THE FD-METHOD FOR SOLVING STURM-LIOUVILLE PROBLEMS WITH SPECIAL SINGULAR DIFFERENTIAL OPERATOR

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ABSTRACT. A superexponentially convergent method for solving the Sturm-Liouville problem with Legendre's differential operator is given. The presented method (called the FD-method) is based on the coefficient approximation methods (CAM) and the homotopy approach. Sufficient convergence conditions of the proposed method are stated and rigorously substantiated. The algorithms for software implementation of the proposed method are described. The numerical examples included in the paper confirm the theoretical conclusions.

1. INTRODUCTION

There is a great range of numerical methods for solving eigenvalue problems for second-order differential operators (see [10]). Known numerical techniques for eigenvalue problems can be divided into two groups: methods based on direct approximation of solutions of the differential equation and methods based on approximation of the coefficients of the differential equation (see [8]).

The methods of the first group include the finite difference, finite elements, and spectral method, which are extensively treated through theoretical investigations as well as developed software. The main idea of this approach is to replace the eigenfunctions by piecewise polynomial functions. That results in an approximation of the differential equation with a system of linear algebraic equations. Due to the different nature of the original differential operator and the approximating algebraic operator the numerical solution approaches the exact solution only for low-indexed eigenvalues and the corresponding eigenfunctions. Hence, such numerical techniques are effective for finding low-indexed eigenvalues but not effective for finding high-indexed eigenvalues [8].

The methods in the second group are based on the general idea of the *coefficient approximation methods* (hereinafter referred to as the CAM), also known as the Pruess methods. The idea is to replace the coefficient functions of the problem with a simpler approximation and then solve the approximating problem (see [10]). It was first proposed by Kryloff and Bogolioubov (1926) in [5] for piecewise constant approximation. S. Pruess provided a thorough convergence and error analysis of such methods when piecewise polynomial approximation is used. Except for Gordon, who uses linear functions, the earlier references all confine themselves to piecewise constant approximations, and this is the most practical choice; otherwise

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the approximating problem may not be easier to solve numerically than the original. It is well known (see [10]) that if a piecewise constant approximation with the maximum step size h is used, the error of the CAM is estimated by $O(h)$. Hence, a crucial problem arises when we use the CAM: to halve the error we have to double the computation time. This problem in many cases can be solved by using the *functional-discrete* method (hereinafter referred to as the *FD-method*) which is a substantial generalization of the CAM for solving Sturm-Liouville problems of different kinds. The algorithm of the FD-method was first described and justified for the linear Sturm-Liouville problem in [7]. Generally speaking, it has two main steps: first, you have to find the initial approximation by applying the CAM; then, you have to calculate a sufficient number of corrections to get the required accuracy. The second step is provided by the *homotopy method* (also known as the *perturbation method*, the *continuation method* or the *successive loading method*; see, for example, [2]). Since the publication of paper [7] the application field of the FD-method has been substantially extended, especially towards the nonlinear Sturm-Liouville problems (see [3]).

In the present paper we consider the following singular Sturm-Liouville problem:

$$(1.1) \quad \frac{d}{dx} \left[(1-x^2) \frac{du(x)}{dx} \right] - q(x)u(x) = \lambda u(x), \quad x \in (-1, 1),$$

$$(1.2) \quad \lim_{x \rightarrow \pm 1} (1-x^2) \frac{du(x)}{dx} = 0.$$

We may encounter problems of this kind when solving partial differential equations via separation of variables as it was, for example, with the hydrogen-molecule ion's equation in [11] (see [11], pp. 167–170).

Since the problem (1.1), (1.2) has singularities at points ± 1 , we cannot directly apply the algorithm of the FD-method proposed in [7]. However, a modified FD-method technique for problems with such singularities was presented in [6]. Let us briefly state the algorithm of the FD-method described in [6].

According to the FD-method we approximate the eigenvalues and eigenfunctions of problem (1.1), (1.2) with the following truncated series:

$$(1.3) \quad \lambda_n = \sum_{j=0}^m \lambda_n^{(j)}, \quad u_n = \sum_{j=0}^m u_n^{(j)}(x),$$

where $\lambda_n^{(j)}$ and $u_n^{(j)}(x)$, $i = 1, 2, \dots, m^1$ can be determined from the following recurrence problems:

$$(1.4) \quad \begin{aligned} & \frac{d}{dx} \left[(1-x^2) \frac{du_n^{(j)}(x)}{dx} \right] + \lambda_n^{(i)} u_n^{(j)}(x) \\ &= - \sum_{s=0}^{j-1} \lambda_n^{(j-s)} u_n^{(s)}(x) + q(x) u_n^{(j-1)}(x) = F_n^{(j)}(x), \quad x \in (-1, 1) \end{aligned}$$

with boundary conditions

$$(1.5) \quad \lim_{x \rightarrow \pm 1} \left[(1-x^2) \frac{du_n^{(j)}(x)}{dx} \right] = 0,$$

¹The positive integer number m is called the *rank* of FD-method.

and orthogonality conditions

$$(1.6) \quad \int_{-1}^1 u_n^{(j)}(x) u_n^{(0)}(x) dx = 0, \quad j = 1, 2, \dots, m.$$

The initial data $\lambda_n^{(0)}, u_n^{(0)}(x)$ needed to start the recurrence process can be found by solving the eigenvalue problem called *the basic problem*:

$$(1.7) \quad \frac{d}{dx} \left[(1-x^2) \frac{du_n^{(0)}(x)}{dx} \right] + \lambda_n^{(0)} u_n^{(0)}(x) = 0, \quad x \in (-1, 1),$$

$$(1.8) \quad \lim_{x \rightarrow \pm 1} \left[(1-x^2) \frac{du_n^{(0)}(x)}{dx} \right] = 0.$$

The solutions to problem (1.7), (1.8) can be expressed in the form of

$$(1.9) \quad \lambda_n^{(0)} = n(n+1), \quad u_n^{(0)}(x) = \sqrt{\frac{2n+1}{2}} P_n(x),$$

where $P_n(x)$ denotes the Legendre polynomial of the first kind of order $n \in \mathbb{N}$ (see [4]). The following statement about the convergence of the FD-method as described above was proved in [6].

Theorem 1.1. *Let the function $q(x)$ belong to the class $Q^0[-1, 1]$ of piecewise continuous functions defined on $[-1, 1]$. Suppose that the positive integer n_0 satisfies the inequality*

$$(1.10) \quad q_{n_0} = \frac{2 \|q(x)\|_{\infty, [-1, 1]}}{n_0} < 1.$$

Then the FD-method (1.4)–(1.9) has a superexponential convergence rate for all indices $n \geq n_0$:

$$(1.11) \quad \left\| u_n(x) - \overset{m}{u}_n(x) \right\| \leq \frac{(q_n)^{m+1}}{1 - q_n} \alpha_{m+1}, \quad \left| \lambda_n - \overset{m}{\lambda}_n \right| \leq \frac{(q_n)^m}{1 - q_n} \|q(x)\|_{\infty} \alpha_m, \\ n = n_0, n_0 + 1, \dots,$$

where

$$\alpha_{m+1} = 2 \frac{(2m-1)!!}{(2m+2)!!} \leq \frac{1}{(m+1)\sqrt{\pi m}}, \quad \|q(x)\|_{\infty, [-1, 1]} = \max_{x \in [-1, 1]} |q(x)|.$$

Theorem 1.1 was proved in [6] for the case when $q(x) = q(-x)$. However, using the same technique one can prove an analogous theorem for the case of an arbitrary function $q(x)$ from $Q^0[-1, 1]$ without considerable difficulties.

If we try to apply the FD-method (1.4)–(1.9) to approximate the eigensolutions $\lambda_n, u_n(x)$ with $n < 2 \|q(x)\|_{\infty, [-1, 1]}$ (see inequality (1.10)) we might find it divergent. The reason for that is the lack of approximation for $q(x)$ in equation (1.7) of the basic problem. To overcome such difficulties we have to apply the general scheme of the FD-method using the approximation of the function $q(x)$ on $[-1, 1]$ with a piecewise constant function $\bar{q}(x)$.

2. THE GENERAL ALGORITHM OF THE FD-METHOD FOR SOLVING THE
STURM-LIOUVILLE PROBLEM WITH LEGENDRE DIFFERENTIAL OPERATOR:
A THEORETICAL JUSTIFICATION

To make the general algorithm of the FD-method easier to understand, let us consider the following general problem:

$$(2.1) \quad \frac{\partial}{\partial x} \left[(1-x^2) \frac{\partial u(x, t)}{\partial x} \right] + [\lambda(t) - \bar{q}(x) - t(q(x) - \bar{q}(x))] u(x, t) = 0,$$

$$x \in (-1, 1), \quad t \in [0, 1],$$

$$(2.2) \quad \lim_{x \rightarrow \pm 1} (1-x^2) \frac{\partial u(x, t)}{\partial x} = 0, \quad \forall t \in [0, 1].$$

To define the function $\bar{q}(x)$, we have to introduce a mesh on the interval $[-1, 1]$:

$$(2.3) \quad \bar{\omega} = \{-1 = x_0 < x_1 < \dots < x_{N-1} < x_N = 1\}, \quad h = \max_{i=1,2,\dots,N} (x_i - x_{i-1}).$$

We also require that all points of discontinuity of the function $q(x)$ on the interval $[-1, 1]$ be contained in the set $\bar{\omega}$.

One can choose $\bar{q}(x)$ in many possible ways. For example,

$$\bar{q}(x) = \bar{q}(x, \bar{\omega}) = q\left(\frac{1}{2}(x_{i-1} + x_i)\right), \quad x \in [x_{i-1}, x_i], \quad i = 1, 2, \dots, N$$

or

$$(2.4) \quad \bar{q}(x) = \bar{q}(x, \bar{\omega}) = \frac{1}{2}(q(x_{i-1}) + q(x_i)), \quad x \in [x_{i-1}, x_i], \quad i = 1, 2, \dots, N,$$

but in any case the piecewise constant function $\bar{q}(x, \bar{\omega})$ must have the following property:

$$\lim_{h \rightarrow 0} \left(\max_{x \in [-1, 1]} |\bar{q}(x, \bar{\omega}) - q(x)| \right) = 0.$$

It is easy to see that if we make $t = 1$, then problem (2.1), (2.2) will be reduced to problem (1.1), (1.2). Assume that the solution to problem (2.1), (2.2) can be expressed in the form of a series:

$$(2.5) \quad \lambda_n = \sum_{j=0}^{\infty} t^{(j)} \lambda_n^{(j)}, \quad u_n(x, t) = \sum_{j=0}^{\infty} t^j u_n^{(j)}(x)$$

and

$$(2.6) \quad \frac{\partial u_n(x, t)}{\partial x} = \sum_{j=0}^{\infty} t^j \frac{\partial u_n^{(j)}(x)}{\partial x}, \quad \frac{\partial^2 u_n(x, t)}{\partial x^2} = \sum_{j=0}^{\infty} t^j \frac{\partial^2 u_n^{(j)}(x)}{\partial x^2},$$

$$\forall x \in (-1, 1), \quad \forall t \in [0, 1].$$

The unknown summands of the series (2.5) are the solutions to the recurrence problems

$$(2.7) \quad \frac{d}{dx} \left[(1-x^2) \frac{du_n^{(j)}(x)}{dx} \right] + [\lambda_n^{(0)} - \bar{q}(x)] u_n^{(j)}(x) = - \sum_{s=0}^{j-1} \lambda_n^{(j-s)} u_n^{(s)}(x) + [q(x) - \bar{q}(x)] u_n^{(j-1)}(x) \equiv F_n^{(j)}(x), \quad x \in (-1, 1),$$

with the boundary conditions

$$(2.8) \quad \lim_{x \rightarrow \pm 1} \left[(1-x^2) \frac{du_n^{(j)}(x)}{dx} \right] = 0,$$

matching conditions

$$(2.9) \quad \left[u_n^{(j)}(x) \right]_{x=x_i} = 0, \quad \left[\frac{du_n^{(j)}(x)}{dx} \right]_{x=x_i} = 0, \quad i = 1, 2, \dots, N-1,$$

and an additional requirement that

$$(2.10) \quad \int_{-1}^1 u_n^{(j)}(x) u_n^{(0)}(x) dx = \delta_{0,j} = \begin{cases} 1, & \text{if } j = 0, \\ 0, & \text{if } j \neq 0, \end{cases}$$

$$j = 0, 1, \dots, m.$$

Here square brackets denote the jump of the function at the point $x = x_i$.

Out of problems (2.7)–(2.9) it is worth accentuating the first one ($j = 0$):

$$(2.11) \quad \frac{d}{dx} \left[(1-x^2) \frac{du_n^{(0)}(x)}{dx} \right] + [\lambda_n^{(0)} - \bar{q}(x)] u_n^{(0)}(x) = 0, \quad x \in (-1, 1),$$

$$(2.12) \quad \lim_{x \rightarrow \pm 1} \left[(1-x^2) \frac{du_n^{(0)}(x)}{dx} \right] = 0, \quad \int_{-1}^1 \left(u_n^{(0)}(x) \right)^2 dx = 1,$$

$$(2.13) \quad \left[u_n^{(0)}(x) \right]_{x=x_i} = 0, \quad \left[\frac{du_n^{(0)}(x)}{dx} \right]_{x=x_i} = 0.$$

As mentioned above, problem (2.11)–(2.13) is called *the basic problem*. It is easy to verify that the differential operator

$$(2.14) \quad L(\cdot) = \frac{d}{dx} \left[(1-x^2) \frac{d(\cdot)}{dx} \right] - \bar{q}(x)(\cdot)$$

is self-adjoint in Sobolev space

$$\overline{W}^{2,1}(-1, 1) = \left\{ f(x) \in W^{2,1}(-1, 1) : \lim_{x \rightarrow \pm 1} (1-x^2) f'(x) = 0 \right\}.$$

This fact implies that the eigenfunctions $u_n^{(0)}(x)$, $n = 0, 1, \dots$ of problem (2.11)–(2.13) generate a complete orthogonal system in $\overline{W}^{2,1}(-1, 1)$ and the corresponding eigenvalues $\lambda_n^{(0)}$, ($\lambda_i^{(0)} < \lambda_j^{(0)}$ when $i < j$, $i, j = 0, 1, 2, \dots$) are simple.

Suppose that the basic problem is solved and the eigenpair $\lambda_n^{(0)}$, $u_n^{(0)}(x)$ is one of its eigensolutions.

It is well known that problems (2.7)–(2.9) for $j = 1, 2, \dots, m$ can be solved if and only if the following equality holds:

$$(2.15) \quad \int_{-1}^1 u_n^{(0)}(x) F_n^{(j)}(x) dx = 0.$$

Combining equality (2.15) with (2.7) we get the formula for computing $\lambda_n^{(j)}$:
(2.16)

$$\lambda_n^{(j)} = - \sum_{s=1}^{j-1} \lambda_n^{(j-s)} \int_{-1}^1 u_n^{(s)}(x) u_n^{(0)}(x) dx + \int_{-1}^1 (q(x) - \bar{q}(x)) u_n^{(j-1)}(x) u_n^{(0)}(x) dx.$$

Taking into account (2.10) we can simplify formula (2.16) as follows:

$$(2.17) \quad \lambda_n^{(j)} = \int_{-1}^1 (q(x) - \bar{q}(x)) u_n^{(j-1)}(x) u_n^{(0)}(x) dx.$$

One can easily verify that the function $u_n^{(j)}(x)$ can be represented in the form of the series

$$(2.18) \quad u_n^{(j)}(x) = \sum_{\substack{p=0 \\ p \neq n}}^{\infty} \frac{\int_{-1}^1 F_n^{(j)}(x) u_p^{(0)}(x) dx}{\lambda_n^{(0)} - \lambda_p^{(0)}} u_p^{(0)}(x).$$

Formulas (2.17) and (2.18) yield estimations

$$(2.19) \quad \left| \lambda_n^{(j)} \right| \leq \|q(x) - \bar{q}(x)\|_{\infty, [-1, 1]} \left\| u_n^{(j-1)}(x) \right\|$$

and

$$\begin{aligned} & \left\| u_n^{(j)}(x) \right\| \leq M_n \left\| F_n^{(j)}(x) \right\| \\ (2.20) \quad & = M_n \left\{ \left\| \sum_{s=1}^{j-1} \lambda_n^{j-s} u_n^{(s)}(x) - (q(x) - \bar{q}(x)) u_n^{(j-1)}(x) \right\|^2 - (\lambda^{j+1})^2 \right\}^{\frac{1}{2}} \\ & \leq M_n \left\{ \sum_{s=1}^{j-1} \left| \lambda_n^{(j-s)} \right| \left\| u_n^{(s)}(x) \right\| + \|q(x) - \bar{q}(x)\|_{\infty, [-1, 1]} \left\| u_n^{(j-1)}(x) \right\| \right\} \\ & \leq M_n \|q(x) - \bar{q}(x)\|_{\infty, [-1, 1]} \sum_{s=0}^{j-1} \left\| u_n^{(j-1-s)} \right\| \left\| u_n^{(s)}(x) \right\|, \end{aligned}$$

respectively, where

$$(2.21) \quad M_n = \max \left\{ \left(\lambda_n^{(0)} - \lambda_{n-1}^{(0)} \right)^{-1}, \left(\lambda_{n+1}^{(0)} - \lambda_n^{(0)} \right)^{-1} \right\}, \quad n = 1, 2, \dots,$$

$$M_0 = \left(\lambda_1^{(0)} - \lambda_0^{(0)} \right)^{-1},$$

and $\|\cdot\|$ is the common L_2 -norm, $\|f(x)\| = \sqrt{\int_{-1}^1 f^2(x) dx}$.

Multiplying both sides of inequality (2.20) by $\left(M_n \|q(x) - \bar{q}(x)\|_{\infty, [-1, 1]} \right)^{-j}$ and using estimation (2.19) we obtain

$$(2.22) \quad \frac{\left\| u_n^{(j)}(x) \right\|}{\left(M_n \|q(x) - \bar{q}(x)\|_{\infty, [-1, 1]} \right)^j} \leq \sum_{s=0}^{j-1} \frac{\left\| u_n^{(j-s-1)}(x) \right\| \left\| u_n^{(s)}(x) \right\|}{\left(M_n \|q(x) - \bar{q}(x)\|_{\infty, [-1, 1]} \right)^{j-1}}.$$

Using the notation

$$(2.23) \quad v_j = \left\| u_n^{(j)}(x) \right\| \left(M_n \|q(x) - \bar{q}(x)\|_{\infty, [-1, 1]} \right)^{-j} = \left\| u_n^{(j)}(x) \right\| \bar{r}_n^{-j},$$

we can represent inequality (2.22) as

$$(2.24) \quad v_j \leq \sum_{s=0}^{j-1} v_{j-s-1} v_s, \quad j = 1, 2, \dots, \quad v_0 = \left\| u_n^{(0)}(x) \right\| = 1.$$

It is easy to see that the sequence $\{V_i\}$ defined by the recurrence formula

$$(2.25) \quad V_j = \sum_{s=0}^{j-1} V_{j-1-s} V_s, \quad V_0 = 1, \quad j = 0, 1, \dots,$$

satisfies inequalities

$$(2.26) \quad V_j \geq v_j, \quad j = 0, 1, \dots$$

Let us consider the generating function $f(z)$ defined by the formula

$$(2.27) \quad f(z) = \sum_{s=0}^{\infty} V_s z^s, \quad z \in \mathbb{R}.$$

From (2.25) we obtain the equation with respect to $f(z)$:

$$(2.28) \quad f(z) = z f^2(z) + 1, \quad z \in (-\rho, \rho),$$

where ρ denotes the convergence radius for the power series (2.27). The solution to equation (2.28) that satisfies the condition $f(0) = 1$ is

$$(2.29) \quad f(z) = (2z)^{-1} (1 - \sqrt{1 - 4z}) = \left(2z + \sum_{s=2}^{\infty} \frac{(2s-3)!!}{(2s)!!} (4z)^s \right) (2z)^{-1}, \quad z \in \left[-\frac{1}{4}, \frac{1}{4} \right]$$

and

$$(2.30) \quad V_j = 2 \frac{(2j-1)!!}{(2j+2)!!} 4^j = \alpha_j 4^j, \quad j = 1, 2, \dots, \quad (-1)!! \stackrel{\text{def}}{=} 1.$$

As follows from (2.29), the convergence radius of the power series (2.27) is equal to $\frac{1}{4}$ and, additionally, the series converges at the points $z = \pm \frac{1}{4}$.

Going back to formula (2.23) and using inequalities (2.26) along with formula (2.30) we get the estimation

$$(2.31) \quad \left\| u_n^{(j)}(x) \right\| = \left(M_n \|q(x) - \bar{q}(x)\|_{\infty, [-1, 1]} \right)^j v_j \leq (4\bar{r}_n)^j \alpha_j = r_n^j \alpha_j$$

and inequality (2.31) gives us the following estimations of the FD-method's convergence rate:

$$(2.32) \quad \left\| u_n(x) - \overset{m}{u}_n(x) \right\| \leq \begin{cases} \frac{r_n^{m+1}}{1 - r_n} \alpha_{m+1}, & \text{if } r_n < 1, \\ \sum_{j=m+1}^{\infty} \frac{1}{(j+1)\sqrt{\pi j}}, & \text{if } r_n = 1, \end{cases}$$

$$(2.33) \quad \left| \lambda_n - \overset{m}{\lambda}_n \right| \leq \begin{cases} \|q(x) - \bar{q}(x)\|_{\infty, [-1, 1]} \frac{r_n^m}{1 - r_n} \alpha_m, & \text{if } r_n < 1, \\ \|q(x) - \bar{q}(x)\|_{\infty, [-1, 1]} \sum_{j=m}^{\infty} \frac{1}{(j+1)\sqrt{\pi j}}, & \text{if } r_n = 1. \end{cases}$$

The infinite sum in the right sides of inequalities (2.32) and (2.33) can be easily estimated as

$$(2.34) \quad \sum_{j=m}^{\infty} \frac{1}{(j+1)\sqrt{\pi j}} \leq \int_{m-1}^{\infty} \frac{1}{(x+1)\sqrt{x\pi}} dx = \frac{2}{\sqrt{\pi}} \arctan \left(\frac{1}{\sqrt{m+1}} \right).$$

Hereby, we have proved the following theorem.

Theorem 2.1. Assume that $q(x)$ belongs to $Q^0[-1, 1]$ and the constant $\bar{r}_n = M_n \|q(x) - \bar{q}(x)\|_{\infty, [-1, 1]}$ satisfies the inequality $\bar{r}_n \leq \frac{1}{4}$. Under this assumption the FD-method (2.5), (2.7)–(2.10) converges faster than a geometric series with denominator $r_n = 4\bar{r}_n$ (which is to say, superexponentially). The estimations of its convergence rate are given by formulas (2.32), (2.33).

Before passing on to the next section let us investigate the question of asymptotical behavior of the parameter M_n in (2.21) as n tends to $+\infty$. It is easy to see that the basic problem (2.11)–(2.13) is equivalent to the auxiliary eigenvalue problem

$$(2.35) \quad \frac{\partial}{\partial x} \left[(1-x^2) \frac{\partial v(x, \tau)}{\partial x} \right] + [\mu(\tau) - \tau \bar{q}(x)] v(x, \tau) = 0, \quad x \in (-1, 1), \tau \in [0, 1],$$

$$(2.36) \quad \lim_{x \rightarrow \pm 1} \left[(1-x^2) \frac{\partial v(x, \tau)}{\partial x} \right] = 0, \quad \int_{-1}^1 (v(x, \tau))^2 dx = 1, \quad \forall \tau \in [0, 1],$$

$$(2.37) \quad [v(x, \tau)]_{x=x_i} = 0, \quad \left[\frac{\partial v(x, \tau)}{\partial x} \right]_{x=x_i} = 0, \quad \forall \tau \in [0, 1], \quad i = 1, 2, \dots, N-1,$$

when $\tau = 1$. On the other hand, from the proof of Theorem 1.1 (see [6]) it follows that for a sufficiently large number n , namely²

$$(2.38) \quad n \geq n_0 = \begin{cases} [2 \|q(x)\|_{\infty, [-1, 1]}] + 1 & \text{if } \|q(x)\|_{\infty, [-1, 1]} \geq \frac{1}{2}, \\ 0 & \text{if } \|q(x)\|_{\infty, [-1, 1]} < \frac{1}{2}, \end{cases}$$

the eigensolution $\mu_n(\tau)$, $v_n(x, \tau)$ to problem (2.35)–(2.37) exists and can be expanded into a power series with respect to τ on the interval $[0, 1]$ for all $x \in (-1, 1)$; furthermore,

$$(2.39) \quad \mu_n(0) = n(n+1), \mu_n(1) = \lambda_n^{(0)}.$$

Hence, from equality (2.35) it is easy to obtain the equality

$$(2.40) \quad \frac{\partial}{\partial x} \left[(1-x^2) \frac{\partial^2 v_n(x, \tau)}{\partial x \partial \tau} \right] + [\mu_n(\tau) - \tau \bar{q}(x)] \frac{\partial}{\partial \tau} v_n(x, \tau) = [\mu'_n(\tau) - \bar{q}(x)] v_n(x, \tau), \\ x \in (-1, 1), \tau \in [0, 1].$$

Then, applying the integral operator $\int_{-1}^1 v_n(x, \tau)(\cdot) dx$, to both sides of equality (2.40) we derive that

$$(2.41) \quad \mu'_n(\tau) = \int_{-1}^1 (v_n(x, \tau))^2 \bar{q}(x) dx, \quad \tau \in [0, 1].$$

²The square brackets in formula (2.38) denote the integer part function.

Taking into account (2.39) and equality (2.41) we obtain the equality

$$(2.42) \quad \lambda_n^{(0)} = n(n+1) + \int_0^1 \int_{-1}^1 (v_n(x, \tau))^2 \bar{q}(x) dx d\tau.$$

Finally, equality (2.42) yields the following estimation:

$$(2.43) \quad \lambda_{n+1}^{(0)} - \lambda_n^{(0)} \geq 2(n+1) - 2 \|q(x)\|_{\infty, [-1, 1]} > 0.$$

Hence, for a sufficiently large n (see (2.38)) we have arrived at an estimation that describes the asymptotical behavior of M_n :

$$(2.44) \quad M_n \leq \left(2n - 2 \|q(x)\|_{\infty, [-1, 1]}\right)^{-1}.$$

Taking into account (2.38) and using formula (2.42) we can easily prove that if

$$(2.45) \quad 0 \leq q(x) < \frac{1}{2}, \quad \forall x \in [-1, 1],$$

then inequality (2.44) can be reduced as follows:

$$(2.46) \quad M_n < (2n - 1)^{-1}, \quad n = 1, 2, \dots, \quad M_0 < 1.$$

Furthermore, assumption (2.45) provides the inequalities

$$\lambda_{n+1}^{(0)} - \lambda_n^{(0)} > \lambda_n^{(0)} - \lambda_{n-1}^{(0)}, \quad n = 1, 2, \dots,$$

which allow us to simplify formula (2.21):

$$M_n = (\lambda_n^{(0)} - \lambda_{n-1}^{(0)})^{-1} < 1, \quad n = 1, 2, \dots, \quad M_0 = M_1.$$

3. THE GENERAL ALGORITHM OF THE FD-METHOD: SOFTWARE IMPLEMENTATION

In this section we discuss the question of implementing the algorithm of the FD-method in detail.

The solution to the basic problem (2.11)–(2.13) can be represented in the form

$$(3.1) \quad u_n^{(0)}(x) = A_i P_{\nu_i}(x) + B_i Q_{\nu_i}(x), \quad x \in [x_{i-1}, x_i], \quad A_i, B_i \in \mathbb{R}, \quad i = \overline{1, N},$$

where

$$(3.2) \quad \nu_i = \nu_i(\lambda_n^{(0)}) = \frac{1}{2} \left(-1 \pm \sqrt{1 + 4(\lambda_n^{(0)} - \bar{q}_i)} \right)^3, \quad \bar{q}_i \equiv \bar{q}(x), \quad x \in [x_{i-1}, x_i]$$

and $Q_{\nu_i}(x)$ denotes a Legendre function of the second kind (see [4]). It is well known that (see, for example, [4])

$$(3.3) \quad \lim_{x \rightarrow -1} (1 - x^2) \frac{dP_\nu(x)}{dx} = \frac{2 \sin(\pi \nu)}{\pi}, \quad \lim_{x \rightarrow 1} (1 - x^2) \frac{dP_\nu(x)}{dx} = 0,$$

$$\lim_{x \rightarrow -1} (1 - x^2) \frac{dQ_\nu(x)}{dx} = \cos(\pi \nu), \quad \lim_{x \rightarrow 1} (1 - x^2) \frac{dQ_\nu(x)}{dx} = 1.$$

To satisfy boundary condition (2.12) we have to require that

$$(3.4) \quad A_1 \frac{2 \sin(\pi \nu_1)}{\pi} + B_1 \cos(\pi \nu_1) = 0, \quad B_N = 0.$$

³Either “+” or “−” can be chosen.

The constants A_i, B_i for $i < N$ have to be found using matching conditions (2.13). This means that they can be found as solutions to the following recurrence systems of linear algebraic equations:

$$(3.5) \quad \begin{cases} A_{i-1}P_{\nu_{i-1}}(x_{i-1}) + B_{i-1}Q_{\nu_{i-1}}(x_{i-1}) &= A_iP_{\nu_i}(x_{i-1}) + B_iQ_{\nu_i}(x_{i-1}), \\ A_{i-1}P'_{\nu_{i-1}}(x_{i-1}) + B_{i-1}Q'_{\nu_{i-1}}(x_{i-1}) &= A_iP'_{\nu_i}(x_{i-1}) + B_iQ'_{\nu_i}(x_{i-1}), \end{cases}$$

$i = 2, 3, \dots, N$.

Let us consider the function

$$\Phi(\lambda) = A_1 \frac{2 \sin(\pi \nu_1)}{\pi} + B_1 \cos(\pi \nu_1), \quad \nu_1 = \nu_1(\lambda),$$

where constants A_1, B_1 are derived from the following recurrence formulas (see (3.5)):

$$(3.6) \quad \begin{aligned} A_{i-1} &= \frac{1}{\delta_{i-1}} \left(C_i Q'_{\nu_{i-1}}(x_{i-1}) - D_i Q_{\nu_{i-1}}(x_{i-1}) \right), \\ B_{i-1} &= \frac{1}{\delta_{i-1}} \left(D_i P_{\nu_{i-1}}(x_{i-1}) - C_i P'_{\nu_{i-1}}(x_{i-1}) \right), \\ \delta_{i-1} &= P_{\nu_{i-1}}(x_{i-1}) Q'_{\nu_{i-1}}(x_{i-1}) - P'_{\nu_{i-1}}(x_{i-1}) Q_{\nu_{i-1}}(x_{i-1}), \\ C_i &= A_i P_{\nu_i}(x_{i-1}) + B_i Q_{\nu_i}(x_{i-1}), \\ D_i &= A_i P'_{\nu_i}(x_{i-1}) + B_i Q'_{\nu_i}(x_{i-1}), \quad i = N, N-1, \dots, 2 \end{aligned}$$

with the initial data

$$(3.7) \quad A_N = 1, \quad B_N = 0,$$

and the values $\nu_i = \nu_i(\lambda)$ are defined by formula (3.2). It is easy to see that the eigenvalues of the basic problem (2.11)–(2.13) coincide with the solutions of the equation

$$(3.8) \quad \Phi(\lambda) = 0.$$

Since the eigenvalues of the basic problem are simple, they can be found with arbitrary precision as the solutions to equation (3.8) using the bisection method described in Algorithm 1 below. It is worth emphasizing that in practice the value ν_i , obtained from formula (3.2) can be complex. Therefore, the value of $\Phi(\lambda)$ computed using formulas (3.6) can be complex too. However, to avoid unnecessary difficulties related to using complex numbers in computation it is sufficient to take real parts of $P_{\nu_i}(x_i)$, $P'_{\nu_i}(x_i)$, $Q_{\nu_i}(x_i)$, $Q'_{\nu_i}(x_i)$ only. This was done in the algorithm for the function **ComputePhi** described below.

Let $\lambda_n^{(0)}$ be a single fixed solution to equation (3.8). Using the coefficients A_i, B_i , $i = 1, 2, \dots, N$, computed from formulas (3.6) with $\lambda = \lambda_n^{(0)}$ (see comments at the end of Algorithm 1) we can compute the corresponding eigenfunction $u_n^{(0)}(x)$ using formula (3.1).

To compute $\lambda_n^{(j)}$ we can use the formula

$$(3.9) \quad \lambda_n^{(j)} = \left[\int_{-1}^1 \left(u_n^{(0)}(x) \right)^2 dx \right]^{-1} \int_{-1}^1 (q(x) - \bar{q}(x)) u_n^{(j-1)}(x) u_n^{(0)}(x) dx.$$

Algorithm 1: The algorithm for solving the basic problem

input : The left (l_1) and the right (l_2) ends of the interval where $\lambda_n^{(0)}$ have to be found, $l_1 < l_2$, $\Phi(l_1)\Phi(l_2) < 0$;
the absolute computational error $\varepsilon > 0$.

output: The arrays of values ν_i, A_i, B_i , $i = 1, 2, \dots, N$;
 λ — the approximation of $\lambda_n^{(0)}$ with the accuracy ε .

begin

```

1   $\phi_1 := \text{ComputePhi}(l_1)$ ;  $\phi_2 := \text{ComputePhi}(l_2)$ 
2  while  $l_2 - l_1 > \varepsilon$  do
3       $\lambda := \frac{l_2 + l_1}{2}$ ;  $\phi := \text{ComputePhi}(\lambda)$ 
4      if  $\phi_1 \phi < 0$  then  $\phi_2 := \phi$ ;  $l_2 := \lambda$ 
5      else if  $\phi_1 \phi > 0$  then
6           $\phi_1 := \phi$ ;  $l_1 := \lambda$ 
7      else if  $\phi := 0$  then
8           $l_1 := \lambda$ ;  $l_2 := \lambda$ 
9      end
10 end
11  $\lambda := \frac{l_1 + l_2}{2}$ 
12 ComputePhi( $\lambda$ ) // The values  $\nu_i, A_i, B_i$ ,  $i = 1, 2, \dots, N$  calculated
    during this call of function ComputePhi can be used in
    formula (3.1) to compute the values of the function  $u_n^{(0)}(x)$ 
13 end

```

To compute the functions $u_n^{(j)}(x)$, $j = 1, 2, \dots, m$, it is convenient to use the formula

$$\begin{aligned}
 (3.10) \quad u_n^{(j)}(x) &= E_j u_n^{(0)}(x) + \int_{-1}^x K(x, \xi) F_n^{(j)}(\xi) d\xi \\
 &= E_j u_n^{(0)}(x) + w_n(x) \int_{-1}^x u_n^{(0)}(\xi) F_n^{(j)}(\xi) d\xi - u_n^{(0)}(x) \int_{-1}^x w_n(\xi) F_n^{(j)}(\xi) d\xi,
 \end{aligned}$$

where

$$\begin{aligned}
 K(x, \xi) &= w_n(x) u_n^{(0)}(\xi) - w_n(\xi) u_n^{(0)}(x), \\
 E_j &= - \left[\int_{-1}^1 \left(u_n^{(0)}(x) \right)^2 dx \right]^{-1} \int_{-1}^1 \left\{ u_n^{(0)}(x) \int_{-1}^x K(x, \xi) F_n^{(j)}(\xi) d\xi \right\} dx,
 \end{aligned}$$

and $w_n(x)$ denotes a function obtained from the formula

$$w_n(x) = A_i P_{\nu_i}(x) + B_i Q_{\nu_i}(x), \quad x \in [x_{i-1}, x_i], \quad i = \overline{1, N}.$$

Here A_i, B_i , $i = 1, 2, \dots, N$ can be computed recursively from formulas (3.6) with the initial data

$$(3.11) \quad A_N = 0, \quad B_N = 1$$

and ν_i are computed by formula (3.2).

Function ComputePhi(λ)

input : $\lambda \in \mathbb{R}$
output: The value of $\Phi(\lambda)$;
The arrays of values $\nu_i, A_i, B_i, i = 1, 2, \dots, N$;

begin

1 $A_N := 1; B_N := 0$

2 $q_1 := q(\frac{x_N + x_{N-1}}{2});$

3 $\nu_N := \frac{1}{2}(-1 + \sqrt{1 + 4(\lambda - q_1)})$

4 **for** $i := N$ **downto** 2 **do**

5 $q_1 := q(\frac{x_{i-1} + x_{i-2}}{2});$

6 $\nu_{i-1} := \frac{1}{2}(-1 + \sqrt{1 + 4(\lambda - q_1)})$

7 $\delta := \Re(P_{\nu_{i-1}}(x_{i-1}))\Re(Q'_{\nu_{i-1}}(x_{i-1})) - \Re(P'_{\nu_{i-1}}(x_{i-1}))\Re(Q_{\nu_{i-1}}(x_{i-1}))$

8 $C := A_i\Re(P_{\nu_i}(x_{i-1})) + B_i\Re(Q_{\nu_i}(x_{i-1}))$

9 $D := A_i\Re(P'_{\nu_i}(x_{i-1})) + B_i\Re(Q'_{\nu_i}(x_{i-1}))$

10 $A_{i-1} := \frac{1}{\delta} \left(C\Re(Q'_{\nu_{i-1}}(x_{i-1})) - D\Re(Q_{\nu_{i-1}}(x_{i-1})) \right)$

11 $B_{i-1} := \frac{1}{\delta} \left(D\Re(P_{\nu_{i-1}}(x_{i-1})) - C\Re(P'_{\nu_{i-1}}(x_{i-1})) \right)$

end

12 **return** $\Re(2A_1 \sin(\pi\nu_1) + B_1\pi \cos(\pi\nu_1))$

end

Taking into account the properties of Legendre functions (see [4]) it is easy to verify that

$$(3.12) \quad \left. \frac{\partial K(x, \xi)}{\partial x} \right|_{\xi=x} = (1 - x^2)^{-1}, \quad x \in (-1, 1)$$

and $K(x, \xi)$ has the properties of a Cauchy operator for inhomogeneous equations (2.7).

Generally speaking, the integrals in formulas (3.9) and (3.10) may not be analytically obtainable. Moreover, the integrands in these formulas are unbounded at the points ± 1 . To compute these integrals it is convenient to use numerical methods. For example, the *tanh* rule (see [9]) and Stenger's formula (see [12]):

$$(3.13) \quad \int_a^b f(x) dx = \int_{-\infty}^{+\infty} f\left(\frac{a + be^\omega}{1 + e^\omega}\right) \frac{(b-a) d\omega}{(e^{-\omega/2} + e^{\omega/2})^2}$$

$$\approx h_{\text{sinc}} \sum_{l=-K}^K f\left(\frac{a + be^{lh_{\text{sinc}}}}{1 + e^{lh_{\text{sinc}}}}\right) \frac{(b-a)}{(e^{-lh_{\text{sinc}}/2} + e^{lh_{\text{sinc}}/2})^2}$$

and

$$(3.14) \quad \int_a^{z_k} f(x) dx \approx h_{\text{sinc}} \sum_{l=-K}^K \delta_{k-l}^{(-1)} f\left(\frac{a + be^{lh_{\text{sinc}}}}{1 + e^{lh_{\text{sinc}}}}\right) \frac{(b-a)}{(e^{-lh_{\text{sinc}}/2} + e^{lh_{\text{sinc}}/2})^2}^4,$$

⁴The function $f(x)$ needs to be sufficiently smooth on (a, b) ; see [12].

where

$$z_k = \frac{a + be^{h_{sinc}k}}{1 + e^{h_{sinc}k}}, \quad k = -K, \dots, K, \quad \delta_k^{(-1)} = \frac{1}{2} + \int_0^k \frac{\sin(\pi t)}{\pi t} dt, \quad k = -2K, \dots, 2K,$$

$$\text{and } h_{sinc} = \sqrt{\frac{2\pi}{K}}.^5$$

Henceforth we require that the function $q(x)$ be analytical on every interval (x_{i-1}, x_i) , $i = 1, 2, \dots, N$. Before we can describe the algorithm we need to introduce some auxiliary notation:

$$(3.15) \quad z_{i,j} = \frac{x_{i-1} + x_i e^{h_{sinc}j}}{1 + e^{h_{sinc}j}}, \quad \mu_{i,j} = \frac{(x_{i-1} - x_i)}{(e^{-jh_{sinc}/2} + e^{jh_{sinc}/2})^2},$$

$$\nu_n = \left[\int_{-1}^1 \left(u_n^{(0)}(x) \right)^2 dx \right]^{-1}, \quad i = 1, 2, \dots, N, \quad j = -K, \dots, K.$$

One possible algorithm for computing $\lambda_n^{(k)}$ and $u_n^{(k)}(x)$ for $k = 1, 2, \dots, r$, $i = 1, 2, \dots, N$, $j = -K, \dots, K$ is described below.

Algorithm 2: The algorithm for computation of λ_n^d , $u_n^d(z_{i,j})$, $d \in \overline{1, r}$, $i \in \overline{1, N}$, $j \in \overline{-K, K}$

input : The arrays of values $u_n^{(0)}(z_{i,j})$, $w_n(z_{i,j})$, $\mu_{i,j}$, $i \in \overline{1, N}$, $j \in \overline{-K, K}$, $\delta_l^{(-1)}$, $l \in \overline{-2K, 2K}$, λ_n^0 , and r — the rank of the FD-method.

output: The arrays of values $u_n^{(d)}(z_{i,j})$, $d \in \overline{1, r}$, $i \in \overline{1, N}$, $j \in \overline{-K, K}$, $\lambda_n^{(d)}$, $d \in \overline{1, r}$.

begin

```

1   for d := 1 to r do
2       ComputeLambda(  $u_n^{(d)}(z_{i,j})$ ,  $u_n^{(0)}(z_{i,j})$  ) // This procedure
           computes  $\lambda_n^{(d)}$ , see description below
3       ComputeF(  $u_n^{(k)}(z_{i,j})$ ,  $\lambda_n^{(k)}$  ) // This procedure computes the
           values  $F_n^{(d)}(z_{i,j})$ , see description below
4       ComputeCorrectionForEigenfunction(  $u_n^{(0)}(z_{i,j})$ ,  $w_n(z_{i,j})$ ,
            $F_n^{(d)}(z_{i,j})$  ) // This procedure computes the values  $u_n^{(d)}(z_{i,j})$ ,
           see description below
5       ProvideOrthogonalityCondition(  $u_n^{(d)}(z_{i,j})$ ,  $u_n^{(0)}(z_{i,j})$  ) // This
           procedure recompute the values  $u_n^{(d)}(z_{i,j})$  to ensure the
           orthogonality condition  $\int_{-1}^1 u_n^{(0)}(x) u_n^{(d)}(x) dx = 0$ 
       end
   end
end
```

⁵For discussion on the optimal choice of value for h_{sinc} see [12].

Procedure ComputeLambda($u_n^{(d)}(z_{i,j}), u_n^{(0)}(z_{i,j})$)

input : The arrays of values $u_n^{(d)}(z_{i,j}), u_n^{(0)}(z_{i,j}), \quad i \in \overline{1, N}, j \in \overline{-K, K}$
output: The value $\lambda_n^{(d)}$
begin

```

1  |  $\lambda_n^{(d)} := 0$ 
2  | for  $i := 1$  to  $N$  do
3  |   | for  $j := -K$  to  $K$  do
4  |   |   |  $\lambda_n^{(d)} := \lambda_n^{(d)} + u_n^{(0)}(z_{i,j})u_n^{(d-1)}(z_{i,j})\{q(z_{i,j}) - \bar{q}(z_{i,j})\}\mu_{i,j}$ 
      |   |   | // Compute  $\lambda_n^{(d)}$  using formulas (3.9) and (3.13)
      |   |   | end
      |   | end
      |   |  $\lambda_n^{(d)} := h_{sinc}\nu_n\lambda_n^{(d)} // \lambda_n^{(d)}$  has been computed
5  | end

```

Procedure ProvideOrthogonalityCondition($u_n^{(d)}(z_{i,j}), u_n^{(0)}(z_{i,j})$)

input : The arrays of values $u_n^{(d)}(z_{i,j}), u_n^{(0)}(z_{i,j}), \quad i \in \overline{1, N}, j \in \overline{-K, K}$
output: The array of values $u_n^{(d)}(z_{i,j}), \quad i \in \overline{1, N}, j \in \overline{-K, K}$, such that the

integral $\int_{-1}^1 u_n^{(0)}(x)u_n^{(d)}(x)$, computed by formula (3.13), is equal to
zero

begin

```

1  |  $\mathcal{I} := 0$ 
2  | for  $i := 1$  to  $N$  do
3  |   | for  $j := -K$  to  $K$  do
4  |   |   |  $\mathcal{I} := \mathcal{I} + u_n^{(d)}(z_{i,j})u_n^{(0)}(z_{i,j})\mu_{i,j}$ 
      |   |   | end
      |   | end
      |   |  $\mathcal{I} := \mathcal{I}h_{sinc}\nu_n$ 
5  |   | for  $i = 1$  to  $N$  do
6  |   |   | for  $j = -K$  to  $K$  do
7  |   |   |   |  $u_n^{(d)}(z_{i,j}) := u_n^{(d)}(z_{i,j}) - \mathcal{I}u_n^{(0)}(z_{i,j})$ 
8  |   |   |   | end
      |   |   | end
      |   | end
      |   |  $\mathcal{I}$ 
end

```

Procedure ComputeF($u_n^{(k)}(z_{i,j}), \lambda_n^{(k)}$)

input : The arrays of values $u_n^{(k)}(z_{i,j}), \quad k \in \overline{0, d}, i \in \overline{1, N}, j \in \overline{-K, K}$ and $\lambda_n^{(k)}, k \in \overline{0, d}$

output: The array of values $F_n^{(d)}(z_{i,j}), i \in \overline{1, N}, j \in \overline{-K, K}$, computed by formula (2.7)

begin

```

1   for  $i := 1$  to  $N$  do
2       for  $j := -K$  to  $K$  do
3            $F_n^{(d)}(z_{i,j}) := 0$ 
4           for  $k := 0$  to  $d - 1$  do
5                $F_n^{(d)}(z_{i,j}) := F_n^{(d)}(z_{i,j}) - \lambda_{d-k} u_n^{(k)}(z_{i,j})$ 
6           end
7            $F_n^{(d)}(z_{i,j}) := F_n^{(d)}(z_{i,j}) + u_n^{(d-1)}(z_{i,j}) \{q(z_{i,j}) - \bar{q}(z_{i,j})\}$ 
8       end
9   end
10  end

```

Procedure ComputeCorrectionForEigenfunction($u_n^{(0)}(z_{i,j}), w_n(z_{i,j}), F_n^{(d)}(z_{i,j})$)

input : The arrays of values $u_n^{(0)}(z_{i,j}), w_n(z_{i,j})$ and $F_n^{(d)}(z_{i,j}), i \in \overline{1, N}, j \in \overline{-K, K}$

output: The array of values $u_n^{(d)}(z_{i,j}), \quad i \in \overline{1, N}, j \in \overline{-K, K}$

begin

```

1    $\mathcal{V}_1 := 0; \mathcal{V}_2 := 0$  // Initialize auxiliary variables
2   for  $i := 1$  to  $N$  do
3       for  $j := -K$  to  $K$  do
4            $\mathcal{I}_1 := 0; \mathcal{I}_2 := 0$ 
5           for  $l := -K$  to  $K$  do
6                $\mathcal{I}_1 := \mathcal{I}_1 + u_n^{(0)}(z_{i,l}) \delta_{j-l}^{(-1)} F^{(d)}(z_{i,l}) \mu_{i,l}$  // Compute the
               integral  $\int_{-1}^{z_{i,j}} u_n^{(0)}(\xi) F_n^{(j)}(\xi) d\xi$  using Stenger's formula
               (3.14)
7                $\mathcal{I}_2 := \mathcal{I}_2 + w_n(z_{i,l}) \delta_{j-l}^{(-1)} F^{(d)}(z_{i,l}) \mu_{i,l}$  // Compute the integral
                $\int_{-1}^{z_{i,j}} w_n(\xi) F_n^{(j)}(\xi) d\xi$  using Stenger's formula (3.14)
8           end
9            $\mathcal{I}_1 := \mathcal{I}_1 h_{sinc} + \mathcal{V}_1$ 
10           $\mathcal{I}_2 := \mathcal{I}_2 h_{sinc} + \mathcal{V}_2$ 
11           $u_n^{(d)}(z_{i,j}) := w_n(z_{i,j}) \mathcal{I}_1 - u_n^{(0)}(z_{i,j}) \mathcal{I}_2$ 
12      end
13  end
14   $\mathcal{V}_1 := \mathcal{I}_1$ 
15   $\mathcal{V}_2 := \mathcal{I}_2$ 
16  end

```

The algorithm described above is not the most efficient one. Nevertheless, the numerical results presented in the next section were successfully obtained using this algorithm. It is evident that the highest accuracy that can be achieved by increasing the rank r of the FD-method is limited by the accuracy of the quadrature formulas (3.13), (3.14) used in the algorithm. The optimal choice of the parameters r and K remains a current issue. In the next section we will present three numerical examples. In the first example we apply quadrature formulas (3.13), (3.14) with $K = 500$ and in the following two we use $K = 350$.

4. NUMERICAL EXAMPLES

Example 1. Let us consider the following Sturm-Liouville problem:

$$(4.1) \quad L(u(x)) = \frac{d}{dx} \left[(1-x^2) \frac{du(x)}{dx} \right] + (\lambda - q(x)) u(x) = 0, \quad x \in (-1, 1),$$

$$\lim_{x \rightarrow \pm 1} \left[(1-x^2) \frac{du(x)}{dx} \right] = 0$$

with

$$(4.2) \quad q(x) = x.$$

Using the FD-method described in [6] and the computer algebra system Maple we have found that

$$\begin{aligned} \lambda_0^{(0)} &= 0, & u_0^{(0)}(x) &= \frac{\sqrt{2}}{2}, \\ \lambda_0^{(1)} &= 0, & u_0^{(1)}(x) &= -\frac{\sqrt{2}x}{4}, \\ \lambda_0^{(2)} &= -\frac{1}{6}, & u_0^{(2)}(x) &= \frac{\sqrt{2}x^2}{24} - \frac{\sqrt{2}}{72}, \\ \lambda_0^{(3)} &= 0, & u_0^{(3)}(x) &= -\frac{\sqrt{2}x^3}{288} + \frac{5\sqrt{2}x}{259200}, \\ \lambda_0^{(4)} &= \frac{11}{1080}, & u_0^{(4)}(x) &= \frac{\sqrt{2}x^4}{5760} - \frac{\sqrt{2}x^2}{270} + \frac{311\sqrt{2}}{259200}, \\ \lambda_0^{(5)} &= 0, & u_0^{(5)}(x) &= -\frac{\sqrt{2}x^5}{172800} + \frac{2880}{518400} - \frac{1181\sqrt{2}x}{21772800}, \\ \lambda_0^{(6)} &= -\frac{47}{34020}, & u_0^{(6)}(x) &= \frac{\sqrt{2}x^6}{7257600} - \frac{\sqrt{2}x^4}{53760} + \frac{11237\sqrt{2}x^2}{21772800} - \frac{76967\sqrt{2}}{457228800}. \end{aligned}$$

To control the accuracy of the results obtained by the FD-method we use the following functionals:

$$(4.3) \quad {}^m\eta_n = \left[\int_{-1}^1 \left[(1-x^2) \frac{d {}^m u_n(x)}{dx} + \int_{-1}^x (\lambda_n - q(\xi)) {}^m u_n(\xi) d\xi \right]^2 dx \right]^{\frac{1}{2}},$$

$$(4.4) \quad {}^m\bar{\eta}_n = \left[\int_{-1}^1 \left[\frac{d}{dx} \left[(1-x^2) \frac{d {}^m u_n(x)}{dx} \right] + (\lambda_n - q(x)) {}^m u_n(x) \right]^2 dx \right]^{\frac{1}{2}}.$$

We also compare the results of the FD-method with eigenvalues obtained by SLEIGN2 (see [1]).

TABLE 1. Example 1. The data obtained by SLEIGN2

n	$\lambda_{n,sl2}$	TOL	$IFLAG$
0	$-0.157663483D + 00$	$0.23950D - 13$	1
1	$0.209076065D + 01$	$0.19238D - 13$	1
2	$0.602403235D + 01$	$0.20718D - 08$	1
3	$0.120111226D + 02$	$0.10472D - 12$	1
4	$0.200064954D + 02$	$0.70433D - 13$	1

In all of the examples presented in this section we apply the FD-method with a uniform mesh on interval $[-1, 1]$; N denotes the number of subintervals. The results obtained by the FD-method are presented in Tables 2– 4 and Figure 1 below. The results obtained by the SLEIGN2 are presented in Table 1.

Analyzing the data in Tables 2, 3 and 4 we can conclude that the FD-method with $N = 1$ converges much slower than the FD-method with $N = 3$. This fact is in good agreement with the results of Theorem 2.1. Furthermore, as follows from Table 4, the convergence rate of the FD-method increases as the index n of the trial eigenvalue increases. Figure 1 illustrates the exponential nature of the FD-method's convergence.

FIGURE 1. Example 1. The graphs of the functions $\ln(\|u_n^{(m)}(x)\|)$ (left) and $\ln(\eta_n^{(m)})$ (right); $N = 3$.

Example 2. As the second example, let us consider problem (4.1) with

$$(4.5) \quad q(x) = \ln \left(\left| \left(\frac{5}{12} - x \right) \left(\frac{1}{3} + x \right) \right| \right).$$

The results obtained with SLEIGN2 are presented in Table 5 below.

It is worth emphasizing that the problem under consideration does not satisfy the conditions of Theorem 2.1. However, the results presented in Table 6 and Figure 2 show that the FD-method has successfully handled this problem. This is unlike to SLEIGN2 which gives results with substantial errors (see the leftmost colon in Table 6).

TABLE 2. Example 1. The 0-th eigenvalue, FD-method with $N = 1$

m	$\overset{m}{\lambda}_0$	$\ u_0^{(m)}(x)\ $	$\overset{m}{\eta}_n$	$ \overset{m}{\lambda}_0 - \lambda_{0,sl2} $
0	0.0	1.0	0.56	0.157663483
1	0.0	0.29	0.22	0.157663483
2	-0.1666666667	$0.24e - 1$	0.36×10^{-1}	$0.90031875 \times 10^{-2}$
3	-0.1666666667	0.18×10^{-1}	0.16×10^{-1}	$0.90031875 \times 10^{-2}$
4	-0.1564814815	0.21×10^{-2}	0.48×10^{-2}	$0.11819977 \times 10^{-2}$
5	-0.1564814815	0.24×10^{-2}	0.23×10^{-2}	$0.11819977 \times 10^{-2}$
6	-0.1578630218	0.30×10^{-3}	0.78×10^{-3}	0.1995426×10^{-3}
7	-0.1578630218	0.41×10^{-3}	0.41×10^{-3}	0.1995426×10^{-3}
8	-0.1576253633	0.52×10^{-4}	0.15×10^{-3}	0.381159×10^{-4}
9	-0.1576253633	0.79×10^{-4}	0.78×10^{-4}	0.381159×10^{-4}
10	-0.1576713252	0.10×10^{-4}	0.31×10^{-4}	0.78460×10^{-5}

TABLE 3. Example 1. The 0-th eigenvalue, FD-method with $N = 1$

m	$\overset{m}{\lambda}_0$	$\ u_0^{(m)}(x)\ $	$\overset{m}{\eta}_n$	$ \overset{m}{\lambda}_0 - \lambda_{0,sl2} $
51	-0.15766348313775096746	0.12×10^{-16}	0.13×10^{-16}	0.39×10^{-8}
52	-0.15766348313775096031	0.16×10^{-17}	0.59×10^{-17}	0.39×10^{-8}
53	-0.15766348313775096031	0.33×10^{-17}	0.33×10^{-17}	0.39×10^{-8}
54	-0.15766348313775096218	0.39×10^{-18}	0.15×10^{-17}	0.39×10^{-8}
55	-0.15766348313775096218	0.84×10^{-18}	0.86×10^{-18}	0.39×10^{-8}
56	-0.15766348313775096169	0.11×10^{-18}	0.40×10^{-18}	0.39×10^{-8}
57	-0.15766348313775096169	0.22×10^{-18}	0.22×10^{-18}	0.39×10^{-8}
58	-0.15766348313775096182	0.28×10^{-19}	0.11×10^{-18}	0.39×10^{-8}
59	-0.15766348313775096182	0.58×10^{-19}	0.61×10^{-19}	0.39×10^{-8}
60	-0.15766348313775096178	0.74×10^{-20}	0.29×10^{-19}	0.39×10^{-8}

TABLE 4. Example 1. FD-method with $N = 3$

n	m	$\overset{m}{\lambda}_n$	$\frac{ \overset{m}{\lambda}_n }{10^{-20}}$	$\frac{\ u_n^{(m)}(x)\ }{10^{-20}}$	$\frac{\overset{m}{\eta}_n}{10^{-21}}$	$ \overset{m}{\lambda}_n - \lambda_{n,sl2} $
0	15	-0.1576634831377509617898	0.03	0.033	0.095	0.39×10^{-8}
1	15	2.090760648363956948786	0.7	0.13	0.26	0.1482×10^{-5}
2	15	6.024031655336352711291	0.7	0.094	0.22	0.5035×10^{-5}
3	13	12.01112256362987127625	1	0.24	0.20	0.4×10^{-7}
4	11	20.00649533292656299628	1	3.1	1.5	0.7×10^{-7}

TABLE 5. Example 2. The data obtained by SLEIGN2

n	$\lambda_{n,sl2}$	TOL	$IFLAG$
0	-1.98326983D + 00	0.46748D - 08	1
1	0.855187683D + 00	0.73426D - 07	1
2	0.489606686D + 01	0.35447D - 07	1
3	0.104183770D + 02	0.40228D - 07	1
4	0.188163965D + 02	0.61329D - 11	1

TABLE 6. Example 2. FD-method with $N = 24$

n	m	λ_n^m	$\frac{ \lambda_n^{(m)} }{10^{-16}}$	$\frac{\ u_n^{(m)}(x)\ }{10^{-17}}$	$\frac{\eta_n^m}{10^{-17}}$	$ \lambda_n^m - \lambda_{n,sl2} $
0	8	-1.9831442709774408386	0.099	0.26	0.02	0.12556×10^{-3}
1	8	0.85727032837311800023	4.7	7.3	0.44	0.2082645×10^{-2}
2	8	4.8939506826799075597	0.098	0.29	0.051	0.211618×10^{-2}
3	8	10.420511296257433545	0.03	7.4	1.3	0.213430×10^{-2}
4	7	18.816396521508987920	0.11	0.87	0.039	0.2×10^{-7}

FIGURE 2. Example 2. The graphs of the functions $\ln(\|u_n^{(m)}(x)\|)$ (left) and $\ln(\eta_n^m)$ (right); $N = 24$.

Example 3. Finally, let us consider problem (4.1) with

$$(4.6) \quad q(x) = \frac{1}{\sqrt{|x + \frac{1}{3}|}} + \ln\left(x - \frac{1}{3}\right).$$

We find out that SLEIGN2 cannot handle this problem while the FD-method can. The results obtained with the FD-method are presented in Table 7 and Figure 3. As before, Figure 3 confirms the exponential convergence rate of the method.

FIGURE 3. Example 3. The graphs of the functions $\ln(\|u_n^{(m)}(x)\|)$ (left) and $\ln(\eta_n^m)$ (right); $N = 12$.

TABLE 7. Example 3. FD-method with $N = 12$

n	m	λ_n^m	$ \lambda_n^{(m)} $	$\ u_n^{(m)}(x)\ $	η_n^m
0	18	0.40796999146419634	0.42×10^{-15}	0.20×10^{-15}	0.17×10^{-15}
1	18	3.4136861164474333	0.4×10^{-15}	0.17×10^{-15}	0.16×10^{-15}
2	14	6.7759537951814352	0.2×10^{-15}	0.33×10^{-15}	0.10×10^{-15}
3	14	13.323487340142488	0.2×10^{-14}	0.25×10^{-15}	0.28×10^{-15}
4	9	20.8431972121837340	0.1×10^{15}	0.96×10^{-16}	0.13×10^{-16}

5. CONCLUSIONS

In the present paper we have constructed and theoretically justified a generalized algorithm of the FD-method for solving the Sturm-Liouville problem for second-order differential equation (1.1), (1.2) with piecewise continuous functional coefficient $q(x)$. As follows from Theorem 2.1, the generalized FD-method, which uses a piecewise constant approximation of the function $q(x)$, can be applied to approximate eigenvalues and eigenfunctions with any nonnegative index n . The convergence rate of the method can be increased by decreasing the value $\|q(x) - \bar{q}(x)\|_{\infty, [-1, 1]}$. In the case when $\bar{q}(x) \equiv 0$ (this case was considered in [6]) the FD-method, if convergent, allows us to obtain the approximation for the eigensolution analytically. In general, however, when $\bar{q}(x) \neq 0$ the analytical approach fails and it is necessary to use numerical integration methods such as sinc quadratures and Stenger's formula (see [9], [12]).

The problems considered in Examples 2 and 3 do not satisfy the conditions of Theorem 1.1 because of the function $q(x)$ being unbounded on $[-1, 1]$. However, as it has been shown in the examples above, the method successfully converges whereas the well-known and widely applied SLEIGN2 package either provides no more than three correct digits after decimal point (Example 2) or cannot handle the problem at all (Example 3). These examples indicate that the FD-method has considerable potential that is to be investigated in further mathematical works.

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