A NEW VARIANT OF THE HILBERT-WARING PROBLEM

TIANXIN CAI AND DEYI CHEN

ABSTRACT. In this paper, we propose a new variant of Waring’s problem: to express a positive integer n as a sum of s positive integers whose product is a k -th power. We define, in a similar way to that done for $g(k)$ and $G(k)$ in Waring’s problem, numbers $g'(k)$ and $G'(k)$. We obtain $g'(k) = 2k - 1$, $G'(p) \leq p + 1$ for primes p , $G'(2p) \leq 2p + 2$ for odd primes p . Moreover, we obtain several interesting results and make two conjectures about $G'(3)$ and $G'(4)$.

1. INTRODUCTION

In 1770, Lagrange showed that each positive integer could be expressed as a sum of at most four squares of integers. In the same year, Edward Waring in his book *Meditationes Algebraicae* asserted that for every $k \in \mathbb{Z}_+$ there was a number $s(k)$ such that every positive integer n could be expressed as a sum of $s = s(k)$ k -th powers of positive integers, i.e.,

$$(1) \quad n = x_1^k + x_2^k + \cdots + x_s^k.$$

During the next 139 years, special cases of Waring’s problem were solved for $k = 3, 4, 5, 6, 7, 8, 10$. It was in 1909 that Hilbert solved the problem in the affirmative for all k ; see [2]. So Waring’s assertion is also called the Hilbert-Waring Theorem.

Once one knows that the answer to the Hilbert-Waring problem is “yes!” it is natural to ask “How big is $s(k)$?”. Taking $g(k)$ as the smallest value of $s(k)$, we have $g(2) = 4$ (1770, Lagrange), $g(3) = 9$ (1909, Wieferich et al.), $g(4) = 19$ (1986, R. Balasubramanian et al.), $g(5) = 37$ (1964, Chen Jingrun), $g(6) = 73$ (1940, Pillai). In general, there is a conjecture that $g(k) = 2^k + \lfloor (\frac{3}{2})^k \rfloor - 2$, which is true for $k \leq 471, 600, 000$. See [2] and [5] for the history of the problem.

From the work of Hardy and Littlewood it became apparent that a more fundamental number than $g(k)$ was $G(k)$, which is defined to be the least positive integer s such that all sufficiently large integers can be represented as a sum of at most s k -th powers of positive integers. We have $G(2) = 4$, $G(4) = 16$ (1939, Davenport). The precise value of $G(3)$ is not known, the best result to date being $4 \leq G(3) \leq 7$. For $k \geq 5$, we know little about the precise value of $G(k)$, e.g., $6 \leq G(5) \leq 17$, $9 \leq G(6) \leq 24$.

There are a variety of generalizations or variations for the Hilbert-Waring problem, e.g., $g(k, m)$ is defined by C. Small [4] as the smallest positive integer s such that every integer is a sum of s k -th powers mod m , i.e.,

$$(2) \quad n \equiv x_1^k + x_2^k + \cdots + x_s^k \pmod{m}.$$

Received by the editor September 4, 2011 and, in revised form, January 22, 2012, January 27, 2012, February 1, 2012.

2010 *Mathematics Subject Classification.* Primary 11P05; Secondary 11D85, 11D72.

Project supported by China National Science Foundation Grant No.10871169.

In April 2011, the first author proposed a new variant of Waring's problem: given $n \in \mathbb{Z}_+$, write

$$(3) \quad n = x_1 + x_2 + \cdots + x_s$$

with all $x_i \in \mathbb{Z}_+$ such that

$$(4) \quad x_1 x_2 \cdots x_s = x^k.$$

Obviously, each solution of Waring's problem (1) leads to a solution satisfying (3) and (4). From the Hilbert-Waring Theorem, there exists a positive integer $s'(k) \leq s(k)$ such that every positive integer is a sum of $s'(k)$ positive integers and the product of the $s'(k)$ integers is a k -th power. In this paper, we denote by $g'(k)$ (resp. $G'(k)$) the least positive integer such that every integer (resp. all sufficiently large integers) can be represented as a sum of at most $g'(k)$ (resp. $G'(k)$) positive integers, and the product of the $g'(k)$ (resp. $G'(k)$) integers is a k -th power. We have the following theorems:

Theorem 1. *For every positive integer k , $g'(k) = 2k - 1$.*

Theorem 2. *For every prime p , $G'(p) \leq p + 1$.*

Theorem 3. *For every odd prime p , $G'(2p) \leq 2p + 2$.*

Theorem 4. *For every prime $p \geq 7$, $G'(4p) \leq 4p + 3$.*

No prime $p \equiv 3 \pmod{4}$ can be expressed as a sum of two squares, which implies that $p \equiv 3 \pmod{4}$ cannot be represented as a sum of two positive integers such that the product of the two integers is a square, a cube or a k -th power. So we deduce from Theorems 1 and 2 that $G'(2) = 3$, $3 \leq G'(3) \leq 4$, $3 \leq G'(4) \leq 7$, $3 \leq G'(5) \leq 6$, etc.

In order to discuss $G'(3)$, let

$$(5) \quad \mathcal{S} = \{n \mid n = x + y + z, xyz = m^3, n, m, x, y, z \in \mathbb{Z}_+\}.$$

It is obvious that if $n \in \mathcal{S}$, then $kn \in \mathcal{S}$ for any $k \in \mathbb{Z}_+$. So we only need to discuss the special case of (5) when n is prime. We have the following theorems:

Theorem 5. *If p is prime and $p \equiv 1 \pmod{3}$ or $p \equiv \pm 1 \pmod{8}$, then $p \in \mathcal{S}$.*

Theorem 6. *If p is prime, $p \equiv 5 \pmod{24}$, and there exists $(x, y) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ such that*

$$p = 6x^2 - y^2,$$

where

$$(6) \quad \frac{x}{y} \in \left(\frac{7}{17}, \frac{19}{46}\right) \cup \left(\frac{11}{26}, \frac{3}{7}\right) \cup \left(\frac{9}{19}, \frac{7}{13}\right) \cup \left(1, \frac{3}{2}\right).$$

Then $p \in \mathcal{S}$.

By computing, we find that the only primes $p < 10^4$ such that $p \notin \mathcal{S}$ are 2, 5, 11 (cf. Table 1 for $p < 100$). Noting that

$$\begin{aligned} 2^6 &= 7 + 8 + 49, \quad 7 \times 8 \times 49 = 14^3, \\ 5^2 &= 3 + 4 + 18, \quad 3 \times 4 \times 18 = 6^3, \\ 11^2 &= 1 + 45 + 75, \quad 1 \times 45 \times 75 = 15^3, \end{aligned}$$

TABLE 1. One solution of $p = x + y + z, xyz = m^3$ for $p < 100$ with minimum m .

p	x	y	z	m
3	1	1	1	1
7	1	2	4	2
13	1	3	9	3
17	1	8	8	4
19	4	6	9	6
23	2	9	12	6
29	1	1	27	3
31	1	5	25	5
37	9	12	16	12
41	2	3	36	6
43	1	6	36	6

p	x	y	z	m
47	2	20	25	10
53	3	18	32	12
59	1	4	54	6
61	2	27	32	12
67	4	14	49	14
71	1	20	50	10
73	1	8	64	8
79	2	28	49	14
83	3	8	72	12
89	5	9	75	15
97	1	24	72	12

we deduce that $2^\alpha 5^\beta 11^\gamma \in \mathcal{S}$ for $\alpha \geq 6$ or $\beta \geq 2$ or $\gamma \geq 2$. So there are only finitely many numbers of the type $2^\alpha 5^\beta 11^\gamma$ not in $\mathcal{S}$. By computing and checking, we make the following conjecture.

Conjecture 1. *Except for 1, 2, 4, 5, 8, 11, 16, 22, 32, 44, 88, 176, every positive integer can be expressed as a sum of three positive integers where the product of those integers is a cube. In particular, $G'(3) = 3$.*

If we allow negative integers in the discussion, we obtain the following interesting theorem:

Theorem 7. *For every positive integer $n > 1$, there exists $(x, z) \in \mathbb{Z}_+ \times \mathbb{Z}_+$, $(y, m) \in \mathbb{Z} \times \mathbb{Z}$ with $\gcd(x, y, z) < n$ such that*

$$n = x + y + z$$

and

$$xyz = m^3.$$

Similarly, in order to discuss $G'(4)$, let

$$(7) \quad \mathcal{S} = \{n \mid n = x + y + z + w, xyzw = m^4, n, m, x, y, z, w \in \mathbb{Z}_+\}.$$

It is obvious that we only need to discuss the special case of (7) when n is prime. By computing, we find that the only primes $p < 10^4$ such that $p \notin \mathcal{S}$ are 2, 3, 5, 7, 11, 13, 17, 23 (cf. Table 2 for $p < 100$). Noting that

$$p^2 = 1 + (p-1) + (p-1) + (p-1)^2,$$

$$1 \times (p-1) \times (p-1) \times (p-1)^2 = (p-1)^4,$$

we deduce that $2^a 3^b 5^c 7^d 11^e 13^f 17^g 23^h \in \mathcal{S}$ if at least one of a, b, c, d, e, f, g, h is ≥ 2 . So there are only finite numbers of the type $2^a 3^b 5^c 7^d 11^e 13^f 17^g 23^h$ not in $\mathcal{S}$. By computing and checking, we make the following conjecture.

Conjecture 2. *Except for 1, 2, 3, 5, 6, 7, 11, 13, 14, 15, 17, 22, 23, every positive integer can be expressed as a sum of four positive integers where the product of those integers is a 4-th power. Moreover, $G'(4) = 4$.*

Again, if we allow negative integers in the discussion, we obtain the following.

Theorem 8. *For every positive integer $n > 1$, there exists $(x, z, m) \in \mathbb{Z}_+^3$, $(y, w) \in \mathbb{Z}_- \times \mathbb{Z}_-$ with $\gcd(x, y, z, w) < n$ such that*

$$n = x + y + z + w$$

and

$$xyzw = m^4.$$

TABLE 2. One solution of $p = x + y + z + w$, $xyzw = m^4$ for $p < 100$ with minimum m .

p	x	y	z	w	m						
19	1	1	1	16	2	59	4	12	16	27	12
29	2	6	9	12	6	61	1	3	9	48	6
31	3	3	9	16	6	67	1	16	25	25	10
37	1	6	12	18	6	71	3	8	24	36	12
41	1	4	18	18	6	73	2	12	27	32	12
43	2	2	12	27	6	79	2	9	32	36	12
47	1	3	16	27	6	83	2	9	24	48	12
53	8	9	12	24	12	89	2	2	4	81	6
						97	1	12	36	48	12

2. PRELIMINARIES

Lemma 1 (Gauss, cf. [4], Section 3). *If p is an odd prime, then there exists $(x, y) \in \mathbb{N} \times \mathbb{N}$ such that*

$$n \equiv x^2 + y^2 \pmod{2p}$$

for any $n \in \mathbb{Z}_+$.

Lemma 2. *If p is prime and $p \equiv \pm 1 \pmod{8}$, then there exists $(x, y) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ such that*

$$(8) \quad x^2 - 2y^2 = p$$

and

$$x > 2y.$$

Proof. Any prime $p \equiv \pm 1 \pmod{8}$ can be expressed as $p = x^2 - 2y^2$ [6]. It is obvious that $(u, v) = (3, 2)$ is the fundamental solution of $x^2 - 2y^2 = 1$. Denote $x + y\sqrt{2}$ the fundamental solution of the equation (8). We have (cf. [3], pp. 205-206, Theorem 108)

$$(9) \quad 0 \leq y \leq \frac{v\sqrt{p}}{\sqrt{2}(u+1)} = \sqrt{\frac{p}{2}}.$$

Combining (8) and (9), we have

$$\left(\frac{x}{y}\right)^2 = 2 + \frac{p}{y^2} \geq 2 + \frac{p}{(\sqrt{\frac{p}{2}})^2} = 4.$$

Then

$$\frac{x}{y} \geq 2.$$

But $x \neq 2y$, so the inequality is strict. $\square$

Lemma 3. *If p is prime and $p \equiv 5 \pmod{24}$, then there exists $(x, z) \in \mathbb{Z}_+ \times \mathbb{Z}_+$, $(y, m) \in \mathbb{Z} \times \mathbb{Z}$ and $(x, y, z) = 1$ such that*

$$p = x + y + z$$

and

$$xyz = m^3.$$

Proof. Suppose $p \equiv 5 \pmod{24}$, we have (cf. [1], pp. 50, Exercise 21)

$$p = 6x^2 - y^2$$

for some positive integers x and y . Making the transformations

$$(10) \quad \begin{cases} x = 3a + b, \\ y = 2a + b, \end{cases} \quad \begin{cases} x = 7a + b, \\ y = 13a + 2b, \end{cases} \quad \begin{cases} x = a + 9b, \\ y = 2a + 19b, \end{cases}$$

$$\begin{cases} x = 3a + 11b, \\ y = 7a + 26b, \end{cases} \quad \begin{cases} x = 19a + 7b, \\ y = 46a + 17b, \end{cases}$$

we obtain

$$p = 50a^2 + 32ab + 5b^2, \quad p = 125a^2 + 32ab + 2b^2, \quad p = 2a^2 + 32ab + 125b^2,$$

$$p = 5a^2 + 32ab + 50b^2, \quad p = 50a^2 + 32ab + 5b^2,$$

respectively. The inverse transformations are

$$(11) \quad \begin{cases} a = x - y, \\ b = 3y - 2x, \end{cases} \quad \begin{cases} a = 2x - y, \\ b = 7y - 13x, \end{cases} \quad \begin{cases} a = 19x - 9y, \\ b = y - 2x, \end{cases}$$

$$\begin{cases} a = 26x - 11y, \\ b = 3y - 7x, \end{cases} \quad \begin{cases} a = 17x - 7y, \\ b = 19y - 46x, \end{cases}$$

respectively. The required co-primarity condition is trivial. $\square$

Remark 1. To prove Lemma 3, we only need one of the five transformations (10). The reason we make five transformations is to prepare for the proof of Theorem 6.

Lemma 4. *If p is prime and $p \equiv 11 \pmod{24}$, then there exists $(x, z) \in \mathbb{Z}_+ \times \mathbb{Z}_+$, $(y, m) \in \mathbb{Z} \times \mathbb{Z}$ and $(x, y, z) = 1$ such that*

$$p = x + y + z$$

and

$$xyz = m^3.$$

Proof. Now $p \equiv 11 \pmod{24} \Rightarrow p \equiv 3 \pmod{8} \Rightarrow p = x^2 + 2y^2$ for some positive integers x and y [6]. Making the transformations

$$(12) \quad \begin{cases} x = 5a, \\ y = a + b, \end{cases} \quad \begin{cases} x = 2a + b, \\ y = 5a, \end{cases} \quad \begin{cases} x = a + 4b, \\ y = -a + b, \end{cases} \\ \begin{cases} x = a + 4b, \\ y = a - b, \end{cases} \quad \begin{cases} x = 2a - b, \\ y = a + 2b, \end{cases} \quad \begin{cases} x = a - 2b, \\ y = 2a + b, \end{cases}$$

we obtain

$$p = 27a^2 + 4ab + 2b^2, \quad p = 54a^2 + 4ab + b^2, \quad p = 3a^2 + 4ab + 18b^2, \\ p = 3a^2 + 4ab + 18b^2, \quad p = 6a^2 + 4ab + 9b^2, \quad p = 9a^2 + 4ab + 6b^2,$$

respectively. The inverse transformations are

$$(13) \quad \begin{cases} a = \frac{x}{5}, \\ b = y - \frac{x}{5}, \end{cases} \quad \begin{cases} a = \frac{y}{5}, \\ b = x - \frac{2y}{5}, \end{cases} \quad \begin{cases} a = \frac{x+y}{5} - y, \\ b = \frac{x+y}{5}, \end{cases} \\ \begin{cases} a = \frac{x-y}{5} + y, \\ b = \frac{x-y}{5}, \end{cases} \quad \begin{cases} a = \frac{2x+y}{5}, \\ b = \frac{2(2x+y)}{5} - x, \end{cases} \quad \begin{cases} a = \frac{x+2y}{5}, \\ b = y - \frac{2(x+2y)}{5}. \end{cases}$$

respectively. We now prove that there exists an inverse transformation among the six inverse transformations (13) such that a and b are integers. We only need to prove that one of the following six numbers is an integer:

$$\frac{x}{5}, \frac{y}{5}, \frac{x+y}{5}, \frac{x-y}{5}, \frac{2x+y}{5}, \frac{x+2y}{5}.$$

This follows because

$$xy(x+y)(x-y)(2x+y)(x+2y) = 5x^4y^2 + 2x^5y - 5x^2y^4 - 2xy^5 \\ \equiv 2xy(x^4 - y^4) \equiv 0 \pmod{5}.$$

The required co-primarity condition is trivial. $\square$

3. PROOFS OF THE THEOREMS

Proof of Theorem 1. It is obviously true for $k = 1$. Suppose $k > 1$, and for any positive integer n , define i by $n \equiv i \pmod{k}$, where $0 \leq i \leq k-1$. We have

$$n = \underbrace{1 + \cdots + 1}_i + \underbrace{\frac{n-i}{k} + \cdots + \frac{n-i}{k}}_k.$$

Therefore

$$(14) \quad g'(k) \leq 2k - 1.$$

On the other hand, if $2k - 1$ can be expressed as a sum of s ($s < 2k - 1$) positive integers and the product of the s integers is a k -th power, then there exists a positive integer $m > 1$ among the s integers. Let $q = \min_{p \text{ prime}} \{p \mid m \equiv 0 \pmod{p}\}$. Then

$$2k - 1 \geq q^{\beta_1} + \cdots + q^{\beta_r},$$

where $r \leq s$, $\beta_i \geq 1$ and $\beta_1 + \cdots + \beta_r \equiv 0 \pmod{k}$. Hence

$$2k - 1 \geq q^{\beta_1} + \cdots + q^{\beta_r} \geq \beta_1 q + \cdots + \beta_r q = (\beta_1 + \cdots + \beta_r)q \geq 2k.$$

This contradiction implies that $g'(k) \geq 2k - 1$. Now combine this with (14). $\square$

Proof of Theorem 2. Suppose $n \equiv i \pmod{p}$, $0 \leq i \leq p-1$. By Fermat's Theorem,

$$n \equiv i \equiv i^p \pmod{p}.$$

When $n > p^p$, we obtain

$$n = i^p + \underbrace{\frac{n - i^p}{p} + \cdots + \frac{n - i^p}{p}}_p. \quad \square$$

Proof of Theorem 3. By Fermat's Theorem and Lemma 1, there exist two positive integers x and y , $1 \leq x, y \leq 2p$, such that

$$n \equiv x^2 + y^2 \equiv x^{2p} + y^{2p} \pmod{2p}.$$

When $n > (2p)^{2p} + (2p)^{2p}$, we obtain

$$n = x^{2p} + y^{2p} + \underbrace{\frac{n - x^{2p} - y^{2p}}{2p} + \cdots + \frac{n - x^{2p} - y^{2p}}{2p}}_{2p}. \quad \square$$

Proof of Theorem 4. By [4], we have $g(4, 4p) = 3$ for $p \geq 7$, i.e., there exist three positive integers x, y and z , $1 \leq x, y, z \leq 4p$, such that

$$n \equiv x^4 + y^4 + z^4 \pmod{4p},$$

for any $n \in \mathbb{Z}_+$. By Fermat's Theorem, we have

$$n \equiv x^4 + y^4 + z^4 \equiv x^{4p} + y^{4p} + z^{4p} \pmod{4p}.$$

When $n > (4p)^{4p} + (4p)^{4p} + (4p)^{4p}$, we obtain

$$n = x^{4p} + y^{4p} + z^{4p} + \underbrace{\frac{n - x^{4p} - y^{4p} - z^{4p}}{4p} + \cdots + \frac{n - x^{4p} - y^{4p} - z^{4p}}{4p}}_{4p}. \quad \square$$

Remark 2. By [4], we have $g(3, 3^2) = 4$, $g(5, 5^2) = 3$, with which we deduce $G'(3^2) \leq 3^2 + 4 = 13$, $G'(5^2) \leq 5^2 + 3 = 28$. Also, from [4] we know that $g(4, 4) = 3$, $g(4, 5) = 4$, so $g(4, 20) = 4$, from which we obtain $G'(20) \leq 20 + 4 = 24$.

Proof of Theorem 5. If $p \equiv 1 \pmod{3}$, then there exist two positive integers x, y , $(x, y) = 1$ such that

$$p = x^2 + xy + y^2.$$

If $p \equiv \pm 1 \pmod{8}$, by Lemma 2, there exists $(x, y) \in \mathbb{Z}_+ \times \mathbb{Z}_+$, $x > 2y$, $(x, y) = 1$, such that

$$p = x^2 - 2y^2.$$

Making the transformation

$$\begin{cases} x = 2a + b, \\ y = a, \end{cases}$$

we obtain

$$p = (2a + b)^2 - 2a^2 = 2a^2 + 4ab + b^2. \quad \square$$

Proof of Theorem 6. By the proof of Lemma 3, if the inverse transformations (11) satisfy one of the following

$$(15) \quad \begin{cases} x - y > 0, \\ 3y - 2x > 0, \end{cases} \quad \begin{cases} 2x - y > 0, \\ 7y - 13x > 0, \end{cases} \quad \begin{cases} 19x - 9y > 0, \\ y - 2x > 0, \end{cases}$$

$$\begin{cases} 26x - 11y > 0, \\ 3y - 7x > 0, \end{cases} \quad \begin{cases} 17x - 7y > 0, \\ 19y - 46x > 0, \end{cases}$$

then $a > 0$ and $b > 0$. But having (x, y) satisfies one of (15), which is equivalent to (6). $\square$

Proof of Theorem 7. For any positive integer $n > 1$, there exists a prime p such that

$$n = pn_1,$$

where $n_1 \in \mathbb{Z}_+$. If $p = 2$, then $2 = 3 - 9 + 8$, and $3 \times (-9) \times 8 = (-6)^3$, $\gcd(3, -9, 8) = 1$. If p is an odd prime, then it is easy to deduce that p satisfies one of the four congruences:

$$p \equiv 1 \pmod{3}, \quad p \equiv \pm 1 \pmod{8}, \quad p \equiv 5 \pmod{24}, \quad p \equiv 11 \pmod{24}.$$

By Theorem 5, Lemma 3 and Lemma 4, there exist $(x, z) \in \mathbb{Z}_+ \times \mathbb{Z}_+$, $(y, m) \in \mathbb{Z} \times \mathbb{Z}$ and $(x, y, z) = 1$ such that

$$p = x + y + z$$

and

$$xyz = m^3.$$

Then

$$n = pn_1 = n_1x + n_1y + n_1z$$

and

$$n_1x \cdot n_1y \cdot n_1z = (n_1m)^3$$

$$(n_1x, n_1y, n_1z) = n_1(x, y, z) = n_1 < n. \quad \square$$

Proof of Theorem 8. It is obvious that we only need to discuss the special case when n is a prime p . If $p = 2$, then $2 = 6 - 3 + 8 - 9$, and $6 \times (-3) \times 8 \times (-9) = 6^4$, $\gcd(6, -3, 8, 9) = 1$. If p is an odd prime, then it is easy to deduce that p satisfies one of the four congruences:

$$p \equiv 1 \pmod{3}, \quad p \equiv \pm 1 \pmod{8}, \quad p \equiv 3 \pmod{8}, \quad p \equiv 5 \pmod{24}.$$

When $p = 3k + 1$, $p = 8k + 1$ or $8k + 3$, $p = 8k - 1$, $p = 24k + 5$, there exist two positive integers x, y , $(x, y) = 1$ such that [6]

$$p = x^2 + 3y^2 = 4x^2 - 3x^2 + 12y^2 - 9y^2, \quad (4x^2, -3x^2, 12y^2, -9y^2) = 1,$$

$$p = x^2 + 2y^2 = 2x^2 - x^2 + 4y^2 - 2y^2, \quad (2x^2, -x^2, 4y^2, -2y^2) = 1,$$

$$p = x^2 - 2y^2 = 2x^2 - x^2 + 2y^2 - 4y^2, \quad (2x^2, -x^2, 2y^2, -4y^2) = 1,$$

$$p = 2x^2 - 3y^2 = 6x^2 - 4x^2 + 6y^2 - 9y^2, \quad (6x^2, -4x^2, 6y^2, -9y^2) = 1,$$

respectively. $\square$

ACKNOWLEDGEMENTS

The authors thank the referee for his careful reading of the manuscript and for making it more readable.

REFERENCES

- [1] Alaca S. and Williams K., Introductory algebraic number theory. CUP, Cambridge (2004). MR2031707 (2005d:11152)
- [2] Ellison W. J., Waring's problem, Amer. Math. Monthly. 78, 10-36 (1971). MR0414510 (54:2611)
- [3] Nagell T., Introduction to Number Theory. pp. 205-206. Wiley, New York (1951). MR0043111 (13:207b)
- [4] Small C., Waring's problem mod n . Amer. Math. Monthly. 84, 12-25 (1977); Solution of Waring's problem mod n . Amer. Math. Monthly. 84, 356-359 (1977). MR0424734; MR0439787
- [5] Vaughan R. C. and Wooley T. D., Waring's Problem: Surveys in Number Theory, A. K. Peters, pp. 285-324, edited by M. A. Bennett et al. (2003).
- [6] Weisstein E. W., Prime representation. From MathWorld, A Wolfram Web Resource. <http://mathworld.wolfram.com/PrimeRepresentation.html>

DEPARTMENT OF MATHEMATICS, ZHEJIANG UNIVERSITY, HANGZHOU, 310027, CHINA
E-mail address: `txcai@zju.edu.cn`

DEPARTMENT OF MATHEMATICS, ZHEJIANG UNIVERSITY, HANGZHOU, 310027, CHINA
E-mail address: `chendeyi1986@126.com`