,j \leq N} \frac{|f_{i,j}|}{b_0}, \quad 0 \leq i, j \leq N.$$

Proof. Because of the boundary conditions (3.10)–(3.11), we have

$$\begin{aligned} U_{i,N} + \frac{|f_{i,N}|}{b_0} &\geq 0, & U_{i,0} + \frac{|f_{i,0}|}{b_0} &\geq 0, & U_{i,N} - \frac{|f_{i,N}|}{b_0} &\leq 0, & U_{i,0} - \frac{|f_{i,0}|}{b_0} &\leq 0, \\ & & & & & & i = 0, 1, \dots, N, \\ U_{N,j} + \frac{|f_{N,j}|}{b_0} &\geq 0, & U_{0,j} + \frac{|f_{0,j}|}{b_0} &\geq 0, & U_{N,j} - \frac{|f_{N,j}|}{b_0} &\leq 0, & U_{0,j} - \frac{|f_{0,j}|}{b_0} &\leq 0, \\ & & & & & & j = 0, 1, \dots, N. \end{aligned}$$

Furthermore, because $4 \cosh^2(\frac{\mu_0 h}{2}) > \eta_0 > 0$, it is easy to check that the operator $\mathbb{L}_h$ defined in (3.9) satisfies the *discrete maximum principle*, and

$$\mathbb{L}_h \left(U_{i,j} + \frac{|f_{i,j}|}{b_0} \right) \geq 0, \quad \mathbb{L}_h \left(U_{i,j} - \frac{|f_{i,j}|}{b_0} \right) \leq 0, \quad 1 \leq i, j \leq N-1.$$

We can obtain the estimate (3.12) immediately. $\square$

Let

$$(3.13) \quad E_{i,j} = U_{i,j} - u(P_{i,j}), \quad 0 \leq i, j \leq N,$$

then

$$(3.14) \quad \mathbb{L}_h E = \mathbb{T}_h u = \mathcal{O} \left\{ h^{2+2\gamma_0} + e^{-\beta h^{-\gamma_0}} \right\}.$$

By Theorem 3.2 we have

Theorem 3.3. *Suppose that (3.1) and (3.6) hold, then we have the following error estimate:*

$$(3.15) \quad |E_{i,j}| \leq C \left\{ h + \varepsilon^{-2} e^{-\frac{p_0 h}{2\varepsilon^2}} + e^{-\beta \frac{h}{\varepsilon}} \right\}, \quad 0 \leq i, j \leq N.$$

4. RECOVERY OF THE SOLUTION

FIGURE 2. Local mesh for recovery.

We now study the recovery formulae for our TFPM in the case $\varepsilon \ll h$ (cf. Figure 2). Let

$$\begin{aligned} p_0 &= \frac{1}{|D_0|} \int_{D_0} p(\mathbf{x}) d\mathbf{x}, & q_0 &= \frac{1}{|D_0|} \int_{D_0} q(\mathbf{x}) d\mathbf{x}, & \mu_0 &= \frac{1}{\varepsilon} \sqrt{b_0 + \frac{p_0^2 + q_0^2}{4\varepsilon^2}}, \\ b_0 &= \frac{1}{|D_0|} \int_{D_0} b(\mathbf{x}) d\mathbf{x}, & f_0 &= \frac{1}{|D_0|} \int_{D_0} f(\mathbf{x}) d\mathbf{x}, & \nu_0 &= \frac{\sqrt{p_0^2 + q_0^2}}{2\varepsilon^2}, \end{aligned}$$

where $|D_0| = h^2$, denotes the area of the domain D_0 . Take the center of D_0 as the origin of the local coordinates.

If $b_0 > 0$, we use the recovery formula for u_h in the cell D_0 :

$$(4.1) \quad u_h(x, y) = a_0 + e^{\frac{p_0 x + q_0 y}{2\varepsilon^2}} \left(c_1 e^{\frac{\mu_0(x+y)}{\sqrt{2}}} + c_2 e^{-\frac{\mu_0(x+y)}{\sqrt{2}}} + c_3 e^{\frac{\mu_0(x-y)}{\sqrt{2}}} + c_4 e^{\frac{\mu_0(y-x)}{\sqrt{2}}} \right),$$

with $a_0 = f_0/b_0$. By the equations $u_h(\mathbf{x}^j) = U_j$, $j = 1, 2, 3, 4$, we can get the values of these constants c_1, c_2, c_3, c_4 :

$$(4.2) \quad \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \frac{1}{4 \sinh^2(\frac{\mu_0 h}{\sqrt{2}})} \begin{pmatrix} \exp(-\frac{\mu_0 h}{\sqrt{2}}) & -1 & \exp(\frac{\mu_0 h}{\sqrt{2}}) & -1 \\ \exp(\frac{\mu_0 h}{\sqrt{2}}) & -1 & \exp(-\frac{\mu_0 h}{\sqrt{2}}) & -1 \\ -1 & \exp(\frac{\mu_0 h}{\sqrt{2}}) & -1 & \exp(-\frac{\mu_0 h}{\sqrt{2}}) \\ -1 & \exp(-\frac{\mu_0 h}{\sqrt{2}}) & -1 & \exp(\frac{\mu_0 h}{\sqrt{2}}) \end{pmatrix} \cdot \begin{pmatrix} \tilde{U}_1 \\ \tilde{U}_2 \\ \tilde{U}_3 \\ \tilde{U}_4 \end{pmatrix}$$

where

$$\begin{aligned} \tilde{U}_1 &= (U_1 - a_0) e^{\frac{p_0 + q_0}{4\varepsilon^2} h}, & \tilde{U}_2 &= (U_2 - a_0) e^{\frac{q_0 - p_0}{4\varepsilon^2} h}, \\ \tilde{U}_3 &= (U_3 - a_0) e^{-\frac{p_0 + q_0}{4\varepsilon^2} h}, & \tilde{U}_4 &= (U_4 - a_0) e^{\frac{p_0 - q_0}{4\varepsilon^2} h}. \end{aligned}$$

If $b_0 = 0$, and $p_0^2 + q_0^2 > 0$, we use the following recovery formula for u_h in the cell D_0 :

$$(4.3) \quad u_h(x, y) = \frac{p_0 x + q_0 y}{p_0^2 + q_0^2} f_0 + e^{\frac{p_0 x + q_0 y}{2\varepsilon^2}} \left(c_1 e^{\frac{\nu_0(x+y)}{\sqrt{2}}} + c_2 e^{-\frac{\nu_0(x+y)}{\sqrt{2}}} + c_3 e^{\frac{\nu_0(x-y)}{\sqrt{2}}} + c_4 e^{\frac{\nu_0(y-x)}{\sqrt{2}}} \right).$$

By the equations $u_h(\mathbf{x}^j) = U_j$, $j = 1, 2, 3, 4$, we can get the values of the constants c_1, c_2, c_3, c_4 :

$$(4.4) \quad \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \frac{1}{4 \sinh^2(\frac{\nu_0 h}{\sqrt{2}})} \begin{pmatrix} \exp(-\frac{\nu_0 h}{\sqrt{2}}) & -1 & \exp(\frac{\nu_0 h}{\sqrt{2}}) & -1 \\ \exp(\frac{\nu_0 h}{\sqrt{2}}) & -1 & \exp(-\frac{\nu_0 h}{\sqrt{2}}) & -1 \\ -1 & \exp(\frac{\nu_0 h}{\sqrt{2}}) & -1 & \exp(-\frac{\nu_0 h}{\sqrt{2}}) \\ -1 & \exp(-\frac{\nu_0 h}{\sqrt{2}}) & -1 & \exp(\frac{\nu_0 h}{\sqrt{2}}) \end{pmatrix} \cdot \begin{pmatrix} \tilde{U}_1 \\ \tilde{U}_2 \\ \tilde{U}_3 \\ \tilde{U}_4 \end{pmatrix}$$

where

$$\begin{aligned} \tilde{U}_1 &= (U_1 - a_1) e^{\frac{p_0 + q_0}{4\varepsilon^2} h}, & \tilde{U}_2 &= (U_2 - a_2) e^{\frac{q_0 - p_0}{4\varepsilon^2} h}, \\ \tilde{U}_3 &= (U_3 - a_3) e^{-\frac{p_0 + q_0}{4\varepsilon^2} h}, & \tilde{U}_4 &= (U_4 - a_4) e^{\frac{p_0 - q_0}{4\varepsilon^2} h}, \\ a_1 &= -\frac{p_0 + q_0}{p_0^2 + q_0^2} \cdot \frac{f_0 h}{2}, & a_2 &= \frac{p_0 - q_0}{p_0^2 + q_0^2} \cdot \frac{f_0 h}{2}, \\ a_3 &= \frac{p_0 + q_0}{p_0^2 + q_0^2} \cdot \frac{f_0 h}{2}, & a_4 &= -\frac{p_0 - q_0}{p_0^2 + q_0^2} \cdot \frac{f_0 h}{2}. \end{aligned}$$

If $b_0 = 0$, and $p_0 = q_0 = 0$, we use the following recovery formula for u_h in the cell D_0 :

$$(4.5) \quad u_h(x, y) = \frac{\frac{h^2}{2} - x^2 - y^2}{4\varepsilon^2} f_0 + \frac{U_1 + U_2 + U_3 + U_4}{4} + ax + by + cxy,$$

with

$$a = \frac{-U_1 + U_2 + U_3 - U_4}{2h}, \quad b = \frac{-U_1 - U_2 + U_3 + U_4}{2h}, \quad c = \frac{U_1 - U_2 + U_3 - U_4}{h^2}.$$

5. NUMERICAL EXAMPLES

In this section, we will use some examples to show the accuracy and efficiency of our new method. We will denote the recovered solution of our TFPM (shown in Section 4) by u^{rc} , and

$$E_{i,j} = U_{i,j} - u(P_{i,j}),$$

where $U_{i,j}$ are the value of our approximation at node $P_{i,j} = (x_i, y_j)$ solved by our TFPM.

Example 5.1. First, we consider a simple problem:

$$(5.1) \quad -\varepsilon^2 \Delta u + p(\mathbf{x})u_x + q(\mathbf{x})u_y + b(\mathbf{x})u = f(\mathbf{x}), \quad \forall \mathbf{x} = (x, y) \in \Omega,$$

$$(5.2) \quad u|_{\partial\Omega} = 0,$$

where

$$\Omega = [0, 1]^2, \quad p(\mathbf{x}) = 1, \quad q(\mathbf{x}) = 0, \quad b(\mathbf{x}) = 1, \\ f(x, y) = [2\varepsilon^2 + y(1 - y)] \left[e^{\frac{x-1}{\varepsilon^2}} + (x-1)e^{-\frac{1}{\varepsilon^2}} - x \right] + y(1 - y) \left(e^{-\frac{1}{\varepsilon^2}} - 1 \right).$$

It is easy to check that the exact solution of the problem (5.1)–(5.2) is

$$u(x, y) = y(1 - y) \left[e^{\frac{x-1}{\varepsilon^2}} + (x-1)e^{-\frac{1}{\varepsilon^2}} - x \right].$$

The numerical results are shown in Tables 1–2 and Figure 3. When $h \ll \varepsilon^2$, we have the second order convergence rate as we mentioned in Remark 2.1 (cf. Table 1). When $h \gg \varepsilon^2$, we still have good approximation as we proved in Theorem 3.3 (cf. Table 2). The error between our approximation u^{rc} and the exact solution u is still very small even in the sharp boundary layer when $h \gg \varepsilon$ (cf. Figure 3).

TABLE 1. L^∞ errors of the numerical solutions for Example 5.1 with $\varepsilon = \mathcal{O}(1)$.

mesh size h		1/8	1/16	1/32	1/64
$\varepsilon = 1.0$	$\max_{ij} E_{ij} / \ u\ _{L^\infty}$	1.53E-3	3.85E-4	9.64E-5	2.41E-5
$\varepsilon = 1.0$	$\ u - u^{rc}\ _{L^\infty} / \ u\ _{L^\infty}$	1.90E-2	4.47E-3	1.10E-3	2.72E-4
$\varepsilon = 0.5$	$\max_{ij} E_{ij} / \ u\ _{L^\infty}$	8.18E-3	2.05E-3	5.14E-4	1.29E-4
$\varepsilon = 0.5$	$\ u - u^{rc}\ _{L^\infty} / \ u\ _{L^\infty}$	1.67E-2	3.93E-3	9.62E-4	2.39E-4

TABLE 2. L^∞ errors of the numerical solutions for Example 5.1 with $\varepsilon \ll 1$ and $h = 1/32$.

ε	1.0E-2	1.0E-3	1.0E-4	1.0E-5
$\max_{ij} E_{ij} / \ u\ _{L^\infty}$	9.89E-3	9.67E-3	9.66E-3	9.66E-3
$\ u - u^{rc}\ _{L^\infty} / \ u\ _{L^\infty}$	9.89E-3	9.67E-3	9.66E-3	9.66E-3

FIGURE 3. The graphs of our approximation for Example 5.1 with $\varepsilon = 10^{-5}$, $h = \frac{1}{32}$. (a): u^{rc} , (b): enlarged plot of u^{rc} near the boundary layer, (c) $|u - u^{rc}|$ (near the boundary layer).

Example 5.2. Then we consider a more general problem:

$$(5.3) \quad -\varepsilon^2 \Delta u + p(\mathbf{x})u_x + q(\mathbf{x})u_y + b(\mathbf{x})u = f(\mathbf{x}), \quad \forall \mathbf{x} = (x, y) \in \Omega,$$

$$(5.4) \quad u|_{\partial\Omega} = 0,$$

where

$$\Omega = [0, 1]^2, \quad p(\mathbf{x}) = x, \quad q(\mathbf{x}) = y, \quad b(\mathbf{x}) = 1 + x + y, \quad f(\mathbf{x}) = \sin \pi(x + y).$$

The numerical results are shown in Tables 3–4 and Figure 4. The “exact” solution $u(x, y)$ is solved on very fine mesh with $h = 1/1024$ (when ε is $\mathcal{O}(1)$) or combining with asymptotic expansion (when $\varepsilon \ll 1$). We can see that when ε is $\mathcal{O}(1)$, we have the second order convergence rate as we show in Remark 2.1. When $h \gg \varepsilon$, we still have good approximation as we prove in Theorem 3.3, even the coefficients are not constants. In Figure 4, we can see that our approximation can capture the very sharp boundary layers even when $h \gg \varepsilon$.

TABLE 3. L^∞ errors of the numerical solutions for Example 5.2 with $\varepsilon = \mathcal{O}(1)$.

mesh size h		1/8	1/16	1/32	1/64
$\varepsilon = 1.0$	$\max_{ij} E_{ij} /\ u\ _{L^\infty}$	2.19E-3	5.42E-4	1.34E-4	3.19E-5
$\varepsilon = 1.0$	$\ u - u^{rc}\ _{L^\infty}/\ u\ _{L^\infty}$	6.66E-3	1.76E-3	4.52E-4	1.15E-4
$\varepsilon = 0.5$	$\max_{ij} E_{ij} /\ u\ _{L^\infty}$	9.18E-3	2.30E-3	5.70E-4	1.36E-4
$\varepsilon = 0.5$	$\ u - u^{rc}\ _{L^\infty}/\ u\ _{L^\infty}$	1.11E-2	2.85E-3	7.14E-4	1.72E-4

FIGURE 4. The graphs of our approximation for Example 5.2 with $\varepsilon = 10^{-5}$, $h = \frac{1}{32}$. (a): u^{rc} , (b): enlarged plot of u^{rc} near $x = 1$, (c) $|u - u^{rc}|$ (near $x = 1$).

TABLE 4. L^∞ errors of the numerical solutions for Example 5.2 with $\varepsilon \ll 1$ and $h = 1/32$.

ε	1.0E-2	1.0E-3	1.0E-4	1.0E-5
$\max_{ij} E_{ij} /\ u\ _{L^\infty}$	1.28E-3	1.82E-3	1.42E-3	1.41E-3
$\ u - u^{rc}\ _{L^\infty}/\ u\ _{L^\infty}$	1.29E-3	1.83E-3	1.42E-3	1.41E-3

6. CONCLUSION

In this paper, we present a class of new tailored finite point methods (TFPM) for the numerical solution to a kind of convection-diffusion-reaction problems in the two-dimensional case. First, we take a partition of the domain, and approximate the coefficients by piecewise constants in each cell. Then we use the exponential functions which are the exact solutions of the local reduced equation with constant coefficients to get the tailored scheme. These new schemes can satisfy the *discrete maximum principle* automatically. Furthermore, we study the error estimates of our TFPM. We prove that our TFPM has second order convergence rate as $h \rightarrow 0$, and our method can also achieve good accuracy on all mesh points even when the mesh size $h \gg \varepsilon$ in some cases without any prior knowledge of the boundary layers. All of our numerical results support our mathematical theory. Our numerical results show that the error of our recovered solution is still small even in the boundary layers with $h \gg \varepsilon$. The error analysis for the recovered solution will be studied in the future.

7. ACKNOWLEDGEMENT

The authors thank Professor R. Bruce Kellogg for fruitful discussions on this work.

REFERENCES

1. M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions*, National Bureau of Standards, 1964.
2. A.E. Berger, H.D. Han and R.B. Kellogg, *A priori estimates and analysis of a numerical method for a turning point problem*, Math. Comp., **42** (1984): 465–492. MR736447 (85h:65158)
3. I. Brayanov and I. Dimitrova, *Uniformly convergent high-order schemes for a 2D elliptic reaction-diffusion problem with anisotropic coefficients*, Lecture Notes In Computer Science **2542** (2003): 395–402. MR2053277
4. L.C. Evans, *Partial differential equations*, American Mathematical Society, Providence, RI, 1998. MR1625845 (99e:35001)
5. I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series and products*, 6th Ed., Academic Press, 2000. MR1773820 (2001c:00002)
6. H. Han and Z. Huang, *A tailored finite point method for the Helmholtz equation with high wave numbers in heterogeneous medium*, J. Comp. Math, **26** (2008): 728–739. MR2444729 (2009j:65156)
7. H. Han and Z. Huang, *Tailored finite point method for a singular perturbation problem with variable coefficients in two dimensions*, J. Sci. Comp., **41** (2009): 200–220. MR2550367 (2010m:65249)
8. H. Han and Z. Huang, *Tailored finite point method for steady-state reaction-diffusion equations*, Commun. Math. Sci. **8** (2010), no. 4, 887–899. MR2744911 (2012a:65332)
9. H. Han, Z. Huang and B. Kellogg, *A Tailored finite point method for a singular perturbation problem on an unbounded domain*, J. Sci. Comp. **36** (2008): 243–261. MR2434846 (2010c:65192)
10. H. Han and R. B. Kellogg, *Differentiability properties of solutions of the equation $-\varepsilon \Delta u + ru = f(x, y)$ in a square*, SIAM J. Math. Anal. **21** (1990): 394–408. MR1038899 (91e:35025)
11. Z. Huang, *Tailored finite point method for the interface problem*, Networks and Heterogeneous Media, **4** (2009): 91–106. MR2480424 (2010f:65223)
12. A.M. Il'in, *Differencing scheme for a differential equation with a small parameter affecting the highest derivative*, Math. Notes, **6** (1969): 596–602. MR0260195 (41:4823)
13. R. B. Kellogg and M. Stynes, *Sharpened bounds for corner singularities and boundary layers in a simple convection-diffusion problem*, Applied Mathematics Letters **20** (2007): 539–544. MR2303990 (2007k:35093)

14. J. Li and Y. Chen, *Uniform convergence analysis for singularly perturbed elliptic problems with parabolic layers*, Numer. Math. Theor. Meth. Appl. **1** (2008): 138-149. MR2441969 (2009e:65180)
15. J. Li and I.M. Navon, *Uniformly convergent finite element methods for singularly perturbed elliptic boundary value problems: convection-diffusion*, Comput. Methods Appl. Mech. Engrg. **162** (1998): 49–78. MR1645349 (99f:65172)
16. Z.C. Li, H.S. Tsai, S. Wang and J.J.H. Miller, *Accurate and approximate analytic solutions of singularly perturbed differential equations with two-dimensional boundary layers*, Comput. Math. Appl., **55** (2008): 2602-2622. MR2416029 (2009e:65106)
17. J.M. Melenk, *hp-finite element methods for singular perturbations. Lecture Notes in Mathematics*, 1796. Springer-Verlag, Berlin, 2002. MR1939620 (2003i:65108)
18. J.J.H. Miller, *On the convergence, uniformly in ε , of difference schemes for a two-point boundary singular perturbation problem*, Numerical analysis of singular perturbation problems, eds., Hernker P. W. and Miller J. J, H., Academic Press, 467–474, 1979. MR556537 (81f:65061)
19. K.W. Morton, *Numerical solution of convection-diffusion problems*, Applied Mathematics and Mathematical Computation **12**, Chapman and Hall, London, 1996. MR1445295 (98b:65004)
20. E. O’Riordan and G.I. Shishkin, *Parameter uniform numerical methods for singularly perturbed elliptic problems with parabolic boundary layers*, Appl. Numer. Math., **58** (2008): 1761-1772. MR2464809 (2009j:65366)
21. H.-G. Roos, M. Stynes and L. Tobiska, *Robust numerical methods for singularly perturbed differential equations*, Springer, Berlin, 2nd ed., 2008. MR2454024 (2009f:65002)
22. G. I. Shishkin, *A finite difference scheme on a priori adapted meshes for a singularly perturbed parabolic convection-diffusion equation*, Numer. Math. Theor. Meth. Appl. **1** (2008): 214-234. MR2441974 (2009e:65124)
23. M. Stynes, *Steady-state convection-diffusion problems*, Acta Numerica, **14** (2005): 445–508. MR2170509 (2007d:35050)

DEPARTMENT OF MATHEMATICAL SCIENCES, TSINGHUA UNIVERSITY, BEIJING 100084, CHINA
E-mail address: `hhan@math.tsinghua.edu.cn`

DEPARTMENT OF MATHEMATICAL SCIENCES, TSINGHUA UNIVERSITY, BEIJING 100084, CHINA
E-mail address: `zhuang@math.tsinghua.edu.cn`