A NOVEL SERIES EXPANSION FOR THE MULTIVARIATE NORMAL PROBABILITY INTEGRALS BASED ON FOURIER SERIES

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ABSTRACT. In this article, we derive a series expansion of the multivariate normal probability integrals based on Fourier series. The basic idea is to transform the limits of each integral from h_i to ∞ to be from $-\infty$ to ∞ by multiplying the integrand by a periodic square wave that approximates the domain of the integral. This square wave is expressed by its Fourier series expansion. Then a Cholesky decomposition of the covariance matrix is applied to transform the integrand to a simple one that can be easily evaluated. The resultant formula has a simple pattern that is expressed as multiple series expansion of trigonometric and exponential functions.

1. INTRODUCTION

A common problem that arises in many statistics computations is computing the complementary integral of the multivariate normal distribution which is given by

$$L(h, \Sigma) = \frac{1}{\sqrt{(2\pi)^m |\Sigma|}} \int_{h_1}^{\infty} \cdots \int_{h_m}^{\infty} \exp \left\{ -\frac{1}{2} x^T \Sigma^{-1} x \right\} dx,$$

where $x = (x_1, x_2, \dots, x_m)$ and Σ is an $m \times m$ symmetric positive definite covariance matrix.

The problem has received considerable attention in the literature [14–16, 18]. Efficient formulas exist for $m = 1$ and $m = 2$. For $m = 2$, Pearson [21] derived the following simple tetrachoric series for the bivariate normal CDF:

$$\Phi(h_1, h_2, \rho) = \Phi(h)\Phi(k) + \frac{1}{2\pi} \exp \left\{ -\frac{(h_1^2 + h_2^2)}{2} \right\} \sum_{j=0}^{\infty} \frac{2^{-j} \rho^{j+1}}{j+1!} H_j \left(\frac{h_1}{\sqrt{2}} \right) H_j \left(\frac{h_2}{\sqrt{2}} \right),$$

where

$$L(h_1, h_2, \rho) = \Phi(-h_1, -h_2, \rho)$$

and $H_j(x)$ is the Hermite polynomial defined by Abramowitz and Stegun [1, p. 775]:

$$H_j(x) = (-1)^j \exp(x^2) D^j \exp(-x^2) = j! \sum_{k=0}^{[j/2]} \frac{(-1)^k}{k!(j-2k)!} (2x)^{j-2k}$$

$$[j/2] = \begin{cases} j/2 & j \text{ even,} \\ (j-1)/2 & j \text{ odd,} \end{cases}$$

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and $\Phi(x)$ is the standard univariate CDF given by

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x \exp\left(-\frac{t^2}{2}\right) dt.$$

However, this series converges too slowly for large values of $|\rho|$ and thus is not very satisfactory. Owen [20] expressed the bivariate normal CDF in terms of the T-function as

$$\Phi(h_1, h_2, \rho) = \frac{1}{2}\Phi(h_1) - T\left(h_1, \frac{h_2 - \rho h_1}{h_1 \sqrt{1-\rho^2}}\right) + \frac{1}{2}\Phi(h_2) - T\left(h_2, \frac{h_1 - \rho h_2}{h_2 \sqrt{1-\rho^2}}\right) + \eta,$$

where

$$\eta = \begin{cases} 0 & \text{if } h_1 h_2 > 0 \text{ or if } h_1 h_2 = 0 \text{ but } h_1 + h_2 \geq 0, \\ -\frac{1}{2} & \text{otherwise,} \end{cases}$$

$$T(h, a) = \frac{1}{2\pi} \int_0^a \frac{1}{1+x^2} \exp\left\{-\frac{h^2}{2}(1+x^2)\right\} dx$$

$$= \frac{\arctan a}{2\pi} - \frac{1}{2\pi} \sum_{j=0}^{\infty} \frac{(-1)^j a^{2j+1}}{2j+1} \left[1 - \exp\left(-\frac{h^2}{2}\right) \sum_{i=0}^j \frac{h^{2i}}{2^i i!}\right].$$

Note that the T-function can also be written in terms of the normalized incomplete Gamma function [1, pp. 260, 262]:

$$P(a, x) = \frac{1}{\Gamma(a)} \int_0^x t^{a-1} \exp(-t) dt = 1 - \exp(-x) \sum_{i=0}^{a-1} \frac{x^i}{i!}$$

as

$$T(h, a) = \frac{\arctan a}{2\pi} - \frac{1}{2\pi} \sum_{j=0}^{\infty} \frac{(-1)^j a^{2j+1}}{2j+1} P(j+1, \frac{h^2}{2}).$$

The series expansion of the T-function converges rapidly for small values of a and h but converges quite slowly when h is large and a is close to 1. Donnelly [5] directly coded Owen's method and formed the Owen-Donnelly algorithm, which has been widely used as it gives the values of the bivariate normal CDF to 15 digit accuracy. Fayed and Atiya [8] derived two formulas that converge fast for large values of $|\rho|$. One of them, for $\rho < 0$, is given by

$$L(h_1, h_2, \rho) = L_{20} + \sum_{u=0}^{\infty} [\psi_u(b_2) + \psi_u(c_2)]$$

$$\cdot \begin{cases} (-1)^u [1 - P(\frac{u}{2} + 1, z_{10}^2)] & h_1 + h_2 \geq 0, \\ [(-1)^u - P(\frac{u}{2} + 1, z_{10}^2)] & h_1 + h_2 < 0, \end{cases}$$

where

$$\psi_u(x) = \frac{1}{2\sqrt{\pi}\Gamma(\frac{u}{2} + \frac{3}{2})} \left(\frac{a_2}{2}\right)^{u+1} \exp(-x^2) H_u(x),$$

$$a_2 = \sqrt{\frac{1+\rho}{1-\rho}}, b_2 = \frac{-h_2}{\sqrt{1-\rho}}, c_2 = \frac{-h_1}{\sqrt{1-\rho}}, z_{10} = \frac{h_1 + h_2}{2\sqrt{1+\rho}},$$

$$L_{20} = \frac{1}{2} \left[\left(1 - \text{sign}(z_{10})(2\Phi(\sqrt{2}|z_{10}|) - 1)\right) \left(1 - \Phi(-\sqrt{2}b_2) - \Phi(-\sqrt{2}c_2)\right) \right],$$

and another similar formula for $\rho > 0$. There is another group of methods in the literature that are based on numerically approximating the integral. Among those methods, Divgi [4] estimated the complementary integral in polar coordinates. Drezner [6] provided different formulas based on a direct computation of the double integral using the Gauss quadrature method. Drezner and Wesolowsky [7] presented a simple method that achieves single precision accuracy with less computation time than Divgi's algorithm. Genz [13] presented a modified form of the Drezner and Wesolowsky algorithm that is slightly more accurate for double precision.

For the multivariate case, $m > 2$, Kendall [18] gives a generalization of the tetrachoric series, for the trivariate case,

$$L(h_1, h_2, h_3, \rho_{12}, \rho_{13}, \rho_{23}) = \frac{\exp\left\{-\frac{h_1^2 + h_2^2 + h_3^2}{2}\right\}}{(\sqrt{2\pi})^3} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{2^{-i-j-k-1} \rho_{12}^i \rho_{23}^j \rho_{13}^k}{i!j!k!} H_{i+j-1}\left(\frac{h_1}{\sqrt{2}}\right) H_{i+k-1}\left(\frac{h_2}{\sqrt{2}}\right) H_{j+k-1}\left(\frac{h_3}{\sqrt{2}}\right)$$

and for the quadrivariate case it can be written as

$$L(h_1, h_2, h_3, h_4, \rho_{12}, \rho_{13}, \rho_{14}, \rho_{23}, \rho_{24}, \rho_{34}) = \frac{\exp\left\{-\frac{h_1^2 + h_2^2 + h_3^2 + h_4^2}{2}\right\}}{(2\pi)^2} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{o=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{2^{-l-m-n-o-p-q-2} \rho_{12}^l \rho_{13}^m \rho_{14}^n \rho_{23}^o \rho_{24}^p \rho_{34}^q}{l!m!n!o!p!q!} \cdot H_{l+m+n-1}\left(\frac{h_1}{\sqrt{2}}\right) H_{l+o+p-1}\left(\frac{h_2}{\sqrt{2}}\right) H_{m+o+q-1}\left(\frac{h_3}{\sqrt{2}}\right) H_{n+p+q-1}\left(\frac{h_4}{\sqrt{2}}\right).$$

However, these series require computation of a number of summations of $O(m^2)$. Moreover, they are found to converge slowly when $\rho_{ij} > 0.3$ and therefore are not satisfactory from a practical point of view (see numerical experiments section). A more computable case occurs when the components of x are equicorrelated. If $\rho_{ij} = \rho$ for all $i \neq j$, $\rho_{ii} = 1$, then the integral can be found by the formula [25, p. 192]:

$$L(h, \Sigma) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{t^2}{2}\right) \prod_{i=1}^m \Phi\left(\frac{-h_i + t\sqrt{\rho}}{\sqrt{1-\rho}}\right) dt.$$

There exist several numerical methods for the general case based on multivariate integration techniques that rely on ordinary Monte-Carlo methods along with some common variance reduction techniques. These include a method introduced by Deák [2, 3], which decomposes the multivariate normal distribution into two components, an $(m-1)$ -dimensional direction and a one-dimensional length component, which can be integrated out explicitly. Another group of algorithms due to Szantai is based on computing upper and lower bounds on the probability (see [10, 12] for a survey of these methods). Besides, Shephard [23] derived a framework to compute the distribution function from the characteristic function and developed numerical integration rules for the evaluation of the resultant integrals. Recently, Miwa et al. [19] proposed a method that is considered among the most efficient methods for $m \leq 7$. In this method, evaluation of the multiple integral is transformed into recursive evaluation of a one-dimensional integration over a fine grid of points.

2. FOURIER SERIES APPROACH

Assume that $h_j \in [-h_{\max}, h_{\max}]$ (for example $h_{\max} = 4$), $j = 1, \dots, m$, then the multivariate probability integral can be approximated as

$$L(h, \Sigma) = \frac{1}{\sqrt{(2\pi)^m |\Sigma|}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left[\prod_{j=1}^m g(x_j - h_j) \right] \exp \left\{ -\frac{1}{2} x^T \Sigma^{-1} x \right\} dx,$$

where $g(x_j)$ is a periodic square wave function with period $2T$ (where $T \geq 2h_{\max}$ for good results) and the fundamental period is given by

$$g(x_j) = \begin{cases} 0 & -T < x_j < 0, \\ 1 & 0 \leq x_j \leq T. \end{cases}$$

Using Cholesky decomposition of the covariance matrix as employed by Deák and Genz [2, 11]; that is, $\Sigma = CC^T$, and then using the transformation $x = Cy$ and thereby $dx = |C| dy = |\Sigma|^{\frac{1}{2}} dy$ leads to

$$(2.1) \quad L(h, \Sigma) = \frac{1}{\sqrt{(2\pi)^m}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left[\prod_{j=1}^m g(c_j^T y - h_j) \right] \exp \left\{ -\frac{1}{2} y^T y \right\} dy,$$

where $C^T = [c_1 \ c_2 \ \cdots \ c_m]$.

Expanding $g(x_j)$ in the Fourier series

$$\begin{aligned} g(x_j) &= \frac{1}{2} + \frac{2}{\pi} \sum_{n_j=0}^{\infty} \frac{1}{(2n_j+1)} \sin \left(\frac{(2n_j+1)\pi(x_j)}{T} \right) \\ &= \frac{1}{2} - \frac{i}{\pi} \sum_{n_j=0}^{\infty} \frac{1}{(2n_j+1)} \left[e^{\frac{(2n_j+1)i\pi(x_j)}{T}} - e^{\frac{-(2n_j+1)i\pi(x_j)}{T}} \right] \end{aligned}$$

and substituting in (2.1) leads to

$$\begin{aligned} L(h, \Sigma) &= \frac{1}{\sqrt{(2\pi)^m}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp \left\{ -\frac{1}{2} y^T y \right\} \\ &\quad \cdot \prod_{j=1}^m \left\{ \frac{1}{2} - \frac{i}{\pi} \sum_{n_j=0}^{\infty} \frac{1}{(2n_j+1)} \left[e^{\frac{(2n_j+1)i\pi(c_j^T y - h_j)}{T}} - e^{\frac{-(2n_j+1)i\pi(c_j^T y - h_j)}{T}} \right] \right\} dy. \end{aligned}$$

The above multiple integral can be expressed as sums of products of single integrals for each variable y_j where each one can be evaluated using the following integral [22, p. 459]:

$$(2.2) \quad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{-x^2}{2} + i(qx+r)} dx = e^{\frac{-q^2}{2} + ir}.$$

Note that the complexity of the resultant terms ranges from a constant term which is $\frac{1}{2^m}$ to the most complicated term which is composed of m nested summations.

3. BIVARIATE CASE

For the bivariate case, we have

$$\Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{bmatrix}.$$

Therefore, the complementary integral can be written as

$$\begin{aligned} L(h_1, h_2, \rho) = & \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\frac{-y_1^2 - y_2^2}{2}} \\ & \cdot \left\{ \frac{1}{2} - \frac{i}{\pi} \sum_{n_1=0}^{\infty} \frac{1}{(2n_1+1)} \right. \\ & \left[e^{\frac{(2n_1+1)i\pi(y_1-h_1)}{T}} - e^{\frac{-(2n_1+1)i\pi(y_1-h_1)}{T}} \right] \Big\} \\ & \cdot \left\{ \frac{1}{2} - \frac{i}{\pi} \sum_{n_2=0}^{\infty} \frac{1}{(2n_2+1)} \right. \\ & \left[e^{\frac{(2n_2+1)i\pi(\rho y_1 + \sqrt{1-\rho^2}y_2 - h_2)}{T}} - e^{\frac{-(2n_2+1)i\pi(\rho y_1 + \sqrt{1-\rho^2}y_2 - h_2)}{T}} \right] \Big\} \\ & dy_1 dy_2. \end{aligned}$$

Using (2.2), it can be shown that

$$\begin{aligned} L(h_1, h_2, \rho) = & \frac{1}{4} \\ & - \frac{1}{\pi} \sum_{n_1=0}^{\infty} \frac{e^{\frac{-\pi^2}{2T^2}(2n_1+1)^2}}{(2n_1+1)} \left\{ \sin\left(\frac{(2n_1+1)\pi h_1}{T}\right) + \sin\left(\frac{(2n_1+1)\pi h_2}{T}\right) \right\} \\ & - \frac{2}{\pi^2} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \frac{e^{\frac{-\pi^2}{2T^2}\{(2n_1+1)^2 + (2n_2+1)^2\}}}{(2n_1+1)(2n_2+1)} \\ & \cdot \left\{ e^{\frac{-\pi^2}{T^2}(2n_1+1)(2n_2+1)\rho} \cos\left(\frac{\pi}{T}[(2n_1+1)h_1 + (2n_2+1)h_2]\right) \right. \\ & \left. - e^{\frac{\pi^2}{T^2}(2n_1+1)(2n_2+1)\rho} \cos\left(\frac{\pi}{T}[(2n_1+1)h_1 - (2n_2+1)h_2]\right) \right\}. \end{aligned}$$

Note that, for the special case when $\rho = 0, h_2 = 0$, we have

$$L(h_1, 0, 0) = \frac{1}{4} - \frac{1}{\pi} \sum_{n_1=0}^{\infty} \frac{e^{\frac{-\pi^2}{2T^2}(2n_1+1)^2}}{(2n_1+1)} \sin\left(\frac{(2n_1+1)\pi h_1}{T}\right),$$

which is known to be [21]:

$$L(h_1, 0, 0) = \frac{1}{2}\Phi(-h_1) = \frac{1}{4} - \frac{1}{4}\operatorname{erf}\left(\frac{h_1}{\sqrt{2}}\right).$$

So we can conclude that

$$(3.1) \quad \operatorname{erf}\left(\frac{h_1}{\sqrt{2}}\right) = \frac{4}{\pi} \sum_{n_1=0}^{\infty} \frac{e^{\frac{-\pi^2}{2T^2}(2n_1+1)^2}}{(2n_1+1)} \sin\left(\frac{(2n_1+1)\pi h_1}{T}\right),$$

which is actually the Fourier series expansion of a periodic function of $\operatorname{erf}(h_1/\sqrt{2})$ with a fundamental period $[-T, T]$. Therefore the bivariate complementary integral can be written as

$$(3.2) \quad \begin{aligned} L(h_1, h_2, \rho) = & \frac{1}{4} - \frac{1}{4} \operatorname{erf}\left(\frac{h_1}{\sqrt{2}}\right) - \frac{1}{4} \operatorname{erf}\left(\frac{h_2}{\sqrt{2}}\right) \\ & - \frac{2}{\pi^2} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \frac{e^{\frac{-\pi^2}{2T^2}\{(2n_1+1)^2+(2n_2+1)^2\}}}{(2n_1+1)(2n_2+1)} \\ & \cdot \left\{ e^{\frac{-\pi^2}{T^2}(2n_1+1)(2n_2+1)\rho} \cos\left(\frac{\pi}{T}[(2n_1+1)h_1 + (2n_2+1)h_2]\right) \right. \\ & \left. - e^{\frac{\pi^2}{T^2}(2n_1+1)(2n_2+1)\rho} \cos\left(\frac{\pi}{T}[(2n_1+1)h_1 - (2n_2+1)h_2]\right) \right\} \end{aligned}$$

Since the bivariate normal CDF can be expressed as [1, 9]:

$$\begin{aligned} \Phi(h_1, h_2, \rho) = & \Phi\left(h_1, 0, \frac{(\rho h_1 - h_2) \operatorname{sign}(h_1)}{\sqrt{h_1^2 - 2\rho h_1 h_2 + h_2^2}}\right) + \Phi\left(h_2, 0, \frac{(\rho h_2 - h_1) \operatorname{sign}(h_2)}{\sqrt{h_1^2 - 2\rho h_1 h_2 + h_2^2}}\right) \\ & - \begin{bmatrix} 0 & \text{if } h_1 h_2 > 0 \\ \frac{1}{2} & \text{if } h_1 h_2 < 0 \end{bmatrix}. \end{aligned}$$

Therefore we can study the special case when $h_2 = 0$, so for the above Fourier series, we have

$$(3.3) \quad \begin{aligned} L(h_1, 0, \rho) = & \frac{1}{4} \operatorname{erfc}\left(\frac{h_1}{\sqrt{2}}\right) \\ & + \frac{4}{\pi^2} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \frac{e^{\frac{-\pi^2}{2T^2}\{(2n_1+1)^2+(2n_2+1)^2\}}}{(2n_1+1)(2n_2+1)} \sinh\left[\frac{\pi^2 \rho}{T^2}(2n_1+1)(2n_2+1)\right] \\ & \cdot \cos\left[\frac{\pi h_1}{T}(2n_1+1)\right]. \end{aligned}$$

It is found that convergence of the above series is good only for small values of $|\rho|$. In the following subsection we will derive other forms of this series, one of them converges fast for all values of ρ .

3.1. Different forms of the bivariate case. Let us investigate other forms of the above formula. First, note that for the special case when $h_1 = h_2 = 0$, and comparing with the well-known formula of Johnson and Kotz [17], we can conclude that

$$\arcsin \rho = \frac{8}{\pi} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \frac{e^{\frac{-\pi^2}{2T^2}\{(2n_1+1)^2+(2n_2+1)^2\}}}{(2n_1+1)(2n_2+1)} \sinh\left[\frac{\pi^2 \rho}{T^2}(2n_1+1)(2n_2+1)\right].$$

Using the Maclaurin series of the inverse sine and hyperbolic sine functions,

$$\sum_{j=0}^{\infty} \frac{2j!}{2^{2j} j!^2 (2j+1)} \rho^{2j+1} = \frac{8}{\pi} \sum_{j=0}^{\infty} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \frac{e^{\frac{-\pi^2}{2T^2} \{ (2n_1+1)^2 + (2n_2+1)^2 \}}}{(2n_1+1)(2n_2+1)(2j+1)!} \cdot \left[\frac{\pi^2 \rho}{T^2} (2n_1+1)(2n_2+1) \right]^{2j+1}.$$

By comparing the coefficient of ρ^{2j+1} in both sides, we have

$$(3.4) \quad \sum_{n_1=0}^{\infty} (2n_1+1)^{2j} e^{\frac{-\pi^2}{2T^2} (2n_1+1)^2} = \left(\frac{T}{\pi} \right)^{2j+1} 2^{j-\frac{3}{2}} \Gamma(j+\frac{1}{2}).$$

By differentiating (3.1), $2j+1$ times, we can conclude that

$$\begin{aligned} \sum_{n_1=0}^{\infty} (-1)^j (2n_1+1)^{2j} e^{\frac{-\pi^2}{2T^2} (2n_1+1)^2} \cos \left(\frac{(2n_1+1)\pi h_1}{T} \right) \\ = \frac{1}{2} \sqrt{\frac{\pi}{2}} \left(\frac{T}{\pi} \right)^{2j+1} \frac{d^{2j}}{dh_1^{2j}} \left\{ \exp \left(-\frac{h_1^2}{2} \right) \right\}. \end{aligned}$$

Using the definition of the Hermite polynomial [1, p. 785], it can be written as

$$(3.5) \quad \begin{aligned} \sum_{n_1=0}^{\infty} (-1)^j (2n_1+1)^{2j} e^{\frac{-\pi^2}{2T^2} (2n_1+1)^2} \cos \left(\frac{(2n_1+1)\pi h_1}{T} \right) \\ = \sqrt{\pi} \left(\frac{T}{\pi} \right)^{2j+1} 2^{-j-\frac{3}{2}} \exp \left(-\frac{h^2}{2} \right) H_{2j} \left(\frac{h}{\sqrt{2}} \right). \end{aligned}$$

Substituting by (3.4) and (3.5) in (3.3), we get

$$L(h_1, 0, \rho) = \frac{1}{4} \operatorname{erfc} \left(\frac{h_1}{\sqrt{2}} \right) + \frac{1}{2\pi} \exp \left(-\frac{h^2}{2} \right) \sum_{j=0}^{\infty} \frac{(-1)^j 2^{-2j} \rho^{2j+1}}{j!(2j+1)} H_{2j} \left(\frac{h_1}{\sqrt{2}} \right),$$

which is actually the tetrachoric series [21]. It is also known that it converges fast for small values of $|\rho|$ but it is very slow when $|\rho|$ is close to 1.

Another form can be obtained as follows:

$$\begin{aligned} L(h_1, 0, \rho) &= \frac{1}{4} \operatorname{erfc} \left(\frac{h_1}{\sqrt{2}} \right) \\ &+ \frac{4}{\pi^2} \sum_{n_1=0}^{\infty} \frac{e^{\frac{-\pi^2}{2T^2} (2n_1+1)^2}}{(2n_1+1)} \cos \left[\frac{\pi h_1}{T} (2n_1+1) \right] \\ &\cdot \sum_{j=0}^{\infty} \frac{2^{j-\frac{3}{2}} \Gamma(j+\frac{1}{2})}{(2j+1)!} \left[\frac{\pi \rho}{T} (2n_1+1) \right]^{2j+1}. \end{aligned}$$

Using the definition of Dawson function, [1, p. 298],

$$\operatorname{Daw}(x) = \exp(-x^2) \int_0^x \exp(t^2) dt = \exp(-x^2) \sum_{j=0}^{\infty} \frac{x^{2j+1}}{j!(2j+1)},$$

we get

$$\begin{aligned}
 L(h_1, 0, \rho) &= \frac{1}{4} \operatorname{erfc}\left(\frac{h_1}{\sqrt{2}}\right) \\
 (3.6) \quad &+ \frac{2}{\pi\sqrt{\pi}} \sum_{n_1=0}^{\infty} \frac{e^{\frac{-\pi^2(1-\rho^2)}{2T^2}(2n_1+1)^2}}{(2n_1+1)} \operatorname{Daw}\left[\frac{\pi\rho}{\sqrt{2}T}(2n_1+1)\right] \\
 &\cdot \cos\left[\frac{\pi h_1}{T}(2n_1+1)\right].
 \end{aligned}$$

Note that the above series is convergent, since the non-alternating series (say S)

$$\begin{aligned}
 |S| &\leq \frac{2}{\pi\sqrt{\pi}} \sum_{n_1=0}^{\infty} \frac{e^{\frac{-\pi^2(1-\rho^2)}{2T^2}(2n_1+1)^2}}{(2n_1+1)} \left| \operatorname{Daw}\left[\frac{\pi\rho}{\sqrt{2}T}(2n_1+1)\right] \right| \\
 &\leq \frac{2(0.55)}{\pi\sqrt{\pi}} \sum_{n_1=0}^{\infty} \frac{e^{\frac{-\pi^2(1-\rho^2)}{2T^2}(2n_1+1)^2}}{(2n_1+1)} \\
 &\leq \frac{1.1}{\pi\sqrt{\pi}} \sum_{n_1=0}^{\infty} \frac{e^{\frac{-\pi^2(1-\rho^2)}{2T^2}(2n_1+1)}}{(2n_1+1)} = \frac{0.55}{\pi\sqrt{\pi}} \ln \left[\frac{1 + e^{\frac{-\pi^2(1-\rho^2)}{2T^2}}}{1 - e^{\frac{-\pi^2(1-\rho^2)}{2T^2}}} \right],
 \end{aligned}$$

where we have used $|\operatorname{Daw}(x)| \leq 0.55$ and $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} = \frac{1}{2} \ln \left[\frac{1+x}{1-x} \right]$, $|x| < 1$.

Note also that $\operatorname{Daw}(0) = 0$ and it increases until reaching its maximum value (which is less than 0.55) and then decreases and approaches zero as x tends to ∞ [1, p. 297]; see Figure 1. Therefore it converges faster than the tetrachoric series for large values for $|\rho|$.

4. MULTIVARIATE CASE

For the trivariate case, using the special shape of the Cholesky decomposition, it can be shown that

$$\begin{aligned}
 L(h_1, h_2, h_3, \rho_{12}, \rho_{13}, \rho_{23}) &= L_0 + L_1 + L_2 + L_3, \\
 L_0 &= \frac{1}{8}, \quad L_1 = -\frac{1}{8} \left[\operatorname{erf}\left(\frac{h_1}{\sqrt{2}}\right) + \operatorname{erf}\left(\frac{h_2}{\sqrt{2}}\right) + \operatorname{erf}\left(\frac{h_3}{\sqrt{2}}\right) \right], \\
 L_2 &= -\frac{1}{\pi^2} \sum_{j>i} \sum_{n_i=0}^{\infty} \sum_{n_j=0}^{\infty} \frac{e^{\frac{-\pi^2}{2T^2} \{ (2n_i+1)^2 + (2n_j+1)^2 \}}}{(2n_i+1)(2n_j+1)} \\
 &\cdot \sum_{s=0}^1 \left\{ (-1)^s e^{\frac{-(-1)^s \pi^2}{T^2} \{ \rho_{ij}(2n_i+1)(2n_j+1) \}} \right. \\
 &\quad \left. \cos\left(\frac{\pi}{T} [(2n_i+1)h_i + (-1)^s(2n_j+1)h_j]\right) \right\}
 \end{aligned}$$

FIGURE 1. Dawson function

$$\begin{aligned}
 L_3 = & \frac{2}{\pi^3} \sum_{k>j>i} \sum_{n_i=0}^{\infty} \sum_{n_j=0}^{\infty} \sum_{n_k=0}^{\infty} \frac{e^{\frac{-\pi^2}{2T^2} \{ (2n_i+1)^2 + (2n_j+1)^2 + (2n_k+1)^2 \}}}{(2n_i+1)(2n_j+1)(2n_k+1)} \\
 & \cdot \sum_{s_i, s_j, s_k = \{0,1\}, s_i + s_j + s_k \leq 1} (-1)^{\sum_{a \in \{i,j,k\}} s_a} e^{\frac{-\pi^2}{T^2} \sum_{a, b \in \{i,j,k\}, a \neq b} (-1)^{s_a + s_b} (2n_a+1)(2n_b+1) \rho_{ab}} \\
 & \cdot \sin \left(\frac{\pi}{T} \sum_{a \in \{i,j,k\}} (-1)^{s_a} (2n_a+1) h_a \right) \Bigg\}
 \end{aligned}$$

and for the quadrivariate case

$$\begin{aligned}
 L(h_1, h_2, h_3, h_4, \rho_{12}, \rho_{13}, \rho_{14}, \rho_{23}, \rho_{24}, \rho_{34}) &= L_0 + L_1 + L_2 + L_3 + L_4, \\
 L_0 &= \frac{1}{16}, \quad L_1 = -\frac{1}{16} \left[\operatorname{erf}\left(\frac{h_1}{\sqrt{2}}\right) + \operatorname{erf}\left(\frac{h_2}{\sqrt{2}}\right) + \operatorname{erf}\left(\frac{h_3}{\sqrt{2}}\right) + \operatorname{erf}\left(\frac{h_4}{\sqrt{2}}\right) \right],
 \end{aligned}$$

$$\begin{aligned}
L_2 = & -\frac{1}{2\pi^2} \sum_{j>i} \sum_{n_i=0}^{\infty} \sum_{n_j=0}^{\infty} \frac{e^{\frac{-\pi^2}{2T^2} \{ (2n_i+1)^2 + (2n_j+1)^2 \}}}{(2n_i+1)(2n_j+1)} \\
& \cdot \sum_{s=0}^1 \left\{ (-1)^s e^{\frac{-(-1)^s \pi^2}{T^2} \{ \rho_{ij} (2n_i+1)(2n_j+1) \}} \right. \\
& \quad \left. \cos \left(\frac{\pi}{T} [(2n_i+1)h_i + (-1)^s (2n_j+1)h_j] \right) \right\}, \\
L_3 = & \frac{1}{\pi^3} \sum_{k>j>i} \sum_{n_i=0}^{\infty} \sum_{n_j=0}^{\infty} \sum_{n_k=0}^{\infty} \frac{e^{\frac{-\pi^2}{2T^2} \{ (2n_i+1)^2 + (2n_j+1)^2 + (2n_k+1)^2 \}}}{(2n_i+1)(2n_j+1)(2n_k+1)} \\
& \cdot \sum_{s_i, s_j, s_k = \{0,1\}, s_i+s_j+s_k \leq 1} (-1)^{\sum_{a \in \{i,j,k\}} s_a} e^{\frac{-\pi^2}{T^2} \sum_{a,b \in \{i,j,k\}, a \neq b} (-1)^{s_a+s_b} (2n_a+1)(2n_b+1) \rho_{ab}} \\
& \cdot \sin \left(\frac{\pi}{T} \sum_{a \in \{i,j,k\}} (-1)^{s_a} (2n_a+1) h_a \right) \Bigg\}, \\
L_4 = & \frac{2}{\pi^4} \sum_{l>k>j>i} \sum_{n_i=0}^{\infty} \sum_{n_j=0}^{\infty} \sum_{n_k=0}^{\infty} \sum_{n_l=0}^{\infty} \frac{e^{\frac{-\pi^2}{2T^2} \{ (2n_i+1)^2 + (2n_j+1)^2 + (2n_k+1)^2 + (2n_l+1)^2 \}}}{(2n_i+1)(2n_j+1)(2n_k+1)(2n_l+1)} \\
& \cdot \sum_{s_i, s_j, s_k, s_l = \{0,1\}, s_i+s_j+s_k+s_l \leq 2} (-1)^{\sum_{a \in \{i,j,k,l\}} s_a} e^{\frac{-\pi^2}{T^2} \sum_{a,b \in \{i,j,k,l\}, a \neq b} (-1)^{s_a+s_b} (2n_a+1)(2n_b+1) \rho_{ab}} \\
& \cdot \cos \left(\frac{\pi}{T} \sum_{a \in \{i,j,k,l\}} (-1)^{s_a} (2n_a+1) h_a \right),
\end{aligned}$$

where either $s_a = s_b = 1$ or $s_a = s_b = 0$ for $a, b \in \{i, j, k, l\}$ is selected when $s_i + s_j + s_k + s_l = 2$ in L_4 . Following the same procedure, it can be extended for any dimension.

5. NUMERICAL EXPERIMENTS

We compared the well-known tetrachoric series (*Tetrachoric*) with the newly derived Dawson series (*Dawson*), (3.6), and Fourier series (*Fourier*), (3.2), for the bivariate complementary integral. As our benchmark, We selected the Owen-Donnelly algorithm (*Owen*), since it has been considered the most widely implemented with 15 significant digits accuracy [5], and the Drezner-Wesolowsky method [7] with slightly improved version implemented by Genz [13] which is one of the most efficient methods currently available in double precision. A sample of 50 sets of values of h_1 is chosen uniformly from $[0, \sqrt{2}]$ for each ρ ranges from $\{0.1, 0.2, \dots, 0.9\}$ while setting $h_2 = 0$. For *Dawson* and *Fourier* series, the half period length, T , is set to 10. For *Tetrachoric*, *Dawson* and *Owen* methods, we examined the partial sums until an accuracy of 10^{-15} is obtained or the maximum number of terms (100 in our experiments) is reached. For *Fourier* series, we examined partial sums of the first $N = 8$ and $N = 12$ terms. We used C language in our implementation on Windows XP with SP2 running on Pentium 2.4GHz PC with 3GB RAM. Dawson function is obtained from “Mathematics Source Library” available

at: <http://www.mymathlib.com>. The average absolute error ($Error_{av}$) and the average number of terms (N_{av}) and their standard deviations required in *Tetrachoric*, *Dawson* and *Owen* methods over the experimental range of h_1 is reported in Table 1. The average time for each algorithm was less than 10^{-4} , so it is not reported. It can be noticed from the results that Dawson series converges for all values of ρ . Moreover, it requires fewer number of terms to converge than the tetrachoric series especially for large values of $|\rho|$. On the other hand, Fourier series attains an average error less than 10^{-6} for $\rho \leq 0.6$ using $N = 8$ terms and of order 10^{-12} for $\rho \leq 0.7$ using $N = 12$.

TABLE 1. Bivariate results. Average absolute error ($Error_{av}$) and the average number of added terms (N_{av}) for *Tetrachoric*, *Dawson* and *Owen* methods. N is the preset number of added terms for *Fourier* methods. Second row for each value of ρ represents the standard deviation of the corresponding entry.

ρ	<i>Tetrachoric</i>		<i>Dawson</i>		<i>Fourier</i>		<i>Genz</i>	<i>Owen</i>
	$Error_{av}$	N_{av}	$Error_{av}$	N_{av}	$Error_{av}$	N	$Error_{av}$	
0.10	2.4E-017	15	1.7E-017	13	4.3E-009	1.50E-15	2.1E-016	7
	2.3E-017	0	1.8E-017	0	2.9E-009	1.10E-15	6.3E-016	1
0.20	2.9E-017	22	1.4E-017	13	9.5E-009	2.90E-15	9.7E-017	8
	2.8E-017	1	1.7E-017	0	8.1E-009	1.90E-15	2.1E-016	2
0.30	2.9E-017	29	3.1E-017	13	2.3E-008	2.00E-15	1.2E-016	8
	3.6E-017	0	3.2E-017	0	1.6E-008	1.90E-15	2.8E-016	2
0.40	3.7E-017	39	2.2E-017	13	4.3E-008	3.40E-15	1.4E-016	9
	4.1E-017	1	2.4E-017	0	3.5E-008	2.80E-15	2.5E-016	2
0.50	3.3E-017	51	3.2E-017	15	1.2E-007	1.40E-14	1.2E-015	10
	3.2E-017	0	3.7E-017	0	8.2E-008	1.50E-14	8.3E-015	3
0.60	3.8E-017	70	3.7E-017	15	4.8E-007	6.80E-14	1.4E-016	11
	3.9E-017	1	3.8E-017	0	4.0E-007	1.40E-13	4.2E-016	4
0.70	4.4E-017	100	3.5E-017	17	2.9E-006	3.20E-12	4.1E-017	13
	3.9E-017	0	3.6E-017	0	1.8E-006	2.90E-12	3.2E-017	4
0.80	2.9E-014	100	6.2E-017	21	2.5E-005	1.20E-09	9.9E-017	13
	1.6E-014	0	5.4E-017	1	1.3E-005	1.10E-09	1.4E-016	4
0.90	7.3E-009	100	6.9E-017	28	2.5E-004	6.50E-07	1.1E-016	13
	4.2E-009	0	4.9E-017	1	1.4E-004	4.60E-07	1.2E-016	4

We also compared the Fourier series approach for higher dimension (*FORMVN*). We applied it for $3 \leq m \leq 7$ cases and compared it with the tetrachoric series (*TETMVN*), Genz's subregion adaptive partitioning algorithm (*SADMVN*) and Genz's implementation of randomized Korobov rules (*KROMVN*) and Miwa's algorithm (*Miwa*). As a benchmark, we used the Gauss-Kronrod (7,15) pair quadrature integration method for the equicorrelated case [25]. We implemented *FORMVN* and *TETMVN* in C language, and obtained the C code of Miwa's algorithm and the FORTRAN code of *SADMVN* and *KROMVN* from Genz's homepage (<http://www.math.wsu.edu/faculty/genz/homepage>). For the trivariate case, we compared also with Plackett's formula implemented by Genz [13]. A sample of

50 sets of values of h'_i s are chosen uniformly from $[0, \sqrt{m}]$ for each ρ ranges from $\{0.1, 0.2, \dots, 0.9\}$. For *FORMVN*, *TETMVN* partial sums of the first $N = 8$ and $N = 12$ terms are considered for each summation and the half period length, T , is set to 10 for *FORMVN*. For *SADMVN* and *KROMVN*, the required accuracy is set to 10^{-8} . For Miwa's algorithm, grid sizes examined are $G = 128$ and $G = 4096$ (which is the maximum that could be used due to memory limitation). The average absolute error, the average elapsed time and their standard deviations over the 50 runs are reported in Table 2 to Table 11. It can be noticed that *TETMVN* is quite fast and has a good accuracy only for $m = 3$ when $\rho \leq 0.4$ and for $m = 4$ when $\rho \leq 0.3$ and in general *FORMVN* and *Miwa*'s algorithms are significantly faster than *SADMVN* and *KROMVN* where *FORMVN* beats for $3 \leq m \leq 5$ when $\rho \leq 0.5$ while *Miwa*'s algorithm beats for the other cases. Specifically, for $m = 3$, *Plackett* method has the best accuracy and speed followed by *FORMVN* with $N = 12$ for $\rho \leq 0.5$ and *Miwa* with $G = 4096$ for $\rho > 0.5$. For $m = 4$ and $m = 5$, *FORMVN* with $N = 12$ has the best accuracy and speed for $\rho \leq 0.5$ followed by *Miwa* with $G = 4096$ for $\rho > 0.5$. For $m = 6$ and $m = 7$, *Miwa* with $G = 4096$ has the best accuracy and speed followed by *FORMVN* with $N = 12$ for $\rho \leq 0.5$ for the same accuracy level. Note that, for $m \geq 4$, *TETMVN* series fails to converge when $\rho > 0.5$ while *FORMVN* series still converges but with a slower rate as ρ increases. Moreover, higher accuracy can be achieved using *FORMVN* by increasing N , but it will be at the expense of higher computational time.

TABLE 2. Average time over 50 runs for the trivariate case. Second row for each value of ρ represents the standard deviation.

	<i>TETMVN</i>		<i>FORMVN</i>		<i>SADMVN</i>	<i>KROMVN</i>	<i>Miwa</i>		<i>Plackett</i>
	$N = 8$	$N = 12$	$N = 8$	$N = 12$			$G = 128$	$G = 4096$	
0.1	0.00	0.00	0.00	0.00	0.01	2.14	0.06	0.07	0.00
	0.00	0.00	0.01	0.01	0.01	0.92	0.01	0.02	0.00
0.2	0.00	0.00	0.00	0.00	0.01	2.42	0.05	0.08	0.00
	0.00	0.00	0.00	0.01	0.01	1.25	0.01	0.02	0.00
0.3	0.00	0.00	0.00	0.00	0.01	1.71	0.05	0.07	0.00
	0.00	0.00	0.01	0.01	0.01	0.85	0.01	0.02	0.00
0.4	0.00	0.00	0.00	0.00	0.01	0.98	0.05	0.07	0.00
	0.00	0.00	0.01	0.01	0.01	0.51	0.01	0.01	0.00
0.5	0.00	0.00	0.00	0.00	0.01	0.49	0.05	0.07	0.00
	0.00	0.00	0.01	0.01	0.01	0.41	0.01	0.02	0.00
0.6	0.00	0.00	0.00	0.00	0.01	0.38	0.06	0.08	0.00
	0.00	0.00	0.01	0.01	0.01	0.30	0.01	0.02	0.00
0.7	0.00	0.00	0.00	0.00	0.01	0.32	0.06	0.07	0.00
	0.00	0.00	0.01	0.01	0.01	0.26	0.01	0.01	0.00
0.8	0.00	0.00	0.00	0.00	0.01	0.27	0.06	0.08	0.00
	0.00	0.00	0.01	0.01	0.01	0.21	0.01	0.02	0.00
0.9	0.00	0.00	0.00	0.00	0.01	0.20	0.06	0.08	0.00
	0.00	0.00	0.00	0.01	0.01	0.20	0.01	0.02	0.00

TABLE 3. Average absolute error over 50 runs for the trivariate case. Second row for each value of ρ represents the standard deviation. ‘x’ values in *TETMVN* mean that the series does not converge.

	<i>TETMVN</i>		<i>FORMVN</i>		<i>SADMVN</i>	<i>KROMVN</i>	<i>Miwa</i>		<i>Plackett</i>
	<i>N</i> = 8	<i>N</i> = 12	<i>N</i> = 8	<i>N</i> = 12			<i>G</i> = 128	<i>G</i> = 4096	
0.1	2.10E-11	1.20E-15	2.20E-10	6.30E-15	2.10E-09	6.10E-09	2.40E-10	8.30E-14	1.30E-16
0.2	1.90E-11	9.60E-16	1.80E-10	6.50E-15	2.00E-10	4.00E-09	1.60E-10	1.50E-13	1.00E-16
	6.60E-09	5.10E-12	1.90E-10	1.30E-14	1.60E-09	4.80E-09	2.00E-10	1.40E-13	1.50E-16
	7.30E-09	3.50E-12	1.40E-10	1.20E-14	5.90E-10	3.60E-09	1.20E-10	1.50E-13	1.20E-16
0.3	2.20E-07	1.40E-09	1.50E-10	1.00E-14	1.20E-09	3.20E-09	1.10E-10	2.00E-13	1.00E-15
0.4	2.70E-07	1.20E-09	1.00E-10	9.00E-15	9.10E-10	3.30E-09	7.10E-11	2.20E-13	5.50E-16
	2.50E-06	1.30E-07	1.00E-10	4.90E-14	1.10E-09	3.50E-09	7.10E-11	1.80E-13	5.50E-15
	2.70E-06	7.00E-08	9.80E-11	3.00E-14	6.70E-10	3.30E-09	6.90E-11	1.70E-13	4.10E-15
0.5	4.10E-05	1.50E-05	6.50E-11	7.20E-13	1.40E-09	4.20E-09	7.20E-11	2.10E-13	2.00E-14
0.6	4.90E-05	1.30E-05	6.70E-11	5.00E-13	6.60E-10	2.60E-09	5.50E-11	1.50E-13	4.50E-14
	1.00E-03	2.30E-03	1.70E-10	2.00E-11	2.00E-09	4.00E-09	1.80E-10	3.70E-13	3.80E-14
	7.60E-04	1.70E-03	9.70E-11	1.40E-11	1.00E-09	4.10E-09	8.20E-11	2.30E-13	1.90E-13
0.7	1.10E-02	2.50E-01	4.20E-10	8.80E-10	2.30E-09	4.20E-09	3.80E-10	3.70E-13	2.50E-12
0.8	1.00E-02	1.60E-01	2.30E-10	9.20E-10	4.60E-09	3.90E-09	2.00E-10	2.70E-13	1.40E-11
	x	x	8.30E-10	4.40E-08	1.50E-09	3.30E-09	1.00E-09	3.70E-13	6.70E-15
	x	x	4.40E-10	4.50E-08	1.20E-09	3.30E-09	5.70E-10	2.80E-13	1.10E-14
0.9	x	x	2.60E-09	5.00E-06	5.00E-09	2.20E-09	3.20E-09	4.40E-13	1.80E-13
	x	x	1.90E-09	5.60E-06	3.30E-09	2.70E-09	2.00E-09	3.50E-13	3.70E-13

TABLE 4. Average time over 50 runs for the quadrivariate case. Second row for each value of ρ represents the standard deviation.

	<i>TETMVN</i>		<i>FORMVN</i>		<i>SADMVN</i>	<i>KROMVN</i>	<i>Miwa</i>	
	<i>N</i> = 8	<i>N</i> = 12	<i>N</i> = 8	<i>N</i> = 12			<i>G</i> = 128	<i>G</i> = 4096
0.1	0.01	0.05	0.01	0.03	0.12	4.69	0.01	0.10
0.2	0.01	0.01	0.01	0.01	0.02	1.91	0.01	0.01
	0.01	0.05	0.01	0.03	0.13	6.39	0.00	0.10
	0.01	0.01	0.01	0.01	0.03	3.26	0.01	0.01
0.3	0.01	0.05	0.01	0.03	0.10	4.64	0.01	0.10
0.4	0.01	0.01	0.01	0.01	0.03	1.75	0.01	0.01
	0.01	0.05	0.01	0.03	0.07	3.70	0.01	0.10
	0.01	0.01	0.01	0.01	0.02	1.34	0.01	0.01
0.5	0.01	0.05	0.01	0.03	0.04	3.12	0.01	0.10
0.6	0.01	0.01	0.01	0.01	0.01	1.89	0.01	0.01
	0.01	0.05	0.01	0.03	0.03	2.29	0.00	0.11
	0.01	0.01	0.01	0.01	0.02	1.33	0.01	0.01
0.7	0.01	0.05	0.02	0.03	0.02	1.78	0.01	0.11
0.8	0.01	0.01	0.02	0.01	0.01	1.05	0.01	0.01
	0.01	0.05	0.02	0.03	0.02	1.82	0.01	0.11
	0.01	0.01	0.02	0.01	0.01	1.70	0.01	0.01
0.9	0.01	0.06	0.02	0.03	0.02	1.54	0.01	0.11
	0.01	0.01	0.02	0.01	0.01	1.39	0.01	0.01

TABLE 5. Average absolute error over 50 runs for the quadrivariate case. Second row for each value of ρ represents the standard deviation. ‘x’ values in *TETMVN* mean that the series does not converge.

	<i>TETMVN</i>		<i>FORMVN</i>		<i>SADMVN</i>	<i>KROMVN</i>	<i>Miwa</i>	
	<i>N</i> = 8	<i>N</i> = 12	<i>N</i> = 8	<i>N</i> = 12			<i>G</i> = 128	<i>G</i> = 4096
0.1	2.80E-11	3.60E-15	5.30E-10	5.50E-14	1.40E-08	3.60E-09	1.60E-09	4.10E-13
	2.90E-11	2.90E-15	4.30E-10	9.10E-14	5.40E-09	2.90E-09	6.20E-10	4.50E-13
0.2	6.10E-09	4.10E-12	1.00E-09	4.50E-14	1.40E-08	3.50E-09	1.30E-09	5.30E-13
	4.30E-09	3.70E-12	7.30E-10	4.60E-14	5.00E-09	2.50E-09	5.10E-10	4.30E-13
0.3	2.00E-07	2.10E-09	2.20E-09	5.90E-14	6.10E-09	1.80E-09	1.10E-09	7.50E-13
	1.70E-07	2.30E-09	1.70E-09	5.80E-14	2.10E-09	2.00E-09	4.20E-10	6.10E-13
0.4	1.20E-05	1.40E-04	4.50E-09	6.60E-14	2.80E-09	2.40E-09	6.80E-10	8.20E-13
	9.50E-06	1.10E-04	3.40E-09	6.20E-14	1.40E-09	2.10E-09	3.00E-10	5.70E-13
0.5	4.80E-02	x	3.00E-08	1.40E-12	1.80E-09	2.80E-09	6.10E-10	8.80E-13
	3.40E-02	x	2.20E-08	1.20E-12	1.30E-09	2.70E-09	3.60E-10	5.90E-13
0.6	x	x	1.60E-07	3.50E-11	4.50E-09	2.60E-09	4.50E-10	1.40E-12
	x	x	1.10E-07	2.20E-11	3.20E-09	2.00E-09	2.80E-10	6.70E-13
0.7	x	x	1.20E-06	1.40E-09	2.80E-09	2.00E-09	4.00E-10	1.10E-12
	x	x	9.00E-07	1.10E-09	1.80E-09	1.70E-09	2.10E-10	5.30E-13
0.8	x	x	1.20E-05	9.70E-08	5.60E-09	2.80E-09	8.60E-10	1.80E-12
	x	x	8.90E-06	8.40E-08	4.90E-09	2.60E-09	5.20E-10	2.00E-12
0.9	x	x	1.50E-04	9.40E-06	5.90E-09	2.30E-09	1.90E-09	2.10E-12
	x	x	1.20E-04	6.70E-06	4.00E-09	2.30E-09	1.50E-09	4.00E-12

TABLE 6. Average time over 50 runs for the 5-variate case. Second row for each value of ρ represents the standard deviation.

	<i>TETMVN</i>		<i>FORMVN</i>		<i>SADMVN</i>	<i>KROMVN</i>	<i>Miwa</i>	
	<i>N</i> = 8	<i>N</i> = 12	<i>N</i> = 8	<i>N</i> = 12			<i>G</i> = 128	<i>G</i> = 4096
0.1	23.46	-	0.08	0.26	55.49	7.55	0.02	0.52
	0.75	-	0.01	0.01	48.27	2.90	0.00	0.05
0.2	23.99	-	0.08	0.27	35.76	12.32	0.02	0.52
	0.94	-	0.01	0.01	37.74	9.21	0.00	0.04
0.3	24.29	-	0.08	0.28	32.33	16.72	0.02	0.51
	0.95	-	0.01	0.01	38.36	15.26	0.00	0.03
0.4	25.12	-	0.08	0.29	11.01	15.86	0.02	0.52
	0.54	-	0.01	0.01	21.59	21.93	0.01	0.02
0.5	25.00	-	0.08	0.32	1.44	10.35	0.02	0.52
	0.32	-	0.01	0.01	1.38	13.46	0.00	0.02
0.6	25.05	-	0.09	0.37	1.90	8.20	0.02	0.53
	0.45	-	0.01	0.01	6.05	10.11	0.01	0.03
0.7	25.02	-	0.09	0.41	0.35	8.65	0.02	0.53
	0.18	-	0.01	0.01	0.44	12.67	0.01	0.02
0.8	25.12	-	0.09	0.46	0.16	6.84	0.02	0.55
	0.48	-	0.01	0.01	0.22	10.10	0.00	0.02
0.9	24.96	-	0.09	0.50	0.08	3.23	0.02	0.54
	0.37	-	0.01	0.02	0.16	3.31	0.00	0.02

TABLE 7. Average absolute error over 50 runs for the 5-variate case. Second row for each value of ρ represents the standard deviation. 'x' values in *TETMVN* mean that the series does not converge. '-' values mean that the algorithm takes more than 200 sec.

	<i>TETMVN</i>		<i>FORMVN</i>		<i>SADMVN</i>	<i>KROMVN</i>	<i>Miwa</i>	
	<i>N</i> = 8	<i>N</i> = 12	<i>N</i> = 8	<i>N</i> = 12			<i>G</i> = 128	<i>G</i> = 4096
0.1	2.60E-11	-	5.90E-10	4.80E-13	1.00E-07	5.20E-09	3.60E-09	7.90E-13
	1.90E-11	-	5.60E-10	8.00E-13	3.50E-08	6.10E-09	1.00E-09	7.80E-13
0.2	7.10E-09	-	1.10E-09	2.30E-13	9.10E-08	3.40E-09	3.00E-09	1.30E-12
	5.30E-09	-	8.60E-10	2.10E-13	3.90E-08	2.90E-09	1.10E-09	1.30E-12
0.3	2.80E-07	-	3.00E-09	2.40E-13	7.50E-08	4.40E-09	2.60E-09	1.20E-12
	2.30E-07	-	1.90E-09	2.60E-13	2.90E-08	3.90E-09	1.00E-09	9.40E-13
0.4	x	-	1.10E-08	2.80E-13	6.20E-08	2.90E-09	2.10E-09	2.20E-12
	x	-	6.50E-09	2.10E-13	3.30E-08	2.10E-09	9.30E-10	1.50E-12
0.5	x	-	3.70E-08	2.30E-12	3.50E-08	2.40E-09	1.70E-09	2.20E-12
	x	-	3.40E-08	2.00E-12	1.80E-08	2.10E-09	8.60E-10	1.40E-12
0.6	x	-	2.00E-07	5.50E-11	2.10E-08	3.20E-09	1.00E-09	2.40E-12
	x	-	1.40E-07	3.50E-11	2.10E-08	3.00E-09	8.40E-10	1.30E-12
0.7	x	-	1.70E-06	2.00E-09	1.40E-08	3.20E-09	9.80E-10	3.00E-12
	x	-	1.20E-06	1.80E-09	1.30E-08	3.10E-09	8.90E-10	1.50E-12
0.8	x	-	1.50E-05	1.60E-07	6.60E-09	2.80E-09	1.10E-09	2.80E-12
	x	-	1.30E-05	1.20E-07	5.70E-09	2.90E-09	6.90E-10	1.90E-12
0.9	x	-	1.80E-04	9.10E-06	8.70E-09	3.30E-09	2.40E-09	2.80E-12
	x	-	1.40E-04	7.40E-06	6.50E-09	3.50E-09	2.10E-09	1.90E-12

TABLE 8. Average time over 50 runs for the 6-variate case. Second row for each value of ρ represents the standard deviation.

	<i>TETMVN</i>		<i>FORMVN</i>		<i>SADMVN</i>	<i>KROMVN</i>	<i>Miwa</i>	
	<i>N</i> = 8	<i>N</i> = 12	<i>N</i> = 8	<i>N</i> = 12			<i>G</i> = 128	<i>G</i> = 4096
0.1	-	-	0.88	4.93	130.15	21.40	0.10	3.38
	-	-	0.01	0.03	23.06	12.04	0.01	0.21
0.2	-	-	0.91	5.09	122.98	51.49	0.10	3.32
	-	-	0.01	0.06	31.09	38.78	0.01	0.17
0.3	-	-	0.96	5.23	119.60	45.97	0.11	3.24
	-	-	0.02	0.04	35.34	42.72	0.01	0.17
0.4	-	-	1.04	5.42	108.55	49.44	0.10	3.30
	-	-	0.01	0.04	49.35	46.49	0.01	0.13
0.5	-	-	1.14	5.69	58.59	50.97	0.10	3.34
	-	-	0.02	0.03	58.30	51.55	0.01	0.11
0.6	-	-	1.29	6.28	38.87	54.76	0.10	3.40
	-	-	0.10	0.15	51.63	51.53	0.01	0.13
0.7	-	-	1.41	7.04	14.59	34.60	0.11	3.43
	-	-	0.02	0.05	33.58	44.23	0.01	0.10
0.8	-	-	1.54	8.35	3.27	16.73	0.11	3.43
	-	-	0.08	0.07	9.84	30.09	0.01	0.10
0.9	-	-	1.58	9.83	0.45	10.21	0.11	3.38
	-	-	0.02	0.11	1.17	19.11	0.01	0.14

TABLE 9. Average absolute error over 50 runs for the 6-variate case. Second row for each value of ρ represents the standard deviation. ‘-’ values mean that the algorithm takes more than 200 sec.

	<i>TETMVN</i>		<i>FORMVN</i>		<i>SADMVN</i>	<i>KROMVN</i>	<i>Miwa</i>	
	<i>N</i> = 8	<i>N</i> = 12	<i>N</i> = 8	<i>N</i> = 12			<i>G</i> = 128	<i>G</i> = 4096
0.1	-	-	6.60E-10	1.10E-12	7.20E-07	3.70E-09	3.30E-09	1.50E-12
	-	-	5.10E-10	1.20E-12	7.60E-07	3.10E-09	1.40E-09	1.20E-12
0.2	-	-	1.30E-09	1.10E-12	8.90E-07	3.40E-09	2.50E-09	1.80E-12
	-	-	8.40E-10	1.70E-12	6.50E-07	2.90E-09	1.20E-09	1.30E-12
0.3	-	-	2.40E-09	2.50E-12	4.60E-07	2.80E-09	1.90E-09	2.50E-12
	-	-	1.80E-09	2.70E-12	3.80E-07	2.60E-09	1.30E-09	1.90E-12
0.4	-	-	7.90E-09	1.40E-12	2.90E-07	2.70E-09	2.10E-09	3.40E-12
	-	-	6.20E-09	1.90E-12	2.20E-07	2.70E-09	1.20E-09	2.30E-12
0.5	-	-	5.80E-08	3.00E-12	1.70E-07	3.20E-09	1.70E-09	5.00E-12
	-	-	3.30E-08	2.10E-12	2.00E-07	4.30E-09	1.60E-09	3.60E-12
0.6	-	-	2.50E-07	6.60E-11	9.30E-08	3.40E-09	1.60E-09	4.80E-12
	-	-	2.00E-07	5.90E-11	7.60E-08	2.70E-09	1.50E-09	2.20E-12
0.7	-	-	1.80E-06	3.10E-09	5.00E-08	3.90E-09	1.50E-09	6.40E-12
	-	-	1.30E-06	3.80E-09	5.70E-08	3.60E-09	1.50E-09	4.30E-12
0.8	-	-	1.50E-05	2.10E-07	2.70E-08	3.90E-09	1.50E-09	7.10E-12
	-	-	1.20E-05	2.00E-07	2.00E-08	3.10E-09	1.50E-09	7.40E-12
0.9	-	-	2.20E-04	1.50E-05	1.50E-08	3.90E-09	1.90E-09	6.90E-12
	-	-	1.80E-04	1.30E-05	1.00E-08	3.50E-09	1.90E-09	6.00E-12

TABLE 10. Average time over 50 runs for the 7-variate case. Second row for each value of ρ represents the standard deviation.

	<i>TETMVN</i>		<i>FORMVN</i>		<i>SADMVN</i>	<i>KROMVN</i>	<i>Miwa</i>	
	<i>N</i> = 8	<i>N</i> = 12	<i>N</i> = 8	<i>N</i> = 12			<i>G</i> = 128	<i>G</i> = 4096
0.1	-	-	9.49	107.49	153.55	136.06	0.94	31.33
	-	-	0.05	0.63	20.64	51.14	0.04	1.27
0.2	-	-	9.89	108.32	149.16	168.21	0.94	31.09
	-	-	0.11	0.31	19.24	0.93	0.04	1.47
0.3	-	-	10.41	109.74	149.12	168.25	0.94	30.15
	-	-	0.08	0.69	19.36	10.53	0.03	1.11
0.4	-	-	11.19	111.81	144.85	167.63	0.95	30.51
	-	-	0.17	0.72	24.58	20.43	0.03	0.86
0.5	-	-	12.31	116.41	116.68	169.81	0.96	30.52
	-	-	0.28	2.71	59.34	41.80	0.03	0.86
0.6	-	-	13.88	120.67	84.07	163.86	1.00	32.94
	-	-	0.17	1.54	63.04	34.83	0.07	2.16
0.7	-	-	16.48	131.77	35.82	124.93	1.03	32.57
	-	-	2.06	0.42	52.86	66.73	0.07	2.33
0.8	-	-	18.18	154.93	15.00	91.25	1.01	31.25
	-	-	0.59	0.85	35.28	75.91	0.05	0.95
0.9	-	-	18.85	198.17	3.89	44.31	1.00	30.73
	-	-	0.14	1.84	23.10	71.90	0.04	1.05

TABLE 11. Average absolute error over 50 runs for the 7-variate case. Second row for each value of ρ represents the standard deviation. ‘-’ values mean that the algorithm takes more than 200 sec.

	<i>TETMVN</i>		<i>FORMVN</i>		<i>SADMVN</i>	<i>KROMVN</i>	<i>Miwa</i>	
	$N = 8$	$N = 12$	$N = 8$	$N = 12$			$G = 128$	$G = 4096$
0.1	-	-	8.70E-10	2.60E-12	1.40E-06	6.20E-09	2.60E-09	2.90E-12
	-	-	5.30E-10	1.90E-12	1.10E-06	5.90E-09	2.40E-09	2.20E-12
0.2	-	-	1.10E-09	6.80E-12	2.40E-06	1.20E-08	3.20E-09	3.90E-12
	-	-	7.70E-10	1.10E-11	2.30E-06	1.00E-08	2.30E-09	2.00E-12
0.3	-	-	3.10E-09	5.30E-12	1.60E-06	1.20E-08	3.40E-09	5.00E-12
	-	-	2.70E-09	4.70E-12	1.20E-06	1.40E-08	1.90E-09	6.30E-12
0.4	-	-	1.10E-08	1.30E-11	1.40E-06	1.60E-08	3.00E-09	6.30E-12
	-	-	1.00E-08	1.50E-11	1.10E-06	1.50E-08	1.60E-09	5.00E-12
0.5	-	-	4.50E-08	1.60E-11	5.50E-07	1.40E-08	2.40E-09	9.00E-12
	-	-	3.30E-08	1.90E-11	5.00E-07	1.90E-08	1.70E-09	9.40E-12
0.6	-	-	3.80E-07	1.10E-10	2.90E-07	1.30E-08	2.40E-09	9.80E-12
	-	-	3.70E-07	1.10E-10	4.50E-07	1.50E-08	2.60E-09	5.90E-12
0.7	-	-	2.20E-06	4.10E-09	1.20E-07	8.70E-09	2.20E-09	1.00E-11
	-	-	1.60E-06	3.10E-09	1.40E-07	9.10E-09	1.80E-09	7.80E-12
0.8	-	-	2.20E-05	2.50E-07	5.00E-08	5.40E-09	2.10E-09	1.30E-11
	-	-	1.90E-05	2.30E-07	4.70E-08	6.70E-09	2.00E-09	8.70E-12
0.9	-	-	2.30E-04	2.00E-05	2.30E-08	5.90E-09	1.50E-09	1.30E-11
	-	-	2.10E-04	1.80E-05	2.30E-08	1.50E-08	1.30E-09	1.60E-11

6. CONCLUSIONS

In this article, we expressed the multivariate normal probability integral using Fourier series expansion. This expansion is found to converge rapidly for small values of the correlation coefficients compared to the well-known tetrachoric series and it is better than *Miwa*’s method for $m = 3$ to $m = 5$ when $\rho \leq 0.5$ and outperforms *SADMVN* and *KROMVN* especially for $\rho \leq 0.5$. Moreover, another series expansion is derived for the bivariate case that converges fast for all values of ρ .

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