NEW RESULTS ON REVERSE ORDER LAW FOR $\{1, 2, 3\}$ - AND $\{1, 2, 4\}$ -INVERSES OF BOUNDED OPERATORS

XIAOJI LIU, SHUXIA WU, AND DRAGANA S. CVETKOVIĆ-ILIĆ

ABSTRACT. In this paper, using some block-operator matrix techniques, we give necessary and sufficient conditions for the reverse order law to hold for $\{1, 2, 3\}$ - and $\{1, 2, 4\}$ -inverses of bounded operators on Hilbert spaces. Furthermore, we present some new equivalents of the reverse order law for the Moore-Penrose inverse.

1. INTRODUCTION

Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces and let $\mathcal{B}(\mathcal{H}, \mathcal{K})$ denote the set of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. For a given $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, the symbols $\mathcal{N}(A)$ and $\mathcal{R}(A)$ denote the null space and the range of A , respectively. For given subsets M, N of $\mathcal{B}(\mathcal{H}, \mathcal{K})$, by MN we denote the set consisting of all products XY , where $X \in M$ and $Y \in N$.

Recall that $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ has the Moore-Penrose inverse if there exists an operator $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that

$$(1.1) \quad \begin{aligned} (1) \quad AXA &= A, & (2) \quad XAX &= X, \\ (3) \quad (AX)^* &= AX, & (4) \quad (XA)^* &= XA. \end{aligned}$$

The Moore-Penrose inverse of an operator $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ exists if and only if A has closed range and in this case it is unique. It is denoted by $A^\dagger$.

For any $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, let $A\{i, j, \dots, k\}$ denote the set of all operators $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ which satisfy equations $(i), (j), \dots, (k)$ of (1.1). In this case X is a $\{i, j, \dots, k\}$ -inverse of A and is denoted by $A^{(i, j, \dots, k)}$. Evidently, $A\{1, 2, 3, 4\} = \{A^\dagger\}$, when A has closed range.

The reverse order law for generalized inverses plays an important role in theoretic research and numerical computations in many areas, including the singular matrix problem, ill-posed problems, optimization problems, and statics problems (see, for instance, [1, 9, 13, 14, 18]). These problems have attracted considerable attention since the mid-1960s, and many interesting results for generalized inverses of products of matrices or operators have been obtained (see [5], [6], [10], [11], [15]–[17]). T.N.E. Greville [8] first proved that $(AB)^\dagger = B^\dagger A^\dagger$ if and only if $\mathcal{R}(A^*AB) \subseteq \mathcal{R}(B)$ and $\mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*)$, for matrices A and B . This result was extended to linear bounded operators on Hilbert spaces in [11]. Later, the reverse order law for the Moore-Penrose inverse was considered in rings with involution (see [12]).

Received by the editor April 22, 2011 and, in revised form, November 9, 2011.

2010 *Mathematics Subject Classification*. Primary 15A09.

Key words and phrases. Block-operator matrix, Moore-Penrose inverse, reverse order law, $\{1, 2, 3\}$ -inverse, $\{1, 2, 4\}$ -inverse.

This work was supported by Grant No. 174007 of the Ministry of Science, Technology and Development, Republic of Serbia.

Xiong and Zheng [20] considered the reverse order law for $\{1, 2, 3\}$ - and $\{1, 2, 4\}$ -generalized inverses of products of two matrices. Their techniques involved expressions for maximal and minimal ranks of the generalized Schur complement. In [2] the authors considered the reverse order law for K -inverses in the cases $K \in \{\{1, 3\}, \{1, 4\}, \{1, 2, 3\}, \{1, 2, 4\}\}$ for elements of a C^* -algebra.

In this paper, using block-operator matrix techniques, we consider the reverse order law for $\{1, 2, 3\}$ - and $\{1, 2, 4\}$ -inverses of bounded linear operators on Hilbert spaces. We give necessary and sufficient conditions for

$$B\{1, 2, 3\} \cdot A\{1, 2, 3\} \subseteq (AB)\{1, 2, 3\}$$

and

$$B\{1, 2, 4\} \cdot A\{1, 2, 4\} \subseteq (AB)\{1, 2, 4\}.$$

We generalized the results from [2] to the case of bounded linear operators on Hilbert spaces. This is of particular importance especially in statistics where bounded linear operators play a very important role. Furthermore, we present new equivalent conditions for the reverse order law for the Moore-Penrose inverse. It should be pointed out that when restricted to the set of matrices, our results for the reverse order law for the Moore-Penrose inverse yield facts not previously known.

2. PRELIMINARIES

Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces and let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ have closed range. The operator A has the following matrix decomposition (see [4], [7])

$$(2.1) \quad A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix},$$

where A_1 is invertible. Also, $A^\dagger$ has the form

$$(2.2) \quad A^\dagger = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix}.$$

If $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ has closed range, then we can explicitly describe the sets $A\{1, 2, 3\}$ and $A\{1, 2, 4\}$ using the representation of A given by (2.1).

Lemma 2.1. *Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces and let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ have closed range. Then*

$$A\{1, 2, 3\} = \left\{ \begin{bmatrix} A_1^{-1} & 0 \\ X_3 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} : X_3 \in \mathcal{B}(\mathcal{R}(A), \mathcal{N}(A)) \right\}$$

and

$$A\{1, 2, 4\} = \left\{ \begin{bmatrix} A_1^{-1} & X_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix} : X_2 \in \mathcal{B}(\mathcal{N}(A^*), \mathcal{R}(A^*)) \right\}.$$

Proof. Suppose that A and $A^\dagger$ are given by (2.1) and (2.2), respectively. Since

$$A\{1, 2, 3\} = \{A^\dagger + (I - A^\dagger A)XAA^\dagger : X \in \mathcal{B}(\mathcal{K}, \mathcal{H})\},$$

we have that $A^{(1,2,3)} \in A\{1, 2, 3\}$ if and only if

$$\begin{aligned} A^{(1,2,3)} &= A^\dagger + (I - A^\dagger A)XAA^\dagger \\ &= \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} A_1^{-1} & 0 \\ X_3 & 0 \end{bmatrix}, \end{aligned}$$

for some $X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A^*) \\ \mathcal{N}(A) \end{bmatrix}$. The proof for the case of $\{1, 2, 4\}$ -inverses follows similarly. $\square$

Lemma 2.2. *Let $\mathcal{H}, \mathcal{K}$ and $\mathcal{L}$ be Hilbert spaces. Let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $B \in \mathcal{B}(\mathcal{L}, \mathcal{H})$ be such that $\mathcal{R}(A), \mathcal{R}(B)$ and $\mathcal{R}(AB)$ are closed. Then $\mathcal{R}(B^*) \cap \mathcal{N}(AB) = \{0\}$ if and only if $\mathcal{R}(B) \cap \mathcal{N}(A) = \{0\}$.*

Proof. First, let us note that

$$(2.3) \quad B|_{\mathcal{R}(B^*)} : \mathcal{R}(B^*) \rightarrow \mathcal{R}(B) \text{ is an invertible operator.}$$

$(\Rightarrow)$: Suppose that $\mathcal{R}(B^*) \cap \mathcal{N}(AB) = \{0\}$ and let $x \in \mathcal{R}(B) \cap \mathcal{N}(A)$. By (2.3), there exists $y \in \mathcal{R}(B^*)$ such that $By = x$. Now, $y \in \mathcal{R}(B^*) \cap \mathcal{N}(AB)$, i.e., $y = 0$, so $x = By = 0$.

$(\Leftarrow)$: If we suppose that $\mathcal{R}(B) \cap \mathcal{N}(A) = \{0\}$ and take $u \in \mathcal{R}(B^*) \cap \mathcal{N}(AB)$, we get that $Bu \in \mathcal{R}(B) \cap \mathcal{N}(A)$, i.e., $Bu = 0$. Using (2.3) it follows that $u = 0$. $\square$

Let us introduce the following notation: if a Hilbert space $\mathcal{H}$ is decomposed as $\mathcal{H} = \mathcal{U}_1 \oplus \cdots \oplus \mathcal{U}_k$ where $\mathcal{U}_i \perp \mathcal{U}_j$ for $i \neq j$, then we shall write $\mathcal{H} = \mathcal{U}_1 \oplus^\perp \cdots \oplus^\perp \mathcal{U}_k$. If $\mathcal{U}$ is a complement space of a Hilbert space $\mathcal{H}$ we shall denote by $\mathcal{H} \ominus^\perp \mathcal{U}$ the unique subspace $\mathcal{V}$ of $\mathcal{H}$ such that $\mathcal{H} = \mathcal{U} \oplus^\perp \mathcal{V}$.

Remark 2.1. Let $\mathcal{H}, \mathcal{K}$ and $\mathcal{L}$ be Hilbert spaces and let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $B \in \mathcal{B}(\mathcal{L}, \mathcal{H})$ be such that $\mathcal{R}(A), \mathcal{R}(B)$ and $\mathcal{R}(AB)$ are closed. Denote by

$$\begin{cases} \mathcal{H}_1 = \mathcal{R}(B) \cap \mathcal{N}(A), \\ \mathcal{H}_2 = \mathcal{R}(B) \ominus^\perp \mathcal{H}_1, \\ \mathcal{H}_3 = \mathcal{N}(B^*) \cap \mathcal{N}(A), \\ \mathcal{H}_4 = \mathcal{N}(B^*) \ominus^\perp \mathcal{H}_3, \end{cases} \quad \begin{cases} \mathcal{K}_1 = \mathcal{R}(AB), \\ \mathcal{K}_2 = \mathcal{R}(A) \ominus^\perp \mathcal{R}(AB), \end{cases} \quad \begin{cases} \mathcal{L}_1 = B^\dagger \mathcal{H}_1, \\ \mathcal{L}_2 = \mathcal{R}(B^*) \ominus^\perp B^\dagger \mathcal{H}_1. \end{cases}$$

Hilbert spaces $\mathcal{H}, \mathcal{K}$ and $\mathcal{L}$ can be decomposed as

$$\mathcal{H} = \mathcal{R}(B) \oplus^\perp \mathcal{N}(B^*), \quad \mathcal{K} = \mathcal{R}(A) \oplus^\perp \mathcal{N}(A^*), \quad \mathcal{L} = \mathcal{R}(B^*) \oplus^\perp \mathcal{N}(B),$$

where

$$\mathcal{R}(B) = \mathcal{H}_1 \oplus^\perp \mathcal{H}_2, \quad \mathcal{N}(B^*) = \mathcal{H}_3 \oplus^\perp \mathcal{H}_4, \quad \mathcal{R}(A) = \mathcal{K}_1 \oplus^\perp \mathcal{K}_2, \quad \mathcal{R}(B^*) = \mathcal{L}_1 \oplus^\perp \mathcal{L}_2.$$

We can prove that $B^\dagger(\mathcal{R}(B) \cap \mathcal{N}(A)) = \mathcal{R}(B^*) \cap \mathcal{N}(AB)$. Indeed let $x \in \mathcal{R}(B^*) \cap \mathcal{N}(AB)$. Then $x \in \mathcal{R}(B^\dagger) = \mathcal{R}(B^\dagger B) = B^\dagger \mathcal{R}(B)$ and $ABx = 0$, i.e., $Bx \in \mathcal{N}(A)$. So we have $x = B^\dagger Bx \in B^\dagger \mathcal{N}(A)$. Finally, $x \in B^\dagger(\mathcal{R}(B) \cap \mathcal{N}(A))$. On the other hand, let $y \in B^\dagger(\mathcal{R}(B) \cap \mathcal{N}(A))$, i.e., $y \in \mathcal{R}(B^\dagger B) = \mathcal{R}(B^*)$, then for some $z \in \mathcal{R}(B) \cap \mathcal{N}(A)$ we have $y = B^\dagger z$. So $AB y = ABB^\dagger z = Az = 0$, i.e., $y \in \mathcal{N}(AB)$. Thus, by Lemma 2.2, we get

$$\mathcal{H}_2 = \mathcal{R}(B) \Leftrightarrow \mathcal{H}_1 = \{0\} \Leftrightarrow \mathcal{N}(AB) = \mathcal{N}(B) \Leftrightarrow \mathcal{L}_1 = \{0\} \Leftrightarrow \mathcal{L}_2 = \mathcal{R}(B^*).$$

Furthermore,

$$\begin{aligned}\mathcal{H}_2 = \{0\} &\Leftrightarrow \mathcal{H}_1 = \mathcal{R}(B) \Leftrightarrow \mathcal{R}(B) \subset \mathcal{N}(A) \Leftrightarrow \mathcal{L}_1 = \mathcal{R}(B^*) \Leftrightarrow \mathcal{L}_2 \\ &= \{0\} \Leftrightarrow \mathcal{K}_1 = \{0\} \Leftrightarrow \mathcal{K}_2 = \mathcal{R}(A).\end{aligned}$$

Throughout the paper we will use the notation from the above remark.

A similar result to the following one but for the case of two Hilbert spaces has been presented in [17]. Now we give a different proof in the case of three Hilbert spaces.

Lemma 2.3. *Let $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ be Hilbert spaces and let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $B \in \mathcal{B}(\mathcal{L}, \mathcal{H})$ be such that $\mathcal{R}(A)$, $\mathcal{R}(B)$ and $\mathcal{R}(AB)$ are closed.*

(1) *If $AB \neq 0$ and $\mathcal{N}(AB) \neq \mathcal{N}(B)$, then A and B have the following operator matrix forms:*

$$(2.4) \quad A = \begin{bmatrix} 0 & A_{12} & 0 & A_{14} \\ 0 & 0 & 0 & A_{24} \\ 0 & 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{N}(A^*) \end{bmatrix}$$

and

$$(2.5) \quad B = \begin{bmatrix} B_{11} & B_{12} & 0 \\ 0 & B_{22} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \\ \mathcal{N}(B) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix},$$

where A_{12}, B_{11}, B_{22} are invertible operators and A_{24} is a surjection.

(2) *If $AB \neq 0$ and $\mathcal{N}(AB) = \mathcal{N}(B)$, then A and B have the following operator matrix forms:*

$$(2.6) \quad A = \begin{bmatrix} A_{12} & 0 & A_{14} \\ 0 & 0 & A_{24} \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{N}(A^*) \end{bmatrix}$$

and

$$(2.7) \quad B = \begin{bmatrix} B_{22} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix},$$

where A_{12}, B_{22} are invertible operators and A_{24} is a surjection.

(3) *If $AB = 0$ and $\mathcal{N}(AB) \neq \mathcal{N}(B)$, then A and B have the following operator matrix forms:*

$$(2.8) \quad A = \begin{bmatrix} 0 & 0 & A_{24} \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(A) \\ \mathcal{N}(A^*) \end{bmatrix}$$

and

$$(2.9) \quad B = \begin{bmatrix} B_{11} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{R}(B^*) \\ \mathcal{N}(B) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{R}(B) \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix},$$

where B_{11} and A_{24} are invertible.

Proof. We will assume that the spaces $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ are decomposed as in Remark 2.1, so that the conclusions of the remark also hold.

(1) Suppose that $AB \neq 0$ and $\mathcal{N}(AB) \neq \mathcal{N}(B)$. We have that B can be represented by

$$B = \begin{bmatrix} B_{11} & B_{12} & 0 \\ B_{21} & B_{22} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \\ \mathcal{N}(B) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix},$$

where $\widehat{B} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} : \begin{bmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{bmatrix}$ is invertible.

Since $B\mathcal{L}_1 \subset \mathcal{H}_1$, we get that $B_{21} = 0$. Now from the invertibility of $\widehat{B}$, we get that $B_{11} : \mathcal{L}_1 \rightarrow \mathcal{H}_1$ and $B_{22} : \mathcal{L}_2 \rightarrow \mathcal{H}_2$ are invertible.

Now, we will prove that A has a matrix form given by (2.4). Suppose that

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \end{bmatrix} : \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{N}(A^*) \end{bmatrix}.$$

The reductions A_{i1} and A_{i3} for $i = 1, 2, 3$ are null operators because $\mathcal{H}_1, \mathcal{H}_3 \subset \mathcal{N}(A)$. The range of the reductions A_{3j} , ($j = 1, 2, 3, 4$) is $\mathcal{N}(A^*)$, so $A_{3j} = 0$.

Now we will prove that $A_{22} = 0$: for any $x \in \mathcal{H}_2 \subset \mathcal{R}(B)$, there exists $y \in \mathcal{K}$ such that $By = x$. Now, $Ax = AB y \in \mathcal{K}_1$ and $Ax = A_{12}x + A_{22}x$. Since $A_{12}x \in \mathcal{K}_1$, we get that $A_{22}x = 0$.

In order to prove that A_{12} is bijective, first we will prove that $\mathcal{N}(A_{12}) = \{0\}$: let $u \in \mathcal{H}_2$ be such that $A_{12}u = 0$. Then $u \in \mathcal{N}(A)$ which implies that $u \in \mathcal{H}_1 \cap \mathcal{H}_2 = \{0\}$.

To prove that $A_{12} : \mathcal{H}_2 \rightarrow \mathcal{K}_1$ is surjective take any $k \in \mathcal{K}_1 = \mathcal{R}(AB)$. There exists $k' \in \mathcal{K}$ such that $ABk' = k$. Since $Bk' \in \mathcal{R}(B) = \mathcal{H}_1 \oplus^\perp \mathcal{H}_2$, there exist $h_1 \in \mathcal{H}_1$ and $h_2 \in \mathcal{H}_2$ such that $Bk' = h_1 + h_2$. Now, $Ah_2 = A(Bk' - h_1) = k$, i.e., $A_{12}h_2 = k$.

The surjective property of $A_{24} : \mathcal{H}_4 \rightarrow \mathcal{K}_2$ follows from the fact that for any $u \in \mathcal{K}_2$, there exists $v \in \mathcal{H}$ such that $Av = u$. Let us decompose $v = \sum_{i=1}^4 v_i$, where $v_i \in \mathcal{H}_i$. It is evident that $A_{24}v_4 = u$.

The proof of (2) and (3) is analogous. $\square$

3. MAIN RESULTS

Z. Xiong and B. Zheng [20] presented necessary and sufficient conditions for

$$(3.1) \quad B\{1, 2, 3\}A\{1, 2, 3\} \subseteq (AB)\{1, 2, 3\},$$

in the case when A and B are matrices. Here, we give another characterization of (3.1) for linear bounded operators on Hilbert spaces using techniques which are completely different from those used in [20]. First, we will give the following remark:

Remark 3.1. Let $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ be Hilbert spaces and let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $B \in \mathcal{B}(\mathcal{L}, \mathcal{H})$ be such that $\mathcal{R}(A), \mathcal{R}(B)$ and $\mathcal{R}(AB)$ are closed, $AB \neq 0$ and $\mathcal{N}(AB) \neq \mathcal{N}(B)$. Then we can suppose that the operators A and B are represented by (2.4) and (2.5),

respectively. By Lemma 2.1, $X \in B\{1, 2, 3\}$ if and only if there exist operators F_{11} and F_{12} such that

$$(3.2) \quad X = \begin{bmatrix} B_{11}^{-1} & -B_{11}^{-1}B_{12}B_{22}^{-1} & 0 & 0 \\ 0 & B_{22}^{-1} & 0 & 0 \\ F_{11} & F_{12} & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \\ \mathcal{N}(B) \end{bmatrix}.$$

To describe the set $A\{1, 2, 3\}$, suppose that an arbitrary $Y \in A\{1, 2, 3\}$ is given by

$$(3.3) \quad Y = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \\ Y_{41} & Y_{42} & Y_{42} \end{bmatrix} : \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{N}(A^*) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix}.$$

Since AY is hermitian, we get that

$$AY = \begin{bmatrix} A_{12}Y_{21} + A_{14}Y_{41} & A_{12}Y_{22} + A_{14}A_{42} & 0 \\ A_{24}Y_{41} & A_{24}Y_{42} & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{N}(A^*) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{N}(A^*) \end{bmatrix},$$

where $A_{12}Y_{22} + A_{14}A_{42} = (A_{24}Y_{41})^*$ and $A_{12}Y_{21} + A_{14}Y_{41}$, $A_{24}Y_{42}$ are hermitian. Since AY is an orthogonal projection on $\mathcal{R}(A)$, from the definition of the subspaces $\mathcal{K}_1$ and $\mathcal{K}_2$ we can conclude that $A_{12}Y_{21} + A_{14}Y_{41} = I$, $A_{24}Y_{42} = I$, $A_{12}Y_{22} + A_{14}A_{42} = 0$ and $A_{24}Y_{41} = 0$. Now, from $YAY = 0$, we get that $Y_{i3} = 0$, for $i = \overline{1, 4}$. Hence, $Y \in A\{1, 2, 3\}$ if and only if

$$(3.4) \quad Y = \begin{bmatrix} Y_{11} & Y_{12} & 0 \\ Y_{21} & Y_{22} & 0 \\ Y_{31} & Y_{32} & 0 \\ Y_{41} & Y_{42} & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{N}(A^*) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix},$$

where the Y_{ij} satisfy the following equalities:

$$(3.5) \quad \begin{cases} Y_{i2}A_{24}Y_{42} = Y_{i2}, \quad i = \overline{1, 4}, \\ A_{12}Y_{21} + A_{14}Y_{41} = I_{\mathcal{K}_1}, \\ A_{12}Y_{22} + A_{14}Y_{42} = 0, \\ A_{24}Y_{42} = I_{\mathcal{K}_2}, A_{24}Y_{41} = 0. \end{cases}$$

Since

$$(3.6) \quad AB = \begin{bmatrix} 0 & A_{12}B_{22} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \\ \mathcal{N}(B) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{N}(A^*) \end{bmatrix}.$$

we get that $Z \in (AB)\{1, 2, 3\}$ if and only if there exist operators N_1 and N_2 such that

$$(3.7) \quad Z = \begin{bmatrix} N_1 & 0 & 0 \\ B_{22}^{-1}A_{12}^{-1} & 0 & 0 \\ N_2 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{N}(A^*) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \\ \mathcal{N}(B) \end{bmatrix}.$$

Theorem 3.1. *Let $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ be Hilbert spaces and let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $B \in \mathcal{B}(\mathcal{L}, \mathcal{H})$ be such that $\mathcal{R}(A), \mathcal{R}(B)$ and $\mathcal{R}(AB)$ are closed and $AB \neq 0$. Then the following statements are equivalent:*

- (i) $B\{1, 2, 3\}A\{1, 2, 3\} \subseteq (AB)\{1, 2, 3\}$.
- (ii) $\mathcal{R}(B) = \mathcal{R}(A^*AB) \oplus^\perp [\mathcal{R}(B) \cap \mathcal{N}(A)]$, $\mathcal{R}(AB) = \mathcal{R}(A)$.

Proof. We use the decompositions of the spaces $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ and the matrix decompositions of operators A , B given in Lemma 2.3. We distinguish two cases:

(1) First, suppose that $\mathcal{N}(AB) \neq \mathcal{N}(B)$. We have that the operators A and B are represented by (2.4) and (2.5), respectively. Also, in Remark 3.1 we have given characterizations of the sets $A\{1, 2, 3\}$, $B\{1, 2, 3\}$ and $(AB)\{1, 2, 3\}$ which we will use in this proof.

(i) $\Rightarrow$ (ii): Let arbitrary $X \in B\{1, 2, 3\}$ and $Y \in A\{1, 2, 3\}$ be given by (3.2) and (3.4), respectively. Then

$$(3.8) \quad \begin{aligned} XY &= \begin{bmatrix} M_1 & M_2 & 0 \\ B_{22}^{-1}Y_{21} & B_{22}^{-1}Y_{22} & 0 \\ F_{11}Y_{11} + F_{12}Y_{21} & F_{11}Y_{12} + F_{12}Y_{22} & 0 \end{bmatrix} \\ &: \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{N}(A^*) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \\ \mathcal{N}(B) \end{bmatrix}, \end{aligned}$$

where $M_1 = B_{11}^{-1}Y_{11} - B_{11}^{-1}B_{12}B_{22}^{-1}Y_{21}$, $M_2 = B_{11}^{-1}Y_{12} - B_{11}^{-1}B_{12}B_{22}^{-1}Y_{22}$.

Since $XY \in (AB)\{1, 2, 3\}$, we conclude that XY must be of the form Z given by (3.7) for some operators N_1 and N_2 . If we compare (3.8) and (3.7), we get that $Y_{12} = 0$, $Y_{22} = 0$ and $Y_{21} = A_{12}^{-1}$. Hence, it follows that the system of the operator equations (3.5) is such that Y_{12} , Y_{22} and Y_{21} are uniquely determined. Since A_{24} is surjective and $A_{24}Y_{42} = I_{\mathcal{K}_2}$ we get that $Y_{12} = 0$ if and only if $\mathcal{K}_2 = \{0\}$, i.e., $A_{24} = 0$. If this were not true, then Y_{12} could be taken to be an arbitrary operator on an appropriate subspace, which is not the case. Now, from the first equation of (3.5), we get that $Y_{i2} = 0$, $i = \overline{1, 4}$. Since Y_{41} can be arbitrary, to avoid that $Y_{21} = A_{12}^{-1}$ it must be that $A_{14} = 0$. It is evident that $A_{24} = 0$ is equivalent to $\mathcal{R}(AB) = \mathcal{R}(A)$.

Now, simple computation shows that

$$A^*AB = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_{12}^*A_{12}B_{22} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \\ \mathcal{N}(B) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix},$$

and finally we get $\mathcal{R}(A^*AB) = \mathcal{R}(B) \ominus^\perp [\mathcal{R}(B) \cap \mathcal{N}(A)]$.

(ii) $\Rightarrow$ (i): Suppose $\mathcal{R}(A^*AB) = \mathcal{R}(B) \ominus^\perp [\mathcal{R}(B) \cap \mathcal{N}(A)]$ and $\mathcal{R}(AB) = \mathcal{R}(A)$. We must show that for arbitrary $X \in B\{1, 2, 3\}$ and $Y \in A\{1, 2, 3\}$ there exists $Z \in (AB)\{1, 2, 3\}$ such that $XY = Z$.

From $\mathcal{R}(AB) = \mathcal{R}(A)$, we get that $\mathcal{K}_2 = \{0\}$, i.e., $A_{24} = 0$. Also by $\mathcal{R}(A^*AB) = \mathcal{R}(B) \ominus^\perp [\mathcal{R}(B) \cap \mathcal{N}(A)]$ and the fact that

$$A^*AB = \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_{12}^*A_{12}B_{22} & 0 \\ 0 & 0 & 0 \\ 0 & A_{14}^*A_{12}B_{22} & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \\ \mathcal{N}(B) \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix},$$

where A_{12} and B_{22} are invertible, we have $A_{14} = 0$.

Now, we get that $Y \in A\{1, 2, 3\}$ if and only if

$$(3.9) \quad Y = \begin{bmatrix} Y_{11} & 0 & 0 \\ A_{12}^{-1} & 0 & 0 \\ Y_{31} & 0 & 0 \\ Y_{41} & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{N}(A^*) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix},$$

where Y_{11}, Y_{31}, Y_{41} are arbitrary. It is evident that for arbitrary $X \in B\{1, 2, 3\}$ and $Y \in A\{1, 2, 3\}$ there exists $Z \in (AB)\{1, 2, 3\}$ such that $XY = Z$, i.e.,

$$B\{1, 2, 3\}A\{1, 2, 3\} \subseteq (AB)\{1, 2, 3\}.$$

(2) When $\mathcal{N}(AB) = \mathcal{N}(B)$, the operators A and B are represented by (2.6) and (2.7), respectively, and the proof is analogous to case (1). $\square$

Remark 3.2. 1° If $AB = 0$, then $(AB)\{1, 2, 3\} = \{0\}$. In the case when $A = 0$ or $B = 0$, evidently $B\{1, 2, 3\}A\{1, 2, 3\} \subseteq (AB)\{1, 2, 3\}$. If it is not the case, we have that $AB = 0 \Leftrightarrow \mathcal{H}_2 = \{0\} \Leftrightarrow \mathcal{H}_1 = \mathcal{R}(B) \Leftrightarrow \mathcal{L}_2 = \{0\} \Leftrightarrow \mathcal{L}_1 = \mathcal{R}(B^*) \Leftrightarrow \mathcal{K}_1 = \{0\} \Leftrightarrow \mathcal{K}_2 = \mathcal{R}(A)$. Also, A and B are represented by (2.8) and (2.9), respectively, so arbitrary $X \in B\{1, 2, 3\}$ and $Y \in A\{1, 2, 3\}$ are represented by

$$X = \begin{bmatrix} B_{11}^{-1} & 0 & 0 \\ F_1 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{L}_1 \\ \mathcal{N}(B) \end{bmatrix}$$

and

$$Y = \begin{bmatrix} F_2 & 0 \\ F_3 & 0 \\ A_{24}^{-1} & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{K}_2 \\ \mathcal{N}(A^*) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix},$$

for some operators F_1, F_2 and F_3 .

By simple computation, we observe that

$$XY = \begin{bmatrix} B_{11}^{-1}F_2 & 0 \\ F_1F_2 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{L}_1 \\ \mathcal{N}(B) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{K}_2 \\ \mathcal{N}(A^*) \end{bmatrix} \neq 0,$$

i.e., $B\{1, 2, 3\}A\{1, 2, 3\} \neq \{0\}$.

Hence,

$$AB = 0, A \neq 0, B \neq 0 \Rightarrow B\{1, 2, 3\}A\{1, 2, 3\} \not\subseteq (AB)\{1, 2, 3\}.$$

2° From Theorem 3.1 we conclude that the condition

$$(ABB^\dagger)^\dagger ABB^\dagger = BB^\dagger \quad \text{or} \quad (AB)(AB)^\dagger = AA^\dagger$$

from [2, Theorem 3.3] can be replaced by the sole condition $(AB)(AB)^\dagger = AA^\dagger$, i.e., $\mathcal{R}(AB) = \mathcal{R}(A)$.

A similar result in the case $K = \{1, 2, 4\}$ follows from Theorem 3.1 by reversal of products:

Theorem 3.2. *Let $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ be Hilbert spaces and let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $B \in \mathcal{B}(\mathcal{L}, \mathcal{H})$ be such that $\mathcal{R}(A), \mathcal{R}(B)$ and $\mathcal{R}(AB)$ are closed and $AB \neq 0$. Then the following statements are equivalent:*

- (i) $B\{1, 2, 4\}A\{1, 2, 4\} \subseteq (AB)\{1, 2, 4\}$.
- (ii) $\mathcal{R}(A^*) = \mathcal{R}(BB^*A^*) \oplus^\perp [\mathcal{R}(A^*) \cap \mathcal{N}(B^*)]$, $\mathcal{N}(AB) = \mathcal{N}(B)$.

Remark 3.3. Let $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ be Hilbert spaces and let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $B \in \mathcal{B}(\mathcal{L}, \mathcal{H})$ be such that $\mathcal{R}(A), \mathcal{R}(B), \mathcal{R}(AB)$ are closed, $AB \neq 0$ and $\mathcal{N}(AB) \neq \mathcal{N}(B)$. We have that operator B is represented by (2.5), so

$$(3.10) \quad B^\dagger = \begin{bmatrix} B_{11}^{-1} & -B_{11}^{-1}B_{12}B_{22}^{-1} & 0 & 0 \\ 0 & B_{22}^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \\ \mathcal{N}(B) \end{bmatrix}.$$

Also, if we suppose that the operator A is represented by (2.5), by Remark 3.1 we get that $A^\dagger = Y$ is represented by (3.4), where Y_{ij} satisfy (3.5). Now, since YA is an orthogonal projection on $\mathcal{N}(A)^\perp$, we get that

$$YA = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} : \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix},$$

which implies that $Y_{21}A_{12} = I_{\mathcal{H}_2}$, $Y_{41}A_{12} = 0$, $Y_{21}A_{14} + Y_{22}A_{24} = 0$, $Y_{41}A_{14} + Y_{42}A_{24} = I_{\mathcal{H}_4}$. Now by $YAY = 0$ we get that $Y_{11} = 0$, $Y_{12} = 0$, $Y_{31} = 0$ and $Y_{32} = 0$. Hence,

$$(3.11) \quad A^\dagger = \begin{bmatrix} 0 & Y_{12} & 0 \\ Y_{21} & Y_{22} & 0 \\ 0 & Y_{32} & 0 \\ Y_{41} & Y_{42} & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{N}(A^*) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix}$$

where

$$(3.12) \quad \begin{cases} Y_{i2}A_{24}Y_{42} = Y_{i2}, i = \overline{1, 4}, \\ A_{12}Y_{21} + A_{14}Y_{41} = I_{\mathcal{K}_1}, \\ A_{12}Y_{22} + A_{14}Y_{42} = 0, \\ A_{24}Y_{42} = I_{\mathcal{K}_2}, A_{24}Y_{41} = 0, \\ Y_{21}A_{14} + Y_{22}A_{24} = 0, \\ Y_{41}A_{14} + Y_{42}A_{24} = I_{\mathcal{H}_4}, \\ Y_{21}A_{12} = I_{\mathcal{H}_2}, Y_{41}A_{12} = 0. \end{cases}$$

Simple computation shows that

$$(3.13) \quad B^\dagger A^\dagger = \begin{bmatrix} -B_{11}^{-1}B_{12}B_{22}^{-1}Y_{21} & M_3 & 0 \\ B_{22}^{-1}Y_{21} & B_{22}^{-1}Y_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{N}(A^*) \end{bmatrix} \rightarrow \begin{bmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \\ \mathcal{N}(B) \end{bmatrix},$$

where $M_3 = B_{11}^{-1}Y_{12} - B_{11}^{-1}B_{12}B_{22}^{-1}Y_{22}$.

Using the previous remark, we obtain the following result:

Theorem 3.3. *Let $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ be Hilbert spaces and let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $B \in \mathcal{B}(\mathcal{L}, \mathcal{H})$ be such that $\mathcal{R}(A), \mathcal{R}(B), \mathcal{R}(AB)$ are closed and $AB \neq 0$. Then the following statements are equivalent:*

- (i) $B^\dagger A^\dagger \in (AB)\{1, 2, 3\}$.
- (ii) $B\{1, 2, 3\}A^\dagger \subseteq (AB)\{1, 2, 3\}$.
- (iii) $\mathcal{R}(A^*AB) = \mathcal{R}(B) \ominus^\perp [\mathcal{R}(B) \cap \mathcal{N}(A)]$.

Proof. We will use the same decompositions of spaces $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ as in Theorem 3.1. We distinguish two cases:

(1) Let $\mathcal{N}(AB) \neq \mathcal{N}(B)$.

(i) $\Rightarrow$ (iii) If $B^\dagger A^\dagger \in (AB)\{1, 2, 3\}$, then there exists an operator $Z \in (AB)\{1, 2, 3\}$ such that $B^\dagger A^\dagger = Z$, where Z is represented by (3.7). Comparing (3.7) with (3.13), we obtain $Y_{21} = A_{12}^{-1}$, $Y_{22} = 0$, $Y_{12} = 0$.

We have that (3.12) implies $Y_{21} = A_{12}^{-1}$ only if $A_{14} = 0$ which implies the invertibility of A_{24} . Hence

$$A = \begin{bmatrix} 0 & A_{12} & 0 & 0 \\ 0 & 0 & 0 & A_{24} \\ 0 & 0 & 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{N}(A^*) \end{bmatrix}.$$

It is easy to get $\mathcal{R}(A^*AB) = \mathcal{R}(B) \ominus^\perp [\mathcal{R}(B) \cap \mathcal{N}(A)]$.

(iii) $\Rightarrow$ (i) Since $\mathcal{R}(A^*AB) = \mathcal{R}(B) \ominus^\perp [\mathcal{R}(B) \cap \mathcal{N}(A)]$ is equivalent to $A_{14} = 0$, we obtain from (3.12) that $Y_{21} = A_{12}^{-1}$, $Y_{22} = 0$, $Y_{12} = 0$. Hence, $B^\dagger A^\dagger \in (AB)\{1, 2, 3\}$.

(i) $\Leftrightarrow$ (ii) Using the representation of an arbitrary $X \in B\{1, 2, 3\}$ given by (3.2), we get that $XA^\dagger \in (AB)\{1, 2, 3\}$ if and only if $B^\dagger A^\dagger \in (AB)\{1, 2, 3\}$.

(2) If $\mathcal{N}(AB) = \mathcal{N}(B)$, the proof is analogous to case (1). $\square$

The case $K = \{1, 2, 4\}$ is treated completely analogously, and the corresponding result follows by taking adjoints, or by reversal of products:

Theorem 3.4. *Let $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ be Hilbert spaces and let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $B \in \mathcal{B}(\mathcal{L}, \mathcal{H})$ be such that $\mathcal{R}(A), \mathcal{R}(B), \mathcal{R}(AB)$ are closed and $AB \neq 0$. Then the following statements are equivalent:*

- (i) $B^\dagger A^\dagger \in (AB)\{1, 2, 4\}$.
- (ii) $B^\dagger A\{1, 2, 4\} \subseteq (AB)\{1, 2, 4\}$.
- (iii) $\mathcal{R}(BB^*A^*) = \mathcal{R}(A^*) \ominus^\perp [\mathcal{R}(A^*) \cap \mathcal{N}(B^*)]$.

From the above two theorems, we get the following equivalent condition for the reverse order law for the Moore-Penrose inverse.

Theorem 3.5. *Let $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ be Hilbert spaces and let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $B \in \mathcal{B}(\mathcal{L}, \mathcal{H})$ be such that $\mathcal{R}(A), \mathcal{R}(B), \mathcal{R}(AB)$ are closed and $AB \neq 0$. Then the following statements are equivalent:*

- (i) $(AB)^\dagger = B^\dagger A^\dagger$.
- (ii) $\mathcal{R}(A^*AB) = \mathcal{R}(B) \ominus^\perp [\mathcal{R}(B) \cap \mathcal{N}(A)]$ and $\mathcal{R}(BB^*A^*) = \mathcal{R}(A^*) \ominus^\perp [\mathcal{R}(A^*) \cap \mathcal{N}(B^*)]$.

Remark 3.4. The conditions (ii) from Theorem 3.5 are equivalent to the conditions $\mathcal{R}(A^*AB) \subseteq \mathcal{R}(B)$ and $\mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*)$ given in the paper by Greville [8] for matrices. Also, they are equivalent to those given in [5, Theorem 2.2 (c)] in the case of bounded linear operators on Hilbert space.

ACKNOWLEDGEMENT

The authors wish to thank the anonymous reviewers for very valuable comments and suggestions concerning an earlier version of this paper.

REFERENCES

- [1] A. Ben-Israel, T. N. E. Greville, *Generalized Inverse: Theory and Applications*, 2nd Edition, Springer, New York, 2003. MR1987382 (2004b:15008)
- [2] D. S. Cvetković-Ilić, R. Harte, *Reverse order laws in C^* -algebras*, Linear Algebra Appl., 434(2011), 1388-1394. MR2763596 (2012b:15006)
- [3] D. S. Cvetković-Ilić, *Reverse order laws for $\{1, 3, 4\}$ -generalized inverse in C^* -algebras*, Applied Mathematics Letters, 24(2011), 210-213. MR2735143
- [4] S. L. Campbell, C.D. Meyer, Jr., *Generalized inverses of linear transformations*, Dover Publications, Inc. New York, 1991. MR1105324 (92a:15003)
- [5] D. S. Djordjević, N.Č. Dinčić, *Reverse order law for the Moore-Penrose inverse*, J. Math. Anal. Appl. 361(2010), 252-261. MR2567299 (2010i:15007)
- [6] D. S. Djordjević, *Further results on the reverse order law for generalized inverses*, SIAM J. Matrix Anal. Appl. 29(4)(2007), 1242-1246. MR2369293 (2009g:47003)
- [7] D. S. Djordjević, V. Rakočević, *Lectures on generalized inverses*, Faculty of Sciences and Mathematics, University of Niš, Niš, 2008. MR2472376 (2009h:47002)
- [8] T.N.E. Greville, *Note on the generalized inverse of a matrix product*, SIAM Rev. 8(1966), 518-512. MR0210720 (35:1606)
- [9] G.H. Golub, C.F. Van Loan, *Matrix Computations*, The John Hopkins University Press, Baltimore, MD, 1983. MR733103 (85h:65063)
- [10] R. E. Hartwig, *The reverse order law revisited*, Linear Algebra Appl., 76(1986), 241-246. MR830343 (87j:15009)
- [11] S. Izumino, *The product of operators with closed range and an extension of the reverse order law*, Tohoku Math. J. 34(1982), 43-52. MR651705 (83d:47005)
- [12] J. J. Koliha, D. S. Djordjević, D. Cvetković Ilić, *Moore-Penrose inverse in rings with involution*, Linear Algebra Appl., 426(2007), 371-381. MR2350664 (2008g:47007)
- [13] C.R. Rao, S.K. Mitra, *Generalized Inverse of Matrices and Its Applications*, John Wiley, New York, 1971. MR0338013 (49:2780)
- [14] W. Sun, Y. Yuan, *Optimization Theory and Methods*, Science Press, Beijing, 1996.
- [15] Y. Tian, *Reverse order laws for the generalized inverses of multiple matrix products*, Linear Algebra Appl., 211(1994), 85-100. MR1295873 (95g:15005)
- [16] H. J. Werner, *When is B^-A^- a generalized inverse of AB ?*, Linear Algebra Appl., 210(1994), 255-263. MR1294779 (95h:15010)
- [17] J. Wang, H. Zhang, G. Ji, *A generalized reverse order law for the products of two operators*, Journal of Shaanxi Normal University, 38(4)(2010), 13-17. MR2743096
- [18] H. J. Werner, *G-inverse of matrix products*, Data Analysis and Statistical Inference, Eul-Verlag, Bergisch-Gladbach (1992), 531-546. MR1248855
- [19] M. Wei, *Equivalent conditions for generalized inverses of products*, Linear Algebra App., 266(1997), 347-363. MR1473209 (98i:15010)
- [20] Z. Xiong, B. Zheng, *The reverse order laws for $\{1, 2, 3\}$ - and $\{1, 2, 4\}$ -inverses of a two-matrix product*, Applied Mathematics Letters, 21(7)(2008), 649-655. MR2423040 (2009c:15004)

COLLEGE OF MATHEMATICS AND COMPUTER SCIENCE, GUANGXI UNIVERSITY FOR NATIONALITIES, NANNING 530006, PEOPLE'S REPUBLIC OF CHINA

E-mail address: xiaojiliu72@yahoo.com.cn

COLLEGE OF MATHEMATICS AND COMPUTER SCIENCE, GUANGXI UNIVERSITY FOR NATIONALITIES, NANNING 530006, PEOPLE'S REPUBLIC OF CHINA

E-mail address: anita623482950@yahoo.com.cn

UNIVERSITY OF NIŠ, DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCES AND MATHEMATICS, 18000 NIŠ, SERBIA

E-mail address: dragana@pmf.ni.ac.rs