MÖBIUS INVERSION FORMULAE FOR APOSTOL-BERNOULLI TYPE POLYNOMIALS AND NUMBERS

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ABSTRACT. In this paper, we establish Möbius inversion formulae for the Fourier expansions of the Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials. As an application, by specializing our formulae at some special values we obtain interesting number-theoretical relations. We derive explicit formulae for Apostol-Bernoulli numbers. These formulae involve Stirling numbers of the second kind and powers of cotangent. Our proofs are very simple.

1. INTRODUCTION

For $\omega \in \mathbb{C} \setminus \{0\}$ and x variable, the n^{th} Apostol-Bernoulli polynomial $B_n(x; \omega)$ is defined by the generating function

$$(1) \quad \sum_{n \geq 0} B_n(x; \omega) \frac{t^n}{n!} = \frac{te^{xt}}{\omega e^t - 1}, \quad (|t + \log(\omega)| < 2\pi),$$

where

$$(2) \quad \omega = |\omega|e^{i\theta}, \quad -\pi \leq \theta < \pi \text{ and } \log(\omega) = \log|\omega| + i\theta.$$

For $\omega = 1$, $B_n(x; 1) = B_n(x)$ is the n^{th} Bernoulli polynomial and $B_n(0; 1) = B_n$ is the n^{th} Bernoulli number.

The Möbius inversion formula for the Bernoulli polynomials $B_n(x)$ has been studied in [5] for $n \geq 2$. Hence, the purpose of this paper is to investigate the case $\omega \neq 1$.

It is well known that the Apostol-Bernoulli polynomials (see [2] or [4]) has the following Fourier expansion,

$$(3) \quad -\frac{\omega^x}{n!} B_n(x; \omega) = \sum_{k \in \mathbb{Z}} \frac{e^{2i\pi kx}}{(2i\pi k - \log(\omega))^n},$$

for $0 < x < 1$ if $n = 1$, $0 \leq x \leq 1$ if $n \geq 2$.

Remark 1. The convergence, of the Fourier series (3), is absolute and uniform in $[0, 1]$ except for $B_1(x; \omega)$. In this paper, the case $n = 1$ is not treated.

The paper is organized as follows. In section 2, we establish a Möbius inversion type formula for Apostol-Bernoulli polynomials. For that, we shall slightly modify

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the expression (3). Similarly, in section 3, we obtain Möbius inversion formulae for Apostol-Euler and Apostol-Genocchi polynomials.

2. MÖBIUS INVERSION FOR APOSTOL-BERNOULLI POLYNOMIALS

With a view towards stating our main results, we transform the Fourier expansion (3) in sum over positive integers. Let us define a 1-periodic, in the variable x , function by

$$F_n(x; \omega) = -\frac{\omega^x}{n!} B_n(x; \omega) - \frac{1}{(-\log(\omega))^n},$$

for $0 < x < 1$ if $n = 1$, and $0 \leq x \leq 1$ if $n \geq 2$.

We set

$$f_n(k; x; \omega) = \frac{e^{2i\pi kx}}{(2i\pi k - \log(\omega))^n} + \frac{e^{-2i\pi kx}}{(-2i\pi k - \log(\omega))^n},$$

then from (3) we observe that

$$(4) \quad F_n(x; \omega) = \sum_{k=1}^{\infty} f_n(k; x; \omega),$$

and its Möbius inversion is given by the following interesting and useful theorem.

Theorem 1. *For any positive integer $n \geq 2$, real x and complex $\omega \in \mathbb{C} \setminus \{0, 1\}$, we have*

$$(5) \quad f_n(1; x; \omega) = \sum_{m=1}^{\infty} \frac{\mu(m)}{m^n} F_n(mx; \omega^{1/m}),$$

where μ is the arithmetical Möbius function.

Proof. We first note that the function f_n satisfies the remarkable property,

$$f_n(k; mx; \omega^{1/m}) = \frac{e^{2i\pi mkx}}{(2i\pi k - \log(\omega)/m)^n} + \frac{e^{-2i\pi mkx}}{(-2i\pi k - \log(\omega)/m)^n} = m^n f_n(mk; x; \omega).$$

Hence for any integer $n \geq 2$, real x and complex $\omega \in \mathbb{C} \setminus \{0, 1\}$, we have

$$\begin{aligned} & \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \left| \frac{\mu(m)}{m^n} f_n(k; mx; \omega^{1/m}) \right| \\ & \leq \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} |\mu(m)| \left(\frac{1}{|2imk\pi - \log(\omega)|^n} + \frac{1}{|2imk\pi + \log(\omega)|^n} \right). \end{aligned}$$

Using the elementary inequalities,

$$(6) \quad |2imk\pi \pm \log(\omega)| \geq 2mk\pi - |\arg(\omega)| \geq 2mk\pi - \pi \geq mk\pi,$$

$$(7) \quad |\mu(m)| \leq 1, (\forall m \geq 1),$$

we obtain

$$\sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \left| \frac{\mu(m)}{m^n} f_n(k; mx; \omega^{1/m}) \right| \leq \frac{2}{\pi^n} \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{m^n k^n} \leq \frac{2}{\pi^n} \zeta^2(n).$$

The convergence is then absolute and uniform. Substituting the expansion (4) of F_n in the right-hand of (5), we obtain

$$\begin{aligned} \sum_{m=1}^{\infty} \frac{\mu(m)}{m^n} F_n(mx; \omega^{1/m}) &= \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \frac{\mu(m)}{m^n} f_n(k; mx; \omega^{1/m}) = \sum_{m,k=1}^{\infty} \mu(m) f_n(mk; x; \omega) \\ &= \sum_{m=1}^{\infty} \left(\sum_{d|m} \mu(d) \right) f_n(m; x; \omega). \end{aligned}$$

To end the proof of the theorem we use the following analytical-number property: for every positive integer m we have

$$\sum_{d|m} \mu(d) = \begin{cases} 1 & \text{if } m = 1, \\ 0 & \text{if } m \geq 2. \end{cases}$$

Remark 2. Theorem 1 seems to fit into the general framework for the Möbius inversion outlined in [3]. In fact, we observe that Fourier series (3) has the form

$$(8) \quad F_n(x; \omega) = \sum_{k \geq 1} \alpha(k) f_n(\phi(k; x, \omega)),$$

where the $\alpha(k) = k^{-n}$ is a *completely multiplicative arithmetical function*, and $\phi : (k; x, \omega) \mapsto (kx, \omega^k)$ is a flow, i.e., $\phi(m; \phi(k; x, \omega)) = \phi(mk; x, \omega)$.

Now we can rewrite the Theorem 1 in terms of Apostol-Bernoulli polynomials.

Theorem 2. *Let $\omega \in \mathbb{C} \setminus \{0, 1\}$, x a real number and $n \geq 2$ an integer. Then we have*

$$(9) \quad \begin{aligned} &\frac{e^{2i\pi x}}{(2i\pi - \log(\omega))^n} + \frac{e^{-2i\pi x}}{(-2i\pi - \log(\omega))^n} \\ &= - \sum_{m=1}^{\infty} \frac{\mu(m)}{m^n} \left(\frac{\omega^{\{px\}}}{n!} B_n(\{mx\}; \omega^{1/m}) + \frac{m^n}{(-\log(\omega))^n} \right), \end{aligned}$$

where $\{mx\}$ denote the fractional part of mx .

3. THE APOSTOL-EULER AND APOSTOL-GENOCCHI POLYNOMIALS

For $\omega \in \mathbb{C} \setminus \{0\}$ and x variable, the Apostol-Euler polynomial $E_n(x; \omega)$ and the Apostol-Genocchi polynomial $G_n(x; \omega)$ are given by the generating functions

$$(10) \quad \sum_{n \geq 0} E_n(x; \omega) \frac{t^n}{n!} = \frac{2e^{xt}}{\omega e^t + 1}, \quad (|t + \log(-\omega)| < 2\pi),$$

$$(11) \quad \sum_{n \geq 0} G_n(x; \omega) \frac{t^n}{n!} = \frac{2te^{xt}}{\omega e^t + 1}, \quad (|t + \log(-\omega)| < 2\pi).$$

They also have Fourier type expansions (see [2]). In fact, for any complex $\omega \in \mathbb{C} \setminus \{-1, 0\}$, we have

$$(12) \quad \frac{\omega^x}{n!} E_n(x; \omega) = 2 \sum_{k \in \mathbb{Z}} \frac{e^{2i\pi(k + \frac{1}{2})x}}{(2i\pi(k + \frac{1}{2}) - \log(\omega))^{n+1}},$$

for $0 < x < 1$ if $n = 0$ and $0 \leq x \leq 1$ if $n \geq 1$, and

$$(13) \quad \frac{\omega^x}{n!} G_n(x; \omega) = 2 \sum_{k \in \mathbb{Z}} \frac{e^{2i\pi(k+\frac{1}{2})x}}{(2i\pi(k+\frac{1}{2}) - \log(\omega))^n},$$

for $0 < x < 1$ if $n = 1$ and $0 \leq x \leq 1$ if $n \geq 2$. We note that

$$E_n(x; \omega) = -\frac{2}{n+1} B_{n+1}(x; -\omega) \quad \text{and} \quad G_{n+1}(x; \omega) = (n+1) E_n(x; \omega).$$

Then we get immediately, from the Theorem 1, the following identities:

Theorem 3. *Let $\omega \in \mathbb{C} \setminus \{0, -1\}$, and x a real number. Then, for any $n \geq 1$ we have*

$$\begin{aligned} & \frac{e^{2i\pi x}}{(2i\pi - \log(-\omega))^{n+1}} + \frac{e^{-2i\pi x}}{(-2i\pi - \log(-\omega))^{n+1}} \\ &= \sum_{m=1}^{\infty} \frac{\mu(m)}{m^{n+1}} \left(\frac{(-\omega)^{\frac{\{px\}}{m}}}{2(n!)} E_n(\{mx\}; -(-\omega)^{1/m}) - \frac{m^{n+1}}{(-\log(-\omega))^{n+1}} \right), \end{aligned}$$

and for any $n \geq 2$ we have

$$\begin{aligned} & \frac{e^{2i\pi x}}{(2i\pi - \log(-\omega))^n} + \frac{e^{-2i\pi x}}{(-2i\pi - \log(-\omega))^n} \\ &= \sum_{m=1}^{\infty} \frac{\mu(p)}{m^n} \left(\frac{(-\omega)^{\frac{\{px\}}{m}}}{2(n!)} G_n(\{mx\}; -(-\omega)^{1/m}) - \frac{m^n}{(-\log(-\omega))^n} \right). \end{aligned}$$

Remark 3. Note that we can write the last theorem in terms of $\log(\omega)$, thanks to the equalities $\log(-\omega) = \log \omega + i\pi$ if $-\pi \leq \arg \omega < 0$ and $\log(-\omega) = \log \omega - i\pi$ if $0 \leq \arg \omega < \pi$.

4. APOSTOL-BERNOULLI NUMBERS AND APPLICATIONS

4.1. Apostol-Bernoulli numbers. Let n be an integer ≥ 1 and $\omega \neq 1$. It is well known that

$$B_n(x; \omega) = \sum_{k=1}^n \binom{n}{k} B_k(0; \omega) x^{n-k},$$

with

$$B_k(0; \omega) = \frac{k}{\omega} \sum_{j=0}^{k-1} (-1)^j j! S(k-1, j) \left(\frac{\omega}{\omega-1} \right)^{j+1} \quad \text{for } k \geq 1 \text{ (see [1])},$$

where $S(k, j)$ are Stirling numbers of the second kind defined by $S(k, j) = \frac{\Delta^{(j)} 0^k}{j!}$, with

$$\Delta^{(j)} 0^n = (\Delta^{(j)} x^n)_{x=0}, \quad \Delta^{(j)} 0^n = 0 \text{ if } j > n, \quad \Delta^{(0)} 0^0 = 1,$$

in the usual notation of finite differences. We call the numbers $B_k(0; \omega)$ the Apostol-Bernoulli numbers.

Now, we set $\omega = e^{2i\pi z}$ with $z \notin \mathbb{Z}$. Then, for $k \geq 1$, we have

$$B_k(0; e^{2i\pi z}) = k \sum_{j=0}^{k-1} (-1)^j j! S(k-1, j) e^{-2i\pi z} \left(\frac{e^{2i\pi z}}{e^{2i\pi z} - 1} \right)^{j+1}.$$

On the other hand, an elementary computation allows us to write

$$e^{-2i\pi z} \left(\frac{e^{2i\pi z}}{e^{2i\pi z} - 1} \right)^{j+1} = \frac{1}{(2i)^{j+1}} (1 + \cot(\pi z)^2)(i + \cot(\pi z))^{j-1}.$$

Finally, we get the formula:

Theorem 4. *For any integer $k \geq 1$, we have*

$$(14) \quad \begin{aligned} & B_k(0; e^{2i\pi z}) \\ &= k \sum_{j=0}^{k-1} (-1)^j j! S(k-1, j) \frac{1}{(2i)^{j+1}} (1 + \cot(\pi z)^2)(i + \cot(\pi z))^{j-1}. \end{aligned}$$

The formula (14) shows that, for $k \geq 1$, the k^{th} Apostol-Bernoulli number $B_k(0; e^{2i\pi z})$ is a polynomial in terms of $\cot(\pi z)$ of degree k . Let us give the first values of $B_k(0; e^{2i\pi z})$:

$$\begin{aligned} B_1(0; e^{2i\pi z}) &= \frac{1}{2i} (\cot(\pi z) - i), \quad B_2(0; e^{2i\pi z}) = \frac{1}{2} (1 + \cot(\pi z)^2), \quad B_3(0; e^{2i\pi z}) \\ &= \frac{3i}{4} \cot(\pi z)(1 + \cot^2(\pi z)). \end{aligned}$$

From Theorem 2 and Theorem 4, with $x = 0$ and $\omega = e^{2i\pi z}$, we obtain the following Möbius inversion formula:

Proposition 5. *Let $n \geq 2$ be an integer and $z \notin \mathbb{Z}$ with $-1/2 \leq \Re(z) < 1/2$. Then, we have*

$$\begin{aligned} & \frac{z^n}{(1-z)^n} + \frac{(-z)^n}{(1+z)^n} \\ &= - \sum_{m=1}^{\infty} \mu(m) \left(1 + (\pi z/m)^n \sum_{j=1}^{n-1} A_{(n-1,j)} \left(\frac{d}{dz} (i + \cot)^j \right) (\pi z/m) \right) \end{aligned}$$

where $A_{(n,j)} = (-1)^{n-j} \frac{(-2i)^{n-j} j!}{n!} S(n, j)$.

4.2. Euler's formulas for the zeta function at the even positive integers.

Recall that the Riemann zeta function is given by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad (\Re(s) > 1).$$

As an application of this section, we prove Euler's formulas for the zeta function at the even positive integers

$$(15) \quad \zeta(2k) = \frac{(-1)^{k+1} 2^{2k-1}}{(2k)!} \pi^{2k} B_{2k}.$$

From the Proposition 5, we obtain for $n = 2$ the formula

$$(16) \quad \frac{z^2}{(1-z)^2} + \frac{z^2}{(1+z)^2} = - \sum_{m=1}^{\infty} \mu(m) (1 + (\pi z/m)^2 \cot'(\pi z/m)).$$

On the other hand, for $|z| < 1$, we have the expansions

$$(17) \quad \frac{z^2}{(1-z)^2} + \frac{z^2}{(1+z)^2} = 2 \sum_{k=1}^{\infty} (2k-1) z^{2k}$$

and

$$(18) \quad z^2 \cot'(z) = \sum_{k \geq 0} (2k-1) \beta_{2k} z^{2k}, \quad \text{where} \quad \beta_{2k} = (-1)^k \frac{2^{2k}}{(2k)!} B_{2k}.$$

Hence, from (18), after switching the order of summation in the double series, we have

$$(19) \quad \begin{aligned} & \sum_{m=1}^{\infty} \mu(m) (1 + (\pi z/m)^2 \cot'(\pi z/m)) \\ &= \sum_{k=1}^{\infty} (2k-1) \pi^{2k} \beta_{2k} \left(\sum_{m=1}^{\infty} \frac{\mu(m)}{m^{2k}} \right) z^{2k}. \end{aligned}$$

Using the equalities (16), (17) and (19) we obtain

$$(20) \quad 2 \sum_{k=1}^{\infty} (2k-1) z^{2k} = - \sum_{k=1}^{\infty} (2k-1) \pi^{2k} \beta_{2k} \left(\sum_{m=1}^{\infty} \frac{\mu(m)}{m^{2k}} \right) z^{2k}.$$

By comparing the coefficients of z^{2k} in the right and left sides of formula (20), we obtain

$$(21) \quad -2 = \pi^{2k} \beta_{2k} \sum_{m=1}^{\infty} \frac{\mu(m)}{m^{2k}}.$$

Now, using the classical Möbius inversion

$$\frac{1}{\zeta(s)} = \sum_{m=1}^{\infty} \frac{\mu(m)}{m^s}, \quad (\Re(s) > 1),$$

we recover, from (21) and (18), Euler's formulae (15).

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