AN ALGORITHM TO COMPUTE RELATIVE CUBIC FIELDS

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ABSTRACT. Let K be an imaginary quadratic number field with class number 1. We describe a new, essentially linear-time algorithm, to list all isomorphism classes of cubic extensions L/K up to a bound X on the norm of the relative discriminant ideal. The main tools are Taniguchi's [18] generalization of Davenport-Heilbronn parametrisation of cubic extensions, and reduction theory for binary cubic forms over imaginary quadratic fields. Finally, we give numerical data for $K = \mathbb{Q}(i)$, and we compare our results with ray class field algorithm results, and with asymptotic heuristics, based on a generalization of Roberts' conjecture [19].

1. INTRODUCTION

Given a number field K , a positive integer n and $X > 0$, we define $\mathcal{F}_{K,n}(X)$ to be the set of isomorphism classes of extensions L/K such that

$$[L : K] = n \quad \text{and} \quad \mathcal{N}_{K/\mathbb{Q}}(\mathfrak{d}(L/K)) \leq X,$$

where $\mathfrak{d}(L/K)$ is the relative discriminant ideal of the extension L/K . Sets of this type may be enumerated algorithmically (usually over $\mathbb{Q}$) using the geometry of numbers, following the theorem of Hunter and Martinet [14]. Asymptotically, their cardinality as X tends to infinity is the subject of folklore conjectures, predicting for instance that it should be of the order of X , strikingly refined by Malle [13] who also fixes the Galois group of the Galois closure of L/K . Small values of n are of particular interest, since computer tests become comparatively easier and more theoretical results are available; see [2] for a recent survey.

In the present paper, we will focus on the case $n = 3$. Belabas's algorithm [1] lists all representatives of $\mathcal{F}_{\mathbb{Q},3}(X)$, in time $O_\varepsilon(X^{1+\varepsilon})$, essentially linear in the size of the output. We consider the problem of generalizing this algorithm to other base fields and we will solve it completely when K is imaginary quadratic, with class number 1. Our main result is as follows:

Theorem. *Let K be an imaginary quadratic number field with class number $h_K = 1$. There exists an algorithm which lists all cubic extensions in $\mathcal{F}_{K,3}(X)$ in time $O_\varepsilon(X^{1+\varepsilon})$, for all $\varepsilon > 0$.*

For an arbitrary fixed number field K , Datskovsky and Wright [8, Theorem I.1] proved that the cardinality of $\mathcal{F}_{K,3}(X)$ is asymptotic to a constant (depending on K) times X as $X \rightarrow \infty$. It follows:

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Corollary. *The algorithm runs in time essentially linear in the size of the output.*

The algorithm uses two main ingredients: 1) a general description of isomorphism classes of cubic extensions L/K as classes of suitable binary quadratic forms in $K[x, y]$ modulo a GL_2 action; 2) classical reduction theory in the special case where K is imaginary quadratic. Enumerating cubic extensions then amounts to enumerating integer points in an explicit fundamental domain, cut out by the extra condition $\mathcal{N}_{K/\mathbb{Q}}(\mathfrak{d}(L/K)) \leq X$.

It is interesting to compare our algorithm with the classical one, using class field theory (see Section 9.2.3 of [4]): the latter works in time $O_\varepsilon(X^{3/2+\varepsilon})$, unless we assume the Generalized Riemann Hypothesis to obtain $O_\varepsilon(X^{1+\varepsilon})$. So our algorithm has better *unconditional* complexity. Moreover, even assuming GRH, as we did in our PARI/GP implementation, the ray class field algorithm is slower than ours (see Section 6).

Section 2 is devoted to our two ingredients: Taniguchi's theorem [18], which generalizes the Davenport-Heilbronn bijection used by Belabas [1], and general facts about reduction theory for integral binary cubic forms over imaginary quadratic fields. In Section 3 we further assume that K has class number 1 and study the action of $\mathrm{GL}_2(\mathcal{O}_K)$ on binary cubic forms and obtain a specific fundamental domain, as well as explicit numerical bounds for the coefficients of reduced forms. Section 4 describes the core of our algorithm and Section 5 explores in detail the technical issues encountered during the implementation of the algorithm. Finally, Section 6 presents some timings for our PARI/GP implementation, over $K = \mathbb{Q}(i)$.

2. NOTATIONS AND PRELIMINARY RESULTS

In this section, we recall known results, needed for our algorithm.

2.1. Taniguchi's theorem.

Definition 2.1. Let $\mathcal{O}$ be a Dedekind domain, and let K be its quotient field.

- Let $\mathcal{C}(\mathcal{O})$ be the set of “cubic algebras”, that is, isomorphism classes of $\mathcal{O}$ -algebras that are projective of rank 3 as $\mathcal{O}$ -modules.
- For every fractional ideal $\mathfrak{a}$ of $\mathcal{O}$ we define

$$\mathcal{C}(\mathcal{O}, \mathfrak{a}) = \{R \in \mathcal{C}(\mathcal{O}) \mid \mathrm{St}(R) = \text{the ideal class of } \mathfrak{a}\},$$

where $\mathrm{St}(R) \in \mathrm{Cl}(\mathcal{O})$ is the Steinitz class of R , thus R is of the form $\omega_1 \mathcal{O} \oplus \omega_2 \mathcal{O} \oplus \omega_3 \mathfrak{a}$, for appropriate $\omega_1, \omega_2, \omega_3 \in \mathrm{Frac}(R) := R \otimes_{\mathcal{O}} K$. We define the discriminant ideal $\mathfrak{d}(R) = \mathrm{disc}(\omega_1, \omega_2, \omega_3) \mathfrak{a}^2$, where as usual $\mathrm{disc}(\omega_1, \omega_2, \omega_3) = \det \mathrm{Tr}_{\mathrm{Frac}(R)/K}(\omega_i \omega_j)$.

- Further, let

$$G_{\mathfrak{a}} = \left\{ \begin{pmatrix} \alpha \in \mathcal{O} & \beta \in \mathfrak{a}^{-1} \\ \gamma \in \mathfrak{a} & \delta \in \mathcal{O} \end{pmatrix} \mid \alpha\delta - \beta\gamma \in \mathcal{O}^\times \right\},$$

$$V_{\mathfrak{a}} = \{F = (a, b, c, d) \mid a \in \mathfrak{a}, b \in \mathcal{O}, c \in \mathfrak{a}^{-1}, d \in \mathfrak{a}^{-2}\}.$$

If $F \in V_{\mathfrak{a}}$, its discriminant $\mathrm{disc}(F) = b^2 c^2 - 27 a^2 d^2 + 18 abcd - 4ac^3 - 4b^3 d$ belongs to $\mathfrak{a}^{-2}$.

- We consider elements of $V_{\mathfrak{a}}$ as binary cubic forms, under the identification $(a, b, c, d) = ax^3 + bx^2y + cxy^2 + dy^3$ and we define a left-action of $G_{\mathfrak{a}}$ on $V_{\mathfrak{a}}$ by

$$M \cdot F = (\det M)^{-1} F(\alpha x + \beta y, \gamma x + \delta y),$$

$$\text{where } M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G_{\mathfrak{a}}.$$

The following theorem generalizes the Davenport-Heilbronn [9] theory, corresponding to the special case $\mathcal{O} = \mathbb{Z}$, to cubic algebras over an arbitrary Dedekind domain $\mathcal{O}$:

Theorem 2.2 (Taniguchi [18]). *There exists a canonical bijection between $\mathcal{C}(\mathcal{O}, \mathfrak{a})$ and $V_{\mathfrak{a}}/G_{\mathfrak{a}}$ such that the following diagram is commutative:*

$$\begin{array}{ccc} V_{\mathfrak{a}}/G_{\mathfrak{a}} & \longrightarrow & \mathcal{C}(\mathcal{O}, \mathfrak{a}) \\ \text{disc} \downarrow & & \downarrow \mathfrak{d} \\ \mathfrak{a}^{-2}/(\mathcal{O}^{\times})^2 & \xrightarrow{\times \mathfrak{a}^2} & \{\text{integral ideals of } \mathcal{O}\} \end{array},$$

where $\mathfrak{d}$ is the relative discriminant ideal map.

Remarks.

- A computation proves that the vertical “disc” is well defined. The other vertical map $\mathfrak{d}$ is well defined since an $\mathcal{O}$ -algebra isomorphism preserves the discriminant.
- We slightly changed the notation from Taniguchi’s paper, to keep consistent with the notation of the following sections (Taniguchi’s action $M * F$ is given by $(M^t) \cdot F$).

Corollary 2.3. *Let K be a number field with class number $h_K = 1$. Let $\mathcal{O} = \mathcal{O}_K$ be its ring of integers. Then Taniguchi’s bijection simplifies to a bijection between binary cubic forms with coefficients in $\mathcal{O}$ modulo $\text{GL}_2(\mathcal{O})$ and cubic $\mathcal{O}$ -algebras.*

To enumerate relative cubic extensions L/K , we shall select only the cubic $\mathcal{O}$ -algebras R which are both domains and integrally closed: those algebras are exactly the classes of the $\mathcal{O}_L$. The algebra R is a domain if and only if F is irreducible over K . Being integrally closed is a local property; it is equivalent to $\mathfrak{p}$ -maximality at all prime ideals $\mathfrak{p} \subset \mathcal{O}_K$ such that $\mathfrak{p}^2 \mid \mathfrak{d}(R)$ and this can be tested using Dedekind’s criterion [4, Theorem 2.4.8]. As was done in [1], it is possible to use sieve methods to control the complexity of this step by avoiding costly discriminant factorizations.

2.2. Fundamental domains in hyperbolic 3-space. In this section, we describe fundamental domains for the action of Bianchi groups on hyperbolic 3-space, which underlie the reduction of binary Hermitian and cubic forms (to be dealt with in the next two sections).

Definition 2.4. Let $\mathbb{H} = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$ be the algebra of quaternions, let $\mathbb{C} = \mathbb{R} + \mathbb{R}i$ be the subfield of complex numbers, and let

$$\begin{aligned}\mathcal{H}_3 &= \{z + tj \mid z \in \mathbb{C}, t \in \mathbb{R}_+^*\} \\ &= \{h = z + tj \mid h \in \mathbb{H}, \text{ such that the } k\text{-component is } 0, t > 0\},\end{aligned}$$

denote hyperbolic 3-space. We define the action of $\mathrm{PGL}_2(\mathbb{C})$ on $\mathcal{H}_3$ by $M \cdot (z + tj) = (z' + t'j)$, with

$$(2.1) \quad \begin{cases} z' = \frac{\rho^2 A \overline{C} + z A \overline{D} + \overline{z} B \overline{C} + B \overline{D}}{\rho^2 |C|^2 + z C \overline{D} + \overline{z} C \overline{D} + |D|^2}, \\ t' = \frac{|\det(M)| t}{\rho^2 |C|^2 + z C \overline{D} + \overline{z} C \overline{D} + |D|^2}, \end{cases}$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{PGL}_2(\mathbb{C})$ and $\rho^2 = |z|^2 + t^2$.

Remark. With the quaternion notations (and operations), this translates to the neater formula

$$M \cdot h = (Ah + B)(Ch + D)^{-1}.$$

Definition 2.5. Let $K = \mathbb{Q}(\sqrt{d_K})$ be an imaginary quadratic field of discriminant $d_K < 0$ and class number 1. We define

$$\begin{aligned}F_{\mathbb{Q}(i)} &= \left\{ z \in \mathbb{C} \mid 0 \leq \operatorname{Re}(z) \leq \frac{1}{2}, 0 \leq \operatorname{Im}(z) \leq \frac{1}{2} \right\}, \\ F_{\mathbb{Q}(\sqrt{-2})} &= \left\{ z \in \mathbb{C} \mid -1/2 \leq \operatorname{Re}(z) \leq \frac{1}{2}, 0 \leq \operatorname{Im}(z) \leq \frac{\sqrt{2}}{4} \right\}, \\ F_{\mathbb{Q}(\sqrt{-3})} &= \left\{ z \in \mathbb{C} \mid 0 \leq \operatorname{Re}(z) \leq \frac{1}{2}, -\frac{\sqrt{3}}{3} \operatorname{Re}(z) \leq \operatorname{Im}(z) \leq \frac{\sqrt{3}}{3} \operatorname{Re}(z) \right\}, \\ F_K &= \left\{ z \in \mathbb{C} \mid |\operatorname{Re}(z)| \leq 1/2, 0 \leq \operatorname{Im}(z) \leq \sqrt{|d_K|}/4 \right\}, \text{ when } d_K \neq -2, -3, -4.\end{aligned}$$

Moreover, we set

$$\mathcal{B}_K = \left\{ z + tj \in \mathcal{H}_3 \mid z \in F_K \text{ and } |z|^2 + t^2 \geq 1 \right\}.$$

Let

$$\begin{aligned}\mathcal{F}_{\mathbb{Q}(i)} &= \{z + tj \in \mathcal{B}_K \mid \operatorname{Re}(z) \leq \operatorname{Im}(z) \text{ if } z + tj \in \partial \mathcal{B}_K\}, \\ \mathcal{F}_{\mathbb{Q}(\sqrt{-2})} &= \{z + tj \in \mathcal{B}_K \mid \operatorname{Re}(z) \geq 0 \text{ if } z + tj \in \partial \mathcal{B}_K\}, \\ \mathcal{F}_{\mathbb{Q}(\sqrt{-3})} &= \{z + tj \in \mathcal{B}_K \mid \operatorname{Im}(z) \geq 0 \text{ if } z + tj \in \partial \mathcal{B}_K\},\end{aligned}$$

where $\partial \mathcal{B}_K$ denotes the boundary of $\mathcal{B}_K$.

Finally, for K such that $d_K \neq -2, -3, -4$, we define

$$\mathcal{F}_K = \left\{ z + tj \in \mathcal{B}_K \mid \begin{array}{l} \operatorname{Re}(z) \leq 1/4 \text{ if } \left(\operatorname{Im}(z) = \sqrt{|d_K|}/4 \text{ and } |\operatorname{Re}(z)| \leq |z|^2 + t^2 - 3/4 \right) \\ \operatorname{Re}(z) \geq 0 \text{ if } (|z|^2 + t^2 = 1 \text{ or } |\operatorname{Re}(z)| = 1/2 \text{ or } \operatorname{Im}(z) = 0) \end{array} \right\}.$$

Theorem 2.6. Let K be an imaginary quadratic number field of class number 1, let $\mathcal{O}$ be its maximal order, and let $\mathcal{F}_K$ be as defined above.

- (1) $\mathcal{F}_K$ is a fundamental domain for the action of $\mathrm{PGL}_2(\mathcal{O})$ on $\mathcal{H}_3$. No two points in $\mathcal{F}_K$ are $\mathrm{PGL}_2(\mathcal{O})$ -equivalent.

- (2) There exists a constant $t_K > 0$ such that $t \geq t_K$ for every $z + tj \in \mathcal{F}_K$. The value of t_K^2 is given in the following tables:

D	1	2	3	7	11
t_K^2	1/2	1/4	2/3	3/7	2/11

D	19	43	67	163
t_K^2	2/19	2/43	2/67	2/163

Proof.

- (1) Since $\mathrm{PGL}_2(\mathcal{O})/\mathrm{PSL}_2(\mathcal{O}) \simeq \mathcal{O}^\times/(\mathcal{O}^\times)^2$ our hypotheses imply that its cardinality is 2. Using the well-known fundamental domains for the $\mathrm{PSL}_2(\mathcal{O})$ action on $\mathcal{H}_3$ (see for example [10]) we construct $\mathcal{B}_K$.

It remains to show that points on the boundary of the fundamental domain are counted only once (modulo $\mathrm{PSL}_2(\mathcal{O})$) in $\mathcal{F}_K$.

The action of $\mathrm{PGL}_2(\mathcal{O})$ on $\mathcal{H}_3$ is generated by the following matrices:

- (a) Either $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ acting on points $z + jt$ such that $|z|^2 + t^2 = 1$.
 (b) Or translations of the form $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$, for an appropriate $\alpha \in \mathcal{O}$.

A tedious computation yields the result.

- (2) See [5] and [20] for details. □

Remark. Thanks to Definition 2.5 and Theorem 2.6, we have explicit bounds for z and t -components of elements in a fundamental domain of $\mathcal{H}_3$ modulo $\mathrm{GL}_2(\mathcal{O})$, when $\mathcal{O}$ is principal. Unfortunately, when $h_K \neq 1$, we do not have a lower bound for t (there are points in the boundary of the fundamental domain such that $t = 0$), and this will prevent us from bounding the coefficients of reduced forms. This is the reason why we will restrict our work to the class number 1 case.

2.3. Reduction of binary Hermitian forms. Before tackling cubic forms, we recall the classical reduction theory of binary Hermitian forms modulo $\mathrm{GL}_2(\mathcal{O})$, where $\mathcal{O}$ is the maximal order of an imaginary quadratic field.

Definition 2.7. Let $\mathcal{P}$ be the set of *positive definite* binary quadratic Hermitian forms over $\mathbb{C}$; in other words,

$$\mathcal{P} = \{(P, Q, R) : P, R \in \mathbb{R}^+, Q \in \mathbb{C}, \mathrm{disc}(P, Q, R) < 0\},$$

where (P, Q, R) denotes the binary quadratic Hermitian form

$$H(x, y) = P|x|^2 + Q\bar{x}y + \overline{Q}x\bar{y} + R|y|^2,$$

of discriminant $\mathrm{disc}(H) := -\Delta = |Q|^2 - PR$.

The group $\mathrm{PGL}_2(\mathcal{O})$ acts on $\mathcal{P}$ via

$$M \cdot H(x, y) = H(\alpha x + \beta y, \gamma x + \delta y), \quad \text{where } M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{PGL}_2(\mathcal{O}).$$

Remark. It is customary to identify the Hermitian form

$$\begin{pmatrix} \bar{x} & \bar{y} \end{pmatrix} \begin{pmatrix} P & Q \\ \overline{Q} & R \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

with the Hermitian matrix $H = \begin{pmatrix} P & Q \\ \overline{Q} & R \end{pmatrix}$; the $\mathrm{PGL}_2(\mathcal{O})$ action is then

$$M \cdot H = M^* \times H \times M, \quad \text{where } M^* = (\overline{M})^t.$$

Lemma 2.8. *Let $\tilde{\Phi} : \mathcal{P}/\mathbb{R}_+^* \rightarrow \mathcal{H}_3$ be defined by:*

$$(2.2) \quad \Phi((P, Q, R)) = -\frac{Q}{P} + \frac{\sqrt{\Delta}}{P}j.$$

$\tilde{\Phi}$ is a bijection which commutes with the action of $\mathrm{PGL}_2(\mathcal{O})$.

This defines natural representatives for orbits of Hermitian forms modulo $\mathrm{PGL}_2(\mathcal{O})$. Namely

Definition 2.9. Let $H \in \mathcal{P}$ be a binary Hermitian form. H is called *reduced* if and only if $\tilde{\Phi}(H) \in \mathcal{F}_K$.

Lemma 2.10. *Let $(P, Q, R) = P|x|^2 + Qx\overline{y} + \overline{Q}\overline{x}y + R|y|^2$ be a reduced Hermitian form in $\mathcal{P}$, with discriminant $-\Delta = |Q|^2 - PR$. We have*

$$(2.3) \quad P \leq \frac{\sqrt{\Delta}}{t_K},$$

$$(2.4) \quad |Q|^2 \leq c_K P^2,$$

and

$$(2.5) \quad PR \leq \left(1 + \frac{c_K}{t_K^2}\right) \Delta,$$

where c_K is a constant depending only on the number field K , defined as follows:

$$c_K = \begin{cases} 1/2 & \text{if } K = \mathbb{Q}(i), \\ 7/12 & \text{if } K = \mathbb{Q}(\sqrt{-3}), \\ \left(\frac{1+|d_K|}{4}\right) & \text{otherwise.} \end{cases}$$

Proof. For (2.3) just recall that $t = \sqrt{\Delta}/P$ by the definition of $\tilde{\Phi}$ in (2.2) and $t \geq t_K$.

Thanks to the bounds on $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ given in the description of the fundamental domain $\mathcal{F}_K$ (in Definition 2.5) we get

- $0 \leq |\mathrm{Re}(Q)| \leq P/2$, $0 \leq \mathrm{Im}(-Q) \leq 1/2$, and so $|Q|^2 \leq P^2/2$ when $K = \mathbb{Q}(i)$;
- $0 \leq \mathrm{Re}(-Q) \leq P/2$, $-\sqrt{3}/6P \leq \mathrm{Im}(-Q) \leq \sqrt{3}/3P$ and then $|Q|^2 \leq 7/12P^2$, when $K = \mathbb{Q}(\sqrt{-3})$;
- $0 \leq \mathrm{Re}(-Q) \leq P/2$, $0 \leq \mathrm{Im}(-Q) \leq \frac{\sqrt{|d_K|}}{2}P$ and then $|Q|^2 \leq \left(\frac{1+|d_K|}{4}\right)P^2$.

In all cases we have

$$|Q|^2 \leq c_K P^2 \leq c_K \frac{\Delta}{t_K^2}.$$

Recalling that $PR - |Q|^2 = \Delta$, we obtain

$$PR \leq \left(1 + \frac{c_K}{t_K^2}\right) \Delta. \quad \square$$

2.4. Julia's covariant. From now on, let K be an imaginary quadratic field, let $\mathcal{O}$ be its ring of integers, and let $V_{\mathcal{O}}$ be the set of binary cubic forms in $\mathcal{O}[x, y]$. We want to define a canonical representative (or *reduced form*) in each orbit $\mathrm{GL}_2(\mathcal{O}) \cdot F$, $F \in V_{\mathcal{O}}$.

Definition 2.11. We consider binary cubic forms in $V_{\mathcal{O}}$,

$$F(x, y) = ax^3 + bx^2y + cxy^2 + dy^3, \quad a, b, c, d \in \mathcal{O}$$

modulo the action of $\mathrm{GL}_2(\mathcal{O})$ given by

$$M \cdot F = (\det(M))^{-1} F(Ax + By, Cx + Dy), \quad \text{for each } M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}).$$

Remark. As we saw in Corollary 2.3, this is the restriction of the action used in Taniguchi's theorem, when $h_K = 1$.

Julia [11] gives us a covariant for this action:

Definition 2.12. Let $F \in V_{\mathcal{O}}$ be irreducible over K , factoring over $\mathbb{C}$ as $F(x, y) = a(x - \alpha_1 y)(x - \alpha_2 y)(x - \alpha_3 y)$, with $a \neq 0$. We associate to F the positive definite binary Hermitian form

$$H_F(x, y) = t_1^2 |x - \alpha_1 y|^2 + t_2^2 |x - \alpha_2 y|^2 + t_3^2 |x - \alpha_3 y|^2,$$

where

$$t_i^2 = |a|^2 |\alpha_j - \alpha_k|^2, \quad i, j, k \text{ pairwise distinct.}$$

The following three lemmas follow from a direct computation:

Lemma 2.13. *We have*

$$H_F(x, y) = P|x|^2 + Q\overline{x}y + \overline{Q}x\overline{y} + R|y|^2,$$

where

$$\begin{cases} P = t_1^2 + t_2^2 + t_3^2 \in \mathbb{R}^+, \\ Q = -(\alpha_1 t_1^2 + \alpha_2 t_2^2 + \alpha_3 t_3^2) \in \mathbb{C}, \\ R = |\alpha_1|^2 t_1^2 + |\alpha_2|^2 t_2^2 + |\alpha_3|^2 t_3^2 \in \mathbb{R}^+. \end{cases}$$

Lemma 2.14. *We have*

$$(2.6) \quad (t_1 t_2 t_3)^2 = |a|^2 |\mathrm{disc}(F)|.$$

Lemma 2.15. *Let $\Delta = -\mathrm{disc}(H_F) = PR - |Q|^2$ and $D = \mathrm{disc}(F)$. Then*

$$(2.7) \quad \Delta = 3|D|.$$

Proposition 2.16. *The application which sends F to H_F is covariant, i.e.,*

$$H_{M \cdot F} = M \cdot H_F,$$

for all $M \in \mathrm{GL}_2(\mathcal{O})$.

Thanks to this property we can translate our problem of defining a unique reduced F to the problem of finding a unique reduced covariant H_F plus some extra conditions as we will see in Section 3.2.

Definition 2.17 (Julia reduction). Let $F = (a, b, c, d) \in V_{\mathcal{O}}$ be a binary cubic form with coefficients in $\mathcal{O}$. We say that F is Julia-reduced (modulo $\mathrm{GL}_2(\mathcal{O})$) if its covariant H_F is reduced, in the sense of Definition 2.9.

3. REDUCTION OF BINARY CUBIC FORMS

3.1. Bounds for binary cubic forms. Let F be a binary cubic form and let H_F be its covariant hermitian form. Starting from bounds on H_F coefficients it is possible to directly bound F coefficients and then to loop over all reduced binary cubic forms in time $\tilde{O}(X)$, but the coefficients involved in the complexity of this algorithm are quite big (see [15] for details), so we chose another method, suggested by John Cremona, which can be found in [6, 21, 7].

Definition 3.1. For any $k \in \mathcal{O}$, we note $\tau_k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$.

Definition 3.2. For any $a_0 \in \mathcal{O}$, we fix once and for all a system of representatives $\mathcal{P}_{a_0}$ for $\mathcal{O}/3a_0\mathcal{O}$. This is a finite set with $9|a_0|^2$ elements.

Definition 3.3. Let F_K be as in Definition 2.5.

We define $\mathcal{P}_K$ to be a fundamental region for $\mathbb{C}/\mathcal{O}$ such that $F_K \subset \mathcal{P}_K$.

Proposition 3.4. *Let $F = (a, b, c, d)$ be a binary cubic form. There exists a unique $k \in \mathcal{O}$ such that τ_k sends F to an equivalent binary cubic form $F_0 = (a_0, b_0, c_0, d_0)$ such that $b_0 \in \mathcal{P}_{a_0}$. We will call this F_0 τ -reduced.*

Moreover, if F is Julia-reduced, then we also have the following properties:

$$(3.1) \quad \begin{aligned} |a_0| &\leq 3^{-3/4} t_K^{-3/2} D^{1/4}, \\ |c_0| &\leq \frac{|b_0|^2 + c_H D^{1/2}}{3|a_0|}, \\ \text{either } |d_0 - x_1| &\leq \frac{X^{1/4}}{\sqrt{|A|}} \text{ or } |d_0 - x_2| \leq \frac{X^{1/4}}{\sqrt{|A|}}, \end{aligned}$$

where $c_H = 3^{1/2} 2^{-1/3} t_K^{-1}$, $A = -27a_0^2$, $B = 18a_0b_0c_0 - 4b_0^3$, $C = b_0^2c_0^2 - 4a_0c_0^3$, and we call x_1 and x_2 the roots of the quadratic polynomial $Ax^2 + Bx + C$.

Proof. As regards the first assertion, just remark that τ_k sends (a, b, c, d) to

$$(a_0, b_0, c_0, d_0) = (a, b + 3ak, 3ak^2 + 2bk + c, ak^3 + bk^2 + ck + d).$$

Now, assume that F is Julia reduced.

Let us consider the seminvariants associated to F_0 :

$$P_H = b_0^2 - 3a_0c_0 \quad \text{and} \quad U_H = 2b_0^3 + 27a_0^2d_0 - 9a_0b_0c_0$$

(note that P_H is the first coefficient of the Hessian of F_0 , but it is not in general equal to P_0 , the first coefficient of the covariant associated to F_0).

τ_k leaves P_H and U_H unchanged and, as shown in Womack's thesis [21], we have

$$(3.2) \quad |a_0| \leq 3^{-3/4} t_K^{-3/2} X^{1/8}$$

and

$$|U_H| \leq 3^{3/4} t_K^{-3/2} X^{3/8}$$

so from the syzygy

$$4P_H^3 = U_H^2 + 27 \operatorname{disc}(F_0) a_0^2$$

we obtain

$$(3.3) \quad P_H \leq c_H X^{1/4},$$

where $c_H = 3^{1/2} 2^{-1/3} t_K^{-1}$, and we easily obtain the bound for $|c_0|$.

Finally, since $\operatorname{disc}(F_0) = Ad_0^2 + Bd_0 + C$ and $|\operatorname{disc}(F_0)| \leq \sqrt{X}$ we have

$$|d_0 - x_1| |d_0 - x_2| \leq \sqrt{X} / |A|,$$

and this inequality implies that $|d_0 - x_1|$ and $|d_0 - x_2|$ cannot both be bigger than $\frac{X^{1/4}}{\sqrt{|A|}}$. $\square$

Corollary 3.5. *It is possible to list all the reduced binary cubic forms (a, b, c, d) (modulo $\text{GL}_2(\mathcal{O})$), with $\mathcal{N}(\text{disc}(F)) \leq X$ in time $O(X^{1+\varepsilon})$, for all $\varepsilon > 0$.*

Proof. The number of τ -reduced binary cubic forms (a_0, b_0, c_0, d_0) which are equivalent to Julia-reduced ones (i.e. satisfying all properties enumerated in Proposition 3.4) is

$$N \ll \sum_{|a_0| \ll X^{1/8}} \sum_{b_0 \in \mathcal{P}_{a_0}} \sum_{c_0 \ll X^{1/4}/|a_0|} \sum_{\substack{|d_0 - x_1| \ll X^{1/4}/|a_0| \\ \text{or } |d_0 - x_2| \ll X^{1/4}/|a_0|}} 1.$$

Thus

$$N \ll \sum_{|a_0| \ll X^{1/8}} |a_0|^2 \cdot \frac{X^{1/2}}{|a_0|^2} \cdot \frac{X^{1/2}}{|a_0|^2} = X \cdot \sum_{|a_0| \ll X^{1/8}} \frac{1}{|a_0|^2}$$

and

$$\sum_{|a_0| \ll X^{1/8}} \frac{1}{|a_0|^2} \ll \sum_{n=1}^{X^{1/4}} \frac{\#\{a_0 \in \mathcal{O} : |a_0|^2 = n\}}{n}.$$

Since $\#\{a_0 \in \mathcal{O} : |a_0|^2 = n\} = O(n^\varepsilon) = O(X^\varepsilon)$ for all $\varepsilon > 0$, and $\sum_{n=1}^{X^{1/4}} \frac{1}{n}$ is $O(\log(D))$, we can conclude. $\square$

Proposition 3.6. *Let F be a Julia-reduced binary cubic form, let F_0 be the corresponding τ -reduced form, and let $H_{F_0} = (P_0, Q_0, R_0)$ be the binary Hermitian form associated to F_0 . Then $F = \tau_{-k} \cdot F_0$, for a unique $k \in \mathcal{O}$.*

Proof. The action of τ_k sends $H_F = (P, Q, R)$ to $H_{F_0} = (P_0, Q_0, R_0)$ such that $P_0 = P$ and $Q_0 = Q + kP$. Dividing by P , we obtain $z_0 = z - k$, with $z \in F_K \subset \mathcal{P}_K$, but this uniquely determines k , so we can conclude. $\square$

Algorithm 3.7 (τ -reduction). *Let K be an imaginary quadratic number field of class number 1. This algorithm loops over all τ -reduced binary cubic forms $F_0 = (a_0, b_0, c_0, d_0)$ satisfying the conditions in Proposition 3.4 with $\mathcal{N}(\text{disc}(F_0)) \leq X$, and associates the equivalent binary cubic form $F = (a, b, c, d) = \tau_{-k} \cdot F_0$, such that $z \in F_K$ (as explained in Proposition 3.6).*

For each a_0, b_0, c_0, d_0 in $\mathcal{O}$ satisfying the following properties:

- $|a_0| \leq \left(\frac{1}{t_K \sqrt{3}}\right)^{3/2} X^{1/8}$,
- b_0 belongs to $\mathcal{P}_{a_0}$,
- $|c_0| \leq \frac{|b_0|^2 + c_H X^{1/4}}{3|a_0|}$,
- either $|d_0 - x_1| \leq X^{1/4}/\sqrt{|A|}$ or $|d_0 - x_2| \leq X^{1/4}/\sqrt{|A|}$.

Do the following operations:

- (1) compute the first two coefficients P_0, Q_0 of the covariant H_{F_0} of the cubic form $F_0 = (a_0, b_0, c_0, d_0)$.
- (2) Compute k such that $z_0 + k \in \mathcal{P}_K$ ($z_0 = -Q_0/P_0$).
- (3) Compute $F = (a, b, c, d) = \tau_{-k}(a_0, b_0, c_0, d_0)$.

3.2. Automorphism matrices. In this section we are going to study automorphism matrices for binary hermitian forms.

Proposition 3.8. *Let $F = (a, b, c, d)$ be Julia-reduced. Let $H = H_F$, and $\Delta = PR - |Q|^2$.*

Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{GL}_2(\mathcal{O})$ such that $M \cdot H = H$. Then we have the following bounds on the coefficients of M :

$$(3.4) \quad |A|^2 \leq \frac{PR}{\Delta}, \quad |C| \leq \frac{P}{\sqrt{\Delta}}, \quad |D|^2 \leq \frac{PR}{\Delta}$$

and

$$(3.5) \quad \begin{cases} |B| \leq \frac{PR}{\Delta} + 1 & \text{if } B \neq 0, \\ |B| \leq 2\sqrt{c_K} & \text{if } B = 0. \end{cases}$$

Proof. Let us write $H(x, y) = P|x|^2 + Q\bar{x}y + \bar{Q}x\bar{y} + R|y|^2$. We have

$$(3.6) \quad PH(x, y) = |xP + yQ|^2 + \Delta|y|^2,$$

$$(3.7) \quad RH(x, y) = |Ry + \bar{Q}x|^2 + \Delta|x|^2.$$

Thanks to formula (3.6) we can give upper bounds for $|A|$, $|B|$, and $|D|$. Let us write more explicitly the relation $M \cdot H = H$:

$$(3.8) \quad \begin{aligned} M \cdot H &= \begin{pmatrix} \bar{A} & \bar{C} \\ \bar{B} & \bar{D} \end{pmatrix} \begin{pmatrix} P & Q \\ \bar{Q} & R \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \\ &= \begin{pmatrix} |A|^2P + \bar{A}CQ + A\bar{C}\bar{Q} + |C|^2R & \bar{A}BP + \bar{A}DQ + B\bar{C}\bar{Q} + \bar{C}DR \\ A\bar{B}P + C\bar{B}Q + A\bar{D}\bar{Q} + C\bar{D}R & |B|^2P + \bar{B}DQ + B\bar{D}\bar{Q} + |D|^2R \end{pmatrix} \\ &= \begin{pmatrix} H(A, C) & \dots \\ \dots & H(B, D) \end{pmatrix}. \end{aligned}$$

By imposing this matrix to be equal to M we have

$$\begin{aligned} |AP + CQ|^2 + \Delta|C|^2 &= P^2 \Rightarrow |C| \leq \frac{P}{\sqrt{\Delta}}, \\ |BP + DQ|^2 + \Delta|D|^2 &= PR \Rightarrow |D|^2 \leq \frac{PR}{\Delta}, \\ |CR + A\bar{Q}|^2 + \Delta|A|^2 &= PR \Rightarrow |A|^2 \leq \frac{PR}{\Delta}. \end{aligned}$$

When $C = 0$ the third equation becomes

$$A\bar{B}P + A\bar{D}\bar{Q} = \bar{Q},$$

with $|A| = |D| = 1$, so it is easy to check that $|A\bar{C}|P \leq 2Q \leq 2\sqrt{c_K}P$ and we obtain the formula. Finally, when $C \neq 0$, since $|AD - BC| = 1$ we get

$$|B| \leq \frac{1 + |AD|}{|C|}$$

and we easily conclude. $\square$

The bounds of the previous proposition are completely explicit when $h_K = 1$, since we know t_K and c_K .

Definition 3.9. Let $M \in \mathrm{PGL}_2(\mathcal{O})$. We define

$$S(M) = \{H \in \mathcal{P}/\mathbb{R}_+^* \mid M \cdot H = H \text{ and } H \text{ reduced}\},$$

that is, the set of reduced binary Hermitian forms which are stabilized by the action of M .

The following algorithm lists the finite set of automorphism matrices. It needs to be run only once for each of our 9 imaginary quadratic fields of class number 1.

Algorithm 3.10. *Computes the set $\mathcal{M}$ of all matrices M stabilizing some reduced binary Hermitian form, and for each M outputs also the corresponding set $S(M)$.*

Set $\mathcal{M} = \emptyset$.

For each triple (A, C, D) satisfying the bounds of Proposition 3.8, do the following operations:

- (1) For each $B \in \mathcal{O}$ such that $|AD - BC| = 1$ (if $C = 0$, take only the set $|B| \leq 2\sqrt{c_K}$), let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and do the following.
- (2) Consider the following 4×4 matrix, with coefficients in $\mathcal{O}$:

$$W(M) = \begin{pmatrix} (|A|^2 - 1) & \overline{AC} & A\overline{C} & |C|^2 \\ \overline{AB} & (\overline{AD} - 1) & B\overline{C} & \overline{CD} \\ A\overline{B} & C\overline{B} & (A\overline{D} - 1) & C\overline{D} \\ |B|^2 & \overline{BD} & B\overline{D} & (|D|^2 - 1) \end{pmatrix}.$$

- (3) Compute the rank r of $W(M)$ (over the field K).
- (4) If $r = 1$ or $r = 4$, **skip** to the following quadruple (A, B, C, D) .
- (5) If $r = 0$ **output** $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and $S(M) = \{H \in \mathcal{P}/\mathbb{R}_+^* \mid H \text{ reduced}\}$. $\mathcal{M} = \mathcal{M} \cup \{M\}$.
- (6) If $r = 2$ or $r = 3$, set $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, compute the set $S(M) = \{H = (P, Q, R) \in \mathcal{P}/\mathbb{R}_+^* \mid W \cdot (P, Q, \overline{Q}, R)^t = 0 \text{ and } (P, Q, R) \text{ reduced}\}$. If $S(M) \neq \emptyset$, **output** $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and $S(M)$. $\mathcal{M} = \mathcal{M} \cup \{M\}$.

Output $\mathcal{M}$.

Remarks.

- We could also loop only on A, D and replace step (1) by:
 - (1) Solve $|AD - BC| = 1$ for $B, C \in \mathcal{O}$. This time BC belongs to an explicit finite set, and we enumerate divisors.
- It is possible to write (once for all) explicit conditions to associate to any binary Hermitian form H its set of automorphism matrices, just looking at the sets $S(M)$ computed in Algorithm 3.10
- For an example of application of the above algorithm, Appendix A contains the list of all automorphism matrices for $K = \mathbb{Q}(i)$ and the corresponding conditions on binary Hermitian forms.

Remark. Running the algorithm on all the 9 possible number fields we noticed a property holding for all $K \neq \mathbb{Q}(\sqrt{-3})$:

- For each matrix M found at step 6 (that is, they are not trivial automorphisms) $W(M)$ has rank 2 and $S(M)$ is a subset of the boundary of the fundamental domain.

In the case $K = \mathbb{Q}(\sqrt{-3})$ we have explicit counterexamples.

The proof of Algorithm 3.10 is given by the following proposition.

Proposition 3.11. *Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{PGL}_2(\mathcal{O})$ belong to the stabilizer of H_F , where H_F is the Hessian of some reduced cubic form F . If r is the rank of the matrix W constructed in the above algorithm, then*

- $r = 0$ if and only if $B = C = 0$ and $A = D$ are units. Then M is an automorphism for all Hermitian quadratic forms in $\mathcal{F}$.
- $r = 1$ is impossible.
- $r = 2$ or $r = 3$ then M is an automorphism for some linear subspace of $\mathcal{P}$, defined by explicit equations in the variables $P, Q, \overline{Q}, R$.
- $r = 4$ is impossible.

Proof. The condition $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Aut}(H)$ translates to the linear system $W(M) \cdot X = 0$, with $X = (P, Q, \overline{Q}, R)^t$.

- If $r = 4$, the only solution of the system is $(0, 0, 0, 0)$, but this is not allowed since $P, R > 0$.
- Assume that $r \leq 1$: the matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ has rank 2 so the two 2 by 2 matrices on the lower-left and upper-right corners of $W(M)$ have rank 2 unless $B = C = 0$. In this case $W(M)$ is diagonal

$$\begin{pmatrix} |A|^2 - 1 & & & \\ & \overline{A}D - 1 & & \\ & & A\overline{D} - 1 & \\ & & & |D|^2 - 1 \end{pmatrix}.$$

Since $B = C = 0$, and $AD - BC$ is a unit, we must have $|A| = |D| = 1$, so this matrix has either rank 2 or 0 (when $\overline{A}D = A\overline{D} = 1$). $\square$

4. THE ALGORITHM

Algorithm 4.1. *Given a bound $X = D^2$, output the list of reduced binary cubic forms modulo $\mathrm{GL}_2(\mathcal{O})$, such that $\mathcal{N}(\mathrm{disc}(F)) \leq X$.*

Use sub-Algorithm 3.7 to loop over quadruples $F = (a, b, c, d) \in \mathcal{O}^4$ satisfying all the properties in Section 3.4. Do the following operations:

- (1) Approximate the complex roots of F , $(\alpha_1, \alpha_2, \alpha_3)$ to a sufficient accuracy. Then approximate $H = H_F = (P, Q, R)$ the associated Hermitian form.
- (2) Check if H is in the fundamental domain modulo $\mathrm{PGL}_2(\mathcal{O})$ (i.e. it is reduced), (see Definition 2.5). In particular, if H_F is “near” to the boundary of the fundamental domain use Algorithm 5.2 (see below) to check exactly the boundary condition. If not **skip** to the following F .
- (3) Check whether F is irreducible in $K[x, y]$. If not **skip** to the following F .
- (4) Apply Dedekind criterion to check whether F describes a maximal ring. If not **skip** to the following F .
- (5) Apply sub-Algorithm 3.10 to compute $\mathcal{M}$, the set of all automorphism matrices for H .
- (6) Compute the set $\{M \cdot F \mid M \in \mathcal{M}\}$ and check if F is the minimal element of this set (for some order, for instance, the lexicographic one). If not **skip** to the following F .
- (7) **print** F .

Remarks.

- For the precision needed in step (1) refer to Appendix C of [15].
- In step (5), we compute a list of automorphs for F to decide whether F is minimal among the reduced forms in its orbit with respect to the lexicographic order (in this case F should be kept, otherwise not). Another way to deal with this problem would be to store all those F and then check $\mathrm{GL}_2(\mathcal{O})$ -equivalence once we have all the forms with a fixed discriminant D . The problem is that our algorithm does not output forms ordered by discriminant, so we could apply this test only at the end, and this would increase dramatically the space complexity. (Remember that we output the “good” binary cubic forms as we find them, so we do not keep in memory the list of representatives of cubic extensions).

5. IMPLEMENTATION PROBLEMS

5.1. Checking rigorously the boundary conditions. As the computation of P, Q, R involves floating point approximations of the complex roots of a polynomial in $\mathcal{O}[X]$, it will not give, of course, exact results. Those floating point computations will in general be sufficient to test whether the Hermitian form is strictly inside or outside the fundamental domain. But if it is very near the boundary (or worse *on* the boundary), this approach fails.

To get rid of this problem we use the following formulas:

$$(5.1) \quad P = -\frac{|b|^2}{|a|^2} + 3(|\alpha_1|^2 + |\alpha_2|^2 + |\alpha_3|^2),$$

$$(5.2) \quad Q = \frac{\bar{b}c}{|a|^2} + 3(\overline{\alpha_1}\alpha_2\alpha_3 + \alpha_1\overline{\alpha_2}\alpha_3 + \alpha_1\alpha_2\overline{\alpha_3}),$$

$$(5.3) \quad R = -\frac{|c|^2}{|a|^2} + 3(|\alpha_1|^2|\alpha_2|^2 + |\alpha_1|^2|\alpha_3|^2 + |\alpha_2|^2|\alpha_3|^2).$$

Now we consider $\alpha_1, \alpha_2, \alpha_3, \overline{\alpha_1}, \overline{\alpha_2}, \overline{\alpha_3}$ as algebraic numbers, and we let S be the set of the six permutations fixing the α_i , and acting as S_3 on the $\overline{\alpha_i}$. The polynomial

$$g_P = \prod_{\sigma \in S} (X - \sigma(\alpha_1\overline{\alpha_1} + \alpha_2\overline{\alpha_2} + \alpha_3\overline{\alpha_3}))$$

vanishes at $|\alpha_1|^2 + |\alpha_2|^2 + |\alpha_3|^2$, and its coefficients are symmetric in $(\alpha_1, \alpha_2, \alpha_3)$ and $(\overline{\alpha_1}, \overline{\alpha_2}, \overline{\alpha_3})$ independently. They can thus be expressed in terms of $(b/a, c/a, d/a)$ and $(\overline{b/a}, \overline{c/a}, \overline{d/a})$. The polynomial $f_P(X) = g_P\left(\frac{X}{3} - \frac{|b|^2}{3|a|^2}\right)$ vanishes at P and belongs to $K[X]$.

In the same way we can compute polynomials in $K[X]$ vanishing at $Q, R, \mathrm{Re}(Q)$ or $\mathrm{Im}(Q)$. Such polynomials are easily computed using a computer algebra system like Maple (and it is sufficient to compute them once for all).

We want to verify rigorously boundary conditions, for instance, $P = R$: if f_P and f_R have no common factor in $K[X]$, then $P \neq R$. But this is not enough: we also want to check whether $P < R$ or $P > R$, i.e., if the point we are testing is “inside” or “outside” the fundamental domain.

The following theorem of Mahler [12] provides the accuracy we need for our floating point computations:

Theorem 5.1 (Mahler). *Let $f = a_0x^m + a_1x^{m-1} + \cdots + a_m = a_0(x - \alpha_1) \cdots (x - \alpha_m)$ be a separable polynomial of degree $m \geq 2$, and let*

$$\Delta(f) = \min_{1 \leq i < j \leq m} |\alpha_i - \alpha_j|$$

be the minimal distance between two distinct roots of f . Then

$$\Delta(f) > \sqrt{3}m^{-(m+2)/2} |\text{disc}(f)|^{1/2} M(f)^{-(m-1)},$$

where $\text{disc}(f)$ is the discriminant of f , and $M(f) = |a_0| \prod_{h=1}^m \max(1, |\alpha_h|)$.

This translates to the following algorithm:

Algorithm 5.2 (Checking an algebraic identity). *Let α and $\beta \in \mathbb{R}$ be two algebraic numbers, and let A and $B \in K[X] \setminus 0$ that vanish at α , and β , respectively. Assume we can compute floating point approximations $\hat{\alpha}$ and $\hat{\beta}$ such that $|\alpha - \hat{\alpha}| < \varepsilon$, $|\beta - \hat{\beta}| < \varepsilon$, for any fixed $\varepsilon > 0$.*

We want to decide whether $\alpha < \beta$, $\alpha > \beta$ or $\alpha = \beta$.

- (1) Let $C = AB$ and $f = C/\text{gcd}(C, C')$.
- (2) If the degree of f is 1, then **answer** $\alpha = \beta$.
- (3) Compute a good approximation $\hat{\Delta}$ of

$$\Delta(f) = \sqrt{3}m^{-(m+2)/2} |\text{disc}(f)|^{1/2} M(f)^{-(m-1)},$$

where $\text{disc}(f)$ and $M(f)$ are defined in Theorem 5.1 such that $\hat{\Delta} \leq \Delta(f)$.

- (4) Compute α and β at precision $\varepsilon = \hat{\Delta}/4$, i.e., $\hat{\alpha}$ and $\hat{\beta}$ such that

$$|\alpha - \hat{\alpha}| < \varepsilon, \quad |\beta - \hat{\beta}| < \varepsilon.$$

- (5) If $|\hat{\alpha} - \hat{\beta}| < 2\varepsilon$, **answer** $\alpha = \beta$.
- (6) If $\hat{\alpha} < \hat{\beta}$, **answer** $\alpha < \beta$.
- (7) If $\hat{\alpha} > \hat{\beta}$, **answer** $\alpha > \beta$.

Proof. The polynomial f is nonconstant and has α and β among its roots. If its degree is 1, then $\alpha = \beta$. Otherwise, assume first that $|\hat{\alpha} - \hat{\beta}| < 2\varepsilon$. Then

$$|\alpha - \beta| \leq |\alpha - \hat{\alpha}| + |\beta - \hat{\beta}| + |\hat{\alpha} - \hat{\beta}| < 4\varepsilon \leq \Delta(f).$$

Hence $\alpha = \beta$ by Mahler's theorem in this case, proving (5).

We now assume that $|\hat{\alpha} - \hat{\beta}| \geq 2\varepsilon$; since

$$\alpha - \beta = \hat{\alpha} - \hat{\beta} + (\alpha - \hat{\alpha}) - (\beta - \hat{\beta})$$

and

$$|(\alpha - \hat{\alpha}) - (\beta - \hat{\beta})| < 2\varepsilon,$$

$\alpha - \beta$ and $\hat{\alpha} - \hat{\beta}$ have the same sign. □

Proposition 5.3. *The smallest ε that we can obtain in step (4) of the above algorithm (i.e., the maximal precision needed) is $\gg X^{-\beta}$, for some positive constant β .*

Remark. That means that for our computation we will need at most $\Omega(\log X)$ significant digits.

Proof. Algorithm 4.1 loops over reduced integral cubic forms $F = (a, b, c, d) \in V_{\mathcal{O}}$ with discriminant $\text{disc}(F)$ satisfying $\mathcal{N}(\text{disc}(F)) \leq X$. In particular, Proposition 3.4 implies that $|a| \ll X^{1/8}$.

For each such form, we may compute various separable polynomials f with coefficients in $a^{-u}\mathcal{O}_K$, for some bounded integer u . Then $\text{disc}(f)$ is nonzero, in $a^{-4u}\mathcal{O}_K$. Its norm is a nonzero rational integer divided by $|a|^{-8u}$, hence $\gg X^{-u}$. Thus $\text{disc}(f) \gg X^{-u/2}$.

Landau's theorem (see [3, Proof of Theorem 13.1] for example) tells us that

$$M(f) \leq \|f\|_2$$

and the coefficients of f are monomials in $e_1, e_2, e_3, f_1, f_2, f_3$ (see Appendix D of [15]). Each one of these is bounded by $c \cdot X^\alpha$, for an appropriate constant c and exponent α .

We have

$$\Delta(f) \gg M(f)^{-(m-1)}.$$

So we obtain

$$\|f\|_2 \ll X^\beta,$$

but then we can conclude that $\Delta(f) \gg X^{-\beta}$. □

5.2. An idea to count only half of the extensions. Let K be an imaginary quadratic number field, with class number $h_K = 1$ and discriminant $d_K \neq -3, -4$. It is easy to remark that if $H = (P, Q, R)$ is in the fundamental domain, then $H' = (P, -\overline{Q}, R)$ is also. And, in general, these two Hermitian forms are not equivalent modulo $\text{PGL}_2(\mathcal{O})$.

In particular, if $F = (a, b, c, d)$ has $H_F = H$, then $F' = (\overline{a}, -\overline{b}, \overline{c}, -\overline{d})$ gives $H_{F'} = H'$.

So we can loop only on half of the c satisfying the given bounds, then construct both the forms $F = (a, b, c, d)$ and $F' = (\overline{a}, -\overline{b}, \overline{c}, -\overline{d})$ and check if they are equivalent (comparing F' with the list of automorphic functions to F). If not we verify also the list of automorphic functions to F' to see if one of them will be found in our loops, and if both answers are no, we add this second form F' to our output list.

6. RESULTS

In this section we present results obtained for the case $K = \mathbb{Q}(i)$ via an implementation of our algorithm in Pari/GP [16], running on an Intel Xeon 5160 dual core, 3.0 GHz.

Let X be the bound on $\mathcal{N}(\mathfrak{d}(L/K))$ and $N(X)$ the number of isomorphism classes of cubic extensions of $\mathbb{Q}(i)$ up to that bound.

In the following table, we will compare the time needed to list all $N(X)$ representatives of cubic extensions of $\mathbb{Q}(i)$ with two algorithms: ray class field and ours. t denotes the running time of our algorithm, t' the running time of ray class field one (see Section 9.2.3 of [4]).

Remarks.

- These computations have allowed us to check the correctness of our results. In fact, we compared $N(X)$ up to $X = 9 \cdot 10^6$ with the results of the ray class field algorithm and all results matched.
- The last line of the table would have involved very long computations with ray class field algorithm, so we skipped it, and we give only an estimate on the running time needed.

X	$N(X)$	t	t'
10^4	276	5 s	16 s
$4 \cdot 10^4$	1339	19 s	1mn 18 s
$9 \cdot 10^4$	3305	56 s	3mn 45 s
10^6	42692	24 mn 1 s	2h 52mn 9 s
$4 \cdot 10^6$	181944	2 h 49 mn	34h 24 mn 8 s
$9 \cdot 10^6$	421559	9 h 37 mn	> 134 h
10^8	4990974	359 h 25 mn	> 2720 h

6.1. Roberts' conjecture and asymptotic predictions for $N(X)$.

Frank Thorne compared our numerical results with heuristic asymptotic developments derived from the Datskovsky-Wright method [8], in the spirit of Roberts' conjecture (see [19]). Starting from Roberts' conjecture, Taniguchi and Thorne worked out the formula in the particular case when k is an imaginary quadratic number field:

$$N(X) = \frac{1}{12} \frac{\text{Res}_{s=1} \zeta_K(s)}{\zeta_K(3)} X + \frac{1}{40} d_K^{-1/2} \text{Res}_{s=1} \zeta_K(s) \frac{\sqrt{3} \Gamma(1/3)^6}{\pi^2} \frac{\zeta_K(1/3)}{\zeta_K(2) \zeta_K(5/3)} X^{5/6}.$$

The following table compares our values for $N(X)$ with Thorne's asymptotic data. The results are strikingly similar.

X	$N(X)$ (Morra)	$N(X)$ (Taniguchi-Thorne)
10^4	276	270.2
10^6	42692	42655.6
$9 \cdot 10^6$	421559	421260
10^8	4990974	4990962

APPENDIX A. AUTOMORPHISM MATRICES FOR $\mathbb{Q}(i)$

M (modulo multiplication by $\varepsilon \in \{+1, -1, i, -i\}$)	conditions for $S(M)$
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	for all P, Q, R
$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	if $P = R$ and $\operatorname{Re}(Q) = 0$
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	if $P = R$ and $\operatorname{Im}(Q) = 0$
$\begin{pmatrix} 0 & 1 \\ -i & 0 \end{pmatrix}$	if $P = R$ and $\operatorname{Re}(Q) = \operatorname{Im}(Q)$
$\begin{pmatrix} 0 & 1 \\ i & 0 \end{pmatrix}$	if $P = R$ and $\operatorname{Re}(Q) = -\operatorname{Im}(Q)$
$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$	if $P = R$ and $\operatorname{Re}(Q) = P/2$
$\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$	if $P = R$ and $\operatorname{Re}(Q) = -P/2$
$\begin{pmatrix} 0 & 1 \\ 1 & i \end{pmatrix}, \begin{pmatrix} -i & 1 \\ 1 & 0 \end{pmatrix}$	if $P = R$ and $\operatorname{Im}(Q) = P/2$
$\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$	if $P = R, \operatorname{Re}(Q) = P/2$ and $\operatorname{Im}(Q) = 0$
$\begin{pmatrix} i & 0 \\ -i & 1 \end{pmatrix}, \begin{pmatrix} -1 & -1-i \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & -1-i \\ -i & 1 \end{pmatrix},$ $\begin{pmatrix} -i & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & i \\ 1 & -i+1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -i & -i-1 \end{pmatrix},$ $\begin{pmatrix} 1 & -i+1 \\ -i-1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -i-1 & -1 \end{pmatrix},$ $\begin{pmatrix} 1 & 1 \\ -i-1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -i \\ -i-1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -i \\ -1 & i-1 \end{pmatrix}$	if $P = R, \operatorname{Re}(Q) = P/2$ and $\operatorname{Im}(Q) = P/2$
$\begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}$	if $P = R, \operatorname{Re}(Q) = -P/2$ and $\operatorname{Im}(Q) = 0$
$\begin{pmatrix} -i & 0 \\ -i & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1-i \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1-i \\ -i & 1 \end{pmatrix},$ $\begin{pmatrix} i & 0 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ 1 & -i-1 \end{pmatrix}, \begin{pmatrix} 1 & -i \\ 1 & -i-1 \end{pmatrix},$ $\begin{pmatrix} 1 & -1-i \\ 1-i & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1-i & -1 \end{pmatrix},$ $\begin{pmatrix} 1 & -i \\ 1-i & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1-i & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -i & i-1 \end{pmatrix}$	if $P = R, \operatorname{Re}(Q) = -P/2$ and $\operatorname{Im}(Q) = P/2$
$\begin{pmatrix} -1 & 0 \\ -i & 1 \end{pmatrix}$	if $P = R, \operatorname{Re}(Q) = 0$ and $\operatorname{Im}(Q) = P/2$
$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$	if $Q = 0$
$\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$	if $\operatorname{Re}(Q) = P/2$ and $\operatorname{Im}(Q) = 0$
$\begin{pmatrix} 1 & i \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 1 & 1+i \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & i \end{pmatrix}$	if $\operatorname{Re}(Q) = P/2$ and $\operatorname{Im}(Q) = P/2$
$\begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$	if $\operatorname{Re}(Q) = -P/2$ and $\operatorname{Im}(Q) = 0$
$\begin{pmatrix} 1 & i \\ 0 & i \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 1 & -1+i \\ 0 & -1 \end{pmatrix}$	if $\operatorname{Re}(Q) = -P/2$ and $\operatorname{Im}(Q) = P/2$
$\begin{pmatrix} 1 & i \\ 0 & -1 \end{pmatrix}$	if $\operatorname{Re}(Q) = 0$ and $\operatorname{Im}(Q) = P/2$

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