UNIFORM l^1 CONVERGENCE IN THE CRANK-NICOLSON METHOD OF A LINEAR INTEGRO-DIFFERENTIAL EQUATION FOR VISCOELASTIC RODS AND PLATES

DA XU

ABSTRACT. We study the numerical approximation of a certain Volterra integro-differential equation in Hilbert space which arises in the linear theory of isotropic viscoelastic rods and plates. The equation is discretized in time using a method based on the trapezoidal rule: while the time derivative is approximated by the trapezoidal rule in a two-step method, a convolution quadrature rule, constructed again from the trapezoidal rule, is used to approximate the integral term. The resulting scheme is shown to be convergence in the $l_t^1(0, \infty; H) \cap l_t^\infty(0, \infty; H)$ norm.

1. INTRODUCTION

Let $\mathbf{H}$ be a Hilbert space and L a positive self-adjoint linear operator in $\mathbf{H}$ with dense domain $D(L)$, and a complete eigensystem $\{\lambda_m, \varphi_m\}_{m=1}^\infty$. We shall consider the time discretization for the integro-differential equation

$$(1.1) \quad u_t + \int_0^t A(t-s) L u(s) ds = 0, \quad t > 0, \\ u(0) = u_0.$$

Here, $u_t = \partial u / \partial t$ and u_0 belongs to $\mathbf{H}$. The real-value kernel $A : \mathbf{R}^+ = [0, \infty) \rightarrow \mathbf{R}$ is locally absolutely continuous. The equation arises in the linear theory of isotropic viscoelastic rods and plates. Numerical methods for the time discretization of (1.1) have been studied by various authors [17]–[27]. But here the kernel of the equation is made up in a complicated way from the (distinct) moduli of stress relaxation for compression and shear.

Throughout this paper, the Laplace transform $\hat{h}$ is defined for a function h such that $h(t)e^{-\sigma t} \in L^1(\mathbf{R}^+)$ for all $\sigma > 0$ by the formula

$$\hat{h}(z) = \int_0^\infty e^{-zt} h(t) dt \quad (Re\, z > 0), \\ \hat{h}(i\tau_0) = \lim_{z \rightarrow i\tau_0, Re\, z > 0} \hat{h}(z) \quad (\tau_0 \in \mathbf{R}),$$

whenever the limit exists.

Received by the editor January 22, 2011 and, in revised form, June 5, 2012.

2010 *Mathematics Subject Classification.* Primary 65J08, 65D32; Secondary 45K05.

Key words and phrases. Volterra integro-differential equation, time discretization, trapezoidal rule, l_t^1 convergence.

This work was supported in part by the National Natural Science Foundation of China, contract grant numbers 11271123, 10971062.

The hypotheses on A will be stated in terms of its Laplace transform $\widehat{A}$; the fact that A is locally absolutely continuous and of positive type is known from Theorem 1.1(i) and (2.11) of [4], respectively.

The assumptions on A are as follows:

$$(1.2) \quad \widehat{A}(z) = \widehat{a}(z)f(z) \quad (\operatorname{Re} z > 0),$$

where

$$(1.3) \quad f(z) = F(\widehat{b}(z)/\widehat{a}(z)), \quad F(w) = \frac{w+m}{pw+q},$$

$$(1.4) \quad m, p, \text{ and } q \text{ are nonnegative, } p > 0, q > mp,$$

and

$$(1.5) \quad \begin{aligned} &a(t) \text{ and } b(t) \text{ are continuous, nonnegative, nonincreasing,} \\ &\text{convex, and not constant on } \mathbf{R}^+, \text{ with } a(0) + b(0) < \infty. \end{aligned}$$

The equation for a dynamic problem in linear viscoelasticity [1], [2] can be obtained by a “correspondence principle” from the corresponding problem in the purely elastic case. Formal differentiation of (1.1), with $L = \Delta^2$, $u(t) = u(., t)$, followed by an application of the Laplace transform (with $m = p = 1/2$, $q = 1$ in (1.3)) yields the transformed equation for transverse vibrations in a viscoelastic plate (see [3], [4]). Similarly, the equations for waves in a rod come from (1.1), (1.2), (1.3) with $m = 0$, $p = 2$, $q = 1$; for longitudinal waves, $L = -\partial^2/\partial x^2$, while for bending waves, $L = \partial^4/\partial x^4$. In all cases we must take self-adjoint boundary conditions. In the applied literature, the moduli $a(t)$ and $b(t)$ are often assumed to be positive linear combinations of decaying exponentials. There are indications that for certain viscoelastic materials the assumption that $A(0)$ or $A'(0)$ is finite may not be valid. Authors have studied (1.1) and equations related to (1.1) assuming that the convolution kernel or its derivative becomes infinite at the origin (see [5]–[12], [15], [16]), and the kernel is finite at the origin (see [13], [14], [18]).

Our purpose in this paper is to study only discretization in time for the initial value problem (1.1). As only the Laplace transform $\widehat{A}(s)$, but not the convolution kernel $A(t)$ is known analytically, we will apply the convolution quadrature as proposed by Lubich [28], [31], [32], [30], [29] [33] for the time discretization of (1.1). Thus, we choose a fixed time step $k > 0$, put $t_n = nk$ for $n \geq 0$, and generate a numerical solution $U^n \simeq u(t_n)$ satisfying,

$$(1.6) \quad \left[zI + \widehat{A}(z)L \right]^{-1} u_0 = k \sum_{n=0}^{\infty} U^n \zeta^n, \quad \text{with } z = \frac{\delta(\zeta)}{k}, \quad \zeta \in \mathbb{C}, \quad |\zeta| < 1,$$

and $\delta(\zeta) = 2(1 - \zeta)/(1 + \zeta)$ (see Lubich [28, 4.1] or [32, 1.3]).

Let $\omega_j(k)$ be the coefficients of the power series

$$(1.7) \quad \widehat{A}\left(\frac{\delta(\zeta)}{k}\right) = \sum_{j=0}^{\infty} \omega_j(k) \zeta^j, \quad \zeta \in \mathbb{C}, \quad |\zeta| < 1.$$

We see from (1.6) and (1.7) that

$$(1.8) \quad \begin{aligned} \frac{1}{2k} u_0 + \frac{1}{2k} u_0 \zeta &= \frac{1}{k} \sum_{n=0}^{\infty} U^n \zeta^n - \frac{1}{k} \sum_{n=0}^{\infty} U^n \zeta^{n+1} \\ &+ \frac{1}{2} \left[\sum_{n=0}^{\infty} \left(\sum_{j=0}^n \omega_j(k) L U^{n-j} \right) \zeta^n + \sum_{n=0}^{\infty} \left(\sum_{j=0}^n \omega_j(k) L U^{n-j} \right) \zeta^{n+1} \right] \end{aligned}$$

with $\zeta \in \mathbb{C}$, $|\zeta| < 1$. Comparing the coefficients of the power series in both sides of (1.8) shows that U^n satisfies

$$(1.9) \quad \frac{U^n - U^{n-1}}{k} + \frac{1}{2} \left(\sum_{j=0}^n \omega_j(k) L U^{n-j} + \sum_{j=0}^{n-1} \omega_j(k) L U^{n-1-j} \right) = 0, \\ \text{for } n \geq 2,$$

$$(1.10) \quad \begin{aligned} U^1 &= (2I + k\omega_0(k)L)^{-1} (4I - k\omega_1(k)L) U^0, \\ U^0 &= (2I + k\omega_0(k)L)^{-1} u_0. \end{aligned}$$

Our first result is an extension of the results of [28, Thm. 4.1] or [32, Thm. 2.1] for a scalar function to a function in an infinite-dimensional Hilbert space $\mathbf{H}$, where it is a solution of (1.1).

Theorem 1.1. *Suppose (1.4) and (1.5) hold. If, in addition,*

$$(1.11) \quad (i) \quad -a'(t) \text{ and } -b'(t) \text{ are convex on } (0, \infty),$$

$$(1.12) \quad (ii) \quad a''(t) \text{ and } b''(t) \text{ are convex on } (0, \infty),$$

then we have, for the solutions of (1.6) and (1.1) and for k sufficiently small,

$$(1.13) \quad \|U^n - u(t_n)\| \leq C k t_{n-1}^{-1} \|u_0\|_3, \text{ for } t_{n-1} > 0,$$

where C is a positive constant independent of $u(t)$ and k , $\|\cdot\|$ denotes the norm in $\mathbf{H}$, and $\|v\|_\nu = \|L^{\nu/2} v\|$ denotes the norm in the space $\mathbf{H}^\nu = D(L^{\nu/2})$ (see [23, 2.18]).

The proof will be given in Section 4.

We recall some results concerning the asymptotic behavior of the solution of problem (1.1). Let $u(t, \lambda)$ denote the solution of the real equation

$$(1.14) \quad \frac{d u(t, \lambda)}{dt} + \lambda \int_0^t A(t-s) u(s, \lambda) ds = 0, \text{ for } t > 0, \\ u(0, \lambda) = 1,$$

with parameter λ ($\lambda_1 \leq \lambda < \infty$, where λ_1 is the smallest eigenvalue of L).

In [3], Hannsgen gives sufficient conditions in order that the following hold:

$$(1.15) \quad \begin{aligned} (i) \quad &\int_0^\infty \sup_{\lambda \geq \lambda_1} |u(t, \lambda)| dt < \infty, \text{ and} \\ (ii) \quad &\int_0^\infty \sup_{\lambda \geq \lambda_1} |u'(t, \lambda) \lambda^{-1/2}| dt < \infty, \quad (t = \frac{d}{dt}). \end{aligned}$$

In particular, (1.15) holds when $A(t)$ satisfies (1.4), (1.5) and (1.11) with the condition $a(0) + b(0) < \infty$ in (1.5) replaced by $-a'(0) - b'(0) < \infty$. Also, (1.15)(i) is shown by Noren [4] for the kernel $A(t)$ assuming (1.4), (1.5), (1.11) and

$$(1.16) \quad \limsup_{x \downarrow 0} \frac{x^5 [C(x) + C_b(x)]}{[B(x) + B_b(x)]^3} < \infty,$$

where

$$(1.17) \quad C(x) = \int_0^x -a'(s) ds, \quad C_b(x) = \int_0^x -b'(s) ds,$$

$$(1.18) \quad B(x) = \int_0^x -s a'(s) ds, \quad B_b(x) = \int_0^x -s b'(s) ds.$$

(See [4, Theorem 1.1]). We also note that (1.15)(i) is a result, from Theorem 1.2 of [4], with the assumptions (1.4), $b(t) = t^{-\beta}$, $0 < \beta < 1$, and $a(t)$ as (1.5) and (1.11).

From [3], [4] we know that the formula

$$(1.19) \quad u(t) = \sum_{j=1}^{\infty} u(t, \lambda_j) (u_0, \varphi_j) \varphi_j,$$

gives a solution of (1.1) in the Hilbert space $\mathbf{H}$, where $(\cdot, \cdot)$ denotes the inner product in $\mathbf{H}$. With

$$\|u(t)\| \leq \sup_{\lambda_1 \leq \lambda < \infty} |u(t, \lambda)| \|u_0\|,$$

and using (1.15)(i) and (1.15)(ii), it follows that

$$(1.20) \quad \int_0^{\infty} \|u(t)\| dt \leq C \|u_0\|,$$

$$(1.21) \quad \int_0^{\infty} \|u'(t)\| dt \leq C \|u_0\|_1,$$

respectively. The numerical analogue of (1.14) follows such that, for $\lambda > 0$, assume that the real sequence $\{U^n(\lambda)\}$ is the solution

$$(1.22) \quad k \sum_{n=0}^{\infty} U^n(\lambda) \zeta^n = \widehat{u} \left(\frac{\delta(\zeta)}{k}, \lambda \right), \quad \zeta \in \mathbb{C}, |\zeta| < 1,$$

where $\widehat{u}(z, \lambda)$ is the Laplace transform of the solution of (1.14) with $\widehat{u}(z, \lambda) = [zI + A(z)\lambda]^{-1}$. Then it is easy to see that the solution of (1.6) can be represented as

$$(1.23) \quad U^n = \sum_{j=1}^{\infty} U^n(\lambda_j) (u_0, \varphi_j) \varphi_j.$$

This is a numerical analogue of (1.19). Also, we represent the solutions of (1.22) as the analogue of (1.9)-(1.10),

$$(1.24) \quad \frac{U^n(\lambda) - U^{n-1}(\lambda)}{k} + \frac{\lambda}{2} \left(\sum_{j=0}^n \omega_j(k) U^{n-j}(\lambda) + \sum_{j=0}^{n-1} \omega_j(k) U^{n-1-j}(\lambda) \right) \\ = 0, \text{ for } n \geq 2,$$

$$(1.25) \quad U^1(\lambda) = (2 + k\omega_0(k)\lambda)^{-1} (4 - k\omega_1(k)\lambda) U^0(\lambda),$$

$$(1.26) \quad U^0(\lambda) = (2 + k \omega_0(k) \lambda)^{-1}.$$

We introduce the notation

$$\tau(k, y) = \frac{2 \sin(k y)}{k(1 + \cos(k y))}, \quad \frac{\delta(e^{-ik y})}{k} = i \tau(k, y) = s(k, y) = s, \quad 0 \leq k y < \pi$$

and

$$D(z, \lambda) = D(\sigma + i \tau, \lambda) = \frac{z}{\lambda} + \widehat{A}(z) = \frac{\sigma + i \tau}{\lambda} + \widehat{A}(\sigma + i \tau).$$

We use the representation (cf. [34, 4.4])

$$(1.27) \quad U^n(\lambda) = \frac{k^{-1}}{2\pi i} \int_{|\zeta|=\gamma} \zeta^{-n-1} \widehat{u}\left(\frac{\delta(\zeta)}{k}, \lambda\right) d\zeta, \quad 0 < \gamma < 1.$$

We now change variables $\zeta = e^{-k} z$ and deform the resulting contour

$$\{z = k^{-1} \log(1/\gamma) + i y; |y| \leq \frac{\pi}{k}\}$$

to

$$\Gamma_k = \{z = i y, |y| \leq \frac{\pi}{k}\},$$

(note that $\widehat{u}\left(\frac{\delta(e^{-z k})}{k}; \lambda\right)$ is analytic in $\Pi^+ = \{z : \operatorname{Re} z > 0\}$, and continuous on $\overline{\Pi^+} = \{z; \operatorname{Re} z \geq 0\}$, and the contributions from the line segments on $\operatorname{Im} z = \pm \frac{\pi}{k}$ cancel by periodicity), and get

$$(1.28) \quad U^n(\lambda) = \frac{1}{2\pi} \int_{-\frac{\pi}{k}}^{\frac{\pi}{k}} e^{i y t_n} \widehat{u}\left(\frac{\delta(e^{-i y k})}{k}, \lambda\right) dy.$$

As in [34], we factor λ^{-1} , integrate by parts and change the integration variable on $(-\frac{\pi}{k}, 0)$ to obtain

$$(1.29) \quad U^n(\lambda) = \operatorname{Re} \left\{ \frac{1}{\pi t_n \lambda} \int_0^{\frac{\pi}{k}} e^{i y t_{n-1}} \frac{D_s(s(k, y), \lambda)}{D^2(s(k, y), \lambda)} \frac{4}{(1 + e^{-i y k})^2} dy \right\}.$$

Here and below, $D_s(s, \lambda)$ denotes the derivative with respect to s , thus $D_s(s, \lambda) = \frac{\partial D(s, \lambda)}{\partial s}$. Following [34, 4.7] or [3, 2.23] (also see [4, 2.21]) we can write (1.29) as

$$(1.30) \quad U^n(\lambda) = \operatorname{Re} [\lambda^{-1} U_1^n + \lambda^{-2} U_2^n + \lambda^{-3} U_3^n + U_4^n(\lambda) + U_5^n(\lambda) + U_6^n(\lambda)],$$

where

(1.31)

$$\begin{aligned}
 U_1^n &= \frac{1}{\pi t_n} \int_0^\rho e^{iy t_{n-1}} \frac{\widehat{A}'(s)}{[\widehat{A}(s)]^2} \frac{4}{(1 + e^{-iyk})^2} dy, \\
 U_2^n &= \frac{1}{\pi t_n} \int_0^\rho e^{iy t_{n-1}} \frac{1}{[\widehat{A}(s)]^2} \left(1 - \frac{2s \widehat{A}'(s)}{\widehat{A}(s)} \right) \frac{4}{(1 + e^{-iyk})^2} dy, \\
 U_3^n &= - \frac{1}{\pi t_n} \int_0^\rho e^{iy t_{n-1}} \frac{2s}{[\widehat{A}(s)]^3} \frac{4}{(1 + e^{-iyk})^2} dy, \\
 U_4^n(\lambda) &= \frac{1}{\pi t_n \lambda^3} \int_0^\rho e^{iy t_{n-1}} \frac{s^2 D_s(s, \lambda)}{[\widehat{A}(s)]^2 D(s, \lambda)} \left(\frac{2}{\widehat{A}(s)} + \frac{1}{D(s, \lambda)} \right) \frac{4}{(1 + e^{-iyk})^2} dy, \\
 U_5^n(\lambda) &= \frac{1}{\pi t_n \lambda} \int_{\frac{\pi}{2k}}^{\frac{\pi}{2k}} e^{iy t_{n-1}} \frac{D_s(s, \lambda)}{[D(s, \lambda)]^2} \frac{4}{(1 + e^{-iyk})^2} dy, \\
 U_6^n(\lambda) &= \frac{1}{\pi t_n \lambda} \int_{\frac{\pi}{2k}}^{\frac{\pi}{2k}} e^{iy t_{n-1}} \frac{D_s(s, \lambda)}{[D(s, \lambda)]^2} \frac{4}{(1 + e^{-iyk})^2} dy,
 \end{aligned}$$

with ρ as in (2.15) below, which is an appropriately small positive constant.

Now we define a new numerical solution of (1.1) by modifying $U_6^n(\lambda)$. We define the numerical solution $\widetilde{U}^n$ by the representation

$$(1.32) \quad \widetilde{U}^n = \sum_{j=1}^{\infty} \widetilde{U}^n(\lambda_j) (u_0, \varphi_j) \varphi_j,$$

where

$$(1.33) \quad \widetilde{U}^n(\lambda) = \operatorname{Re} [\lambda^{-1} U_1^n + \lambda^{-2} U_2^n + \lambda^{-3} U_3^n + U_4^n(\lambda) + U_5^n(\lambda) + \widetilde{U}_6^n(\lambda)],$$

with

$$(1.34) \quad \widetilde{U}_6^n(\lambda) = \frac{1}{\pi t_n \lambda} \int_{\pi/2k}^{\pi/k} e^{iy t_{n-1}} \frac{D_s(s, \lambda)}{[D(s, \lambda)]^2} dy.$$

The error estimate (1.13) is a long time l_t^∞ error estimate away from 0, but it does not belong to $l_t^1(t_n, \infty)$ for t_n away from 0. We shall establish the following long time $l_t^\infty \cap l_t^1(t_n, \infty)$ error estimate with t_n away from 0.

Theorem 1.2. *Let the conditions of Theorem 1.1 be in force. Then, for the solutions of (1.32) and (1.1), and for k sufficiently small, we get*

$$(1.35) \quad \|\widetilde{U}^n - u(t_n)\| \leq C k t_{n-1}^{-2} \|u_0\|_3, \text{ for } t_{n-1} > 0.$$

The proof will be given in Section 6.

Formulas (1.13) and (1.35) tells us that U^n or $\widetilde{U}^n$ approximating the solution $u(t)$ of (1.1) in the point t_n is not appropriate when t_n approximates zero. To remedy this difficulty we consider the following numerical scheme

$$(1.36) \quad U_v^n = u_0 + V^n$$

in which V^n satisfies

$$\widehat{v} \left(\frac{\delta(\zeta)}{k} \right) = k \sum_{n=0}^{\infty} V^n \zeta^n, \quad \zeta \in \mathbb{C}, \quad |\zeta| < 1,$$

where

$$(1.37) \quad \widehat{v}(z) = -\frac{\widehat{A}(z)}{z D(z, L)} u_0 = -\frac{\widehat{A}(z) L}{z(z I + \widehat{A}(z) L)} u_0,$$

is the Laplace transform of the solution $v(t)$ for the problem

$$(1.38) \quad \begin{aligned} v_t + \int_0^t A(t-s) L v(s) ds &= - \int_0^t A(t-s) ds L u_0, \quad t > 0, \\ v(0) &= 0, \end{aligned}$$

with $u_0 \in D(L)$.

We write the solution $u(t)$ of (1.1) as $u(t) = u_0 + v(t)$, we shall establish the following long time l_t^∞ error estimate.

Theorem 1.3. *Assume that (1.4) and (1.5) are satisfied. If, in addition, (1.11) holds, then we have, for the solutions of (1.36) and (1.1), and for k sufficiently small,*

$$(1.39) \quad \|U_v^n - u(t_n)\| \leq C k^2 |\log k| \|u_0\|_4, \quad \text{for } n \geq 0.$$

The proof will be given in Section 3.

The schemes (1.6) and (1.36) tell us that the initial data U^0 does not equal u_0 , which are not consistent with the methods. To remedy this inconsistency we use for the approximation of the integral a quadrature rule of the form

$$q_n(\varphi) = \sum_{j=0}^n \omega_j(k) \varphi^{n-j} - \frac{1}{2} \omega_n(k) \varphi^0,$$

and we thus define our scheme to be

$$(1.40) \quad \begin{aligned} \frac{U_u^n - U_u^{n-1}}{k} + \frac{1}{2} \left[\sum_{j=0}^n \omega_j(k) L U_u^{n-j} - \frac{1}{2} \omega_n(k) L U_u^0 \right. \\ \left. + \sum_{j=0}^{n-1} \omega_j(k) L U_u^{n-1-j} - \frac{1}{2} \omega_{n-1}(k) L U_u^0 \right] &= 0, \quad \text{for } n \geq 2, \\ \frac{U_u^1 - U_u^0}{k} + \frac{1}{2} \left[\sum_{j=0}^1 \omega_j(k) L U_u^{1-j} - \frac{1}{2} \omega_1(k) L U_u^0 \right. \\ \left. + \omega_0(k) L U_u^0 - \frac{1}{2} \omega_0(k) L U_u^0 \right] &= \frac{1}{4} \omega_0(k) L U_u^0, \\ U_u^0 &= u_0. \end{aligned}$$

The particular choice of U_u^1 is motivated by the fact that it will yield the desired error estimates. For the scheme (1.40) we shall derive smooth data error estimate.

Theorem 1.4. *Assume that (1.4) and (1.5) are satisfied. If, in addition, (1.11) holds, then we have, for the solutions of (1.40) and (1.1), and for k sufficiently small, we get*

$$(1.41) \quad \|U_u^n - u(t_n)\| \leq C k^2 |\log k| \|u_0\|_4, \quad \text{for } n \geq 0.$$

The proof will be given in Section 7.

As the results of (1.13) and (1.35), we see that the numerical schemes (1.6) and (1.32) are first-order accurate away from 0, for problems of the form (1.1) with the conditions of Theorem 1.1. In comparison, (1.36) is second-order accurate at any time, by Theorem 1.3.

Define

$$(1.42) \quad U_d^n = U_v^n, \text{ if } t_{n-1} \leq 1, \text{ and } U_d^n = \tilde{U}^n, \text{ if } t_{n-1} > 1.$$

By Theorems 1.2 and 1.3 we have obtained the following long time l_t^1 error estimate.

Theorem 1.5. *The conditions of Theorem 1.1 imply that*

$$(1.43) \quad k \sum_{n=0}^{\infty} \|U_d^n - u(t_n)\| \leq C k \|u_0\|_3 + C k^2 |\log k| \|u_0\|_4,$$

with k sufficiently small.

In Section 2 we collect some preliminary results. Sections 3, 4, 5, 6, and 7 contain the proofs. Finally, some numerical illustrations are presented in Section 8.

2. PRELIMINARY ESTIMATES

Throughout this paper, the letter C denotes a finite, *a priori* constant; the numerical value of C may change from line to line. Unless further restricted, any inequality involving C is valid for $\lambda_1 \leq \lambda < \infty$. We assume throughout that (1.4), (1.5) and (1.11) hold.

Define the Fourier transform of $a(t)$ and $b(t)$ by

$$\tilde{a}(\tau) = \hat{a}(i\tau) = \int_0^{\infty} e^{-i\tau t} a(t) dt = \phi(\tau) - i\tau\theta(\tau) \quad (\text{Im } \tau \leq 0)$$

and

$$\tilde{b}(\tau) = \hat{b}(i\tau) = \int_0^{\infty} e^{-i\tau t} b(t) dt = \phi_1(\tau) - i\tau\theta_1(\tau) \quad (\text{Im } \tau \leq 0),$$

where ϕ , ϕ_1 , θ , and θ_1 are real. We first recall some consequences of (1.5). By [7], the functions $\tilde{a}(\tau)$ and $\tilde{b}(\tau)$ are analytic in $\{\text{Im } \tau < 0\}$ and continuous in $S = \{\tau \in \mathbb{C} : \text{Im } \tau \leq 0, \tau \neq 0\}$. Moreover, if $\text{Im } \tau < 0$ and $\text{Re } \tau > 0$, $\tilde{a}(\tau)$ and $\tilde{b}(\tau)$ lie in $\{-\pi/2 < \arg w < 0\}$; if (1.11) also holds, this conclusion remains true when $\text{Im } \tau = 0$, $\tau > 0$. In this paper, $-\pi < \arg w \leq \pi$ ($w \in \mathbb{C}$).

From [10], we know that $\tilde{a}(\tau)$ and $\tilde{b}(\tau)$ are differentiable for $\tau > 0$ and

$$(2.1) \quad \frac{1}{2\sqrt{2}} \int_0^{\frac{1}{\tau}} a(t) dt \leq |\tilde{a}(\tau)| \leq 4 \int_0^{\frac{1}{\tau}} a(t) dt, \quad \tau > 0,$$

$$(2.2) \quad |\tilde{a}'(\tau)| \leq 40 \int_0^{\frac{1}{\tau}} t a(t) dt, \quad \tau > 0,$$

$$(2.3) \quad \int_0^1 \left(\frac{\int_0^{1/\tau} t a(t) dt}{\left(\int_0^{1/\tau} a(t) dt \right)^2} \right) d\tau < \infty,$$

with similar estimates for b . By [5, Lemma 4.1],

$$(2.4) \quad \theta'(\tau) < 0 \quad (\tau > 0).$$

If (1.11) also holds, [5, Lemma 5.1] shows that $\tilde{a}(\tau)$ and $\tilde{b}(\tau)$ belong to $C^2(0, \infty)$ with

$$(2.5) \quad |\tilde{a}''(\tau)| \leq 6000 \int_0^{1/\tau} t^2 a(t) dt \quad (\tau > 0),$$

and [11, pp. 236] shows that

$$(2.6) \quad \begin{aligned} C B(1/\tau) &\leq \phi(\tau) \leq K B(1/\tau), \quad \tau > 0, \\ C B_b(1/\tau) &\leq \phi_1(\tau) \leq K B_b(1/\tau), \quad \tau > 0, \end{aligned}$$

for some positive constants C and K , where

$$B(x) = \int_0^x -s a'(s) ds \quad \text{and} \quad B_b(x) = \int_0^x -s b'(s) ds.$$

By (2.1)–(2.3) and (2.5),

$$(2.7) \quad \int_0^1 \frac{|\tau \tilde{a}''(\tau)| + |\tilde{a}'(\tau)|}{|\tilde{a}(\tau)|^2} d\tau < \infty.$$

The inequalities for b which are analogous to (2.1)–(2.5) and (2.7) also hold.

The fractional linear transformation F maps $\{Re w \geq 0\}$ onto the disk with diameter $[m/q, 1/p]$. Moreover,

$$(2.8) \quad \begin{aligned} 0 &< \arg F(w) < \arg w \quad (0 < \arg w < \pi/2), \\ F(\overline{w}) &= \overline{F(w)} \quad (w \in \mathbb{C}). \end{aligned}$$

It follows that $\arg(\tilde{a}(\tau) \tilde{f}(\tau))$ lies between $\arg \tilde{a}(\tau)$ and $\arg \tilde{b}(\tau)$ with $\tilde{f}(\tau) = f(i\tau)$, and strictly between them when $\arg \tilde{a}(\tau) \neq \arg \tilde{b}(\tau)$ ($\tau \in S$). In particular, if we write

$$(2.9) \quad \tilde{A}(\tau) = \hat{A}(i\tau) = \Phi(\tau) - i\tau\Theta(\tau),$$

then in $\{\operatorname{Im} \tau < 0\}$ (and in S when (1.11) holds) we have,

$$(2.10) \quad \Phi(\tau) > 0.$$

We recall some results of Hannsgen [3] or Noren [4] in (2.11)–(2.16) below:

$$(2.11) \quad \lim_{\tau \rightarrow 0, \tau \in S} \frac{1}{\tilde{A}(\tau)} \text{ exists and is real and nonnegative.}$$

For each $\lambda > 0$,

$$(2.12) \quad \hat{u}(i\tau, \lambda) \text{ is continuous in } \{\operatorname{Im} \tau \leq 0\}.$$

Since $\tilde{a}(\tau), \tilde{b}(\tau) \in C^2(0, \infty)$ when (1.11) holds, therefore $\tilde{A}(\tau) = \hat{A}(i\tau) \in C^2(0, \infty)$. We see from (1.4), (2.1), (2.5), (2.7) and (2.3) that

$$(2.13) \quad \int_0^1 \frac{|\tau \tilde{A}''(\tau)| + |\tilde{A}'(\tau)|}{|\tilde{A}(\tau)|^2} d\tau < \infty,$$

$$(2.14) \quad \frac{|\tau \tilde{A}'(\tau)| + |\tau^2 \tilde{A}''(\tau)|}{|\tilde{A}(\tau)|} = O(1) \quad (\tau \rightarrow 0^+),$$

$$(2.15) \quad \tau^{j+1} \tilde{A}^{(j)}(\tau) = O(1) \quad (\tau \rightarrow \infty, j = 0, 1, 2).$$

With $D(i\tau, \lambda) = \tilde{A}(\tau) + i\tau/\lambda$, choose $\rho > 0$ such that

$$(2.16) \quad |\tilde{A}(\tau)| \geq 2\tau/\lambda_1 \quad (0 < \tau \leq 2\rho).$$

By (2.11), ρ exists. Using the complex inversion formula for Laplace transforms, together with a contour shift and integration by parts, we have the representation (cf. [3, 2.23] or [4, 2.21] and [35, 2.2])

$$(2.17) \quad u(t, \lambda) = \operatorname{Re}[\lambda^{-1}u_1(t) + \lambda^{-2}u_2(t) + \lambda^{-3}u_3(t) + u_4(t, \lambda) + u_5(t, \lambda) + u_6(t, \lambda)],$$

where (see [5, Eq. (4.34)] for the computation)

$$(2.18) \quad \begin{aligned} u_1(t) &= \frac{1}{\pi t} \int_0^\rho e^{i\tau t} \frac{\hat{A}'(i\tau)}{[\hat{A}(i\tau)]^2} d\tau, \\ u_2(t) &= \frac{1}{\pi t} \int_0^\rho e^{i\tau t} \frac{1}{[\hat{A}(i\tau)]^2} \left(1 - \frac{2i\tau \hat{A}'(i\tau)}{\hat{A}(i\tau)} \right) d\tau, \\ u_3(t) &= -\frac{1}{\pi t} \int_0^\rho e^{i\tau t} \frac{2i\tau}{[\hat{A}(i\tau)]^3} d\tau, \\ u_4(t, \lambda) &= \frac{1}{\pi t \lambda^3} \int_0^\rho e^{i\tau t} \frac{(i\tau)^2 D_s(i\tau, \lambda)}{[\hat{A}(i\tau)]^2 D(i\tau, \lambda)} \left(\frac{2}{\hat{A}(i\tau)} + \frac{1}{D(i\tau, \lambda)} \right) d\tau, \\ u_5(t, \lambda) &= \frac{1}{\pi t \lambda} \int_\rho^{\pi/2k} e^{i\tau t} \frac{D_s(i\tau, \lambda)}{[D(i\tau, \lambda)]^2} d\tau, \\ u_6(t, \lambda) &= \frac{1}{\pi t \lambda} \int_{\pi/2k}^\infty e^{i\tau t} \frac{D_s(i\tau, \lambda)}{[D(i\tau, \lambda)]^2} d\tau. \end{aligned}$$

Here, $\hat{A}'(s)$ denotes the derivative with respect to s , thus

$$\hat{A}'(i\tau) = \frac{d\hat{A}(s)}{ds} \Big|_{s=i\tau}.$$

Writing $\Theta(\tau)$ of (2.9) explicitly in terms of ϕ , ϕ_1 , θ and θ_1 , a straightforward calculation shows that

$$(2.19) \quad \lim_{\tau \rightarrow \infty} \frac{\Theta(\tau)}{\theta(\tau)} = L = \frac{p b(0)^2 + (mp + q)a(0) b(0) + mp a(0)^2}{(pb(0) + qa(0))^2}.$$

Following [4] we need a lower bound on $\operatorname{Im} D(i\tau, \lambda/L) = \tau(L/\lambda - \Theta(\tau))$. Define $\omega_1 = \omega_1(\lambda) > 0$ by $\theta(\omega_1) = \lambda^{-1}$. By (2.4), ω_1 is unique when it exists. Now let $\omega(\lambda) = \omega_1(\lambda)$ when ω_1 exists and $\omega_1 \geq \rho$, and let $\omega(\lambda) = \rho$ otherwise. As shown in [8], ω is continuous and nondecreasing function on $\{\lambda > 0\}$.

(2.31) and (2.33) of [4] imply that there is a positive constant $\rho_6 > \rho$ such that

$$(2.20) \quad \begin{aligned} (i) \quad & \int_0^{1/\tau} -a'(s) - b'(s) ds \leq \frac{L a(0)}{8 C_1} \frac{(4 - \pi)}{4}, \quad \tau \geq \rho_6, \\ (ii) \quad & \frac{\omega^2}{2 a(0)} \leq \lambda \leq \frac{2\omega^2}{a(0)}, \quad \text{for } \omega/2 \geq \rho_6, \\ (iii) \quad & \varepsilon(\lambda) \leq \frac{\omega(4 - \pi)}{4} < \omega/2, \quad \omega/2 \geq \rho_6, \end{aligned}$$

where

$$(2.21) \quad \varepsilon(\lambda) = \frac{8 C_1}{a(0) L} \omega \int_0^{2/\omega} -a'(s) - b'(s) ds.$$

For our estimates on (ρ_6, ∞) we will need the inequalities

$$(2.22) \quad |\Theta(\tau) - \frac{L}{\lambda}| \geq C L \omega^2 / (2 \tau^2 \lambda), \quad \rho_6 \leq \tau \leq \omega/2,$$

$$(2.23) \quad |\Theta(\tau) - \frac{L}{\lambda}| \geq C \frac{L |(\omega - \tau)|}{2 \tau \lambda}, \quad \rho_6 \leq \omega/2 < \tau < \omega - \varepsilon, \text{ or } \omega + \varepsilon < \tau,$$

which are contained [4].

For $\omega - \varepsilon < \tau < \omega + \varepsilon$ we need a lower bound for $\operatorname{Re} D(i\tau, \lambda) = \Phi(\tau)$ which is defined by (2.9), as was done in [4, 2.35] to obtain

$$(2.24) \quad \frac{1}{\Phi} \leq C \left(\frac{1}{\phi(\tau)} + \frac{1}{\phi_1(\tau)} \right) \leq C \left(\frac{1}{B(\tau^{-1})} + \frac{1}{B_b(\tau^{-1})} \right), \quad \tau \geq \rho.$$

From [3, 2.12] we know that

$$(2.25) \quad \lim_{\tau \rightarrow +\infty} \tau^2 \Phi(\tau) = \begin{cases} \frac{b'(0^+)(mp-q) - a'(0^+)[p\alpha^2 + 2mp\alpha + mq]}{(p\alpha+q)^2} & -a'(0^+) - b'(0^+) < \infty, \\ \infty & -a'(0^+) - b'(0^+) = \infty, \end{cases}$$

where $\alpha = b(0^+)/a(0^+)$. Then by (2.25) and (2.20)(ii),

$$(2.26) \quad \sup_{\omega(\lambda) \geq 2\rho_6, |\tau-\omega| \leq \varepsilon_0} \frac{1}{\lambda \Phi(\tau)} < \infty.$$

If in addition (1.12) on the kernels $a(t)$ and $b(t)$ holds, then we have the following estimate.

Lemma 2.1. *Suppose that (1.5), (1.11) and (1.12) hold. Then $\tilde{a}(\tau)$ is three continuously differentiable and*

$$(2.27) \quad |\tilde{a}'''(\tau)| \leq C \int_0^{\tau^{-1}} t^3 a(t) dt, \quad \tau > 0.$$

The analogous results for b are, of course, valid. The proof of this lemma is analogous to that of Lemma 6.1 in [8]. We omit the details.

Therefore $\tilde{A}(\tau) = \hat{A}(i\tau) \in C^3(0, \infty)$ under the hypotheses of Lemma 2.1, and we have

Lemma 2.2. *If the assumptions of Lemma 2.1 hold, then*

$$(2.28) \quad \int_0^1 \frac{|\tau^2 \tilde{A}'''(\tau)|}{|\tilde{A}(\tau)|^2} d\tau < \infty,$$

$$(2.29) \quad \frac{|\tau^3 \tilde{A}'''(\tau)|}{|\tilde{A}(\tau)|} = O(1) \quad (\tau \rightarrow 0^+),$$

$$(2.30) \quad \tau^4 \tilde{A}'''(\tau) = O(1) \quad (\tau \rightarrow \infty).$$

The proof will be given in the Appendix.

Lemma 2.3. *We have*

$$\frac{\delta(e^{-iky})}{k} - iy = O(k^2 y^3) = O(k y^2),$$

when $y \in \mathbb{R}$, $|y| \leq \pi/(2k)$, where $\delta(\zeta) = 2(1 - \zeta)/(1 + \zeta)$, $\zeta \in \mathbb{C}$, $\zeta \neq -1$.

Proof. Set

$$(2.31) \quad h(z) = \delta(e^{-z}) = \frac{2(1 - e^{-z})}{1 + e^{-z}}.$$

Then $h(z)$ is analytic in the open disk $S_\pi = \{z \in \mathbb{C} : |z| < \pi\}$, so that we have

$$(2.32) \quad h(z) = h(0) + h'(0)z + \frac{h''(0)}{2!}z^2 + \frac{h'''(0)}{3!}z^3 + \dots, \text{ for } |z| < \pi.$$

By simple computations we may obtain

$$(2.33) \quad h(0) = 0, \quad h'(0) = 1, \quad h''(0) = 0, \quad h'''(0) = -\frac{1}{2}.$$

From (2.32) and (2.33) we now have established the expansion of the power series

$$(2.34) \quad h(z) = z - \frac{1}{12}z^3 + \dots, \text{ for } |z| < \pi.$$

It is now a clear matter that, from (2.31) and (2.34) we get

$$\delta(e^{-iky}) = iky - \frac{1}{12}(iky)^3 + \dots, \text{ for } y \in R, \quad |y| \leq \frac{\pi}{2k}.$$

This implies that

$$\frac{\delta(e^{-iky})}{k} - iy = O(k^2 y^3) = O(k y^2), \text{ for } y \in R, \quad |y| \leq \frac{\pi}{2k}.$$

This completes the proof of Lemma 2.3. $\square$

3. PROOF OF THEOREM 1.3.

Let us assume that $U_v^n(\lambda) = 1 + V^n(\lambda)$ in which $V^n(\lambda)$ satisfies

$$(3.1) \quad k \sum_{n=0}^{\infty} V^n(\lambda) \zeta^n = \widehat{v}\left(\frac{\delta(\zeta)}{k}, \lambda\right), \quad \zeta \in \mathbb{C}, \quad |\zeta| < 1,$$

where $\widehat{v}(z, \lambda) = -\frac{\widehat{A}(z)}{zD(z, \lambda)}$ is the Laplace transform of the solution of $v(t, \lambda) = u(t, \lambda) - 1$.

This gives us the representation (cf. (1.28))

$$(3.2) \quad V^n(\lambda) = \operatorname{Re} \left\{ \frac{1}{\pi} \int_0^{\frac{\pi}{k}} e^{iyt_n} \widehat{v}\left(\frac{\delta(e^{-iyk})}{k}, \lambda\right) dy \right\}.$$

On the other hand, we apply the Laplace inversion formula, since $\widehat{v}(z, \lambda)$ is analytic in $\overline{\Pi}^+ = \{z : \operatorname{Re} z > 0\}$, and continuous on $\overline{\Pi}^+ = \{z; \operatorname{Re} z \geq 0\}$, to obtain that

$$(3.3) \quad v(t_n, \lambda) = \operatorname{Re} \left\{ \frac{1}{\pi} \int_0^{\infty} e^{iyt_n} \widehat{v}(iy, \lambda) dy \right\}.$$

Subtracting (3.3) from (3.2), we have

$$(3.4) \quad U_v^n(\lambda) - u(t_n, \lambda) = \frac{1}{\pi} \operatorname{Re} \{ E_1^n(\lambda) + E_2^n(\lambda) + V_3^n(\lambda) - v_3(t_n, \lambda) - v_4(t_n, \lambda) \},$$

where

$$(3.5) \quad E_1^n(\lambda) = V_1^n(\lambda) - v_1(t_n, \lambda) = \int_0^{\rho} e^{iyt_n} \left[\widehat{v}\left(\frac{\delta(e^{-iyk})}{k}, \lambda\right) - \widehat{v}(iy, \lambda) \right] dy,$$

$$(3.6) \quad \begin{aligned} E_2^n(\lambda) &= V_2^n(\lambda) - v_2(t_n, \lambda) \\ &= \int_{\rho}^{\frac{\pi}{2k}} e^{iy t_n} \left[\widehat{v} \left(\frac{\delta(e^{-iyk})}{k}, \lambda \right) - \widehat{v}(iy, \lambda) \right] dy, \end{aligned}$$

$$(3.7) \quad V_3^n(\lambda) = \int_{\frac{\pi}{2k}}^{\frac{\pi}{k}} e^{iy t_n} \widehat{v} \left(\frac{\delta(e^{-iyk})}{k}, \lambda \right) dy,$$

$$(3.8) \quad v_3(t_n, \lambda) = \int_{\frac{\pi}{2k}}^{\frac{\pi}{k}} e^{iy t_n} \widehat{v}(iy, \lambda) dy,$$

$$(3.9) \quad v_4(t_n, \lambda) = \int_{\frac{\pi}{k}}^{\infty} e^{iy t_n} \widehat{v}(iy, \lambda) dy.$$

We use the notation

$$(3.10) \quad \tau(k, y) = \frac{2 \sin(ky)}{k(1 + \cos(ky))}, \quad \frac{\delta(e^{-iky})}{k} = i\tau(k, y) = s(k, y) = s, \quad 0 \leq ky < \pi,$$

to obtain, for $0 \leq y \leq \rho$, and k appropriately small,

$$(3.11) \quad \frac{y}{4} \leq \tau \leq 2y.$$

By the mean-value theorem and Lemma 2.3 $s - iy = O(k^2 y^3)$, for $0 \leq y \leq \rho$, we find that

$$(3.12) \quad |\widehat{v}(s, \lambda) - \widehat{v}(iy, \lambda)| \leq C k^2 y^3 |\widehat{v}'(s_1, \lambda)|,$$

with $s_1 = s_1(k, y) = i\tau_1(k, y)$, $\tau_1(k, y)$ lie between y and $\tau(k, y)$, and

$$(3.13) \quad \widehat{v}'(z, \lambda) = -\frac{\widehat{A}'(z)}{z D(z, \lambda)} + \frac{\widehat{A}(z)}{z^2 D(z, \lambda)} + \frac{\widehat{A}(z) D'_z(z, \lambda)}{z D^2(z, \lambda)}.$$

Therefore, we have

$$\begin{aligned} |E_1^n(\lambda)| &\leq C k^2 \int_0^{\rho} y^3 |\widehat{v}'(s_1, \lambda)| dy \\ &\leq C k^2 \int_0^{\rho} \left[\frac{y^2 |\widehat{A}'(i\tau_1)|}{|D(i\tau_1, \lambda)|} + \frac{y |\widehat{A}(i\tau_1)|}{|D(i\tau_1, \lambda)|} + \frac{y^2 |\widehat{A}(i\tau_1)| |D'_z(i\tau_1, \lambda)|}{|D^2(i\tau_1, \lambda)|} \right] dy. \end{aligned}$$

From (1.2)–(1.5) and (2.1) we have

$$(3.14) \quad C_1 |\widetilde{a}(y)| \leq |\widehat{A}(iy)| \leq C_2 |\widetilde{a}(y)|, \quad \text{for } y > 0.$$

We use (2.16) to obtain

$$(3.15) \quad |D(iy, \lambda)| \geq |\widehat{A}(iy)| - \frac{y}{\lambda} \geq \frac{1}{2} |\widehat{A}(iy)|$$

and

$$(3.16) \quad |D(iy, \lambda)| \geq |\widehat{A}(iy)| - \frac{y}{\lambda} \geq \frac{1}{\lambda_1} y,$$

for $\lambda \geq \lambda_1$, $0 < y \leq 2\rho$. Thus, by (3.14)–(3.16), (2.14) and (3.11) we have that

$$(3.17) \quad |E_1^n(\lambda)| \leq C k^2, \quad \text{for } n \geq 0, \lambda \geq \lambda_1.$$

Next we consider the estimate of $E_2^n(\lambda)$. Similarly, applying the mean-value theorem we have that, with $s - iy = O(k^2 y^3)$ for $\rho \leq y \leq \frac{\pi}{2k}$,

(3.18)

$$\begin{aligned} |E_2^n(\lambda)| &\leq C k^2 \int_{\rho}^{\frac{\pi}{2k}} y^3 |\widehat{v}'(s_1, \lambda)| dy \\ &\leq C k^2 \int_{\rho}^{\frac{\pi}{2k}} \left[\frac{y^3 |\widehat{A}'(i\tau_1)|}{\tau_1 |D(i\tau_1, \lambda)|} + \frac{y^3 |\widehat{A}(i\tau_1)|}{\tau_1^2 |D(i\tau_1, \lambda)|} + \frac{y^3 |\widehat{A}(i\tau_1)| |D'_z(i\tau_1, \lambda)|}{\tau_1 |D^2(i\tau_1, \lambda)|} \right] dy, \end{aligned}$$

where $s_1 = i\tau_1(k, y)$, $\tau_1(k, y)$ lies between y and $\tau(k, y)$.

Now, (3.10) implies that

$$(3.19) \quad \frac{2}{\pi} y \leq \tau(k, y) \leq 2y, \text{ for } \rho \leq y \leq \frac{\pi}{2k},$$

for k appropriately small. We use (3.18), (3.19) and (2.15) to obtain

(3.20)

$$|E_2^n(\lambda)| \leq C k^2 \int_{\rho}^{\frac{\pi}{2k}} \left[\frac{1}{|D(i\tau_1, \lambda)|} + \frac{y}{\lambda |D(i\tau_1, \lambda)|^2} + \frac{1}{y |D^2(i\tau_1, \lambda)|} \right] dy.$$

Here we have to bound the various terms on the right-hand side. By (3.19) we see that

$$\frac{2}{\pi} y \leq \tau_1 \leq y \quad \text{or} \quad y \leq \tau_1 \leq 2y, \quad \rho \leq y \leq \frac{\pi}{2k}.$$

Below we only consider the case

$$(3.21) \quad \frac{2}{\pi} y \leq \tau_1 \leq y, \quad \text{for } \rho \leq y \leq \frac{\pi}{2k}.$$

For the case

$$y \leq \tau_1 \leq 2y, \quad \text{for } \rho \leq y \leq \frac{\pi}{2k},$$

the argument is similar to (3.21). We omit the details.

Thus, from (2.22), (2.23), (2.24) and (2.26) we have

$$(3.22) \quad |D(i\tau_1, \lambda/L)| \geq C \frac{\omega^2}{(y\lambda)}, \quad \frac{\pi}{2} \rho_6 \leq y \leq \omega/2,$$

$$(3.23) \quad |D(i\tau_1, \lambda/L)| \geq C \frac{|\omega - y|}{\lambda}, \quad \frac{\pi}{4} \omega \leq y \leq \omega - \varepsilon, \quad \frac{\pi}{2}(\omega + \varepsilon) < \pi \omega \leq y,$$

$$(3.24) \quad \frac{1}{|D(i\tau_1, \lambda)|} \leq C \left(\frac{1}{B(y^{-1})} + \frac{1}{B_b(y^{-1})} \right), \quad \rho \leq y,$$

$$(3.25) \quad \frac{1}{\lambda |D(i\tau_1, \lambda)|} \leq C \frac{1}{\lambda \Phi(\tau_1)} \leq C, \quad |\tau_1 - \omega| \leq \varepsilon_0.$$

For $\frac{\pi}{2} \rho_6 \leq y \leq \omega/2$, we use (3.22) and (2.20)(ii) to obtain

$$\begin{aligned} (3.26) \quad &\int_{\frac{\pi}{2} \rho_6}^{\frac{\omega}{2}} \left[\frac{1}{|D(i\tau_1, \lambda/L)|} + \frac{y}{\lambda |D(i\tau_1, \lambda/L)|^2} + \frac{1}{y |D^2(i\tau_1, \lambda/L)|} \right] dy \\ &\leq C \int_{\frac{\pi}{2} \rho_6}^{\frac{\omega}{2}} \left[\frac{\lambda y}{\omega^2} + \frac{\lambda y^3}{\omega^4} + \frac{\lambda^2 y}{\omega^4} \right] dy \leq C \omega^2 \leq C \lambda. \end{aligned}$$

For $\pi\omega \leq y \leq \frac{\pi}{2k}$ we use (3.23) and (2.20)(ii) to obtain the estimate

$$(3.27) \quad \begin{aligned} & \int_{\pi\omega}^{\frac{\pi}{2k}} \left[\frac{1}{|D(i\tau_1, \lambda/L)|} + \frac{y}{\lambda |D(i\tau_1, \lambda/L)|^2} + \frac{1}{y |D^2(i\tau_1, \lambda/L)|} \right] dy \\ & \leq C \int_{\pi\omega}^{\frac{\pi}{2k}} \left[\frac{\lambda}{(y-\omega)} + \frac{\lambda y}{(y-\omega)^2} + \frac{\lambda^2}{y(y-\omega)^2} \right] dy \leq C \lambda^{\frac{3}{2}} |\log k|. \end{aligned}$$

For $\omega/2 \leq y \leq \pi\omega$ and $\frac{\pi}{2k} < \pi\omega$, by (3.25) and (2.20)(ii) we have the estimate

$$(3.28) \quad \begin{aligned} & \int_{\omega/2}^{\pi\omega} \left[\frac{1}{|D(i\tau_1, \lambda)|} + \frac{y}{\lambda |D(i\tau_1, \lambda)|^2} + \frac{1}{y |D^2(i\tau_1, \lambda)|} \right] dy \\ & \leq C \left(\frac{\omega}{\Phi(\omega)} + \frac{\omega^2}{\lambda \Phi^2(\omega)} + \frac{1}{\Phi^2(\omega)} \right) \leq C \lambda^{\frac{3}{2}} \omega \leq C \lambda^2. \end{aligned}$$

For $\rho \leq y \leq \frac{\pi}{2}\rho_6$, we use (3.24) to obtain

$$(3.29) \quad \begin{aligned} & \int_{\rho}^{\frac{\pi}{2}\rho_6} \left[\frac{1}{|D(i\tau_1, \lambda)|} + \frac{y}{\lambda |D(i\tau_1, \lambda)|^2} + \frac{1}{y |D^2(i\tau_1, \lambda)|} \right] dy \\ & \leq C \int_{\rho}^{\frac{\pi}{2}\rho_6} \left[\frac{1}{\Phi(\tau_1)} + \frac{y}{\lambda \Phi^2(\tau_1)} + \frac{1}{y \Phi^2(\tau_1)} \right] dy \leq C. \end{aligned}$$

Combining (3.20), (3.26), (3.27), (3.28) and (3.29) we have obtained that

$$(3.30) \quad |E_2^n(\lambda)| \leq C k^2 \lambda^2 |\log k|.$$

Returning to (3.4), we observe that

$$\begin{aligned} |v_3(t_n, \lambda) + v_4(t_n, \lambda)| & \leq \int_{\frac{\pi}{2k}}^{\infty} |\widehat{v}(iy, \lambda)| dy \\ & \leq \int_{\frac{\pi}{2k}}^{\infty} \frac{|\widehat{A}(iy)|}{y |D(iy, \lambda)|} dy. \end{aligned}$$

From (2.15) it follows that

$$(3.31) \quad |v_3(t_n, \lambda) + v_4(t_n, \lambda)| \leq \int_{\frac{\pi}{2k}}^{\infty} \frac{dy}{y^2 |D(iy, \lambda)|}.$$

The proof now splits into three cases.

Case 1. If $\rho_6 < \frac{\pi}{2k} < \omega/2$, then $\omega^{-1} \leq Ck$, so (2.22) implies

$$(3.32) \quad \begin{aligned} \int_{\frac{\pi}{2k}}^{\frac{\omega}{2}} \frac{dy}{y^2 |D(iy, \lambda)|} & \leq C \int_{\rho_6}^{\frac{\omega}{2}} \frac{\lambda}{y \omega^2} dy \leq C \frac{\lambda}{\omega^2} |\log \omega| \\ & \leq C k^2 \lambda |\log \omega| \leq C k^2 \lambda^{\frac{3}{2}}. \end{aligned}$$

We use (2.23) to obtain

$$(3.33) \quad \int_{2\omega}^{\infty} \frac{dy}{y^2 |D(iy, \lambda)|} \leq C \int_{2\omega}^{\infty} \lambda \frac{dy}{y^2 (y-\omega)} \leq C k^2 \lambda.$$

In the case $\frac{\omega}{2} < y < 2\omega$, (2.26) and (2.20)(ii) yield

$$(3.34) \quad \int_{\frac{\omega}{2}}^{2\omega} \frac{dy}{y^2 |D(iy, \lambda)|} \leq C \int_{\frac{\omega}{2}}^{2\omega} \frac{dy}{y^2 \Phi(y)} \leq C \frac{1}{\omega \Phi(\omega)} \leq C k^2 \lambda^{\frac{3}{2}}.$$

Combining (3.31)–(3.34) we have established the estimate as in Case 1.

$$(3.35) \quad |v_3(t_n, \lambda) + v_4(t_n, \lambda)| \leq C k^2 \lambda^{\frac{3}{2}}.$$

Case 2. If $\frac{\omega}{2} \leq \frac{\pi}{2k} < 2\omega$, then $\omega^{-1} \leq C k$ so (3.31), (2.23), (2.26) and (2.20)(ii) give us

$$(3.36) \quad |v_3(t_n, \lambda/L) + v_4(t_n, \lambda/L)| \leq \left(\int_{\frac{\omega}{2}}^{2\omega} + \int_{2\omega}^{\infty} \right) \frac{dy}{y^2 |D(iy, \lambda/L)|} \leq C k^2 \lambda^{\frac{3}{2}}.$$

Case 3. If $2\omega \leq \frac{\pi}{2k}$, then (2.23) implies

$$(3.37) \quad |v_3(t_n, \lambda/L) + v_4(t_n, \lambda/L)| \leq \int_{\frac{\pi}{2k}}^{\infty} \frac{dy}{y^2 |D(iy, \lambda)|} \leq C \lambda \int_{\frac{\pi}{2k}}^{\infty} \frac{dy}{y^2 (y - \omega)} \leq C k^2 \lambda.$$

Taken together with (3.35)–(3.37), this shows that

$$(3.38) \quad |v_3(t_n, \lambda) + v_4(t_n, \lambda)| \leq C k^2 \lambda^{\frac{3}{2}}.$$

Finally, we will derive an estimate on

$$(3.39) \quad |V_3^n(\lambda)| \leq \int_{\frac{\pi}{2k}}^{\frac{\pi}{k}} |\widehat{v}(iy, \lambda)| dy,$$

with

$$(3.40) \quad \tau = \frac{2 \sin(ky)}{k(1 + \cos(ky))}, \quad d\tau = \frac{2}{1 + \cos(ky)} dy.$$

Taking the variable substitution (3.40) in the integral of (3.39) we have

$$|V_3^n(\lambda)| \leq C \int_{\frac{2}{k}}^{\infty} |\widehat{v}(iy, \lambda)| dy.$$

Repeating the proof of (3.38) we now obtain

$$(3.41) \quad |V_3^n(\lambda)| \leq C k^2 \lambda^{\frac{3}{2}}.$$

Combining (3.4), (3.17), (3.30), (3.38) and (3.41) yields that

$$(3.42) \quad |U_v^n(\lambda) - u(t_n, \lambda)| \leq C k^2 \lambda^2 |\log k|.$$

This completes the proof of Theorem 1.3.

4. PROOF OF THEOREM 1.1.

To prove Theorem 1.1 we need the expressions of (2.17) and (2.18) for the solutions of (1.14). Similarly, we also need the expressions of (1.30) and (1.31) for the solutions of (1.24)–(1.26).

By subtracting (2.17) from (1.30) we can write

$$(4.1) \quad \begin{aligned} U^n(\lambda) - u(t_n, \lambda) &= Re [\lambda^{-1}(U_1^n - u_1(t_n)) + \lambda^{-2}(U_2^n - u_2(t_n)) \\ &\quad + \lambda^{-3}(U_3^n - u_3(t_n)) + (U_4^n(\lambda) - u_4(t_n, \lambda)) \\ &\quad + (U_5^n(\lambda) - u_5(t_n, \lambda)) + (U_6^n(\lambda) - u_6(t_n, \lambda))]. \end{aligned}$$

We shall estimate the above various terms. To begin with, we establish the following estimate.

Estimate 4.1.

$$(4.2) \quad |U_1^n - u_1(t_n)| \leq C k t_{n-1}^{-2}, \quad (t_{n-1} > 0).$$

Write $u_1(t_n) - U_1^n = e_{1,1}^n + e_{1,2}^n + e_{1,3}^n$, where

$$(4.3) \quad \begin{aligned} e_{1,1}^n &= \frac{1}{\pi t_n} \int_0^\rho e^{iy t_{n-1}} (e^{iky} - 1) \widehat{K}_1(iy) dy, \\ e_{1,2}^n &= \frac{1}{\pi t_n} \int_0^\rho e^{iy t_{n-1}} \left(\widehat{K}_1(iy) - \widehat{K}_1(s) \right) dy, \\ e_{1,3}^n &= \frac{1}{\pi t_n} \int_0^\rho e^{iy t_{n-1}} \widehat{K}_1(s) \frac{-3 + 2e^{-iky} + e^{-2iky}}{(1 + e^{-iky})^2} dy, \\ \widehat{K}_1(s) &= \frac{\widehat{A}'(s)}{[\widehat{A}(s)]^2}, \quad s = \frac{\delta(e^{-iky})}{k} = i \tau(k, y). \end{aligned}$$

Integrate by parts in (4.3) to bring out another factor of t_{n-1}^{-1} ,

$$(4.4) \quad \begin{aligned} e_{1,1}^n &= \frac{1}{\pi t_n i t_{n-1}} e^{iy t_{n-1}} (e^{iky} - 1) \widehat{K}_1(iy) \Big|_{y=0}^{y=\rho} \\ &\quad - \frac{1}{\pi t_n i t_{n-1}} \int_0^\rho e^{iy t_{n-1}} i k e^{iky} \widehat{K}_1(iy) dy \\ &\quad - \frac{i}{\pi t_n i t_{n-1}} \int_0^\rho e^{iy t_{n-1}} (e^{iky} - 1) \widehat{K}_1'(iy) dy \\ &= e_{1,1,1}^n + e_{1,1,2}^n + e_{1,1,3}^n, \\ \widehat{K}_1'(s) &= \frac{\widehat{A}''(s)}{[\widehat{A}(s)]^2} - \frac{2[\widehat{A}'(s)]^2}{[\widehat{A}(s)]^3}. \end{aligned}$$

By (2.11) and (2.14) we find that

$$|e_{1,1,1}^n| \leq C k t_{n-1}^{-2}, \quad (t_{n-1} > 0).$$

Evidently, in view of (2.13),

$$|e_{1,1,2}^n| \leq C k t_{n-1}^{-2}, \quad (t_{n-1} > 0).$$

From (2.13) and (2.14) it follows that

$$|e_{1,1,3}^n| \leq C k t_{n-1}^{-2} \int_0^\rho \left(\frac{y |\widehat{A}''(iy)|}{|\widehat{A}(iy)|^2} + \frac{y |\widehat{A}'(iy)|^2}{|\widehat{A}(iy)|^3} \right) dy \leq C k t_{n-1}^{-2} \quad (t_{n-1} > 0).$$

Therefore, we have

$$(4.5) \quad |e_{1,1}^n| \leq C k t_{n-1}^{-2}, \quad (t_{n-1} > 0).$$

With integration by parts on $e_{1,3}^n$, we find that

$$(4.6) \quad \begin{aligned} e_{1,3}^n &= \frac{1}{\pi t_n i t_{n-1}} e^{iy t_{n-1}} \widehat{K}_1(s) \frac{-3 + 2e^{-iky} + e^{-2iky}}{(1 + e^{-iky})^2} \Big|_{y=0}^{y=\rho} \\ &\quad - \frac{1}{\pi t_n i t_{n-1}} \int_0^\rho e^{iy t_{n-1}} \widehat{K}_1'(s) \frac{4i e^{-iky}}{(1 + e^{-iky})^2} \frac{-3 + 2e^{-iky} + e^{-2iky}}{(1 + e^{-iky})^2} dy \\ &\quad - \frac{1}{\pi t_n i t_{n-1}} \int_0^\rho e^{iy t_{n-1}} \widehat{K}_1(s) (-1) 8ik e^{-iky} \frac{dy}{(1 + e^{-iky})^3} \\ &= e_{1,3,1}^n + e_{1,3,2}^n + e_{1,3,3}^n, \end{aligned}$$

where

$$s = s(k, y) = \frac{\delta(e^{-iky})}{k} = i\tau(k, y), \quad \tau(k, y) = \frac{2 \sin(ky)}{k(1 + \cos(ky))}.$$

From (2.11), (2.14) and (3.11) it follows that

$$|e_{1,3,1}^n| \leq C k t_{n-1}^{-2}, \quad (t_{n-1} > 0).$$

From (3.11), (3.40), (2.13) and (2.14) we have

$$\begin{aligned} |e_{1,3,2}^n| &\leq C k t_{n-1}^{-2} \int_0^\rho |\widehat{K}_1'(s)| y dy \\ &\leq C k t_{n-1}^{-2} \int_0^{2\rho} \frac{\tau |\widehat{A}''(i\tau)| + |\widehat{A}'(i\tau)|}{|\widehat{A}(i\tau)|^2} d\tau \leq C k t_{n-1}^{-2}, \quad (t_{n-1} > 0). \end{aligned}$$

Then (3.11), (3.40), (2.13) yield

$$|e_{1,3,3}^n| \leq C k t_{n-1}^{-2} \int_0^{2\rho} \frac{|\widehat{A}'(i\tau)|}{|\widehat{A}(i\tau)|^2} d\tau \leq C k t_{n-1}^{-2}, \quad (t_{n-1} > 0).$$

Therefore, we have the estimate

$$(4.7) \quad |e_{1,3}^n| \leq C k t_{n-1}^{-2}, \quad (t_{n-1} > 0).$$

For $e_{1,2}^n$, again using integration by parts gives us

$$(4.8) \quad e_{1,2}^n = e_{1,2,1}^n + e_{1,2,2}^n + e_{1,2,3}^n + e_{1,2,4}^n,$$

where

$$\begin{aligned} e_{1,2,1}^n &= \frac{1}{\pi t_n i t_{n-1}} e^{iy t_{n-1}} \left(\widehat{K}_1(iy) - \widehat{K}_1(s) \right) \Bigg|_{y=0}^{y=\rho}, \\ e_{1,2,2}^n &= - \frac{1}{\pi t_n t_{n-1}} \int_0^\rho e^{iy t_{n-1}} [\widehat{K}_1'(iy) - \widehat{K}_1'(s)] dy, \\ e_{1,2,3}^n &= - \frac{1}{\pi t_n t_{n-1}} \int_0^\rho e^{iy t_{n-2}} (e^{iky} - 1) \widehat{K}_1'(s) dy, \\ e_{1,2,4}^n &= - \frac{1}{\pi t_n t_{n-1}} \int_0^\rho e^{iy t_{n-2}} \widehat{K}_1'(s) \frac{-3 + 2e^{-iky} + e^{-2iky}}{(1 + e^{-iky})^2} dy. \end{aligned}$$

By the mean-value theorem and $s - iy = O(ky^2)$ for $0 \leq y \leq \rho$ we have

$$|\widehat{K}_1(s) - \widehat{K}_1(iy)| \leq C k |y|^2 |\widehat{K}_1'(s_1)|,$$

where $s_1 = s_1(k, y) = i\tau_1(k, y)$, $\tau_1(k, y)$ lies between y and $\tau(k, y)$.

Using (3.11), (2.11) and (2.14) we obtain, for $0 \leq y \leq \rho$,

$$y^2 |\widehat{K}_1'(s_1)| \leq C \left(\frac{\tau_1^2 |\widehat{A}''(i\tau_1)|}{|\widehat{A}(i\tau_1)|^2} + \frac{\tau_1^2 |\widehat{A}'(i\tau_1)|^2}{|\widehat{A}(i\tau_1)|^3} \right) \leq C,$$

which implies

$$|\widehat{K}_1(s) - \widehat{K}_1(iy)| \leq C k,$$

and it follows that

$$(4.9) \quad |e_{1,2,1}^n| \leq C k t_{n-1}^{-2}, \quad (t_{n-1} > 0).$$

We know from (3.11), (3.40), (2.13) and (2.14) that

$$\begin{aligned}
 |e_{1,2,3}^n| &\leq C k t_{n-1}^{-2} \int_0^\rho y |\widehat{K}_1'(i\tau)| dy \\
 &\leq C k t_{n-1}^{-2} \int_0^{2\rho} \tau |\widehat{K}_1'(i\tau)| d\tau \\
 (4.10) \quad &\leq C k t_{n-1}^{-2} \int_0^{2\rho} \left(\frac{\tau |\widehat{A}''(i\tau)|}{|\widehat{A}(i\tau)|^2} + \frac{\tau |\widehat{A}'(i\tau)|^2}{|\widehat{A}(i\tau)|^3} \right) d\tau \\
 &\leq C k t_{n-1}^{-2}, \quad (t_{n-1} > 0),
 \end{aligned}$$

and

$$(4.11) \quad |e_{1,2,4}^n| \leq C k t_{n-1}^{-2} \int_0^\rho y |\widehat{K}_1'(i\tau)| dy \leq C k t_{n-1}^{-2}, \quad (t_{n-1} > 0).$$

For $e_{1,2,2}^n$, we first use the mean-value theorem to obtain

$$|\widehat{K}_1'(s) - \widehat{K}_1'(iy)| \leq C k |y|^2 |\widehat{K}_1''(s_1)|,$$

where

$$\widehat{K}_1''(s) = \frac{\widehat{A}'''(s)}{[\widehat{A}(s)]^2} - \frac{6\widehat{A}'(s)\widehat{A}''(s)}{[\widehat{A}(s)]^3} + \frac{6[\widehat{A}'(s)]^3}{[\widehat{A}(s)]^4}.$$

From (3.11), (2.14), it follows that

$$\begin{aligned}
 |e_{1,2,2}^n| &\leq C k t_{n-1}^{-2} \int_0^\rho y^2 |\widehat{K}_1''(s_1)| dy \\
 &\leq C k t_{n-1}^{-2} \int_0^\rho \left(\frac{\tau_1^2 |\widehat{A}'''(i\tau_1)|}{|\widehat{A}(i\tau_1)|^2} + \frac{|\widehat{A}'(i\tau_1)|}{|\widehat{A}(i\tau_1)|^2} + \frac{\tau_1 |\widehat{A}''(i\tau_1)|}{|\widehat{A}(i\tau_1)|^2} \right) dy.
 \end{aligned}$$

Now (A.5), (A.8) and (A.12) (see Appendix below). imply that

$$\begin{aligned}
 |e_{1,2,2}^n| &\leq C k t_{n-1}^{-2} \int_0^\rho y^2 |\widehat{K}_1''(s_1)| dy \\
 (4.12) \quad &\leq C k t_{n-1}^{-2} \int_0^\rho \left(\frac{\tau_1^2 |\widehat{A}'''(i\tau_1)|}{|\widehat{A}(i\tau_1)|^2} + \frac{|\widehat{A}'(i\tau_1)|}{|\widehat{A}(i\tau_1)|^2} + \frac{\tau_1 |\widehat{A}''(i\tau_1)|}{|\widehat{A}(i\tau_1)|^2} \right) dy \\
 &\leq C k t_{n-1}^{-2}, \quad t_{n-1} > 0.
 \end{aligned}$$

Then it follows that

$$(4.13) \quad |e_{1,2}^n| \leq C k t_{n-1}^{-2}, \quad (t_{n-1} > 0).$$

Together the three estimates (4.5), (4.7) and (4.13) complete the proof of **Estimate 4.1.**

Estimate 4.2.

$$(4.14) \quad |U_2^n - u_2(t_n)| \leq C k t_{n-1}^{-2}, \quad (t_{n-1} > 0).$$

As in (4.3), we write the representation

$$(4.15) \quad u_2(t_n) - U_2^n = e_{2,1}^n + e_{2,2}^n + e_{2,3}^n,$$

in which the formula for $e_{2,1}^n$, $e_{2,2}^n$ and $e_{2,3}^n$ is of the same form as in (4.3), with

$$\widehat{K}_2(s) = \frac{1}{[\widehat{A}(s)]^2} \left(1 - \frac{2s\widehat{A}'(s)}{\widehat{A}(s)} \right)$$

instead of $\widehat{K}_1(s)$.

Formally, we differentiate $\widehat{K}_2(s)$ to obtain

$$\widehat{K}_2'(s) = -\frac{4\widehat{A}'(s)}{[\widehat{A}(s)]^3} - \frac{2s\widehat{A}''(s)}{[\widehat{A}(s)]^3} + \frac{6s[\widehat{A}'(s)]^2}{[\widehat{A}(s)]^4},$$

and again differentiate the result to show

$$\widehat{K}_2''(s) = -\frac{6\widehat{A}''(s)}{[\widehat{A}(s)]^3} + \frac{18[\widehat{A}'(s)]^2}{[\widehat{A}(s)]^4} - \frac{2s\widehat{A}'''(s)}{[\widehat{A}(s)]^3} + \frac{18s\widehat{A}'(s)\widehat{A}''(s)}{[\widehat{A}(s)]^4} - \frac{24s[\widehat{A}'(s)]^3}{[\widehat{A}(s)]^5}.$$

We may repeat the arguments of the proof of (4.2) to obtain (4.14). We omit the details.

Estimate 4.3.

$$(4.16) \quad |U_3^n - u_3(t_n)| \leq C k t_{n-1}^{-2}, \quad (t_{n-1} > 0).$$

We argue similarly to the proof of Estimate 4.2. Write

$$(4.17) \quad U_3^n - u_3(t_n) = e_{3,1}^n + e_{3,2}^n + e_{3,3}^n,$$

in which the formula for $e_{3,1}^n$, $e_{3,2}^n$ and $e_{3,3}^n$ is of the same form as in (4.3), with

$$\widehat{K}_3(s) = \frac{2s}{[\widehat{A}(s)]^3}$$

instead of $\widehat{K}_1(s)$.

Now with differentiation corresponding to s in $\widehat{K}_3(s)$ we get that

$$\widehat{K}_3'(s) = \frac{2}{[\widehat{A}(s)]^3} - \frac{6s\widehat{A}'(s)}{[\widehat{A}(s)]^4},$$

and again we differentiate the result to obtain

$$\widehat{K}_3''(s) = -\frac{12\widehat{A}'(s)}{[\widehat{A}(s)]^4} - \frac{6s\widehat{A}''(s)}{[\widehat{A}(s)]^4} + \frac{24s[\widehat{A}'(s)]^2}{[\widehat{A}(s)]^5}.$$

An analogous argument of the proof of (4.2) can be used to obtain the proof of Estimate 4.3.

We now establish an estimate needed to deduce the bound for the fourth term on the right-hand side of (4.1).

Estimate 4.4.

$$(4.18) \quad |U_4^n(\lambda) - u_4(t_n, \lambda)| \leq C k t_{n-1}^{-2} \lambda^{-3}, \quad (t_{n-1} > 0).$$

As in (4.3) we first write

$$u_4(t_n, \lambda) - U_4^n(\lambda) = e_{4,1}^n(\lambda) + e_{4,2}^n(\lambda) + e_{4,3}^n(\lambda),$$

where

$$\begin{aligned}
 e_{4,1}^n(\lambda) &= \frac{1}{\pi t_n \lambda^3} \int_0^\rho e^{iy t_{n-1}} (e^{iky} - 1) \widehat{K}_4(iy, \lambda) dy, \\
 e_{4,2}^n(\lambda) &= \frac{1}{\pi t_n \lambda^3} \int_0^\rho e^{iy t_{n-1}} \left(\widehat{K}_4(iy, \lambda) - \widehat{K}_4(s, \lambda) \right) dy, \\
 e_{4,3}^n(\lambda) &= \frac{1}{\pi t_n \lambda^3} \int_0^\rho e^{iy t_{n-1}} \widehat{K}_4(s, \lambda) \frac{-3 + 2e^{-iky} + e^{-2iky}}{(1 + e^{-iky})^2} dy, \\
 \widehat{K}_4(s, \lambda) &= \frac{s^2 D_s(s, \lambda)}{[\widehat{A}(s)]^2 D(s, \lambda)} \left(\frac{2}{\widehat{A}(s)} + \frac{1}{D(s, \lambda)} \right), \\
 s &= \frac{\delta(e^{-iky})}{k} = i \tau(k, y).
 \end{aligned}
 \tag{4.19}$$

Arguing similarly to (4.4), (4.8) and (4.6), we have that

$$\begin{aligned}
 e_{4,1}^n(\lambda) &= \frac{1}{\pi t_n \lambda^3 i t_{n-1}} e^{iy t_{n-1}} (e^{iky} - 1) \widehat{K}_4(iy, \lambda) \Big|_{y=0}^{y=\rho} \\
 &\quad - \frac{1}{\pi t_n \lambda^3 i t_{n-1}} \int_0^\rho e^{iy t_{n-1}} i k e^{iky} \widehat{K}_4(iy, \lambda) dy \\
 &\quad - \frac{i}{\pi t_n \lambda^3 i t_{n-1}} \int_0^\rho e^{iy t_{n-1}} (e^{iky} - 1) \widehat{K}_4'(iy, \lambda) dy \\
 &= e_{4,1,1}^n(\lambda) + e_{4,1,2}^n(\lambda) + e_{4,1,3}^n(\lambda), \\
 \widehat{K}_4'(s, \lambda) &= \frac{2s D_s(s, \lambda) + s^2 \widehat{A}''(s)}{[\widehat{A}(s)]^2 D(s, \lambda)} \left(\frac{2}{\widehat{A}(s)} + \frac{1}{D(s, \lambda)} \right) \\
 &\quad - \frac{s^2 D_s(s, \lambda)}{[\widehat{A}(s)]^2 D(s, \lambda)} \left(\frac{6 \widehat{A}'(s)}{[\widehat{A}(s)]^2} + \frac{4 \widehat{A}'(s) + 2\lambda^{-1}}{\widehat{A}(s) D(s, \lambda)} + \frac{2 D_s(s, \lambda)}{[D(s, \lambda)]^2} \right),
 \end{aligned}
 \tag{4.20}$$

also

$$\begin{aligned}
 e_{4,2}^n(\lambda) &= \frac{1}{\pi t_n \lambda^3 i t_{n-1}} e^{iy t_{n-1}} \left(\widehat{K}_4(iy, \lambda) - \widehat{K}_4(s, \lambda) \right) \Big|_{y=0}^{y=\rho} \\
 &\quad - \frac{1}{\pi t_n \lambda^3 i t_{n-1}} \int_0^\rho e^{iy t_{n-1}} [\widehat{K}_4'(iy, \lambda) - \widehat{K}_4'(s, \lambda)] dy \\
 &\quad - \frac{1}{\pi t_n \lambda^3 i t_{n-1}} \int_0^\rho e^{iy t_{n-2}} (e^{iky} - 1) \widehat{K}_4'(s, \lambda) dy \\
 &\quad - \frac{1}{\pi t_n \lambda^3 i t_{n-1}} \int_0^\rho e^{iy t_{n-2}} \widehat{K}_4'(s, \lambda) \frac{-3 + 2e^{-iky} + e^{-2iky}}{(1 + e^{-iky})^2} dy \\
 &= e_{4,2,1}^n(\lambda) + e_{4,2,2}^n(\lambda) + e_{4,2,3}^n(\lambda) + e_{4,2,4}^n(\lambda),
 \end{aligned}
 \tag{4.21}$$

and

(4.22)

$$\begin{aligned}
e_{4,3}^n(\lambda) &= \frac{1}{\pi t_n \lambda^3 i t_{n-1}} e^{iy t_{n-1}} \widehat{K}_4(s, \lambda) \frac{-3 + 2e^{-iky} + e^{-2iky}}{(1 + e^{-iky})^2} \Bigg|_{y=0}^{y=\rho} \\
&\quad - \frac{1}{\pi t_n \lambda^3 i t_{n-1}} \int_0^\rho e^{iy t_{n-1}} \widehat{K}_4'(s, \lambda) \frac{4i e^{-iky}}{(1 + e^{-iky})^2} \frac{-3 + 2e^{-iky} + e^{-2iky}}{(1 + e^{-iky})^2} dy \\
&\quad - \frac{1}{\pi t_n \lambda^3 i t_{n-1}} \int_0^\rho e^{iy t_{n-1}} \widehat{K}_4(s, \lambda) (-1) 8ik e^{-iky} \frac{dy}{(1 + e^{-iky})^3} \\
&= e_{4,3,1}^n(\lambda) + e_{4,3,2}^n(\lambda) + e_{4,3,3}^n(\lambda).
\end{aligned}$$

We use the estimates of (3.15) and (3.16) for lower bounds on $|D(iy, \lambda)|$ and $|D(i\tau, \lambda)|$ with (3.11), and of (3.14) for lower bounds on $|\widehat{A}(iy)|$ and $|\widehat{A}(i\tau)|$. (2.14) will give upper bounds on the numerators.

Similar arguments of (4.5), (4.7), (4.10) and (4.11) can be applied to show that

$$(4.23) \quad |e_{4,1}^n(\lambda)|, |e_{4,3}^n(\lambda)|, |e_{4,2,3}^n(\lambda)|, |e_{4,2,4}^n(\lambda)| \leq C k \lambda^{-3} t_{n-1}^{-2}, \quad (t_{n-1} > 0).$$

For $e_{4,2,1}^n(\lambda)$, $e_{4,2,2}^n(\lambda)$, we apply the mean-value theorem, then use the estimates of (3.15) and (3.16) for lower bounds on $|D(i\tau_1, \lambda)|$, of (3.14) and (2.11) for lower bounds on $|\widehat{A}(i\tau_1)|$ with (3.11), and of (2.14), (2.29) for upper bounds on the numerators. Therefore we deduce the estimates

$$(4.24) \quad |e_{4,2,1}^n(\lambda)|, |e_{4,2,2}^n(\lambda)| \leq C k \lambda^{-3} t_{n-1}^{-2}, \quad (t_{n-1} > 0).$$

Thus, the representation (4.19), along with the estimates (4.23) and (4.24), gives us (4.18).

This completes the proof of Estimate 4.4.

In Section 5 we will show that if the conditions of Theorem 1.1 hold. Then

$$(4.25) \quad |U_5^n(\lambda) - u_5(t_n, \lambda)| \leq C k \lambda^{\frac{3}{2}} t_{n-1}^{-2}, \quad (t_{n-1} > 0),$$

$$(4.26) \quad |u_6(t_n, \lambda)| \leq C k^2 \lambda^{\frac{3}{2}} t_n^{-2}, \quad (t_n > 0),$$

and

$$(4.27) \quad |U_6^n(\lambda)| \leq C k \lambda t_n^{-1}, \quad (t_n > 0).$$

Referring to (4.1), (4.2), (4.14), (4.16) and (4.18), along with the estimates (4.25), (4.26) and (4.27), we obtain the proof of Theorem 1.1.

5. PROOFS OF (4.25), (4.26) AND (4.27)

Arguing similarly to (4.3), we can write

$$u_5(t_n, \lambda) - U_5^n(\lambda) = e_{5,1}^n(\lambda) + e_{5,2}^n(\lambda) + e_{5,3}^n(\lambda),$$

where

$$\begin{aligned}
 e_{5,1}^n(\lambda) &= \frac{1}{\pi t_n \lambda} \int_{\rho}^{\frac{\pi}{2k}} e^{iy t_{n-1}} (e^{iky} - 1) \widehat{K}_5(iy, \lambda) dy, \\
 e_{5,2}^n(\lambda) &= \frac{1}{\pi t_n \lambda} \int_{\rho}^{\frac{\pi}{2k}} e^{iy t_{n-1}} \left(\widehat{K}_5(iy, \lambda) - \widehat{K}_5(s, \lambda) \right) dy, \\
 e_{5,3}^n(\lambda) &= \frac{1}{\pi t_n \lambda} \int_{\rho}^{\frac{\pi}{2k}} e^{iy t_{n-1}} \widehat{K}_5(s, \lambda) \frac{-3 + 2e^{-iky} + e^{-2iky}}{(1 + e^{-iky})^2} dy, \\
 \widehat{K}_5(s, \lambda) &= \frac{D_s(s, \lambda)}{[D(s, \lambda)]^2}, \\
 s &= \frac{\delta(e^{-iky})}{k} = i\tau(k, y).
 \end{aligned}
 \tag{5.1}$$

We argue similarly to (4.4), (4.8) and (4.6) to obtain

$$\begin{aligned}
 e_{5,1}^n(\lambda) &= \frac{1}{\pi t_n \lambda i t_{n-1}} e^{iy t_{n-1}} (e^{iky} - 1) \widehat{K}_5(iy, \lambda) \Big|_{y=\rho}^{y=\frac{\pi}{2k}} \\
 &\quad - \frac{1}{\pi t_n \lambda i t_{n-1}} \int_{\rho}^{\frac{\pi}{2k}} e^{iy t_{n-1}} i k e^{iky} \widehat{K}_5(iy, \lambda) dy \\
 &\quad - \frac{i}{\pi t_n \lambda i t_{n-1}} \int_{\rho}^{\frac{\pi}{2k}} e^{iy t_{n-1}} (e^{iky} - 1) \widehat{K}_5'(iy, \lambda) dy \\
 &= e_{5,1,1}^n(\lambda) + e_{5,1,2}^n(\lambda) + e_{5,1,3}^n(\lambda), \\
 \widehat{K}_5'(s, \lambda) &= \frac{\widehat{A}''(s)}{[D(s, \lambda)]^2} - \frac{2[D_s(s, \lambda)]^2}{[D(s, \lambda)]^3},
 \end{aligned}
 \tag{5.2}$$

also

$$\begin{aligned}
 e_{5,2}^n(\lambda) &= \frac{1}{\pi t_n \lambda i t_{n-1}} e^{iy t_{n-1}} \left(\widehat{K}_5(iy, \lambda) - \widehat{K}_5(s, \lambda) \right) \Big|_{y=\rho}^{y=\frac{\pi}{2k}} \\
 &\quad - \frac{1}{\pi t_n \lambda t_{n-1}} \int_{\rho}^{\frac{\pi}{2k}} e^{iy t_{n-1}} [\widehat{K}_5'(iy, \lambda) - \widehat{K}_5'(s, \lambda)] dy \\
 &\quad - \frac{1}{\pi t_n \lambda t_{n-1}} \int_{\rho}^{\frac{\pi}{2k}} e^{iy t_{n-2}} (e^{iky} - 1) \widehat{K}_5'(s, \lambda) dy \\
 &\quad - \frac{1}{\pi t_n \lambda t_{n-1}} \int_{\rho}^{\frac{\pi}{2k}} e^{iy t_{n-2}} \widehat{K}_5'(s, \lambda) \frac{-3 + 2e^{-iky} + e^{-2iky}}{(1 + e^{-iky})^2} dy \\
 &= e_{5,2,1}^n(\lambda) + e_{5,2,2}^n(\lambda) + e_{5,2,3}^n(\lambda) + e_{5,2,4}^n(\lambda),
 \end{aligned}
 \tag{5.3}$$

and

$$\begin{aligned}
 e_{5,3}^n(\lambda) &= \frac{1}{\pi t_n \lambda i t_{n-1}} e^{iy t_{n-1}} \widehat{K}_5(s, \lambda) \frac{-3 + 2e^{-iky} + e^{-2iky}}{(1 + e^{-iky})^2} \Big|_{y=\rho}^{y=\frac{\pi}{2k}} \\
 &\quad - \frac{1}{\pi t_n \lambda i t_{n-1}} \int_{\rho}^{\frac{\pi}{2k}} e^{iy t_{n-1}} \widehat{K}_5'(s, \lambda) \frac{4i e^{-iky}}{(1 + e^{-iky})^2} \frac{-3 + 2e^{-iky} + e^{-2iky}}{(1 + e^{-iky})^2} dy \\
 &\quad - \frac{1}{\pi t_n \lambda i t_{n-1}} \int_{\rho}^{\frac{\pi}{2k}} e^{iy t_{n-1}} \widehat{K}_5(s, \lambda) (-1) 8ik e^{-iky} \frac{dy}{(1 + e^{-iky})^3} \\
 &= e_{5,3,1}^n(\lambda) + e_{5,3,2}^n(\lambda) + e_{5,3,3}^n(\lambda).
 \end{aligned}
 \tag{5.4}$$

From (5.2) it is clear that

$$(5.5) \quad |e_{5,1,1}^n(\lambda)| \leq C k \lambda^{-1} t_{n-1}^{-2} \rho |\widehat{K}_5(i\rho, \lambda)| \\ + C k \lambda^{-1} t_{n-1}^{-2} \frac{\pi}{2k} |\widehat{K}_5(i\frac{\pi}{2k}, \lambda)|, \quad (t_{n-1} > 0).$$

We know from (3.15) that

$$(5.6) \quad \rho |\widehat{K}_5(i\rho, \lambda)| \leq \rho \frac{\lambda^{-1} + |\widehat{A}(i\rho)|}{|D(i\rho, \lambda)|^2} \leq C \frac{\lambda^{-1} \rho + |\widehat{A}'(i\rho)|}{|\widehat{A}(i\rho)|^2} \leq C.$$

In order to obtain a similar estimate on $\lambda^{-1} \frac{\pi}{2k} |\widehat{K}_5(i\frac{\pi}{2k}, \lambda)|$ we split into three cases and let $\tau_k = \frac{\pi}{2k}$.

Case 1. If $\rho_6 \leq \tau_k \leq \omega/2$, then (2.22), (2.15) and (2.20)(ii) imply

$$\lambda^{-1} \tau_k |\widehat{K}_5(i\tau_k, \lambda)| \leq \tau_k \frac{\lambda^{-1} + |\widehat{A}'(i\tau_k)|}{\lambda |D(i\tau_k, \lambda)|^2} \leq C \frac{\lambda^{-1} \tau_k + \tau_k^{-1}}{\lambda [\frac{\omega^2}{\lambda \tau_k}]^2} \\ = C \left[\frac{\tau_k^3}{\omega^4} + \frac{\lambda \tau_k}{\omega^4} \right] \leq C \omega^{-1} \leq C.$$

Case 2. If $\omega/2 < \tau_k < 2\omega$, then (2.26), (2.15) and (2.20)(ii) imply

$$\lambda^{-1} \tau_k |\widehat{K}_5(i\tau_k, \lambda)| \leq \tau_k \frac{\lambda^{-1} + |\widehat{A}'(i\tau_k)|}{\lambda |D(i\tau_k, \lambda)|^2} \leq C \left[\frac{\tau_k}{\lambda^2 \Phi^2(\tau_k)} + \frac{1}{\tau_k \lambda \Phi^2(\tau_k)} \right] \\ \leq C \frac{\omega}{\lambda^2 \Phi^2(\omega)} \leq C \omega \leq C \lambda^{\frac{1}{2}}.$$

Case 3. If $2\omega \leq \tau_k$, then using (2.23), (2.15) and (2.20)(ii), we obtain

$$\lambda^{-1} \tau_k |\widehat{K}_5(i\tau_k, \lambda)| \leq \tau_k \frac{\lambda^{-1} + |\widehat{A}'(i\tau_k)|}{\lambda |D(i\tau_k, \lambda)|^2} \leq C \left[\frac{\tau_k}{|\omega - \tau_k|^2} + \frac{\lambda}{\tau_k |\omega - \tau_k|^2} \right] \\ \leq C \left(\frac{1}{\tau_k} + \frac{\lambda}{\tau_k^3} \right) \leq C \omega^{-1} \leq C.$$

Thus, we have the estimate

$$(5.7) \quad |e_{5,1,1}^n(\lambda)| \leq C k \lambda^{\frac{1}{2}} t_{n-1}^{-2}, \quad (t_{n-1} > 0).$$

Similarly, we have that

$$(5.8) \quad |e_{5,1,2}^n(\lambda)|, |e_{5,3,3}^n(\lambda)| \leq C k \lambda^{\frac{1}{2}} t_{n-1}^{-2}, \quad (t_{n-1} > 0),$$

$$(5.9) \quad |e_{5,1,3}^n(\lambda)|, |e_{5,2,3}^n(\lambda)|, |e_{5,2,4}^n(\lambda)|, |e_{5,3,2}^n(\lambda)| \leq C k \lambda t_{n-1}^{-2}, \quad (t_{n-1} > 0),$$

$$(5.10) \quad |e_{5,3,1}^n(\lambda)| \leq C k \lambda^{\frac{1}{2}} t_{n-1}^{-2}, \quad (t_{n-1} > 0),$$

for example, for $|e_{5,1,2}^n(\lambda)|$,
(5.11)

$$\begin{aligned} |e_{5,1,2}^n(\lambda)| &\leq C k t_{n-1}^{-2} \int_{\rho}^{\frac{\pi}{2k}} \lambda^{-1} |\widehat{K}_5(iy, \lambda)| dy \\ &\leq C k t_{n-1}^{-2} \int_{\rho}^{\frac{\pi}{2k}} \left(\frac{\lambda^{-2}}{|D(iy, \lambda)|^2} + \frac{|\widehat{A}'(iy)|}{\lambda |D(iy, \lambda)|^2} \right) dy \\ &\leq C k t_{n-1}^{-2} \left(\int_{\rho}^{\rho_6} \left(\frac{\lambda^{-2}}{\Phi^2(y)} + \frac{y^{-2}}{\lambda \Phi^2(y)} \right) dy + \int_{\rho_6}^{\omega/2} \left(\frac{y^2}{\omega^4} + \frac{\lambda}{\omega^4} \right) dy \right. \\ &\quad \left. + \int_{\omega/2}^{2\omega} \left(\frac{1}{\lambda^2 \Phi^2(y)} + \frac{1}{y^2 \lambda \Phi^2(y)} \right) dy + \int_{2\omega}^{\infty} \left(\frac{1}{|\omega - y|^2} + \frac{\lambda}{y^2 |\omega - y|^2} \right) dy \right) \\ &\leq C k t_{n-1}^{-2} (\lambda^{-1} + \omega^{-1} + \omega) \leq C k t_{n-1}^{-2} \omega, \end{aligned}$$

also, for $|e_{5,3,2}^n(\lambda)|$, by (3.19) and (3.40) it follows that

$$\begin{aligned} |e_{5,3,2}^n(\lambda)| &\leq C k t_{n-1}^{-2} \int_{\rho}^{\frac{\pi}{2k}} \lambda^{-1} y |\widehat{K}_5'(i\tau, \lambda)| dy \\ &\leq C k t_{n-1}^{-2} \int_{\frac{2}{\pi}\rho}^{\frac{2}{k}} \lambda^{-1} \tau |\widehat{K}_5'(i\tau, \lambda)| d\tau, \end{aligned}$$

and a similar argument can be applied to show that

$$\begin{aligned} &\lambda^{-1} \int_{\frac{2}{\pi}\rho}^{\frac{2}{k}} \tau |\widehat{K}_5'(i\tau, \lambda)| d\tau \\ &\leq C \int_{\frac{2}{\pi}\rho}^{\frac{2}{k}} \left(\frac{\lambda^{-1} \tau^{-2}}{|D(i\tau, \lambda)|^2} + \frac{\tau \lambda^{-2}}{\lambda |D(i\tau, \lambda)|^3} + \frac{\tau^{-3}}{\lambda |D(i\tau, \lambda)|^3} \right) d\tau \\ &\leq C \int_{\frac{2}{\pi}\rho}^{\rho_6} \left(\frac{\tau^{-2}}{\lambda \Phi^2(\tau)} + \frac{\tau}{\lambda^3 \Phi^3(\tau)} + \frac{\tau^{-3}}{\lambda \Phi^3(\tau)} \right) d\tau \\ &\quad + C \int_{\rho_6}^{\omega/2} \left(\frac{\tau^2}{\lambda (\omega^2/\tau \lambda)^2} + \frac{\tau}{\lambda^3 (\omega^2/\tau \lambda)^3} + \frac{\tau^{-3}}{\lambda (\omega^2/\tau \lambda)^3} \right) d\tau \\ &\quad + C \int_{\omega/2}^{2\omega} \left(\frac{\tau^{-2}}{\lambda \Phi^2(\tau)} + \frac{\tau}{\lambda^3 \Phi^3(\tau)} + \frac{\tau^{-3}}{\lambda \Phi^3(\tau)} \right) d\tau \\ &\quad + C \int_{2\omega}^{\infty} \left(\frac{\tau^{-2} \lambda}{|\omega - \tau|^2} + \frac{\tau}{|\omega - \tau|^3} + \frac{\tau^{-3} \lambda^2}{|\tau - \omega|^3} \right) d\tau \\ &\leq C (\lambda^{-1} + \omega^{-1} + \omega + \lambda) \leq C \lambda; \end{aligned}$$

in other words,

$$|e_{5,3,2}^n(\lambda)| \leq C k t_{n-1}^{-2} \lambda.$$

To estimate $e_{5,2,1}^n(\lambda)$ we first use the mean-value theorem to obtain

$$(5.12) \quad |\widehat{K}_5(iy, \lambda) - \widehat{K}_5(s, \lambda)| \leq C k y^2 |\widehat{K}_5'(s_1, \lambda)|,$$

where $s_1 = s_1(k, y) = i\tau_1(k, y)$, $\tau_1(k, y)$ lies between y and $\tau(k, y)$. From (5.3) and (5.12) it follows that

$$(5.13) \quad |e_{5,2,1}^n(\lambda)| \leq C k t_{n-1}^{-2} \lambda^{-1} \rho^2 |\widehat{K}_5'(i\tau_1(k, \rho), \lambda)| \\ + C k t_{n-1}^{-2} \lambda^{-1} \tau_k^2 |\widehat{K}_5'(i\tau_1(k, \tau_k), \lambda)|,$$

with $\tau_k = \pi/2k$.

From (3.11), (3.15), (3.16), (2.14) and (2.11) we find that

$$(5.14) \quad \rho^2 |\widehat{K}_5'(i\tau_1(k, \rho), \lambda)| \leq C \tau_1^2 |\widehat{K}_5'(i\tau_1, \lambda)| \\ \leq C \left[\frac{\tau_1^2 |\widehat{A}''(i\tau_1)|}{|D(i\tau_1, \lambda)|^2} + \frac{\lambda^{-2} \tau_1^2}{|D(i\tau_1, \lambda)|^3} + \frac{\tau_1^2 |\widehat{A}'(i\tau_1)|^2}{|D(i\tau_1, \lambda)|^3} \right] \\ \leq \left(\frac{1}{|\widehat{A}(i\tau_1)|} + \frac{\lambda^{-2}}{|\widehat{A}(i\tau_1)|} \right) \leq C.$$

From (3.19) we have

$$(5.15) \quad \lambda^{-1} \tau_k^2 |\widehat{K}_5'(i\tau_1(k, \tau_k), \lambda)| \leq C \lambda^{-1} \tau_1^2 |\widehat{K}_5'(i\tau_1, \lambda)| \\ \leq C \left[\frac{\tau_1^2 |\widehat{A}''(i\tau_1)|}{\lambda |D(i\tau_1, \lambda)|^2} + \frac{\tau_1^2}{\lambda^3 |D(i\tau_1, \lambda)|^3} + \frac{\tau_1^2 |\widehat{A}'(i\tau_1)|^2}{\lambda |D(i\tau_1, \lambda)|^3} \right],$$

with $\tau_1 = \tau_1(k, \tau_k)$.

If $\rho_6 \leq \tau_1 \leq \omega/2$, then (2.22), (2.15) and (2.20)(ii) imply

$$(5.16) \quad \lambda^{-1} \tau_1^2 |\widehat{K}_5'(i\tau_1, \lambda)| \leq C \left[\frac{\tau_1^{-1}}{\lambda(\omega^2/\lambda\tau_1)^2} + \frac{\tau_1^2}{\lambda^3(\omega^2/\lambda\tau_1)^3} + \frac{\tau_1^{-2}}{\lambda(\omega^2/\lambda\tau_1)^3} \right] \\ \leq C \left[\frac{\lambda\tau_1}{\omega^4} + \frac{\tau_1^5}{\omega^6} + \frac{\lambda^2\tau_1}{\omega^6} \right] \leq C \omega^{-1} \leq C.$$

If $\omega/2 < \tau_1 < 2\omega$, then (2.26), (2.15) and (2.20)(ii) imply

$$(5.17) \quad \lambda^{-1} \tau_1^2 |\widehat{K}_5'(i\tau_1, \lambda)| \leq C \left[\frac{\tau_1^{-1}}{\lambda\Phi^2(\tau_1)} + \frac{\tau_1^2}{\lambda^3\Phi^3(\tau_1)} + \frac{\tau_1^{-2}}{\lambda\Phi^3(\tau_1)} \right] \\ \leq C [\omega + \lambda] \leq C\lambda.$$

If $2\omega \leq \tau_1$, then using (2.23), (2.15) and (2.20)(ii), we obtain

$$(5.18) \quad \lambda^{-1} \tau_1^2 |\widehat{K}_5'(i\tau_1, \lambda)| \leq C \left[\frac{\tau_1^{-1}\lambda}{|\omega - \tau_1|^2} + \frac{\tau_1^2}{|\omega - \tau_1|^3} + \frac{\tau_1^{-2}\lambda^2}{|\omega - \tau_1|^3} \right] \\ \leq C \left[\frac{\lambda}{\tau_1^3} + \frac{1}{\tau_1} + \frac{\lambda^2}{\tau_1^5} \right] \leq C \omega^{-1} \leq C.$$

Then (5.15), (5.16), (5.17) and (5.18) yield

$$(5.19) \quad \lambda^{-1} \tau_k^2 |\widehat{K}_5'(i\tau_1(k, \tau_k), \lambda)| \leq C\lambda.$$

Now (5.13), (5.14) and (5.19) give us

$$(5.20) \quad |e_{5,2,1}^n(\lambda)| \leq C k t_{n-1}^{-2} \lambda.$$

Returning to (5.3), we use the mean-value theorem, and along with (2.15), (2.29) and (3.19) to obtain

$$\begin{aligned}
 |e_{5,2,2}^n(\lambda)| &\leq C k t_{n-1}^{-2} \lambda^{-1} \int_{\rho}^{\frac{\pi}{2k}} y^2 |\widehat{K}_5''(s_1, \lambda)| dy \\
 &\leq C k t_{n-1}^{-2} \int_{\rho}^{\frac{\pi}{2k}} \left(\frac{y^2 |\widehat{A}'''(i\tau_1)|}{\lambda |D(i\tau_1, \lambda)|^2} + \frac{y^2 |\widehat{A}''(i\tau_1)| \lambda^{-1}}{\lambda |D(i\tau_1, \lambda)|^3} \right. \\
 &\quad \left. + \frac{y^2 |\widehat{A}'(i\tau_1)| |\widehat{A}'(i\tau_1)|}{\lambda |D(i\tau_1, \lambda)|^3} + \frac{y^2 \lambda^{-3}}{\lambda |D(i\tau_1, \lambda)|^4} + \frac{y^2 |\widehat{A}'(i\tau_1)|^3}{\lambda |D(i\tau_1, \lambda)|^4} \right) dy \\
 (5.21) \quad &\leq C k t_{n-1}^{-2} \int_{\rho}^{\frac{\pi}{2k}} \left(\frac{y^{-2}}{\lambda |D(i\tau_1, \lambda)|^2} + \frac{y^{-1}}{\lambda^2 |D(i\tau_1, \lambda)|^3} \right. \\
 &\quad \left. + \frac{y^{-3}}{\lambda |D(i\tau_1, \lambda)|^3} + \frac{y^2}{\lambda^4 |D(i\tau_1, \lambda)|^4} + \frac{y^{-4}}{\lambda |D(i\tau_1, \lambda)|^4} \right) dy \\
 &= C k t_{n-1}^{-2} \int_{\rho}^{\frac{\pi}{2k}} I(y, \tau_1, \lambda) dy,
 \end{aligned}$$

where $I(y, \tau_1, \lambda)$ is the integrand in (5.21). We argue similarly to (3.30).

For $\frac{\pi}{2} \rho_6 \leq y \leq \omega/2$, we use (3.22), and (2.20)(ii) to obtain

$$\begin{aligned}
 \int_{\frac{\pi}{2} \rho_6}^{\omega/2} I(y, \tau_1, \lambda/L) dy &\leq C \int_{\frac{\pi}{2} \rho_6}^{\omega/2} \left[\frac{y^{-2}}{\lambda (\omega^2/\lambda y)^2} + \frac{y^{-1}}{\lambda^2 (\omega^2/\lambda y)^3} \right. \\
 (5.22) \quad &\quad \left. + \frac{y^{-3}}{\lambda (\omega^2/\lambda y)^3} + \frac{y^2}{\lambda^4 (\omega^2/\lambda y)^4} + \frac{y^{-4}}{\lambda (\omega^2/\lambda y)^4} \right] dy \leq C \omega^{-1} \leq C.
 \end{aligned}$$

For $\omega/2 < y \leq \pi \omega$, by (3.25), and (2.20)(ii) we have

$$\begin{aligned}
 \int_{\frac{\omega}{2}}^{\pi \omega} I(y, \tau_1, \lambda) dy &\leq C \int_{\frac{\omega}{2}}^{\pi \omega} \left[\frac{y^{-2}}{\lambda \Phi^2(\omega)} + \frac{y^{-1}}{\lambda^2 \Phi^3(\omega)} \right. \\
 (5.23) \quad &\quad \left. + \frac{y^{-3}}{\lambda \Phi^3(\omega)} + \frac{y^2}{\lambda^4 \Phi^4(\omega)} + \frac{y^{-4}}{\lambda \Phi^4(\omega)} \right] dy \leq C \left[\omega + \lambda + \lambda^{\frac{3}{2}} \right] \leq C \lambda^{\frac{3}{2}}.
 \end{aligned}$$

For $\pi \omega \leq y$ we use (3.23) and (2.20)(ii) to obtain

$$\begin{aligned}
 \int_{\pi \omega}^{\infty} I(y, \tau_1, \lambda/L) dy &\leq C \int_{\pi \omega}^{\infty} \left[\frac{y^{-2} \lambda}{|\omega - y|^2} + \frac{y^{-1} \lambda}{|\omega - y|^3} \right. \\
 (5.24) \quad &\quad \left. + \frac{y^{-3} \lambda^2}{|\omega - y|^3} + \frac{y^2}{|\omega - y|^4} + \frac{y^{-4} \lambda^3}{|\omega - y|^4} \right] dy \\
 &\leq C \int_{\pi \omega}^{\infty} \left(\frac{\lambda}{y^4} + \frac{\lambda^2}{y^6} + \frac{1}{y^2} + \frac{\lambda^3}{y^8} \right) dy \leq C \omega^{-1} \leq C.
 \end{aligned}$$

For $\rho \leq y \leq \frac{\pi}{2} \rho_6$, we use (3.24) to obtain

$$\begin{aligned}
 \int_{\rho}^{\frac{\pi}{2} \rho_6} I(y, \tau_1, \lambda) dy &\leq C \int_{\rho}^{\frac{\pi}{2} \rho_6} \left[\frac{y^{-2}}{\lambda \Phi^2(\tau_1)} + \frac{y^{-1}}{\lambda^2 \Phi^3(\tau_1)} \right. \\
 (5.25) \quad &\quad \left. + \frac{y^{-3}}{\lambda \Phi^3(\tau_1)} + \frac{y^2}{\lambda^4 \Phi^4(\tau_1)} + \frac{y^{-4}}{\lambda \Phi^4(\tau_1)} \right] dy \leq C.
 \end{aligned}$$

Combining (5.21) through (5.25) we now have established that

$$(5.26) \quad |e_{5,2,2}^n(\lambda)| \leq C k t_{n-1}^{-2} \lambda^{\frac{3}{2}}, \quad t_{n-1} > 0.$$

Using (5.1) through (5.4), along with (5.7) through (5.10), (5.20) and (5.26), we see that

$$(5.27) \quad |u_5(t_n, \lambda) - U_5^n(\lambda)| \leq C k t_{n-1}^{-2} \lambda^{\frac{3}{2}}, \quad t_{n-1} > 0.$$

This completes the proof of (4.25).

Now integrate by parts on $u_6(t_n, \lambda)$ to obtain

$$(5.28) \quad \begin{aligned} -u_6(t_n, \lambda) &= \frac{1}{i \pi t_n^2 \lambda} e^{iy t_n} \widehat{K}_5(iy, \lambda) \Big|_{y=\frac{\pi}{2k}} \\ &+ \frac{1}{\pi t_n^2 \lambda} \int_{\frac{\pi}{2k}}^{\infty} e^{iy t_n} \widehat{K}_5'(iy, \lambda) dy = e_{6,1}^n(\lambda) + e_{6,2}^n(\lambda). \end{aligned}$$

We argue as in (5.7) to obtain

$$(5.29) \quad |e_{6,1}^n(\lambda)| \leq C k^2 t_n^{-2} \lambda, \quad t_n > 0.$$

A similar argument can be applied to show that

$$(5.30) \quad \begin{aligned} |e_{6,2}^n(\lambda)| &\leq C t_n^{-2} \lambda^{-1} \int_{\frac{\pi}{2k}}^{\infty} |\widehat{K}_5'(iy, \lambda)| dy \\ &\leq C t_n^{-2} \int_{\frac{\pi}{2k}}^{\infty} \left(\frac{y^{-3}}{\lambda |D(iy, \lambda)|^2} + \frac{1}{\lambda^3 |D(iy, \lambda)|^3} + \frac{y^{-4}}{\lambda |D(iy, \lambda)|^3} \right) dy. \end{aligned}$$

If $\rho_6 \leq \frac{\pi}{2k} \leq \omega/2$, then $\frac{1}{\omega} \leq Ck$, and we have

$$(5.31) \quad \begin{aligned} |e_{6,2}^n(\lambda)| &\leq C t_n^{-2} \left(\int_{\rho_6}^{\omega/2} \left[\frac{y^{-3}}{\lambda(\omega^2/\lambda y)^2} + \frac{1}{\lambda^3(\omega^2/\lambda y)^3} + \frac{y^{-4}}{\lambda(\omega^2/\lambda y)^3} \right] dy \right. \\ &+ \int_{\frac{\omega}{2}}^{2\omega} \left[\frac{y^{-3}}{\lambda \Phi^2(y)} + \frac{1}{\lambda^3 \Phi^3(y)} + \frac{y^{-4}}{\lambda \Phi^3(y)} \right] dy \\ &+ \left. \int_{2\omega}^{\infty} \left[\frac{y^{-3}\lambda}{|\omega - y|^2} + \frac{1}{|\omega - y|^3} + \frac{y^{-4}\lambda^2}{|\omega - y|^3} \right] dy \right) \\ &\leq C t_n^{-2} \left(k^2 + k^2 \lambda + k^2 \lambda^{\frac{3}{2}} \right) \leq C k^2 t_n^{-2} \lambda^{\frac{3}{2}}. \end{aligned}$$

If $\omega/2 \leq \frac{\pi}{2k} \leq 2\omega$, then $\frac{1}{\omega} \leq Ck$, and arguing similarly to (5.31),

$$(5.32) \quad \begin{aligned} |e_{6,2}^n(\lambda)| &\leq C t_n^{-2} \int_{\frac{\omega}{2}}^{2\omega} \left[\frac{y^{-3}}{\lambda \Phi^2(y)} + \frac{1}{\lambda^3 \Phi^3(y)} + \frac{y^{-4}}{\lambda \Phi^3(y)} \right] dy \\ &+ C t_n^{-2} \int_{2\omega}^{\infty} \left[\frac{y^{-3}\lambda}{|\omega - y|^2} + \frac{1}{|\omega - y|^3} + \frac{y^{-4}\lambda^2}{|\omega - y|^3} \right] dy \leq C k^2 t_n^{-2} \lambda^{\frac{3}{2}}. \end{aligned}$$

If $2\omega \leq \frac{\pi}{2k}$, then we have that

$$(5.33) \quad \begin{aligned} |e_{6,2}^n(\lambda)| &\leq C t_n^{-2} \int_{\frac{\pi}{2k}}^{\infty} \left[\frac{y^{-3}\lambda}{|\omega - y|^2} + \frac{1}{|\omega - y|^3} + \frac{y^{-4}\lambda^2}{|\omega - y|^3} \right] dy \\ &\leq C t_n^{-2} (k^4 \lambda + k^2) \leq C k^2 t_n^{-2}. \end{aligned}$$

Using (5.31) through (5.33) it follows that

$$(5.34) \quad |e_{6,2}^n(\lambda)| \leq C k^2 t_n^{-2} \lambda^{\frac{3}{2}}, \quad t_n > 0,$$

and together with (5.29) and (5.28) we show that

$$(5.35) \quad |u_6(t_n, \lambda)| \leq C k^2 t_n^{-2} \lambda^{\frac{3}{2}}, \quad t_n > 0.$$

This completes the proof of (4.26).

Referring to (1.31), along with (3.40) we obtain that

$$(5.36) \quad \begin{aligned} |U_6^n(\lambda)| &\leq C t_n^{-1} \lambda^{-1} \int_{\frac{2}{k}}^{\frac{\pi}{k}} |\widehat{K}_5(i\tau, \lambda)| \frac{dy}{1 + \cos(ky)} \\ &\leq C t_n^{-1} \lambda^{-1} \int_{\frac{2}{k}}^{\infty} |\widehat{K}_5(i\tau, \lambda)| d\tau \\ &\leq C t_n^{-1} \int_{\frac{2}{k}}^{\infty} \left[\frac{1}{\lambda^2 |D(i\tau, \lambda)|^2} + \frac{\tau^{-2}}{\lambda |D(i\tau, \lambda)|^2} \right] d\tau. \end{aligned}$$

Repeating the proof of (5.34) we now have

$$|U_6^n(\lambda)| \leq C k t_n^{-1} \lambda, \quad t_n > 0,$$

which is (4.27).

6. PROOF OF THEOREM 1.2.

In view of the proof of Theorem 1.1, we can show that

$$(6.1) \quad |\widetilde{U}_6^n(\lambda)| \leq C k^2 t_{n-1}^{-2} \lambda^{\frac{3}{2}}, \quad t_{n-1} > 0.$$

From this, we obtain the proof of Theorem 1.2. We shall not pursue this line of investigation in further detail.

7. PROOF OF THEOREM 1.4.

Writing again $U_u^n = u_0 + V_u^n$ for $n \geq 0$, we may write our scheme, as

$$(7.1) \quad \begin{aligned} \frac{V_u^n - V_u^{n-1}}{k} + \frac{1}{2} \left[\sum_{j=0}^n \omega_j(k) L V_u^{n-j} + \sum_{j=0}^{n-1} \omega_j(k) L V_u^{n-1-j} \right] \\ = - \sum_{j=0}^{n-1} \omega_j(k) L u_0 - \frac{1}{4} (\omega_n(k) - \omega_{n-1}(k)) L u_0, \quad \text{for } n \geq 2, \\ \frac{V_u^1 - V_u^0}{k} + \frac{1}{2} \left[\sum_{j=0}^1 \omega_j(k) L V_u^{1-j} - \frac{1}{2} \omega_1(k) L V_u^0 + \omega_0(k) L V_u^0 - \frac{1}{2} \omega_0(k) L V_u^0 \right] \\ = - \frac{1}{4} \omega_1(k) L u_0 - \frac{1}{2} \omega_0(k) L u_0, \\ V_u^0 = 0. \end{aligned}$$

Multiplying by $k \zeta^n$ and summing from 0 to ∞ , we obtain for the generating function, $\widetilde{V}_u(\zeta) = k \sum_{n=0}^{\infty} V_u^n \zeta^n$,

$$(7.2) \quad \widetilde{V}_u(\zeta) = \frac{k \omega_0(k)}{2} \left[\frac{\delta(\zeta)}{k} I + \widehat{A} \left(\frac{\delta(\zeta)}{k} \right) L \right]^{-1} L u_0 + \widehat{v} \left(\frac{\delta(\zeta)}{k} \right).$$

Hence, we have

$$(7.3) \quad V_u^n = \frac{k \omega_0(k)}{2} U_{u0}^n + V^n, \quad n \geq 0,$$

where U_{u0}^n and V^n are the solutions of (1.6) with $L u_0$ instead of u_0 and (1.36), respectively. Since $\omega_0(k) = \widehat{A}\left(\frac{2}{k}\right)$,

$$(7.4) \quad \left| \widehat{A}\left(\frac{2}{k}\right) \right| \leq C k,$$

$$(7.5) \quad \|U_{u0}^n\| \leq C \|L u_0\|$$

and using Theorem 1.3, we yield

$$(7.6) \quad \|U_u^n - u(t_n)\| \leq C k^2 |\log k| \|u_0\|_4, \quad \text{for } n \geq 0,$$

this completes the proof.

We remark that the proof of (7.4) needs an estimate which is similar to (2.15) with $j = 0$, and we can analogize the proof of Theorem 1 of [36] or [37] to obtain the proof of (7.5). For these proofs we omit the details.

8. NUMERICAL EXAMPLE

In this section we apply the numerical method (1.36) or $U_v^n = u_0 + V^n$, where V^n satisfies

$$(8.1) \quad \begin{aligned} & \frac{V^n - V^{n-1}}{k} + \frac{1}{2} \left(\sum_{j=0}^n \omega_j(k) L V^{n-j} + \sum_{j=0}^{n-1} \omega_j(k) L V^{n-1-j} \right) \\ &= -\frac{1}{2} \left(\sum_{j=0}^{n-1} \omega_j(k) + \sum_{j=0}^{n-2} \omega_j(k) \right) L u_0 \\ & \quad - \frac{1}{4} (\omega_n(k) + \omega_{n-1}(k)) L u_0, \quad \text{for } n \geq 2, \end{aligned}$$

$$(8.2) \quad \begin{aligned} V^1 &= (2I + k \omega_0(k) L)^{-1} \cdot (2I + k \omega_0(k) L)^{-1} \\ & \quad \cdot (-k \omega_1(k) I - 4k \omega_0(k) I - k^2 \omega_0^2(k) L) u_0, \end{aligned}$$

$$(8.3) \quad V^0 = (2I + k \omega_0(k) L)^{-1} \left(-\frac{k}{2} \omega_0(k) L \right) u_0,$$

to a viscoelastic rod problem of the form (1.1), (1.2) and (1.3) with $m = \frac{1}{4}$, $p = 2$, $q = 1$ in one space dimension. We take

$$(8.4) \quad \begin{aligned} a(t) &= e^{-t}, \quad b(t) = e^{-t}(1 - e^{-t})/t, \\ L u &= -u_{xx}, \quad u_0(x) = \sin(x), \quad x \in (0, \pi), \\ u(0, t) &= u(\pi, t) = 0, \quad t \in (0, 1]. \end{aligned}$$

To compute the numerical solution we must also discretize in space. This is accomplished by centered finite differences with parameter $h = \pi/J$. We can apply Theorem 1.3 to (8.1)–(8.3) to justify that the global error due to the full discretization of (1.1), (1.2) and (1.3) with $m = \frac{1}{4}$, $p = 2$, $q = 1$ and (8.4) behaves like $O(h^2 + k^2 |\log k|)$.

Take $k = \frac{1}{N}$, $J = N$. We present spatial errors in $L_2(0, \pi)$ norms at $t_N = 1, 1/4$, and the corresponding convergence rates determined by

$$\text{convergence rate} \simeq \frac{\log(e_N/e_{N+1})}{\log(h_N/h_{N+1})},$$

where

$$e_N = \|V_{hk} - V_{\frac{h}{2}\frac{k}{2}}\|_{L_2} = \left(\sum_{i=1}^{J-1} \left| V_{hki}^N - V_{\frac{h}{2}\frac{k}{2}2i}^N \right|^2 \frac{\pi}{J} \right)^{\frac{1}{2}}, \quad J = N,$$

is the norm of the corresponding error, and h_N is the step size of the corresponding e_N . Tables 1 and 2 give the errors in $L_2(0, \pi)$ norm, at time $t_N = 1$ and corresponding convergence rates, and Table 1 gives the errors in $L_2(0, \pi)$ norm and corresponding convergence rates at time $t_N = 1/4$. In Table 3, we fix a space mesh size $J = 1002$ then take halved time steps, and give the errors in $L_2(0, \pi)$ norm and corresponding convergence rates at time $t_N = 1$ with $N = 4, 8, 16, 32$. The numerical results shown in the tables indicate that the rate of convergence of (8.1)–(8.3) is consistent with the theory.

TABLE 1. The L_2 errors and convergence rate with $N = 4, 8, 16, 32$ and $J = N$

N	$\ V_{hk} - V_{\frac{h}{2}\frac{k}{2}}\ _{L_2}(t_N = 1)$	Rate ($t_N = 1$)	$\ V_{hk} - V_{\frac{h}{2}\frac{k}{2}}\ _{L_2}(t_N = 1/4)$	Rate ($t_N = 1/4$)
4	$3.9969e - 004$		0.0038	
8	$8.8171e - 005$	2.1805	$9.2935e - 004$	2.0404
16	$2.1252e - 005$	2.0527	$2.3074e - 004$	2.0100
32	$5.2637e - 006$	2.0134	$5.7585e - 005$	2.0025

TABLE 2. The L_2 errors and convergence rate with $N = 4, 5, 8, 10$ and $J = N$

N	$\ V_{hk} - V_{\frac{h}{2}\frac{k}{2}}\ _{L_2}(t_N = 1)$	Rate ($t_N = 1$)
4	$3.9969e - 004$	
5	$2.4225e - 004$	2.2439
8	$8.8171e - 005$	2.1504
10	$5.5457e - 005$	2.0779

TABLE 3. The L_2 errors and convergence rate with $N = 4, 8, 16, 32$ and $J = 1002$

N	$\ V_{hk} - V_{\frac{h}{2}\frac{k}{2}}\ _{L_2}(t_N = 1)$	Rate ($t_N = 1$)
4	$1.6187e - 004$	
8	$3.5729e - 005$	2.1797
16	$8.5781e - 006$	2.0584
32	$2.1225e - 006$	2.0149

APPENDIX. PROOF OF LEMMAS 2.2.

By (1.2) and (1.3) we have the representation

$$(A.1) \quad \begin{aligned} \tilde{A}(\tau) &= \hat{A}(i\tau) = \tilde{a}(\tau) f(i\tau) \\ &= \tilde{a}(\tau) \tilde{f}(\tau) = \tilde{a}(\tau) \frac{\tilde{b}(\tau) + m\tilde{a}(\tau)}{p\tilde{b}(\tau) + q\tilde{a}(\tau)}. \end{aligned}$$

From (2.1) it follows that

$$(A.2) \quad c|\tilde{a}(\tau)| \leq |\tilde{A}(\tau)| \leq C|\tilde{a}(\tau)|, \quad \tau > 0.$$

Differentiation of (A.1) yields

$$(A.3) \quad \tilde{A}'(\tau) = \tilde{a}'(\tau) \tilde{f}(\tau) + \tilde{a}(\tau) \tilde{f}'(\tau), \quad \tau > 0$$

and

$$(A.4) \quad \begin{aligned} \tilde{f}'(\tau) &= \frac{\tilde{b}'(\tau) + m\tilde{a}'(\tau)}{p\tilde{b}(\tau) + q\tilde{a}(\tau)} - \frac{\tilde{b}(\tau) + m\tilde{a}(\tau)}{p\tilde{b}(\tau) + q\tilde{a}(\tau)} \cdot \frac{p\tilde{b}'(\tau) + q\tilde{a}'(\tau)}{p\tilde{b}(\tau) + q\tilde{a}(\tau)} \\ &= \frac{(q - mp)[\tilde{a}(\tau)\tilde{b}'(\tau) - \tilde{a}'(\tau)\tilde{b}(\tau)]}{[p\tilde{b}(\tau) + q\tilde{a}(\tau)]^2} \end{aligned}$$

From (2.1), (2.3), (A.1) through (A.4) and (A.2) we find that

$$(A.5) \quad \begin{aligned} \int_0^1 \frac{|\tilde{A}'(\tau_1)|}{|\tilde{A}(\tau_1)|^2} dy &\leq C \int_0^1 \frac{|\tilde{a}'(\tau_1)|}{|\tilde{a}(\tau_1)|^2} dy + C \int_0^1 \frac{|\tilde{b}'(\tau_1)|}{|\tilde{b}(\tau_1)|^2} dy \\ &\leq C \int_0^1 \left(\frac{|\tilde{a}'(y)|}{|\tilde{a}(y)|^2} + \frac{|\tilde{b}'(y)|}{|\tilde{b}(y)|^2} \right) dy \leq C, \end{aligned}$$

where τ_1 as in (4.12).

Similarly, differentiations of (A.3) and (A.4) show that

$$(A.6) \quad \tilde{A}''(\tau) = \tilde{a}''(\tau) \tilde{f}(\tau) + 2\tilde{a}'(\tau) \tilde{f}'(\tau) + \tilde{a}(\tau) \tilde{f}''(\tau), \quad \tau > 0$$

and

$$(A.7) \quad \begin{aligned} \tilde{f}''(\tau) &= (q - mp) \frac{[\tilde{a}(\tau)\tilde{b}''(\tau) - \tilde{a}''(\tau)\tilde{b}(\tau)]}{[p\tilde{b}(\tau) + q\tilde{a}(\tau)]^2} \\ &\quad - 2(q - mp) \frac{[\tilde{a}(\tau)\tilde{b}'(\tau) - \tilde{a}'(\tau)\tilde{b}(\tau)][p\tilde{b}'(\tau) + q\tilde{a}'(\tau)]}{[p\tilde{b}(\tau) + q\tilde{a}(\tau)]^3}. \end{aligned}$$

We know from (2.1), (2.2), (2.3), (2.5), (A.6) and (A.7) that

$$(A.8) \quad \int_0^1 \frac{\tau_1 |\tilde{A}''(\tau_1)|}{|\tilde{A}(\tau_1)|^2} dy \leq C \int_0^1 \left(\frac{|\tilde{a}''(y)|}{|\tilde{a}(y)|^2} + \frac{|\tilde{b}''(y)|}{|\tilde{b}(y)|^2} \right) dy \leq C,$$

where τ_1 as in (4.12).

Again we differentiate (A.6) and (A.7), which yields

$$(A.9) \quad \tilde{A}'''(\tau) = \tilde{a}'''(\tau) \tilde{f}(\tau) + 3\tilde{a}''(\tau) \tilde{f}'(\tau) + 3\tilde{a}'(\tau) \tilde{f}''(\tau) + \tilde{a}(\tau) \tilde{f}'''(\tau),$$

$$(A.10) \quad \tilde{f}'''(\tau) = f_1(\tau) + f_2(\tau) + f_3(\tau),$$

where

$$\begin{aligned}
 f_1(\tau) &= (q - mp) \frac{\tilde{a}(\tau)\tilde{b}'''(\tau) - \tilde{a}'''(\tau)\tilde{b}(\tau)}{[p\tilde{b}(\tau) + q\tilde{a}(\tau)]^2}, \\
 f_2(\tau) &= (q - mp) \frac{(\tilde{a}'(\tau)\tilde{b}''(\tau) - \tilde{a}''(\tau)\tilde{b}'(\tau))(p\tilde{b}(\tau) + q\tilde{a}(\tau))}{[p\tilde{b}(\tau) + q\tilde{a}(\tau)]^3} \\
 &\quad - 2(q - mp) \frac{(\tilde{a}(\tau)\tilde{b}''(\tau) - \tilde{a}''(\tau)\tilde{b}(\tau))(p\tilde{b}'(\tau) + q\tilde{a}'(\tau))}{[p\tilde{b}(\tau) + q\tilde{a}(\tau)]^3} \\
 f_3(\tau) &= -2(q - mp) \frac{(\tilde{a}(\tau)\tilde{b}''(\tau) - \tilde{a}''(\tau)\tilde{b}(\tau))(p\tilde{b}'(\tau) + q\tilde{a}'(\tau))}{[p\tilde{b}(\tau) + q\tilde{a}(\tau)]^3} \\
 &\quad - 2(q - mp) \frac{(\tilde{a}(\tau)\tilde{b}'(\tau) - \tilde{a}'(\tau)\tilde{b}(\tau))(p\tilde{b}''(\tau) + q\tilde{a}''(\tau))}{[p\tilde{b}(\tau) + q\tilde{a}(\tau)]^3}.
 \end{aligned}$$

We use (2.1) through (2.3), (2.5), (2.27), (A.9), (A.1), (A.2), (A.4), (A.7) and (A.10) to obtain

$$(A.11) \quad \int_0^1 \frac{\tau^2 |\tilde{A}'''(\tau)|}{|\tilde{A}(\tau)|^2} d\tau \leq C,$$

$$(A.12) \quad \int_0^1 \frac{\tau_1^2 |\tilde{A}'''(\tau_1)|}{|\tilde{A}(\tau_1)|^2} dy \leq C,$$

where τ_1 as in (4.12) and

$$(A.13) \quad \frac{|\tau^3 \tilde{A}'''(\tau)|}{|\tilde{A}(\tau)|} = O(1) \quad (\tau \rightarrow 0^+),$$

$$(A.14) \quad \tau^4 |\tilde{A}'''(\tau)| = O(1) \quad (\tau \rightarrow \infty),$$

thus proving Lemma 2.2.

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DEPARTMENT OF MATHEMATICS, HUNAN NORMAL UNIVERSITY, CHANGSHA 410081, HUNAN, PEOPLE'S REPUBLIC OF CHINA

E-mail address: daxu@hunnu.edu.cn