A DISCRETE KPP-THEORY FOR FISHER'S EQUATION

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ABSTRACT. The purpose of this paper is to extend the theory by Kolmogorov, Petrowsky and Piscunov (KPP) for Fisher's equation, to a discrete solution. We approximate the time derivative in Fisher's equation by an explicit Euler scheme and the diffusion operator by a symmetric difference scheme of second order. We prove that the discrete solution converges towards a traveling wave, under restrictions in the time- and space-widths, as the number of time steps increases to infinity. We also prove that the flame velocity can be determined as a solution to an optimization problem.

1. INTRODUCTION

Consider the initial-value problem

$$(1.1) \quad \partial u / \partial t = \partial^2 u / \partial x^2 + F(u), \quad -\infty < x < \infty, \quad t > 0,$$

$$(1.2) \quad u(x, 0) = f(x), \quad -\infty < x < \infty,$$

where $F(u)$ is a smooth function, which satisfies the conditions

$$(1.3) \quad F(0) = F(1) = 0,$$

$$(1.4) \quad F(u) > 0, \quad \text{for } 0 < u < 1,$$

$$(1.5) \quad F'(0) = 1 \text{ and } F(u) \leq F'(0)u = u, \quad \text{for } 0 < u \leq 1.$$

The equation (1.1) was proposed by Fisher [1] as a model for propagation of an advantageous gene population in an infinite region.

A mathematical theory for Fisher's equation, with the initial condition

$$(1.6) \quad f(x) = 0, \quad x < 0 \text{ and } f(x) = 1, \quad x > 0,$$

was formulated by Kolmogorov, Petrowsky and Piscunow [4]. This theory (the KPP-theory) has been used extensively in combustion studies; see Hakberg [2].

The problem of numerical validation of an extended KPP-theory for models in turbulent combustion, motivates the study of the discrete flame velocity for Fisher's equation.

Suppose that we approximate Fisher's equation (1.1) with the explicit difference scheme

$$(1.7) \quad \begin{aligned} (v(x, (n+1)k) - v(x, nk))/k &= (v(x-h, nk) - 2v(x, nk) \\ &+ v(x+h, nk))/h^2 + F(v(x, nk)), \quad -\infty < x < \infty, \quad \forall n \geq 0, \end{aligned}$$

where h and k are mesh-widths in x and t , respectively, and $v(x, nk)$ is an approximation of $u(x, nk)$. The initial-condition (1.6) is approximated by

$$(1.8) \quad v(x, 0) = 0, \quad x < 0; \quad v(x, 0) = x/h, \quad 0 \leq x \leq h \text{ and } v(x, 0) = 1, \quad h < x.$$

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Let c be a fixed, but arbitrary constant, such that $0 < c < 1$, and let φ^n be defined by

$$(1.9) \quad v(\varphi^n, nk) = c, \quad \forall n \geq 0,$$

which is possible, since $v(x, nk)$ is continuous for all x and strictly monotone for $0 < v(x, nk) < 1$, if $2\lambda + k\kappa < 1$, where $\kappa = \sup_{0 < u < 1} |F'(u)|$ and

$$\lambda = k/h^2.$$

See Theorem 1.

The first step in the proof of convergence of the discrete solution $v(x, nk)$ towards a traveling flame, is to show that $v(\varphi^n + ih, nk)$ is a monotone function of n for fixed i . This result is valid according to Theorem 3, if $3\lambda + k\kappa < 1$.

Assume that the conditions in Theorem 3 are satisfied. Define the function $v^*(x, nk)$ by

$$v^*(x, nk) = v(x + \varphi^n, nk),$$

where φ^n is defined by (1.9). Then

$$v^*(0, nk) = v(\varphi^n, nk) = c,$$

and

$$\lim_{n \rightarrow \infty} v^*(ih, nk) = v^*(ih), \quad \forall i \in Z$$

exists. See Theorem 4.

The function

$$v_m(ih, nk) = v(ih + \varphi^m, (n + m)k), \quad \forall i, \quad \forall n \geq 0, \quad \forall m \geq 0,$$

converges as $m \rightarrow \infty$, towards a function $V(ih, nk)$, which satisfies the difference equation

$$(1.10) \quad \begin{aligned} V(ih, (n + 1)k) = & (1 - 2\lambda)V(ih, nk) + \lambda V(ih - h, nk) + \lambda V(ih + h, nk) \\ & + kF(V(ih, nk)), \quad \forall i, \quad \forall n \geq 0, \end{aligned}$$

with initial data

$$V(ih, 0) = v^*(ih), \quad \forall i,$$

where $v^*(ih)$ is defined above. See Theorem 5.

Numerical experiments suggest that the discrete flame velocity, defined as $(\varphi^{n+1} - \varphi^n)/k$, fluctuates and does not converge as $n \rightarrow \infty$. Theorem 6 shows that the mean flame velocity, defined as φ^n/nk , converges as $n \rightarrow \infty$ to a limit

$$\lim_{n \rightarrow \infty} \varphi^n/nk = \gamma h/k, \quad \text{with } h \text{ and } k \text{ fixed.}$$

The result is derived under the restrictions $3\lambda + k\kappa < 1$, where

$$\kappa = \sup_{0 < u < 1} |F'(u)|.$$

Let us first consider the case, when γ is a rational number. Let

$$\gamma = P/Q,$$

where P and $Q > 0$ are integers with no common factor. From the proof of Theorem 7 it follows that

$$V((i + P)h, (j + Q)k) = V(ih, jk), \quad \forall i, \quad \forall j \geq 0.$$

If we introduce a function W , defined by

$$W(ih - jhP/Q) = V(ih, jk), \quad \forall i, \quad \forall j \geq 0,$$

then W satisfies the traveling wave equation

$$(1.11) \quad \begin{aligned} W(ih - (j+1)hP/Q) &= (1 - 2\lambda)W(ih - jhP/Q) + \lambda W(ih - h - jhP/Q) \\ &+ \lambda W(ih + h - jhP/Q) + kF(W(ih - jhP/Q)), \quad \forall i, \forall j \geq 0. \end{aligned}$$

In Theorem 8, we prove a similar result, if γ is an irrational number.

In Theorem 9, we prove that γ can be determined from the optimization problem

$$\gamma = - \inf_{\alpha > 0} \{ \log[1 - 2\lambda + k + \lambda(\exp(\alpha h) + \exp(-\alpha h))] / (\alpha h) \},$$

based upon an upper estimate of the discrete flame velocity, proved in Theorem 2.

The analysis shows that the discrete equation (1.7) has a traveling wave solution $v(x - n\gamma h)$, and no traveling wave solution exists, which has a smaller flame velocity in absolute value.

We therefore conclude that the discrete flame converges towards a traveling flame. The convergence is both in shape or form, defined by Theorem 4 and in velocity, defined by Theorem 6.

A general theory for discrete approximations of Fisher's equation has been formulated by Weinberger [5]. The asymptotic flame velocity agrees with $\gamma h/k$ for the explicit difference scheme (1.7), if $2\lambda + k\kappa < 1$. The initial function in Weinberger [5] is a pulse compared to a step in Kolmogorov, Petrowsky and Piskunow [4]. The theory by Weinberger [5] does not generalize results on convergence in shape obtained by Kolmogorov, Petrowsky and Piskunow [4], which shows that the width of the transition layer between the high and low parts of a spreading pulse remains bounded. This paper generalizes results on convergence in shape as defined in Theorem 5 or as defined by Weinberger.

Zinner et al. [6] have studied the existence of traveling wave solutions to spatial-discretizations of Fisher's equation. The necessary and sufficient condition for the flame velocity agrees with the discrete KPP-theory above in the limit, when $k \rightarrow 0$.

2. A DISCRETE KPP-THEORY

The first theorem gives sufficient conditions, such that $v(x, nk)$, defined by (1.7)–(1.8), is bounded, continuous and strictly monotone.

Theorem 1. *Assume that $F(u)$ is a continuous function on $[0, 1]$, with a bounded and continuous derivative on $(0, 1)$, and which satisfies the conditions (1.3)–(1.5). Let v be the solution of (1.7)–(1.8).*

Assume further that the time step k and space step h satisfy the inequality

$$(2.1) \quad 2\lambda + k\kappa < 1, \text{ where } \lambda = k/h^2 \text{ and } \kappa = \sup_{0 < u < 1} |F'(u)|.$$

Then we have for $n \geq 0$:

(i)

$$\begin{aligned} v(x, nk) &= 0, \quad x \leq -nh, \\ 0 < v(x, nk) &< 1, \quad -nh < x < (n+1)h, \\ v(x, nk) &= 1, \quad (n+1)h \leq x. \end{aligned}$$

(ii) $v(x, nk)$ is a continuous function of x for all x .

(iii) Let $\Delta > 0$. Then

$$(2.2) \quad v(x - \Delta, nk) < v(x, nk) < v(x + \Delta, nk), \quad -nh < x < (n+1)h.$$

Proof. Since the proof is elementary, we only prove (iii). From (1.7), (1.8) follows that the derivative

$$w(x, nk) = \partial v(x, nk) / \partial x,$$

exists for all $x \neq ih$, $\forall i$, and satisfies the difference equation

$$(2.3) \quad \begin{aligned} w(x, (n+1)k) = & (1 - 2\lambda + kF'(v(x, nk)))w(x, nk) + \lambda w(x - h, nk) \\ & + \lambda w(x + h, nk), \quad -\infty < x < \infty, \quad x \neq ih, \quad \forall i. \end{aligned}$$

The initial condition is

$$(2.4) \quad w(x, 0) = 1/h, \quad 0 < x < h; \quad w(x, 0) = 0, \quad x < 0, \quad x > h.$$

Since by (2.1)

$$1 - 2\lambda + kF'(u) > 0, \quad 0 < u < 1,$$

it follows from (2.3) and (2.4),

$$(2.5) \quad w(x, nk) > 0, \quad -nh < x < (n+1)h, \quad x \neq ih, \quad -n < i < n+1.$$

The strict monotonicity (2.2) follows from (2.5) and the mean value theorem. $\square$

The second theorem is a discrete counterpart to Theorem 10 in [4].

Theorem 2. Assume that the conditions in Theorem 1 are satisfied. Let x be fixed. Then

$$(2.6) \quad \lim_{n \rightarrow \infty} v(x - \beta nk, nk) = 0,$$

if

$$(2.7) \quad \beta k/h > \inf_{\alpha > 0} \{\log[1 - 2\lambda + k + \lambda(\exp(\alpha h) + \exp(-\alpha h))]/(\alpha h)\},$$

where $\lambda = k/h^2$.

Proof. Introduce, for $\alpha > 0$, $\beta > 0$,

$$(2.8) \quad w(x, nk) = v(x - \beta nk, nk) \exp(-\alpha x), \quad -\infty < x < \infty,$$

where α and β are constants. Then w satisfies

$$\begin{aligned} w(x, (n+1)k) = & (1 - 2\lambda) \exp(-\alpha \beta k) w(x - \beta k, nk) \\ & + \lambda \exp(\alpha(-h - \beta k)) w(x - \beta k - h, nk) \\ & + \lambda \exp(\alpha(h - \beta k)) w(x - \beta k + h, nk) \\ & + k \exp(-\alpha x) F\{w(x - \beta k, nk) \exp(\alpha(x - \beta k))\}, \quad -\infty < x < \infty, \end{aligned}$$

with

$$(2.9) \quad w(x, 0) = v(x, 0) \exp(-\alpha x), \quad -\infty < x < \infty. \quad \square$$

Lemma 1. Define $z(x, nk)$ by

$$(2.10) \quad \begin{aligned} z(x, (n+1)k) = & (1 - 2\lambda) \exp(-\alpha \beta k) z(x - \beta k, nk) \\ & + \lambda \exp(\alpha(-h - \beta k)) z(x - \beta k - h, nk) \\ & + \lambda \exp(\alpha(h - \beta k)) z(x - \beta k + h, nk) \\ & + k \exp(-\alpha \beta k) z(x - \beta k, nk), \quad -\infty < x < \infty, \end{aligned}$$

where $z(x, 0)$ satisfies

$$(2.11) \quad z(x, 0) = 0, \quad x < 0; \quad z(x, 0) = \exp(-\alpha x), \quad x > 0.$$

Then

$$w(x, nk) \leq z(x, nk), \quad -\infty < x < \infty, \quad \forall n \geq 0.$$

Proof. The result follows by induction from (2.10) and the inequalities

$$\begin{aligned} w(x, 0) &\leq z(x, 0), \\ F(u) &\leq u, \quad 0 \leq u \leq 1, \\ 1 - 2\lambda &> 0, \end{aligned}$$

which follow from (1.5), (2.1), (2.9) and (2.11). $\square$

Proof of Theorem 2 (cont.) We first choose β , such that (2.7) is satisfied. Then from (2.7) it follows that for some $\alpha > 0$,

$$\beta k/h > \log[1 - 2\lambda + k + \lambda(\exp(\alpha h) + \exp(-\alpha h))]/(\alpha h).$$

From (2.8) and Lemma 1 it follows that (2.6) is true if $z(x, nk) \rightarrow 0$, $n \rightarrow \infty$, for fixed x and α, β , chosen as above.

Introduce the Fourier transform

$$z_f(y, nk) = \int_{-\infty}^{+\infty} z(x, nk) \exp(-iyx) dx, \quad -\infty < y < \infty.$$

$z_f(y, 0)$ exists since $\alpha > 0$ in (2.11).

We have from (2.10),

$$(2.12) \quad z_f(y, (n+1)k) = A(y)z_f(y, nk),$$

where

$$A(y) = \exp(-\alpha\beta k - i\beta ky)[1 - 2\lambda + k + \lambda \exp(-\alpha h - ihy) + \lambda \exp(\alpha h + ihy)].$$

Successive application of (2.12) gives

$$z_f(y, nk) = \{A(y)\}^n z_f(y, 0).$$

Inversion of the Fourier transform gives

$$(2.13) \quad z(x, nk) = \exp(-\alpha\beta nk) \left\{ \sum_j c_j^n z(x - \beta nk + jh, 0) \right\},$$

where

$$c_j^n = (1/2\pi) \int_{-\pi}^{\pi} (B(\theta))^n \exp(-ij\theta) d\theta$$

and

$$B(\theta) = 1 - 2\lambda + k + \lambda(\exp(-\alpha h - i\theta) + \exp(\alpha h + i\theta)).$$

(2.13) is proved by showing that $z(x, nk)$ defined in (2.13) satisfies the difference equation (2.10).

We get from (2.11) and (2.13),

$$(2.14) \quad z(x, nk) = \exp(-\alpha\beta nk) \sum_{j > -(x-\beta nk)/h} c_j^n \exp(-\alpha(x - \beta nk + jh)).$$

Let x be fixed. Then from (2.14)

$$(2.15) \quad |z(x, nk)| \leq \exp(-\alpha\beta nk) \left\{ \sup_{|\theta| \leq \pi} |B(\theta)| \right\}^n \sum_{j=0}^{\infty} \exp(-\alpha jh),$$

and

$$\begin{aligned}
 \sup_{|\theta| \leq \pi} |B(\theta)| &= \sup_{|\theta| \leq \pi} \{(1 - 2\lambda + k)^2 \\
 &\quad + 2(1 - 2\lambda + k)\lambda(\exp(\alpha h) + \exp(-\alpha h)) \cos(\theta) \\
 &\quad + \lambda^2(\exp(\alpha h) + \exp(-\alpha h))^2 \cos^2(\theta) \\
 &\quad + \lambda^2(\exp(\alpha h) - \exp(-\alpha h))^2(1 - \cos^2(\theta))\}^{1/2} \\
 &= \sup_{|\theta| \leq \pi} \{(1 - 2\lambda + k)^2 \\
 &\quad + 2(1 - 2\lambda + k)\lambda(\exp(\alpha h) + \exp(-\alpha h)) \cos(\theta) \\
 &\quad + \lambda^2(\exp(\alpha h) - \exp(-\alpha h))^2 + 4\lambda^2 \cos^2(\theta)\}^{1/2} \\
 &= \{(1 - 2\lambda + k)^2 + 2(1 - 2\lambda + k)\lambda(\exp(\alpha h) + \exp(-\alpha h)) \\
 &\quad + \lambda^2(\exp(\alpha h) - \exp(-\alpha h))^2 + 4\lambda^2\}^{1/2},
 \end{aligned}$$

which finally gives

$$(2.16) \quad \sup_{|\theta| \leq \pi} |B(\theta)| = 1 - 2\lambda + k + \lambda(\exp(\alpha h) + \exp(-\alpha h)) = B(0).$$

We have for fixed x , from (2.15) and (2.16),

$$\lim_{n \rightarrow \infty} z(x, nk) = 0,$$

if

$$\exp(-\alpha\beta k)\{1 - 2\lambda + k + \lambda(\exp(\alpha h) + \exp(-\alpha h))\} < 1,$$

or equivalently,

$$\beta k/h > \log[1 - 2\lambda + k + \lambda(\exp(\alpha h) + \exp(-\alpha h))]/(\alpha h),$$

which is true according to our choice of α and β above. $\square$

Theorem 3 is a discrete version of Theorem 11 in [4].

Theorem 3. *Assume that the conditions in Theorem 1 are satisfied. Assume further that*

$$(2.17) \quad 3k/h^2 + k\kappa < 1,$$

where κ is defined in (2.1). Let φ^n be defined by (1.9). Then $v(\varphi^n + ih, nk)$ is a decreasing function of n for all fixed $i > 0$ and an increasing function of n for all fixed $i < 0$.

Proof. Introduce

$$w(x, nk) = v(x, nk) - v(x + C, (n+1)k),$$

where C is a fixed but arbitrary constant. We first observe that $w(x, nk)$ satisfies the difference equation

$$(2.18) \quad \begin{aligned} w(x, (n+1)k) &= (1 - 2\lambda)w(x, nk) + \lambda w(x - h, nk) + \lambda w(x + h, nk) \\ &\quad + kK(x, nk)w(x, nk), \end{aligned}$$

where

$$K(x, nk) = F'(\eta),$$

and η is a number between $v(x, nk)$ and $v(x + C, (n+1)k)$. We first study $w(x, nk)$. $\square$

Lemma 2. *Introduce*

$$x_i = ih + \Delta, \quad \forall i \in Z,$$

where Δ is a fixed but arbitrary constant. Assume that (2.17) is satisfied. Assume that, for some i ,

$$(2.19) \quad w(x_{i-1}, nk) \leq 0, \quad w(x_i, nk) \leq 0, \quad w(x_{i+1}, nk) \geq 0, \quad \text{and} \quad w(x_{i+2}, nk) \geq 0.$$

Then

$$(2.20) \quad w(x_{i+1}, (n+1)k) \geq 0 \quad \text{if} \quad w(x_i, (n+1)k) \geq 0$$

and

$$(2.21) \quad w(x_{i+1}, (n+1)k) > 0 \quad \text{if} \quad w(x_i, (n+1)k) > 0.$$

Proof. Assume that $w(x_{i+1}, (n+1)k) < 0$ and $w(x_i, (n+1)k) \geq 0$. Define

$$a = w(x_{i-1}, nk), \quad b = w(x_i, nk), \quad d = w(x_{i+1}, nk) \quad \text{and} \quad e = w(x_{i+2}, nk).$$

Then

$$w(x_i, (n+1)k) = \lambda a + (1 - 2\lambda)b + \lambda d + kK(x_i, nk)b \geq 0$$

and

$$-w(x_{i+1}, (n+1)k) = -(\lambda b + (1 - 2\lambda)d + \lambda e + kK(x_{i+1}, nk)d) > 0,$$

by assumption. Addition of these inequalities gives

$$(2.22) \quad \lambda a + (1 - 3\lambda + kK(x_i, nk))b + (3\lambda - 1 - kK(x_{i+1}, nk))d - \lambda e > 0.$$

From the assumptions (2.17) and (2.19) follows that $a \leq 0$, $b \leq 0$, $d \geq 0$ and $e \geq 0$. Furthermore,

$$(1 - 3\lambda + kK(x_i, nk)) \geq 1 - 3\lambda - k\kappa > 0,$$

$$(3\lambda - 1 - kK(x_{i+1}, nk)) \leq 3\lambda - 1 + k\kappa = -(1 - 3\lambda - k\kappa) < 0.$$

This contradicts (2.22) and we conclude that (2.20) holds. Similarly, we prove that (2.21) is valid. $\square$

Let i_0 denote the largest integer for which

$$(2.23) \quad v(x_i + C, k) = v(x_i, 0) = 0, \quad \forall i \leq i_0,$$

and i_1 the smallest number for which

$$(2.24) \quad v(x_i + C, k) = v(x_i, 0) = 1, \quad \forall i \geq i_1.$$

Then we derive from (2.17), (2.18), (2.23) and (2.24) that

$$w(x_i, nk) = 0, \quad i \leq i_0 - n, \quad i \geq i_1 + n, \quad \forall n \geq 0.$$

Lemma 3. *There exist three cases:*

Case 1.

$$(2.25) \quad w(x_i, nk) \geq 0, \quad i_0 - n < i < i_1 + n, \quad \forall n \geq 0.$$

Case 2.

$$w(x_i, nk) < 0, \quad i_0 - n < i < i_1 + n, \quad \forall n \geq 0.$$

Case 3. There exist numbers $i(n)$ such that

$$(2.26) \quad i(0) - n \leq i(n) \leq i(0) + n, \quad \forall n \geq 0,$$

$$(2.27) \quad w(x_i, nk) < 0, \quad i_0 - n < i \leq i(n), \quad \forall n \geq 0,$$

$$(2.28) \quad w(x_i, nk) \geq 0, \quad i(n) < i < i_1 + n, \quad \forall n \geq 0.$$

Proof. We consider first the case when $n = 0$. If

$$v(x_{i_0+1}, 0) \geq v(x_{i_0+1} + C, k),$$

then

$$v(x_{i_0+1}, 0) > 0 \text{ and } v(x_{i_0+2}, 0) = 1,$$

by the definition of $v(x_i, 0)$ in (1.8) and i_0 in (2.23). Case 1 is therefore valid. If

$$v(x_{i_0+1}, 0) < v(x_{i_0+1} + C, k),$$

then

$$v(x_{i_0+2}, 0) \geq v(x_{i_0+2} + C, k) \text{ or } v(x_{i_0+2}, 0) < v(x_{i_0+2} + C, k).$$

In the first case we have $i(0) = i_0 + 1$ and case 3 is valid. In the second case, i.e., if

$$v(x_{i_0+2}, 0) < v(x_{i_0+2} + C, k),$$

then

$$v(x_{i_0+3}, 0) \geq v(x_{i_0+3} + C, k) \text{ or } v(x_{i_0+3}, 0) < v(x_{i_0+3} + C, k).$$

In the first case we have $i(0) = i_0 + 2$. The proof is continued in a finite number of steps and proves that Case 1, Case 2 or Case 3 are the only possible cases for $n = 0$.

Case 1 implies, for $n = 0$,

$$w(x_i, 0) \geq 0, \quad i_0 < i < i_1.$$

(2.17), (2.18), (2.23) and (2.24) give by induction,

$$w(x_i, nk) \geq 0, \quad i_0 - n < i < i_1 + n, \quad \forall n \geq 0.$$

Case 2 implies, for $n = 0$,

$$w(x_i, 0) < 0, \quad i_0 < i < i_1$$

and, in general,

$$w(x_i, nk) < 0, \quad i_0 - n < i < i_1 + n, \quad \forall n \geq 0.$$

Case 3 implies for $n = 0$ that there exists a number $i(0)$, such that

$$w(x_i, 0) < 0, \quad i_0 < i \leq i(0),$$

$$w(x_i, 0) \geq 0, \quad i(0) < i < i_1.$$

(2.17), (2.18), (2.23) and (2.24) then give

$$w(x_i, k) = 0, \quad i \leq i_0 - 1, \quad i \geq i_1 + 1,$$

$$w(x_i, k) < 0, \quad i_0 - 1 < i \leq i(0) - 1,$$

$$w(x_i, k) \geq 0, \quad i(0) + 1 < i < i_1 + 1.$$

Lemma 2 gives

$$w(x_{i(0)+1}, k) \geq 0 \text{ if } w(x_{i(0)}, k) \geq 0$$

and

$$w(x_{i(0)}, k) < 0 \text{ if } w(x_{i(0)+1}, k) < 0.$$

We therefore conclude that there exists a number $i(1)$, which is equal to $i(0) - 1$, $i(0)$ or $i(0) + 1$, such that

$$\begin{aligned} w(x_i, k) &< 0, \quad i_0 - 1 < i \leq i(1), \\ w(x_i, k) &\geq 0, \quad i(1) < i \leq i_1 + 1. \end{aligned}$$

By induction it follows that there exists a number $i(n)$, such that

$$\begin{aligned} i(0) - n &\leq i(n) \leq i(0) + n, \quad \forall n \geq 0, \\ w(x_i, nk) &< 0, \quad i_0 - n < i \leq i(n), \quad \forall n \geq 0, \\ w(x_i, nk) &\geq 0, \quad i(n) < i < i_1 + n, \quad \forall n \geq 0, \end{aligned}$$

i.e., Lemma 3 is proved. $\square$

Proof of Theorem 3 (cont.) Let v_0 be a fixed, but arbitrary constant, such that $0 < v_0 < 1$.

In order to prove Theorem 3, we choose a number X_n , depending on n and v_0 , such that

$$v(X_n, nk) = v_0.$$

This is possible since $v(x, nk)$ is continuous and strictly monotone for $0 < v(x, nk) < 1$, by Theorem 1. We then choose C_n such that

$$(2.29) \quad v(X_n + C_n, (n+1)k) = v(X_n, nk) = v_0.$$

Then

$$w(X_n, nk) = 0,$$

which implies that case 2 in Lemma 3 is not valid, since $0 < v_0 < 1$. It then follows from (2.25) or (2.28) that

$$w(X_n + ih, nk) \geq 0, \quad i \geq 1,$$

which, with (2.29), implies

$$v(X_n + ih, nk) - v(X_n, nk) \geq v(X_n + ih + C_n, (n+1)k) - v(X_n + C_n, (n+1)k),$$

which proves the case $i \geq 1$.

The case $i \leq -1$ is proved by replacing Cases 1–3 in Lemma 3, by

Case 1'.

$$w(x_i, nk) > 0, \quad i_0 - n < i < i_1 + n, \quad \forall n \geq 0.$$

Case 2'.

$$w(x_i, nk) \leq 0, \quad i_0 - n < i < i_1 + n, \quad \forall n \geq 0.$$

Case 3'. There exists a number $i(n)$, such that

$$\begin{aligned} i(0) - n &\leq i(n) \leq i(0) + n, \quad \forall n \geq 0, \\ w(x_i, nk) &\leq 0, \quad i_0 - n < i \leq i(n), \quad \forall n \geq 0, \\ w(x_i, nk) &> 0, \quad i(n) < i < i_1 + n, \quad \forall n \geq 0. \end{aligned}$$

The proof of Cases 1'–3' is similar to the proof of Cases 1–3 in Lemma 3. $\square$

Theorem 4. Assume that the conditions in Theorem 1 are satisfied. Define the function $v^*(x, nk)$ by

$$v^*(x, nk) = v(x + \varphi^n, nk),$$

where φ^n is defined by (1.9). Then

$$v^*(0, nk) = v(\varphi^n, nk) = c,$$

and

$$\lim_{n \rightarrow \infty} v^*(ih, nk) = v^*(ih), \forall i \in Z$$

exists.

Proof. From Theorem 3 follows that $v^*(ih, nk)$, $i \geq 1$, is a nonincreasing function of n and $v^*(ih, nk)$, $i \leq -1$, is a nondecreasing function of n .

Since $v^*(ih, nk)$ is bounded below by zero and above by one, it follows that

$$\lim_{n \rightarrow \infty} v^*(ih, nk) = v^*(ih), \forall i,$$

exists. □

Theorem 5 is a discrete analogue to Theorem 14 in [4].

Theorem 5. The function

$$v_m(ih, nk) = v(ih + \varphi^m, (n + m)k), \forall i, \forall n \geq 0, \forall m \geq 0,$$

converges as $m \rightarrow \infty$, towards a function $V(ih, nk)$, which satisfies the difference equation

$$(2.30) \quad \begin{aligned} V(ih, (n + 1)k) &= (1 - 2\lambda)V(ih, nk) + \lambda V(ih - h, nk) + \lambda V(ih + h, nk) \\ &+ kF(V(ih, nk)), \forall i, \forall n \geq 0, \end{aligned}$$

with initial data

$$V(ih, 0) = v^*(ih), \forall i,$$

where $v^*(ih)$ is defined in Theorem 4.

Proof. From Theorem 4 follows that

$$v_m(ih, 0) = v(ih + \varphi^m, mk) = v^*(ih, mk) \rightarrow v^*(ih), m \rightarrow \infty, \forall i.$$

Then

$$V(ih, 0) = v^*(ih), \forall i.$$

From (1.7) follows:

$$\begin{aligned} v_m(ih, k) &= v(ih + \varphi^m, (m + 1)k) = (1 - 2\lambda)v(ih + \varphi^m, mk) \\ &+ \lambda v(ih + \varphi^m - h, mk) + \lambda v(ih + \varphi^m + h, mk) \\ &+ kF(v(ih + \varphi^m, mk)), \forall i, \forall m \geq 0, \end{aligned}$$

which converges, as $m \rightarrow \infty$, towards a solution $V(ih, k)$ to the difference equation (2.30). By induction it follows that $v_m(ih, nk)$ converges towards a function $V(ih, nk)$, which satisfies (2.30). □

In order to study the properties of V , defined in Theorem 5, we first derive an upper estimate for $\varphi^n - \varphi^{n+1}$.

Lemma 4. Assume that h and k satisfy the condition (2.1) in Theorem 1. Then

$$\varphi^n - \varphi^{n+1} \leq h, \forall n \geq 0.$$

Proof. Introduce

$$(2.31) \quad z(x, nk) = v(x - h, (n + 1)k) - v(x, nk), \quad -\infty < x < \infty, \quad \forall n \geq 0.$$

Then, from (1.7),

$$(2.32) \quad \begin{aligned} z(x, (n + 1)k) &= (1 - 2\lambda)z(x, nk) + \lambda z(x - h, nk) + \lambda z(x + h, nk) \\ &\quad + kF'(v^n)z(x, nk), \quad \forall n \geq 0, \end{aligned}$$

where v^n is a number between $v(x - h, (n + 1)k)$ and $v(x, nk)$.

Using $n = 0$ in (2.31) gives

$$z(x, 0) = v(x - h, k) - v(x, 0).$$

(1.7) and (1.8) give, for $0 < x < h$,

$$z(x, 0) = \lambda v(x, 0) - v(x, 0) < 0, \quad 0 < x < h.$$

Also,

$$z(x, 0) \leq 0, \quad x \leq 0 \text{ and } x \geq h.$$

From (2.1) and (2.32) it follows by induction that

$$v(x - h, (n + 1)k) \leq v(x, nk), \quad \forall n \geq 0,$$

which implies

$$\varphi^{n+1} - \varphi^n \geq -h, \quad \forall n \geq 0. \quad \square$$

Lemma 5. $V(ih, nk)$ has the properties

$$(2.33) \quad \text{i) } V(ih, nk) \leq V((i + 1)h, nk), \quad \forall i, \quad \forall n \geq 0,$$

$$(2.34) \quad \text{ii) } 0 < V(ih, nk) < 1, \quad \forall i, \quad \forall n \geq 0,$$

$$(2.35) \quad \text{iii) } \lim_{i \rightarrow \infty} V(ih, nk) = 1, \quad \forall n \geq 0,$$

$$(2.36) \quad \text{iv) } \lim_{i \rightarrow -\infty} V(ih, nk) = 0, \quad \forall n \geq 0.$$

Proof. i) V is monotone since v is monotone.

ii) (2.34) follows from Theorem 3.

iii) Introduce

$$\lim_{i \rightarrow \infty} V(ih, 0) = \alpha,$$

which exists since V is monotone by i) and bounded by ii).

Then $\alpha > 0$ since $V(0, 0) = c$ and $0 < c < 1$.

Assume that $\alpha < 1$. Then

$$\begin{aligned} V(ih, k) &= \lim_{n \rightarrow \infty} v(ih + \varphi^n, (n + 1)k) \\ &= \lim_{n \rightarrow \infty} v(ih + \varphi^n - \varphi^{n+1} + \varphi^{n+1}, (n + 1)k) \\ &\leq \lim_{n \rightarrow \infty} v(ih + h + \varphi^{n+1}, (n + 1)k) = V(ih + h, 0) \leq \alpha < 1, \end{aligned}$$

by Lemma 4. From (2.30)

$$\begin{aligned} V(ih, k) &= (1 - 2\lambda)V(ih, 0) + \lambda V(ih - h, 0) \\ &\quad + \lambda V(ih + h, 0) + kF(V(ih, 0)), \quad \forall i, \end{aligned}$$

which implies

$$\lim_{i \rightarrow \infty} V(ih, k) = (1 - 2\lambda)\alpha + 2\lambda\alpha + kF(\alpha) = \alpha + kF(\alpha) > \alpha,$$

if $0 < \alpha < 1$.

The contradiction implies that

$$\lim_{i \rightarrow \infty} V(ih, 0) = 1.$$

Similarly, we prove by induction that

$$\lim_{i \rightarrow \infty} V(ih, nk) = 1, \quad \forall n \geq 0.$$

iv) Similar arguments can be used to prove

$$\lim_{i \rightarrow -\infty} V(ih, nk) = 0, \quad \forall n \geq 0. \quad \square$$

Numerical experiments suggest that $\varphi^n - \varphi^{n+1}$ fluctuates as n increases. Theorem 6 shows that $\lim_{n \rightarrow \infty} \varphi^n/n$ exists.

Theorem 6. *Assume that the conditions in Theorems 1 and 3 are satisfied. Then*

$$\lim_{n \rightarrow \infty} \varphi^n/n = \gamma h.$$

Proof. The proofs relies on the following lemmas. $\square$

Lemma 6. *Assume that the conditions in Theorems 1 and 3 are satisfied. Let c be a fixed, but arbitrary constant defined in (1.9). Then, there exists for every $m \geq 0$ a unique number j_m , such that*

$$(2.37) \quad V((j_m - 1)h, mk) \leq c < V(j_m h, mk), \quad \forall m \geq 0.$$

Proof. The result follows directly from Lemma 5. $\square$

Lemma 7. *Assume that the conditions of Lemma 6 are satisfied. Let j_m be as in Lemma 6. Then*

$$(2.38) \quad \limsup_{n \rightarrow \infty} (\varphi^n - \varphi^{n+m}) \leq -(j_m - 1 - j_0)h, \quad \forall m \geq 0,$$

$$(2.39) \quad \liminf_{n \rightarrow \infty} (\varphi^n - \varphi^{n+m}) \geq -(j_m + 1 - j_0)h, \quad \forall m \geq 0.$$

Proof. We first prove (2.38).

From the definition of V in Theorem 5 follows:

$$(2.40) \quad \begin{aligned} V((j_m - 1)h, mk) &= \lim_{n \rightarrow \infty} v((j_m - 1)h + \varphi^n, (m + n)k) \\ &= \lim_{n \rightarrow \infty} v((j_m - 1)h + \varphi^n - \varphi^{n+m} + \varphi^{n+m}, (m + n)k) \leq c, \end{aligned}$$

according to (2.37). We then claim that

$$(2.41) \quad (j_m - 1)h + \limsup_{n \rightarrow \infty} (\varphi^n - \varphi^{n+m}) \leq j_0 h, \quad \forall m \geq 0.$$

If the claim is not true, we can find some $m \geq 0$, such that

$$(j_m - 1)h + \limsup_{n \rightarrow \infty} (\varphi^n - \varphi^{n+m}) > j_0 h.$$

Then there exists a subsequence $\{n(i)\}_{i \geq 1}$, such that

$$(j_m - 1)h + (\varphi^{n(i)} - \varphi^{n(i)+m}) > j_0 h, \quad \forall i \geq 1$$

and

$$(2.42) \quad (j_m - 1)h + \lim_{i \rightarrow \infty} (\varphi^{n(i)} - \varphi^{n(i)+m}) > j_0 h.$$

(2.40) and (2.42) give

$$(2.43) \quad \begin{aligned} V((j_m - 1)h, mk) &\geq \lim_{i \rightarrow \infty} v(j_0 h + \varphi^{n(i)+m}, (n(i) + m)k) \\ &= V(j_0 h, 0) > c, \end{aligned}$$

from the definition of j_0 in (2.37).

The contradiction between (2.40) and (2.43) shows that the claim is true, i.e.,

$$(j_m - 1)h + \limsup_{n \rightarrow \infty} (\varphi^n - \varphi^{n+m}) \leq j_0 h, \quad \forall m \geq 0.$$

(2.41) is equivalent to (2.38). Similarly, we prove that

$$j_m h + \liminf_{n \rightarrow \infty} (\varphi^n - \varphi^{n+m}) \geq (j_0 - 1)h, \quad \forall m \geq 0. \quad \square$$

Proof of Theorem 6. Let $m \geq 1$ be a fixed but arbitrary number. Then for every $\varepsilon > 0$, there exists a number $M \geq 0$, such that

$$(2.44) \quad \varphi^n - \varphi^{n-m} \leq \limsup_{n \rightarrow \infty} (\varphi^n - \varphi^{n-m}) + \varepsilon, \quad n - m \geq M.$$

Write

$$\begin{aligned} \varphi^n &= (\varphi^n - \varphi^{n-m}) + (\varphi^{n-m} - \varphi^{n-2m}) + \dots \\ &\quad + (\varphi^{n-(L-1)m} - \varphi^{n-Lm}) + \varphi^{n-Lm}, \end{aligned}$$

where L is the largest number that satisfies $n - Lm \geq M$.

Then, we derive from (2.44),

$$\begin{aligned} \varphi^n &\leq L \{ \limsup_{n \rightarrow \infty} (\varphi^n - \varphi^{n-m}) + \varepsilon \} + C_M, \\ C_M &= \max \{ \varphi^M, \varphi^{M+1}, \dots, \varphi^{M+m-1} \}, \end{aligned}$$

so that

$$\varphi^n \leq ((n - M)/m) \{ \limsup_{n \rightarrow \infty} (\varphi^n - \varphi^{n-m}) + \varepsilon \} + C_M,$$

and

$$\varphi^n/n \leq ((n - M)/(mn)) \{ \limsup_{n \rightarrow \infty} (\varphi^n - \varphi^{n-m}) + \varepsilon \} + C_M/n,$$

which gives

$$\limsup_{n \rightarrow \infty} \varphi^n/n \leq (1/m) \{ \limsup_{n \rightarrow \infty} (\varphi^n - \varphi^{n-m}) + \varepsilon \}.$$

Since ε is arbitrary, we derive

$$(2.45) \quad \limsup_{n \rightarrow \infty} \varphi^n/n \leq (1/m) \{ \limsup_{n \rightarrow \infty} (\varphi^n - \varphi^{n-m}) \}, \quad \forall m \geq 1.$$

Similarly, we derive

$$(2.46) \quad \liminf_{n \rightarrow \infty} \varphi^n/n \geq (1/m) \{ \liminf_{n \rightarrow \infty} (\varphi^n - \varphi^{n-m}) \}, \quad \forall m \geq 1.$$

Since

$$\begin{aligned} \liminf_{n \rightarrow \infty} (\varphi^n - \varphi^{n-m}) &= - \limsup_{n \rightarrow \infty} (\varphi^{n-m} - \varphi^n) \\ &= - \limsup_{n \rightarrow \infty} (\varphi^n - \varphi^{n+m}), \\ \limsup_{n \rightarrow \infty} (\varphi^n - \varphi^{n-m}) &= - \liminf_{n \rightarrow \infty} (\varphi^{n-m} - \varphi^n) \\ &= - \liminf_{n \rightarrow \infty} (\varphi^n - \varphi^{n+m}), \end{aligned}$$

we derive from Lemma 7, (2.45) and (2.46)

$$(2.47) \quad (j_m - 1 - j_0)h/m \leq \liminf_{n \rightarrow \infty} \varphi^n/n \leq \limsup_{n \rightarrow \infty} \varphi^n/n \leq (j_m + 1 - j_0)h/m.$$

Since the difference between the upper and lower estimate in (2.47) differ by $2h/m$, which converges to zero as $m \rightarrow \infty$, it follows that

$$\liminf_{n \rightarrow \infty} \varphi^n/n = \limsup_{n \rightarrow \infty} \varphi^n/n,$$

i.e., $\lim_{n \rightarrow \infty} \varphi^n/n$ exists, which proves Theorem 6. $\square$

Theorem 7. Assume that γ , defined in Theorem 6, is a rational number. Let

$$\gamma = P/Q,$$

where P and $Q > 0$ are integers with no common factor. Then

$$(2.48) \quad \lim_{L \rightarrow \infty} V((i + LP)h, (j + LQ)k) = W(ih - jhP/Q),$$

exists for $\forall i, \forall j \geq 0$, where W is uniquely defined and satisfies the difference equation

$$(2.49) \quad \begin{aligned} W(ih - (j + 1)hP/Q) &= (1 - 2\lambda)W(ih - jhP/Q) + \lambda W(ih - h - jhP/Q) \\ &+ \lambda W(ih + h - jhP/Q) + kF(W(ih - jhP/Q)), \quad \forall i, \forall j \geq 0. \end{aligned}$$

It follows from Lemma 5 and the proof of Theorem 7 that

$$\begin{aligned} \lim_{i \rightarrow -\infty} W(ih - jhP/Q) &= 0, \\ \lim_{i \rightarrow \infty} W(ih - jhP/Q) &= 1. \end{aligned}$$

Proof. The proofs relies on the following lemmas. $\square$

Lemma 8. Introduce

$$\alpha_L^n = \varphi^n - \varphi^{n+QL} + LPh, \quad \forall n \geq 0, \quad \forall L \geq 1.$$

Define

$$\begin{aligned} \alpha_L^+ &= \limsup_{n \rightarrow \infty} \alpha_L^n, \quad \forall n \geq 0, \quad \forall L \geq 1, \\ \alpha_L^- &= \liminf_{n \rightarrow \infty} \alpha_L^n, \quad \forall n \geq 0, \quad \forall L \geq 1. \end{aligned}$$

Then

$$-2h \leq \alpha_L^- \leq 0 \leq \alpha_L^+ \leq 2h, \quad \forall L \geq 1.$$

Proof. Assume that $\alpha_L^- > 0$. Then there exists a number M_L , such that

$$\alpha_L^n > (\alpha_L^-)/2, \quad n \geq M_L$$

or

$$\varphi^{n+QL} - \varphi^n < LPh - (\alpha_L^-)/2, \quad n \geq M_L$$

Summation gives:

$$\varphi^{n+mQL} - \varphi^n < mLPh - m(\alpha_L^-)/2, \quad n \geq M_L$$

and

$$\begin{aligned} &\varphi^{n+mQL}/(n + mQL) - \varphi^n/(n + mQL) \\ &< mLPh/(n + mQL) - m((\alpha_L^-)/2)/(n + mQL), \quad n \geq M_L. \end{aligned}$$

Let $m \rightarrow \infty$. Then it follows that

$$(P/Q)h \leq (P/Q)h - (\alpha_L^-)/(2QL),$$

which is a contradiction, since $Q > 0$ and $(\alpha_L^-) > 0$ by assumptions. Similarly, we prove that $\alpha_L^+ \geq 0$.

Assume that $\alpha_L^- < -2h$. Then there exists two subsequences $\{\varphi^{m(i)}\}$, $\forall i \geq 1$ and $\{\varphi^{n(i)}\}$, $\forall i \geq 1$, such that

$$\begin{aligned}\alpha_L^{m(i)} &< -2h, \quad \forall i \geq 1, \\ \alpha_L^{m(i)} &\rightarrow \alpha_L^-, \quad i \rightarrow \infty, \\ \alpha_L^{n(i)} &> -h, \quad \forall i \geq 1, \\ \alpha_L^{n(i)} &\rightarrow (\alpha_L^+) \geq 0, \quad i \rightarrow \infty.\end{aligned}$$

Lemma 6 states that

$$(2.50) \quad V((j_m - 1)h, mk) \leq c < V(j_m h, mk), \quad \forall m \geq 0.$$

Then, we derive

$$\begin{aligned}&V((j_m + 1)h + LPh, (m + LQ)k) \\&= \lim_{i \rightarrow \infty} v((j_m + 1)h + LPh + \varphi^{m(i)}, (m(i) + m + LQ)k) \\&= \lim_{i \rightarrow \infty} v((j_m + 1)h + (LPh + \varphi^{m(i)} - \varphi^{m(i) + LQ}) + \varphi^{m(i) + LQ}, (m(i) + m + LQ)k) \\&\leq \lim_{i \rightarrow \infty} v((j_m + 1)h - 2h + \varphi^{m(i) + LQ}, (m(i) + m + LQ)k) \\&= V((j_m - 1)h, mk) \leq c,\end{aligned}$$

by (2.50). We also have

$$\begin{aligned}&V((j_m + 1)h + LPh, (m + LQ)k) \\&= \lim_{i \rightarrow \infty} v((j_m + 1)h + LPh + \varphi^{n(i)}, (n(i) + m + LQ)k) \\&= \lim_{i \rightarrow \infty} v((j_m + 1)h + (LPh + \varphi^{n(i)} - \varphi^{n(i) + LQ}) + \varphi^{n(i) + LQ}, (n(i) + m + LQ)k) \\&\geq \lim_{i \rightarrow \infty} v(j_m h + \varphi^{n(i) + LQ}, (n(i) + m + LQ)k) = V(j_m h, mk) > c,\end{aligned}$$

by (2.50). The contradiction proves that $\alpha_L^- \geq -2h$. Similarly, we prove that $\alpha_L^+ \leq 2h$. $\square$

Lemma 9. Assume that

$$(2.51) \quad \alpha_1^- < 0 < \alpha_1^+.$$

Then

$$(2.52) \quad V((i + P)h, (j + Q)k) = V(ih, jk), \quad \forall i, \forall j \geq 0,$$

and W exists by (2.48).

Proof. If (2.51) is satisfied, it follows that there exists two subsequences $\{\varphi^{m(L)}\}$, $\forall L \geq 1$ and $\{\varphi^{n(L)}\}$, $\forall L \geq 1$, such that

$$\begin{aligned}\alpha_1^{m(L)} &< 0, \quad \forall L \geq 1, \\ \alpha_1^{m(L)} &\rightarrow \alpha_1^- < 0, \quad L \rightarrow \infty, \\ \alpha_1^{n(L)} &> 0, \quad \forall L \geq 1, \\ \alpha_1^{n(L)} &\rightarrow \alpha_1^+ > 0, \quad L \rightarrow \infty.\end{aligned}$$

Then

$$\begin{aligned}
 V((i+P)h, (j+Q)k) &= \lim_{L \rightarrow \infty} v((i+P)h + \varphi^{m(L)}, (m(L) + j + Q)k) \\
 (2.53) \quad &= \lim_{L \rightarrow \infty} v(ih + (Ph + \varphi^{m(L)} - \varphi^{m(L)+Q}) + \varphi^{m(L)+Q}, (m(L) + j + Q)k) \\
 &\leq \lim_{L \rightarrow \infty} v(ih + \varphi^{m(L)+Q}, (m(L) + j + Q)k) = V(ih, jk).
 \end{aligned}$$

Similarly, we prove using $\{\varphi^{n(L)}\}$, $\forall L \geq 1$, that

$$(2.54) \quad V((i+P)h, (j+Q)k) \geq V(ih, jk).$$

(2.53) and (2.54) imply

$$V((i+P)h, (j+Q)k) = V(ih, jk), \quad \forall i, \forall j \geq 0.$$

It then follows from the definition of W in (2.48) that

$$W(ih - jhP/Q) = V(ih, jk), \quad \forall i, \forall j \geq 0. \quad \square$$

Proof of Theorem 7. We have from Lemma 8 that

$$\alpha_1^- \leq 0 \leq \alpha_1^+.$$

We next consider the case

$$\alpha_1^- \leq 0 = \alpha_1^+.$$

If there exists two subsequences, such that

$$\begin{aligned}
 \alpha_1^{m(L)} &\leq 0, \quad \forall L \geq 1, \\
 \alpha_1^{m(L)} &\rightarrow \alpha_1^- \leq 0, \quad L \rightarrow \infty, \\
 \alpha_1^{n(L)} &\geq 0, \quad \forall L \geq 1, \\
 \alpha_1^{n(L)} &\rightarrow \alpha_1^+ = 0, \quad L \rightarrow \infty,
 \end{aligned}$$

then it follows, as in Lemma 9, that

$$V((i+P)h, (j+Q)k) = V(ih, jk), \quad \forall i, \forall j \geq 0.$$

It is therefore sufficient to study the case, where

$$\alpha_1^- \leq 0 = \alpha_1^+ \text{ and } \alpha_1^m \leq 0, \quad \forall m \geq 1. \quad \square$$

Lemma 10. Assume that

$$\alpha_1^- \leq 0 = \alpha_1^+,$$

$$(2.55) \quad \alpha_1^m = Ph + \varphi^m - \varphi^{m+Q} \leq 0, \quad \forall m \geq 1.$$

The sequence

$$(2.56) \quad \{V((i+LP)h, (j+LQ)k)\}, \quad \forall L \geq 1,$$

is then nonincreasing, and

$$(2.57) \quad \lim_{L \rightarrow \infty} V((i+LP)h, (j+LQ)k) = W(ih - jhP/Q),$$

exists, and

$$0 < W(ih - jhP/Q) < 1, \quad \forall i, \forall j \geq 0.$$

Proof. We have from (2.55)

$$\begin{aligned} V((i+P)h, (j+Q)k) &= \lim_{m \rightarrow \infty} v((i+P)h + \varphi^m, (m+j+Q)k) \\ &= \lim_{m \rightarrow \infty} v(ih + (Ph + \varphi^m - \varphi^{m+Q}) + \varphi^{m+Q}, (m+j+Q)k) \\ &\leq \lim_{m \rightarrow \infty} v(ih + \varphi^{m+Q}, (m+j+Q)k) = V(ih, jk). \end{aligned}$$

Similarly, we prove that the sequence (2.56) is decreasing. It then follows that the limit (2.57) exists, since the sequence is decreasing and bounded below by zero. Then from (2.57),

$$\begin{aligned} W(ih - jhP/Q) &= \lim_{L \rightarrow \infty} V((i+LP)h, (j+LQ)k) \\ &= \lim_{L \rightarrow \infty} \{ \lim_{m \rightarrow \infty} v((i+LP)h + \varphi^m, (j+LQ+m)k) \} \\ &= \lim_{L \rightarrow \infty} \{ \lim_{m \rightarrow \infty} v(ih + (\alpha_L^m) + \varphi^{m+LQ}, (j+LQ+m)k) \}. \end{aligned}$$

Since by Lemma 8,

$$(2.58) \quad -2h \leq \alpha_L^-,$$

it follows that there exists a subsequence $\{\alpha_L^{m(n)}\}$, $\forall n \geq 1$, such that

$$(2.59) \quad \alpha_L^{m(n)} \geq -3h, \quad \forall n \geq 1,$$

$$(2.60) \quad \alpha_L^{m(n)} \rightarrow \alpha_L^-, \quad n \rightarrow \infty.$$

Then, from (2.58), (2.59) and (2.60),

$$\begin{aligned} W(ih - jhP/Q) &= \lim_{L \rightarrow \infty} \{ \lim_{n \rightarrow \infty} v(ih + (\alpha_L^{m(n)}) + \varphi^{m(n)+LQ}, (j+LQ+m(n))k) \} \\ &\geq \lim_{L \rightarrow \infty} \{ \lim_{n \rightarrow \infty} v(ih - 3h + \varphi^{m(n)+LQ}, (j+LQ+m(n))k) \} \\ &= \lim_{L \rightarrow \infty} \{ V(ih - 3h, jk) \} = V(ih - 3h, jk) > 0, \end{aligned}$$

by Lemma 5. Similarly, we prove that

$$W(ih - jhP/Q) < 1. \quad \square$$

Proof of Theorem 7 (cont). The case when

$$\alpha_1^- = 0 \leq \alpha_1^+,$$

is treated similarly. The difference equation (2.49) follows from (2.48) and (2.30). We finally prove that W is unique. Let

$$i_1 - j_1P/Q = i_2 - j_2P/Q.$$

Then

$$(i_1 - i_2)Q = (j_1 - j_2)P.$$

Since $(P, Q) = 1$, we conclude that there exists a number M , such that

$$(i_1 - i_2) = MP \text{ and } (j_1 - j_2) = MQ.$$

Then

$$\begin{aligned} & \lim_{L \rightarrow \infty} V((i_1 + LP)h, (j_1 + LQ)k) \\ &= \lim_{L \rightarrow \infty} V((i_2 + (L + M)P)h, (j_2 + (L + M)Q)k) \\ &= \lim_{L \rightarrow \infty} V((i_2 + LP)h, (j_2 + LQ)k). \end{aligned} \quad \square$$

Theorem 8. Assume that γ , defined in Theorem 6, is an irrational number. Then

$$\sup_{h(i - \gamma j) < x} V(ih, jk) = W(x), \quad -\infty < x < \infty,$$

exists. W is monotone and satisfies the difference equation

$$W(x - \gamma h) = (1 - 2\lambda)W(x) + \lambda W(x - h) + \lambda W(x + h) + kF(W(x)), \quad -\infty < x < \infty.$$

Proof. We first define U from

$$(2.61) \quad U(ih - jh\gamma) = V(ih, jk), \quad \forall i, \forall j \geq 0.$$

This quantity is well defined, since if

$$i_1 - \gamma j_1 = i_2 - \gamma j_2,$$

then

$$i_1 - i_2 = \gamma(j_1 - j_2),$$

and since γ is irrational, it follows that

$$i_1 = i_2 \text{ and } j_1 = j_2. \quad \square$$

Lemma 11. U defined in (2.61) satisfies

$$\begin{aligned} U(ih - \gamma jh - \gamma h) &= (1 - 2\lambda)U(ih - \gamma jh) + \lambda U(ih - h - \gamma jh) \\ &\quad + \lambda U(ih + h - \gamma jh) + kF(U(ih - \gamma jh)), \quad \forall i, \forall j \geq 0. \end{aligned}$$

Proof.

$$\begin{aligned} U(ih - \gamma jh - \gamma h) &= U(ih - \gamma(j + 1)h) = V(ih, (j + 1)k) \\ &= (1 - 2\lambda)V(ih, jk) + \lambda V(ih - h, jk) + \lambda V(ih + h, jk) + kF(V(ih, jk)) \\ &= (1 - 2\lambda)U(ih - \gamma jh) + \lambda U(ih - h - \gamma jh) + \lambda U(ih + h - \gamma jh) \\ &\quad + kF(U(ih - \gamma jh)). \end{aligned} \quad \square$$

Lemma 12. U is monotone, i.e.,

$$U((i_1 - \gamma j_1)h) \leq U((i_2 - \gamma j_2)h) \text{ if } i_1 - \gamma j_1 < i_2 - \gamma j_2.$$

Proof. We consider first the case

$$i_2 = j_2 = 0.$$

It is then sufficient to prove that

$$V(ih, jk) \leq V(0, 0) = c \text{ if } i < \gamma j, \quad j \geq 0,$$

or that

$$(2.62) \quad \lim_{m \rightarrow \infty} v(ih + \varphi^m, (m + j)k) \leq c = v(\varphi^{m+j}, (m + j)k) \text{ if } i < \gamma j, \quad j \geq 0,$$

(2.62) is satisfied if

$$(2.63) \quad \liminf_{m \rightarrow \infty} (ih + \varphi^m - \varphi^{m+j}) < 0, \quad i < j\gamma, \quad j \geq 0.$$

(2.63) is satisfied if

$$\liminf_{m \rightarrow \infty} (\varphi^m - \varphi^{m+j}) < h\{1 + [-j\gamma]\}, \quad j \geq 0,$$

where $[-j\gamma]$ denotes the integer part of $-j\gamma$, since

$$i \leq -\{1 + [-j\gamma]\} \text{ if } i < j\gamma, \quad j \geq 0,$$

where $\gamma < 0$. If for some i and $j \geq 1$,

$$\liminf_{m \rightarrow \infty} (\varphi^m - \varphi^{m+j}) \geq h\{1 + [-j\gamma]\},$$

then we derive from (2.45) and the definition of γ that

$$-j\gamma \geq 1 + [-j\gamma], \quad j \geq 1,$$

which is a contradiction.

The proof is complete for $i_2 = j_2 = 0$. The case $i_1 = j_1 = 0$ is treated similarly. The general case is proved by reduction either to the case $i_1 = j_1 = 0$ or to the case $i_2 = j_2 = 0$. $\square$

Lemma 13. *Let γ be an irrational number. Then the set*

$$\{i - \gamma j; \forall i, \forall j \geq 0\},$$

is dense in the set of real numbers.

Proof. The result follows from fundamental properties of irrational numbers, see Hardy and Wright [3]. $\square$

We then define

$$(2.64) \quad W(x) = \sup_{h(i-\gamma j) < x} U((i - \gamma j)h) = \sup_{h(i-\gamma j) < x} V(ih, jk),$$

which exist by Lemmas 12 and 13.

Lemma 14. *$W(x)$ defined in (2.64) satisfies the difference equation*

$$(2.65) \quad W(x - \gamma h) = (1 - 2\lambda)W(x) + \lambda W(x - h) + \lambda W(x + h) + kF(W(x)).$$

Proof. Let $\{x_n\}$, $\forall n \geq 1$ be a sequence, such that

$$x_n = h(i_n - \gamma j_n); \quad x_n < x_{n+1}, \quad \forall n \geq 1 \text{ and } x_n \rightarrow x, \quad n \rightarrow \infty.$$

(2.65) then follows from Lemma 11 and the left continuity of W , due to Lemmas 12, 13 and the definition (2.64). $\square$

Proof of Theorem 8 (cont.) The proof is completed by Lemmas 12 and 14. $\square$

Theorem 9. *Assume that the conditions of Theorem 6 are satisfied. Let*

$$\lim_{n \rightarrow \infty} \varphi^n / n = \gamma h.$$

Then

$$\gamma = - \inf_{\alpha > 0} \{\log[1 - 2\lambda + k + \lambda(\exp(-\alpha h) + \exp(\alpha h))]/(\alpha h)\}.$$

Proof. The result follows from Theorems 2, 6, 7, 8 and Lemma 15 below. $\square$

Lemma 15. *Let γ be a real number. The difference equation*

$$(2.66) \quad Z(x - \gamma h) = (1 - 2\lambda)Z(x) + \lambda Z(x - h) + \lambda Z(x + h) + kF(Z(x)),$$

where

$$(2.67) \quad \lim_{x \rightarrow -\infty} Z(x) = 0 \text{ and } \lim_{x \rightarrow \infty} Z(x) = 1,$$

has no measurable solution $Z(x)$, such that $0 < Z(x) < 1$, if

$$0 \leq -\gamma < r,$$

where r is defined by

$$r = \inf_{\alpha > 0} \{ \log[1 - 2\lambda + k + \lambda(\exp(-\alpha h) + \exp(\alpha h))]/(\alpha h) \}.$$

Proof. Let $Z(x)$ be a measurable function, such that $0 < Z(x) < 1$, which satisfies (2.66) and (2.67). First, introduce a new variable t defined by

$$t = -x,$$

and a new function W ,

$$W(t) = Z(-t) = Z(x).$$

Then, from (2.66)

$$(2.68) \quad \begin{aligned} W(t + \gamma h) &= (1 - 2\lambda)W(t) + \lambda W(t - h) + \lambda W(t + h) \\ &\quad + kF(W(t)), \quad -\infty < t < \infty. \end{aligned}$$

Introduce the Laplace transform

$$(2.69) \quad W_L(s) = \int_0^{+\infty} W(t) \exp(-st) dt, \quad s > s_0,$$

where s is a real number. The definition of W_L in (2.69) then implies that $W_L(s)$ is a decreasing and continuous function of s for $s > s_0$, where

$$s_0 = \inf\{s; W(s) < \infty\}.$$

s_0 is either $-\infty$ or finite. Then, it follows that

$$W_L(s) \rightarrow \infty, \quad s_0 < s \rightarrow s_0,$$

since $W_L(s)$ is continuous. We have

$$W_L(s) < \infty \text{ for } s > 0,$$

and we conclude that

$$s_0 \leq 0.$$

Transforming the difference equation (2.68) gives

$$P(s)W_L(s) = Q(s) + R(s), \quad s > s_0,$$

where

$$(2.70) \quad P(s) = \exp(s\gamma h) - (1 - 2\lambda) - k - \lambda(\exp(sh) + \exp(-sh)),$$

$$(2.71) \quad Q(s) = -\exp(s\gamma h) \int_{\gamma h}^0 W(t) \exp(-st) dt \\ + \lambda \{ \exp(-sh) \int_{-h}^0 W(t) \exp(-st) dt - \exp(sh) \int_0^h W(t) \exp(-st) dt \},$$

$$(2.72) \quad R(s) = k \int_0^\infty \{F(W(t)) - W(t)\} \exp(-st) dt.$$

Assume that

$$(2.73) \quad 0 \leq -\gamma < r.$$

Then, it follows that

$$P(s) \neq 0, \quad s < 0,$$

since if

$$P(-s_1) = 0, \quad s_1 > 0,$$

then it follows from (2.70) that

$$\exp(-\gamma s_1 h) = 1 - 2\lambda + k + \lambda(\exp(s_1 h) + \exp(-s_1 h)),$$

which implies

$$-\gamma = \log[1 - 2\lambda + k + \lambda(\exp(-s_1 h) + \exp(s_1 h))]/(s_1 h), \quad s_1 > 0$$

which contradicts the assumption (2.73). We also have from (2.70) that

$$P(0) = -k$$

and

$$\lim_{s \rightarrow -\infty} P(s) = -\infty,$$

since

$$0 \leq -\gamma < r \leq 1.$$

Then, there exists a number $\delta > 0$, such that

$$\sup_{s \leq \delta} P(s) < 0.$$

Division by $P(s) < 0$, $s \leq \delta$, gives

$$W_L(s) = Q(s)/P(s) + R(s)/P(s), \quad s_0 < s \leq \delta,$$

or equivalently,

$$\int_0^\infty \exp(-st) W(t) \{1 + (k/P(s))[1 - F(W(t))/W(t)]\} dt \\ = Q(s)/P(s), \quad s_0 < s \leq \delta,$$

since $W(t) > 0$ by assumption. There exists a constant $T > 0$, such that

$$\{1 + (k/P(s))[1 - F(W(t))/W(t)]\} > 0.5, \quad T < t, \quad s \leq \delta,$$

since

$$\lim_{t \rightarrow +\infty} F(W(t))/W(t) = 1$$

and

$$\sup_{s \leq \delta} P(s) < 0.$$

Then for s_0 finite

$$\int_T^\infty \exp(-st)W(t)\{1 + (k/P(s))[1 - F(W(t))/W(t)]\} dt \\ \rightarrow \infty, \quad s_0 < s \rightarrow s_0,$$

Since $Q(s)/P(s)$ is bounded, we have found a contradiction. Similarly, we find a contradiction if $s_0 = -\infty$, since

$$\{1 + (k/P(s))[1 - F(W(t))/W(t)]\} > 0.5$$

for s sufficiently small, since

$$P(s) \rightarrow -\infty, s \rightarrow -\infty. \quad \square$$

3. DISCUSSION AND CONCLUSIONS

- (1) The discrete analysis of the flame velocity, in particular Theorem 9, does not require a complete analysis of the existence of traveling wave solutions to the discrete equation. The reason is that a discrete version of Theorem 12 in Kolmogorov, Petrowsky and Piskunow [4], which implies strict monotonicity for V , is replaced by a weaker condition (Lemma 6).
- (2) The KPP-theory was proved for a simple model problem (Fisher's equation). In turbulent flame propagation we have models for the mean reaction rate, which satisfy the KPP-conditions (1.3)–(1.5); see Hakberg [2]. We may therefore expect that an extended KPP-theory is valid but the flame equations are too complex for a mathematical analysis. Validation of an extended KPP-theory is difficult for several reasons. First, we have errors due to finite values of h and k . Second, numerical experiments suggest that $(\varphi^{n+1} - \varphi^n)/nk$ fluctuates and does not converge as n tends to infinity. This paper suggests that $\lim_{n \rightarrow \infty} \varphi^n/nk$ is determined from numerical experiments and compared with a discrete KPP-theory as part of a validation procedure for the continuous equations.

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