A DWINDLING FILTER LINE SEARCH METHOD FOR UNCONSTRAINED OPTIMIZATION

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ABSTRACT. A dwindling multidimensional filter is proposed and applied to a second-order line search framework for unconstrained optimization. Usually, the multidimensional filter is built up with a fixed envelope, which is not well-suited to line search frameworks. In this paper, we propose the dwindling multidimensional filter, whose envelope is dwindling as the step-length of line search decreasing. Combining the dwindling multidimensional filter and a second-order line search, the new algorithm globally converges to a second-order critical point, when the negative curvature direction is exploited. Detailed numerical results on small and large CUTE test problems indicate that the new algorithm is more competitive than some classical line search methods.

1. INTRODUCTION

We consider the unconstrained optimization problem

$$(1.1) \quad \min_{x \in \mathbb{R}^n} f(x),$$

where f is a real-valued function on $\mathbb{R}^n$. We assume that both the gradient $g(x) := \nabla_x f(x)$ and the Hessian $H(x) := \nabla_{xx}^2 f(x)$ of f exist and are continuous. In this paper, we use the following notations: $\|\cdot\|$ is the Euclidean norm, $|\cdot|$ is the absolute value, $\{x_k\}$ a sequence of points generated by an algorithm, $f_k := f(x_k)$, $g_k := g(x_k)$, $H_k := H(x_k)$, and so on.

The filter method was first introduced for constrained nonlinear programming by Fletcher and Leyffer [9], which is a method without any penalty function and guarantees global convergence [8, 10, 12, 18, 29, 30, 34, 35]. For unconstrained optimization, there are mainly two types of filter methods. The first approach is a straightforward application of the filter technique; see [21, 26, 36]. In particular, Wang and Zhu [36] transform the simple unconstrained optimization problem (1.1) to a nonlinear programming problem with an equality constraint

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad g(x) = 0,$$

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and then the filter technique for general nonlinear programming is applied. The second approach is the multidimensional filter technique, which is proposed for nonlinear equations and nonlinear least-squares in [13] and for unconstrained optimization in [16]. The multidimensional filter has more freedom in accepting trial points, and has been extensively studied in [17, 19, 24, 37, 38]. A brief history and review of filter methods are given in [11, 31].

A multidimensional filter $\mathcal{F}$ is defined as a list of n -tuples $(g_{k,1}, \dots, g_{k,n})$, where $g_{k,i} := g_i(x_k)$, such that if g_k and g_l belong to $\mathcal{F}$, then $|g_{k,j}| < |g_{l,j}|$ holds for at least one $j \in \{1, \dots, n\}$. A trial point x_k^+ , whose gradient is denoted as $g_k^+ := (g_{k,1}^+, \dots, g_{k,n}^+)$, is acceptable for the multidimensional filter $\mathcal{F}$ if and only if

$$(1.2) \quad \begin{aligned} &\text{for all } g_l \in \mathcal{F}, \text{ there exists a } j \in \{1, \dots, n\} \\ &\text{such that } |g_{k,j}^+| - |g_{l,j}| \leq -\gamma \|g_l\|, \end{aligned}$$

where $\gamma \in (0, 1/\sqrt{n})$ is a small positive constant. We say that $\gamma \|g_l\|$ is a fixed measurement of the envelope of the multidimensional filter $\mathcal{F}$.

Next, many line search algorithms [32] aim to attain a second-order critical point x^* satisfying $g(x^*) = 0$, and $H(x^*)$ is positive semidefinite. The key idea of these second-order line search algorithms is to determine, at each iterate x_k , a pair of descent directions (s_k, d_k) . Here, s_k is a gradient-related direction satisfying

$$(1.3) \quad s_k^\top g_k \leq -c_1 \|g_k\|^2 \quad \text{and} \quad \|s_k\| \leq c_2 \|g_k\|,$$

where c_1 and c_2 are two positive constants. The negative curvature direction d_k satisfies

$$(1.4) \quad \begin{aligned} &d_k^\top g_k \leq 0, \quad d_k^\top H_k d_k \leq 0, \\ &d_k^\top H_k d_k \rightarrow 0 \quad \text{implies} \quad \min(0, \lambda_{\min}(H_k)) \rightarrow 0, \end{aligned}$$

where $\lambda_{\min}(H)$ is the smallest eigenvalue of the matrix H . Traditionally, second-order line search algorithms use these two directions simultaneously, and process a curvilinear line search [23, 25],

$$x_{k+1} = x_k + \alpha_k^2 s_k + \alpha_k d_k,$$

where $\alpha_k > 0$ is a step-length. Combining the second-order line search method and a nonmonotone technique, several efficient algorithms have been proposed for small and large scale optimization problems [2, 7, 22, 33, 39, 40]. The scaling of two directions s_k and d_k is taken into account in [14], which selects the more promising one of these two directions instead of using two directions synchronously.

In our point of view, the fixed envelope of a multidimensional filter keeps an acceptable trial point far from iterates in the filter, which works well in a trust region framework. However, a tiny step-length may take place in a line search iteration. Then, suppose the current iterate has been added into the multidimensional filter, the trial point with a tiny step-length must get close to the current iterate and be rejected by the multidimensional filter due to its fixed envelope. So we think that the fixed envelope of a multidimensional filter is not suitable well to the line search process.

We propose a dwindling multidimensional filter which does not require the envelope involved being fixed. Under the dwindling filter's rule and some mild conditions, the trial point with a sufficiently small step-length cannot be rejected by the current iterate if it was added to the filter. So the dwindling filter proposed here is meaningful and important. Therefore, we apply the dwindling filter to a

second-order line search algorithm, and give a new algorithm to solve unconstrained optimization.

This paper is organized as follows. We present the dwindling multidimensional filter and its motivation in Section 2. The dwindling filter line search algorithm is proposed in Section 3. The corresponding convergence analysis to a second-order critical point is proved in Section 4. In Section 5, we report some preliminary numerical experiments on a collection of CUTE test problems. Some conclusions are finally described in Section 6.

2. THE DWINDLING FILTER

Before giving the definition of the dwindling multidimensional filter, we compare the rule of the original multidimensional filter with a descent condition for the inexact line search, and introduce our motivation.

2.1. Motivation. First, we examine the inexact line search method. Suppose that the current iterate is x_k , the line search direction is a gradient-related direction s_k that satisfies (1.3), and the trial point is

$$x_k^+(\alpha) := x_k + \alpha s_k.$$

We denote its objective function as $f_k^+(\alpha) := f(x_k^+(\alpha))$ and its gradient as $g_k^+(\alpha) := g(x_k^+(\alpha))$ with components $g_{k,j}^+(\alpha)$ for $j = 1, \dots, n$. The trial point $x_k^+(\alpha)$ is acceptable for the inexact line search only if

$$(2.1) \quad f_k^+(\alpha) - f_k \leq \mu \cdot \alpha g_k^\top s_k, \quad \mu \in \left(0, \frac{1}{2}\right).$$

Note that, from the condition (1.3) and the Cauchy-Schwarz inequality, we have

$$-c_2 \|g_k\|^2 \leq g_k^\top s_k \leq -c_1 \|g_k\|^2.$$

Hence, the line search rule (2.1) has two properties.

- (i) The least decrease of the objective function $\mu \alpha g_k^\top s_k$ is proportional to the step-length α and $\mathcal{O}(\|g_k\|^2)$.
- (ii) Inequality (2.1) holds when α is sufficiently small; see [28, 32].

Second, we consider a simple example which shows that the multidimensional filter (1.2) is not well-suited in a line search framework.

Example 2.1. Suppose that $f(x)$ is a one-variable function, i.e., $n = 1$, and the current iterate x_k , whose gradient $g_k \neq 0$, belongs to the original multidimensional filter. By the rule (1.2), the trial point $x_k^+(\alpha)$ is acceptable if and only if

$$(2.2) \quad |g_k^+(\alpha)| - |g_k| \leq -\gamma |g_k|.$$

Then, we find that

- (i) The least decrease $\gamma |g_k|$ in (2.2) is irrelevant to the line search step-length α .
- (ii) The inequality (2.2) does not hold if the step-length α is tiny.

This fact means that according to the rule of multidimensional filter (1.2), the trial point $x_k^+(\alpha)$ must be rejected if the current iterate x_k is included in the filter $\mathcal{F}$ and the step-length α is sufficiently small.

In our viewpoint, compared with the rule (2.1) of the inexact line search, the multidimensional filter's rule (1.2) is crude for line search methods. Then, we give a modification and improvement, which is the dwindling multidimensional filter.

2.2. The dwindling multidimensional filter. Our idea is to introduce the step-length factor into the rule of multidimensional filter and make the envelope of the filter become thinner and thinner as the step-length approaches zero. For a two-variable problem, we illustrate the idea in Figure 1.

Now, we formally present the new rule for the dwindling multidimensional filter. A trial point $x_k^+(\alpha)$ is acceptable to the dwindling filter $\mathcal{F}$ if and only if it satisfies

$$(2.3) \quad \begin{aligned} &\text{for all } g_l \in \mathcal{F}, \text{ there exists a } j \in \{1, \dots, n\} \\ &\text{such that } |g_{k,j}^+(\alpha)| - |g_{l,j}| \leq -\gamma\phi(\alpha)\|g_l\|, \end{aligned}$$

where $\gamma \in (0, \frac{1}{\sqrt{n}}]$ is a constant and $\phi(\alpha)$ is a dwindling function defined as follows.

FIGURE 1. The envelopes of the dwindling multidimensional filter vary as a line search step-length α decreases. (a) When α is the unit step-length, the envelope is substantial and is equal to the original multidimensional filter's envelope. The envelopes become thinner as $\alpha = 0.5$ in (b) and $\alpha = 0.25$ in (c). (d) If α vanishes, the envelope vanishes, too.

Definition 2.2. A continuous function $\phi(\alpha) : [0, 1] \mapsto \mathbb{R}^+$ is a dwindling function if it is monotone increasing and satisfies $\phi(\alpha) = 0$ if and only if $\alpha = 0$. Moreover, we suppose that

$$(2.4) \quad \lim_{\alpha \rightarrow 0^+} \frac{\phi(\alpha)}{\alpha} = 0.$$

Obviously, if $\phi(\alpha) \equiv 1$, the dwindling filter reduces to the original multidimensional filter.

For large-scale problems, checking the dwindling filter's rule (2.3) in each component is a time-consuming work. Next, we give a sufficient condition for (2.3). This lemma also explains why we set $\gamma \in (0, \frac{1}{\sqrt{n}}]$ in a multidimensional filter.

Lemma 2.3. *Suppose that the trial point $x_k^+(\alpha)$ satisfies*

$$(2.5) \quad \frac{\|g_k^+(\alpha)\|}{\|g_l\|} \leq 1 - \sqrt{n}\gamma\phi(\alpha),$$

where x_l is a point included in the dwindling filter. Then, by the dwindling filter's rule (2.3), $x_k^+(\alpha)$ could not be rejected by x_l .

Proof. For contradiction, we suppose that $x_k^+(\alpha)$ is rejected by x_l , then

$$|g_{k,j}^+(\alpha)| - |g_{l,j}| > -\gamma\phi(\alpha)\|g_l\|, \quad \text{for all } j = 1, \dots, n.$$

Hence, we have

$$\begin{aligned} \|g_k^+(\alpha)\| - \|g_l\| &= \frac{\|g_k^+(\alpha)\|^2 - \|g_l\|^2}{\|g_k^+(\alpha)\| + \|g_l\|} \\ &= \frac{\sum_{j=1}^n (|g_{k,j}^+(\alpha)| - |g_{l,j}|)(|g_{k,j}^+(\alpha)| + |g_{l,j}|)}{\|g_k^+(\alpha)\| + \|g_l\|} \\ &> \frac{-\gamma\phi(\alpha)\|g_l\| \sum_{j=1}^n (|g_{k,j}^+(\alpha)| + |g_{l,j}|)}{\|g_k^+(\alpha)\| + \|g_l\|} \\ &= \frac{-\gamma\phi(\alpha)\|g_l\|(\|g_k^+(\alpha)\|_1 + \|g_l\|_1)}{\|g_k^+(\alpha)\| + \|g_l\|} \\ &\geq -\sqrt{n}\gamma\phi(\alpha)\|g_l\|, \end{aligned}$$

where $\|g\|_1$ is the 1-norm satisfying $\|g\|_1 \leq \sqrt{n}\|g\|$. This inequality contradicts (2.5). Hence, this lemma is proved. $\square$

It is easy to see that the condition (2.5) in Lemma 2.3 is strong. Therefore, we present a mild condition under which a trial point $x_k^+(\alpha)$ cannot be rejected by the current iterate x_k when the dwindling filter is employed.

Lemma 2.4. *Suppose that there exist an index $j \in \{1, \dots, n\}$ and a small positive upper bound $\bar{\alpha}$ such that*

$$g_{k,j}^+(\alpha) \cdot g_{k,j} > 0, \quad |g_{k,j}^+(\alpha)| < |g_{k,j}| \quad \text{and} \quad |e_j^\top H_k s_k| \neq 0$$

hold for all $\alpha \in (0, \bar{\alpha}]$, where e_j is the j -th column of the identity matrix, and the dwindling function $\phi(\alpha)$ satisfies (2.4). Then, under the dwindling filter's rule (2.3), the trial point $x_k^+(\alpha)$ could not be rejected by the current iterate x_k if the step-length α is sufficiently small.

Proof. Since $|g_{k,j}| > 0$, we have $\|g_k\| \neq 0$. Then, from the assumptions and Taylor's theorem, we could obtain

$$\begin{aligned} |g_{k,j}^+(\alpha)| - |g_{k,j}| &= -|g_{k,j}^+(\alpha) - g_{k,j}| \\ &= -|e_j^\top (g_k^+(\alpha) - g_k)| \\ &= -|\alpha e_j^\top H_k s_k + o(\alpha)| \\ &= -\alpha |e_j^\top H_k s_k| + o(\alpha). \end{aligned}$$

Since $|e_j^\top H_k s_k| \neq 0$ and $\bar{\alpha}$ is small, without loss of generality, we suppose

$$|g_{k,j}^+(\alpha)| - |g_{k,j}| \leq -\frac{1}{2}\alpha |e_j^\top H_k s_k|$$

holds for all $\alpha \in (0, \bar{\alpha}]$. By the definition (2.4) of $\phi(\cdot)$, we have

$$\lim_{\alpha \rightarrow 0^+} \frac{-\gamma \phi(\alpha) \|g_k\|}{|g_{k,j}^+(\alpha)| - |g_{k,j}|} = \lim_{\alpha \rightarrow 0^+} \frac{\gamma \phi(\alpha) \|g_k\|}{|g_{k,j}| - |g_{k,j}^+(\alpha)|} \leq \lim_{\alpha \rightarrow 0^+} \frac{\phi(\alpha) \gamma \|g_k\|}{\alpha |e_j^\top H_k s_k|/2} = 0 < 1.$$

Therefore, once the step-length α is small enough, the dwindling filter's rule (2.3) holds and $x_k^+(\alpha)$ cannot be rejected by x_k . $\square$

3. THE DWINDLING FILTER LINE SEARCH ALGORITHM

Now, we combine the dwindling multidimensional filter with a second-order line search and give a new algorithm for unconstrained optimization.

The main improvement is on the switching condition, which selects one descent direction from a gradient-related direction and a negative curvature direction for the following line search process. We employ a common local merit function

$$(3.1) \quad m_k(p) := g_k^\top p + \frac{1}{2} \min(p^\top H_k p, 0)$$

and formulate the new switching condition as

$$(3.2) \quad m_k(s_k) \leq \tau m_k(d_k),$$

where τ is a positive constant. If the switching condition (3.2) holds, the local quadratic model is approximately convex. So we select the gradient-related direction. Otherwise, we perform a line search along a negative curvature direction.

It is worthwhile to note that these two directions used in the switching condition (3.2) are not normalized to the same unit norm. In this way, the decrease of the local merit function (3.1) in each selected direction has a negative upper bound if the current iterate is not critical. That is to say, whether the selected direction $p_k = s_k$ or $p_k = d_k$, we have

$$m_k(p_k) \leq -\min(\tau^{-1}, 1) c_1 \|g_k\|^2.$$

For the purpose of avoiding an abnormal norm of the negative curvature direction d_k , we scale d_k by its Rayleigh quotient

$$(3.3) \quad d_k \leftarrow \frac{|d_k^\top H_k d_k|}{\|d_k\|^3} d_k.$$

Obviously, the scaled d_k also satisfies (1.4).

The dwindling filter line search algorithm is described in Algorithm 1. A trial point is acceptable by the dwindling filter only if the local quadratic model is convex.

Algorithm 1 The dwindling filter line search algorithm. (DFLS)

Step 0. Initialization. Given a $x_0 \in \mathbb{R}^n$. Initialize the dwindling filter $\mathcal{F} \leftarrow \emptyset$. Set $\mu \in (0, \frac{1}{2})$, $\beta \in (0, 1)$, $\tau > 0$, $F_0 = f(x_0)$, and $k \leftarrow 0$.

Step 1. Test for convergence. If $\|g_k\|_\infty \leq \varepsilon$ (e.g., $\varepsilon = 10^{-6}$), stop.

Step 2. Compute search directions. Compute s_k and d_k that satisfy (1.3) and (1.4) respectively. If $d_k \neq 0$, set $d_k \leftarrow \frac{|d_k^T H_k d_k|}{\|d_k\|^3} d_k$ and CONVEX = False; Otherwise, set CONVEX = True.

Step 3. Choose the better search direction.

If $m_k(s_k) \leq \tau \cdot m_k(d_k)$, execute **Step 4**. Otherwise, execute **Step 5**.

Step 4. Line search along the direction s_k . Set $\alpha \leftarrow 1$ and $x_k^+(\alpha) \leftarrow x_k + \alpha s_k$.

Step 4.1. The truncated Newton step. If $f(x_k^+(\alpha)) - f(x_k) \leq \mu \cdot m_k(\alpha s_k)$, set $\alpha_k \leftarrow \alpha$, $x_{k+1} = x_k^+(\alpha_k)$, $F_{k+1} = F_k$, and $k \leftarrow k + 1$. Go to **Step 1**.

Step 4.2. The dwindling filter step. If three conditions, CONVEX = True, $f_k^+(\alpha) \leq F_k$ and $g_k^+(\alpha)$ satisfies (2.3), hold simultaneously, we add $g_k^+(\alpha)$ to $\mathcal{F}$ and remove all redundant points. Moreover, set $\alpha_k \leftarrow \alpha$, $x_{k+1} = x_k^+(\alpha_k)$, $F_{k+1} = F_k$, and $k \leftarrow k + 1$. Go to **Step 1**.

Step 4.3. Set $\alpha \leftarrow \alpha\beta$, $x_k^+(\alpha) \leftarrow x_k + \alpha s_k$, and go to **Step 4.1**.

Step 5. Line search along the direction d_k . If $f(x_k + d_k) - f(x_k) > \mu \cdot m_k(d_k)$, choose $\alpha_k = \beta^l$, where l is the smallest positive integer such that

$$(3.4) \quad f(x_k + \alpha_k d_k) - f(x_k) \leq \mu \cdot m_k(\alpha_k d_k).$$

Otherwise, choose $\alpha_k = \beta^{l'}$, where l' is the largest nonpositive integer satisfying (3.4). Furthermore, set $x_{k+1} = x_k + \alpha_k d_k$, $F_{k+1} = f(x_{k+1})$, $\mathcal{F} \leftarrow \emptyset$, and $k \leftarrow k + 1$. Go to **Step 1**.

If the trial point $x_k^+(\alpha_k)$ is accepted by the dwindling filter, we add $g_k^+(\alpha_k)$ to the filter and remove from the filter each point g_l satisfying

$$|g_{l,j}| > |g_{k,j}^+(\alpha_k)| \quad \text{for all } j \in \{1, \dots, n\}.$$

Additionally, when the current local quadratic model is seriously nonconvex, i.e., the switching condition (3.2) fails, we clear out the dwindling filter.

From Lemma 2.1 of reference [14], the line search process along either the gradient-related direction or the negative curvature direction is well defined. That is to say, there is no endless loop.

4. CONVERGENCE ANALYSIS

At the beginning of the analysis, we give a fundamental assumption.

Assumption 4.1. *The objective function $f : \mathbb{R}^n \mapsto \mathbb{R}$ is twice continuously differentiable, and the level set $\Omega := \{x \in \mathbb{R}^n \mid f(x) \leq f(x_0)\}$ is compact for the given initial point x_0 .*

Owing to Assumption 4.1, all the iterates $\{x_k\}_{k=0}^\infty$, objective functions $\{f_k\}_{k=0}^\infty$, gradients $\{g_k\}_{k=0}^\infty$, and Hessians $\{H_k\}_{k=0}^\infty$ generated by the DFLS algorithm are

bounded. Additionally, by (1.3) and (3.3), directions $\{s_k\}_{k=0}^\infty$ and $\{d_k\}_{k=0}^\infty$ are bounded, too.

For the convenience of the following analysis, we formally set a (sufficiently large) positive constant M such that

$$(4.1) \quad \|H_k\| \leq M \quad \text{for all } k.$$

We suppose that there are infinite iterations produced by the DFSL algorithm. According to the switching condition (3.2), either the gradient-related direction s_k or the negative curvature direction d_k is used in each iteration. First, we examine the case in which d_k is employed, infinitely many times.

Theorem 4.2. *Suppose Assumption 4.1 holds. Let*

$$(4.2) \quad \mathcal{K}_d := \{k \mid m_k(s_k) > \tau m_k(d_k)\}$$

be an infinite index set. Then, every limit point of $\{x_k\}_{k \in \mathcal{K}_d}$ is a critical point, i.e.,

$$(4.3) \quad \lim_{k \rightarrow \infty, k \in \mathcal{K}_d} \|g_k\| = 0.$$

Proof. Let $\mathcal{K}_d = \{k_i\}_{i=1}^\infty$. According to Step 5 of the DFSL algorithm, we have

$$f(x_{k_i}) > f(x_{k_i}) + \mu m_{k_i}(\alpha_{k_i} d_{k_i}) \geq f(x_{k_{i+1}}) = F_{k_{i+1}} = \cdots = F_{k_{i+1}} \geq f(x_{k_{i+1}}).$$

Hence,

$$(4.4) \quad f(x_{k_{i+1}}) - f(x_{k_i}) \leq \mu m_{k_i}(\alpha_{k_i} d_{k_i}) \leq \mu \cdot \min(\alpha_{k_i}, \alpha_{k_i}^2) m_{k_i}(d_{k_i}) < 0.$$

Since $f(x)$, $x \in \Omega$, is bounded below, the difference $f(x_{k_{i+1}}) - f(x_{k_i})$ must tend to zero as $i \rightarrow \infty$. So either $\alpha_{k_i} \rightarrow 0$ or $m_{k_i}(d_{k_i}) \rightarrow 0$ as $i \rightarrow \infty$ and $k_i \in \mathcal{K}_d$.

Suppose that there is an index subset $\mathcal{K}_d^0 \subseteq \mathcal{K}_d$ such that

$$\lim_{k \rightarrow \infty, k \in \mathcal{K}_d^0} \alpha_k = 0.$$

Without loss of generality, we assume $\alpha_k \leq \beta$ for all $k \in \mathcal{K}_d^0$. For convenience, we define $\hat{\alpha}_k := \frac{\alpha_k}{\beta}$. Owing to the line search's rule, when $k \in \mathcal{K}_d^0$, we have

$$f(x_k + \hat{\alpha}_k d_k) - f(x_k) > \mu m_k(\hat{\alpha}_k d_k).$$

By the mean-value theorem, we get

$$(4.5) \quad \hat{\alpha}_k g_k^\top d_k + \frac{1}{2} \hat{\alpha}_k^2 d_k^\top \tilde{H}_k d_k > \mu \hat{\alpha}_k g_k^\top d_k + \frac{\mu}{2} \hat{\alpha}_k^2 d_k^\top H_k d_k,$$

where $\tilde{H}_k = H(x_k + \theta \hat{\alpha}_k d_k)$ with $\theta \in (0, 1)$. On the one hand, inequality (4.5) could be rewritten as

$$g_k^\top d_k > \frac{\hat{\alpha}_k}{2(1-\mu)} (\mu d_k^\top H_k d_k - d_k^\top \tilde{H}_k d_k).$$

Since $g_k^\top d_k \leq 0$, and $\hat{\alpha}_k \rightarrow 0$ as $k \rightarrow \infty$, $k \in \mathcal{K}_d^0$, we obtain

$$(4.6) \quad \lim_{k \rightarrow \infty, k \in \mathcal{K}_d^0} g_k^\top d_k = 0.$$

On the other hand, inequality (4.5) could also be rewritten as

$$\frac{d_k^\top \tilde{H}_k d_k - \mu d_k^\top H_k d_k}{1-\mu} > -\frac{2}{\hat{\alpha}_k} g_k^\top d_k \geq 0.$$

When $k \rightarrow \infty$, $k \in \mathcal{K}_d^0$, we have $\hat{\alpha}_k \rightarrow 0$ and $\tilde{H}_k \rightarrow H_k$. So we have

$$\lim_{k \rightarrow \infty, k \in \mathcal{K}_d^0} \frac{d_k^\top \tilde{H}_k d_k - \mu d_k^\top H_k d_k}{1 - \mu} = d_k^\top H_k d_k \leq 0.$$

Hence,

$$(4.7) \quad \lim_{k \rightarrow \infty, k \in \mathcal{K}_d^0} d_k^\top H_k d_k = 0.$$

Combining (4.6) and (4.7), we have

$$\lim_{k \rightarrow \infty, k \in \mathcal{K}_d^0} m_k(d_k) = 0.$$

Further, recalling (4.4), we obtain

$$(4.8) \quad \lim_{k \rightarrow \infty, k \in \mathcal{K}_d} m_k(d_k) = 0.$$

When $k \in \mathcal{K}_d$, we have

$$m_k(d_k) < \tau^{-1} m_k(s_k) \leq \tau^{-1} g_k^\top s_k \leq -\tau^{-1} c_1 \|g_k\|^2 < 0.$$

Hence, $\|g_k\| \rightarrow 0$ as $k \rightarrow \infty$ and $k \in \mathcal{K}_d$. The proof is complete. $\square$

Second, if the gradient-related direction is employed, the corresponding step-length has a positive lower bound.

Lemma 4.3. *Suppose Assumption 4.1 holds. As the gradient-related direction s_k is employed, the corresponding step-length satisfies*

$$(4.9) \quad \alpha_k \geq \alpha_s^{\min} := \frac{(1 - \mu)c_1}{2Mc_2^2}, \quad \text{for all } k \notin \mathcal{K}_d.$$

Proof. By the mean-value theorem, we have

$$g(x + s) - g(x) = H(x + \theta x)s, \quad \theta \in (0, 1).$$

By (4.1), we obtain

$$(4.10) \quad \|g(x + s) - g(x)\| \leq \|H(x + \theta x)\| \|s\| \leq M \|s\|.$$

The inequality (4.10) means that the gradient is Lipschitz continuous.

When $k \notin \mathcal{K}_d$, by (4.10), (4.1), (1.3), and the mean-value theorem, we get

$$\begin{aligned} & |f(x_k + \alpha s_k) - f(x_k) - m_k(\alpha s_k)| \\ &= \left| \int_0^1 g(x_k + t\alpha s_k)^\top (\alpha s_k) dt - \alpha g_k^\top s_k - \frac{1}{2} \alpha^2 \min(s_k^\top H_k s_k, 0) \right| \\ &\leq \left| \alpha \int_0^1 (g(x_k + t\alpha s_k) - g_k)^\top s_k dt \right| + \frac{1}{2} \alpha^2 |s_k^\top H_k s_k| \\ &\leq \alpha \int_0^1 \|g(x_k + t\alpha s_k) - g_k\| \|s_k\| dt + \frac{1}{2} \alpha^2 \|H_k\| \|s_k\|^2 \\ &\leq \alpha \int_0^1 t M \alpha \|s_k\|^2 dt + \frac{1}{2} \alpha^2 M \|s_k\|^2 \\ &= \alpha^2 M \|s_k\|^2 \\ (4.11) \quad &\leq \alpha^2 M c_2^2 \|g_k\|^2. \end{aligned}$$

In addition, from (1.3), we have

$$(4.12) \quad |m_k(\alpha s_k)| = \left| \alpha g_k^\top s_k + \frac{1}{2} \alpha^2 \min(s_k^\top H_k s_k, 0) \right| \geq |\alpha g_k^\top s_k| \geq \alpha c_1 \|g_k\|^2.$$

Combining (4.11) and (4.12), we obtain

$$\frac{|f(x_k + \alpha s_k) - f(x_k) - m_k(\alpha s_k)|}{|m_k(\alpha s_k)|} \leq \alpha M c_1^{-1} c_2^2.$$

Hence, when $\alpha M c_1^{-1} c_2^2 \leq (1 - \mu)$, i.e., $\alpha \leq \frac{(1-\mu)c_1}{M c_2^2}$, the inequality $f(x_k^+(\alpha)) - f(x_k) \leq \mu m_k(\alpha s_k)$ is certainly valid. Therefore, this lemma holds. $\square$

Now, we consider the other case that $\mathcal{K}_d$ is a finite index set. Then, there exists an index

$$(4.13) \quad k_d^{\max} := \max\{k \mid k \in \mathcal{K}_d\}.$$

When $k > k_d^{\max}$, the gradient-related direction is definitely exploited. According to the DFLS algorithm, the dwindling filter will never be cleared out when $k > k_d^{\max}$. In the following theorem, we examine a case in which there are infinite dwindling filter steps.

Theorem 4.4. *Suppose Assumption 4.1 holds and*

$$\mathcal{K}_f := \{k \mid k > k_d^{\max}, g_k^+(\alpha_k) \text{ is added to } \mathcal{F}\}$$

is an infinite index set. Then, every limit point of the sequence $\{x_k\}_{k \in \mathcal{K}_f}$ is a critical point, i.e.,

$$(4.14) \quad \lim_{k \rightarrow \infty, k \in \mathcal{K}_f} \|g_k\| = 0.$$

Proof. Suppose that there is an infinite subset $\{k_i\}_{i=1}^\infty \subseteq \mathcal{K}_f$ such that $\{x_{k_i}\}_{i=1}^\infty$ converges. So gradients $\{g_{k_i}\}_{i=1}^\infty$ converge, too. Therefore, we have

$$(4.15) \quad \lim_{i \rightarrow \infty} g_{k_{i+1}, j} - g_{k_i, j} = 0, \quad \text{for all } j = 1, \dots, n.$$

According to the dwindling filter's rule, if

$$\text{for all } j \in \{1, \dots, n\}, \quad |g_{k_{i+1}, j}| - |g_{k_i, j}| > -\gamma \phi(\alpha_{k_{i+1}-1}) \|g_{k_i}\|,$$

then $x_{k_{i+1}}$ is not acceptable and cannot be added to the dwindling filter. So there must exist $j_i \in \{1, \dots, n\}$ such that

$$|g_{k_{i+1}, j_i}| - |g_{k_i, j_i}| \leq -\gamma \phi(\alpha_{k_{i+1}-1}) \|g_{k_i}\| \leq -\gamma \phi(\alpha_s^{\min}) \|g_{k_i}\| < 0.$$

By (4.15), we have $\|g_{k_i}\| \rightarrow 0$ as $i \rightarrow \infty$. We complete the proof. $\square$

Next, we consider the case that the index set $\mathcal{K}_f$ is also finite. Then, when k is large enough, all the iterations are truncated Newton steps, and the sequence of objective values decreases monotonously.

Theorem 4.5. *Suppose Assumption 4.1 holds and the index*

$$k_f^{\max} := \max\{k \mid k \in \mathcal{K}_f\}$$

is a finite integer. Then, every limit point of the sequence $\{x_k\}_{k > k_f^{\max}}$ is a critical point, i.e.,

$$(4.16) \quad \lim_{k \rightarrow \infty} \|g_k\| = 0.$$

Proof. Owing to the DFSL algorithm, when $k > k_f^{\max}$, we have

$$f_{k+1} - f_k \leq \mu m_k(\alpha_k s_k) \leq \mu \alpha_k g_k^\top s_k \leq -\mu \alpha_s^{\min} c_1 \|g_k\|^2 < 0.$$

Since $f(x)$, $x \in \Omega$ is bounded below, so the difference $f_{k+1} - f_k \rightarrow 0$ as $k \rightarrow \infty$. Therefore, $\|g_k\| \rightarrow 0$ as $k \rightarrow \infty$. $\square$

Obviously, taking Theorems 4.2, 4.4 and 4.5 into account, we claim that the DFSL algorithm could find at least one first-order critical point.

Theorem 4.6. *Suppose Assumption 4.1 holds. Then*

$$\liminf_{k \rightarrow \infty} \|g_k\| = 0.$$

Finally, we give the second-order convergence result for DFSL algorithm.

Theorem 4.7. *Suppose Assumption 4.1 holds and there is a unique limit point of the sequence of iterates $\{x_k\}_{k=0}^\infty$. Then, the unique limit point is a second-order critical point.*

Proof. From Theorem 4.6, we get

$$\lim_{k \rightarrow \infty} \|g_k\| = 0.$$

By (1.3) and (4.1), we have

$$|m_k(s_k)| \leq |g_k^\top s_k| + |\min(s_k^\top H_k s_k, 0)| \leq \|g_k\| \|s_k\| + M \|s_k\|^2 \leq (c_2 + M c_2^2) \|g_k\|^2.$$

So $m_k(s_k) \rightarrow 0$ as $k \rightarrow \infty$.

We prove this theorem by contradiction. Suppose that there is a positive constant ϵ such that

$$(4.17) \quad \frac{d_k^\top H_k d_k}{\|d_k\|^2} \leq -\epsilon \quad \text{for all } k \text{ large enough.}$$

Then, by the scaling (3.3) of d_k , we have

$$(4.18) \quad m_k(d_k) \leq \frac{1}{2} d_k^\top H_k d_k = \frac{1}{2} \frac{d_k^\top H_k d_k}{\|d_k\|^2} \|d_k\|^2 = \frac{1}{2} \left(\frac{d_k^\top H_k d_k}{\|d_k\|^2} \right)^3 \leq -\frac{\epsilon^3}{2}.$$

Hence, when k is sufficiently large, we have $m_k(d_k) < \tau^{-1} m_k(s_k)$, i.e., $k \in \mathcal{K}_d$.

However, when $k \in \mathcal{K}_d$, recalling the limit (4.8) in Theorem 4.2, we find that the assumption (4.17) could not hold. Hence,

$$\lim_{k \rightarrow \infty} d_k^\top H_k d_k = \lim_{k \rightarrow \infty} \left(\frac{d_k^\top H_k d_k}{\|d_k\|^2} \right)^3 = 0.$$

Owing to (1.4), we get $\min(0, \lambda_{\min}(H_k)) \rightarrow 0$ as $k \rightarrow \infty$. Therefore, this theorem is valid. $\square$

5. NUMERICAL EXPERIMENTS

In order to implement the DFSL algorithm, we employ the planar conjugate gradient method [4–6] to compute simultaneously a gradient-related direction and a negative curvature direction that satisfy (1.3) and (1.4) respectively. The planar conjugate gradient method is a sort of Krylov subspace iterative method which does not require an explicit Hessian matrix. Instead, a subroutine, which computes the product of the Hessian and an arbitrary vector, is enough. So it is suitable for large-scale problems. Compared with the Lanczos method, the planar conjugate gradient

method does not need to store or reproduce any matrix of Lanczos orthogonal basis. If the Hessian is sufficiently positive definite, the planar conjugate gradient method reduces to the general conjugate gradient method. It is cheap and fast.

The planar conjugate gradient method is terminated if the step s_k of the local quadratic model satisfies

$$\|H_k s_k + g_k\| \leq \max \left(\min(0.1, \sqrt{\|g_k\|}) \|g_k\|, \sqrt{\epsilon_{\text{mach}}} \right)$$

or the number of inner conjugate gradient iterations exceeds $\min(n, 30)$, where ϵ_{mach} is the machine precision. The parameters involved in the DFSL algorithm are set as follows:

$$\mu = 10^{-4}, \quad \beta = 0.5, \quad \tau = 1, \quad \gamma = \min \left(0.001, \frac{1}{2\sqrt{n}} \right), \quad \phi(\alpha) = \alpha^{10}.$$

We stop the algorithm if

$$\|g_k\|_{\infty} \leq 10^{-6} \quad \text{or} \quad k \geq 10^5.$$

To show the validity of the DFSL algorithm, we compare it with the adaptive line search (ALS) algorithm [14] and the state-of-the-art Limited memory BFGS (LBFGS) algorithm [20, 27]. In our experiments, the code of DFSL is written in Fortran 95 language. In the same language, we rewrite the code of ALS, where the two descent directions are also computed by the planar conjugate gradient method. The LBFGS code is downloaded from <http://users.eecs.northwestern.edu/~nocedal/software.html#lbfgs>.

In the remainder of this section, We illustrate numerical results by performance profiles, which is introduced by Dolan and Moré [3]. The key point of performance profiles is the relative σ -optimum, where $\sigma \geq 1$ is an interesting parameter. Without loss of generality, we take the number of function evaluations, for example. Suppose that there are N test problems. For each problem $i \in \{1, 2, \dots, N\}$, Algorithms DFSL, ALS and LBFGS solve it with the number of function evaluations $P_{(i)}^D$, $P_{(i)}^A$ and $P_{(i)}^L$, respectively. We say that DFSL for the problem i is the relative σ -optimum if

$$\sigma \geq \frac{P_{(i)}^D}{\min(P_{(i)}^D, P_{(i)}^A, P_{(i)}^L)} := r_{(i)}^D.$$

Obviously, an algorithm is excellent for the problem i once $r_{(i)}^D$ takes a small value. Otherwise, if σ is large and cannot be reduced, the algorithm for problem i is disappointing. We define an identity function to show whether DFSL for problem i is the relative σ -optimum

$$\pi_{\sigma}^D(i) := \begin{cases} 1 & \text{if } r_{(i)}^D \leq \sigma, \\ 0 & \text{otherwise.} \end{cases}$$

Then, the possibility of DFSL for all problems being the relative σ -optimum is approximately

$$p^D(\sigma) = \frac{\sum_{i=1}^N \pi_{\sigma}^D(i)}{N}.$$

TABLE 1. The set of small-scale CUTE problems and their dimensions.

No.	Problem	n	No.	Problem	n	No.	Problem	n
1	AKIVA	2	28	DQDRTIC	100	55	NONDIA	100
2	ALLINITU	4	29	ENGVAL1	100	56	NONDQUAR	100
3	ARWHEAD	100	30	ENGVAL2	3	57	OSBORNEA	5
4	BARD	3	31	ERRINROS	50	58	OSBORNEB	11
5	BDQRTIC	100	32	EXPFIT	2	59	PALMER5C	6
6	BEALE	2	33	EXTROSNB	100	60	PENALTY1	100
7	BOX	100	34	FLETCHBV2	100	61	PENALTY2	200
8	BOX3	3	35	FLETCHCR	100	62	PENALTY3	200
9	BRKMCC	2	36	FREUROTH	100	63	POWELLSG	100
10	BROWNAL	200	37	GENROSE	500	64	POWER	100
11	BROWNBS	2	38	GULF	3	65	ROSENBR	2
12	BROWNDEN	4	39	HAIRY	2	66	S308	2
13	BRYBND	100	40	HATFLDD	3	67	SCHMVETT	100
14	CHNROSNB	50	41	HATFLDE	3	68	SINEVAL	2
15	CLIFF	2	42	HATFLDFL	3	69	SISSER	2
16	COSINE	100	43	HELIX	3	70	SPARSINE	100
17	CUBE	2	44	HILBERTA	2	71	SPARSQUR	100
18	CURLY10	100	45	HILBERTB	10	72	TOINTGOR	50
19	CURLY20	100	46	HIMMELBB	2	73	TOINTGSS	100
20	CURLY30	100	47	HIMMELBF	4	74	TOINTPSP	50
21	DENSCHNA	2	48	HIMMELBG	2	75	TOINTQOR	50
22	DENSCHNB	2	49	HUMPS	2	76	TQUARTIC	100
23	DENSCHNC	2	50	KOWOSB	4	77	TRIDIA	100
24	DENSCHND	3	51	LIARWHD	100	78	VAREIGVL	50
25	DENSCHNE	3	52	MARATOSB	2	79	WATSON	12
26	DENSCHNF	2	53	MEXHAT	2	80	YFITU	3
27	DIXON3DQ	100	54	MOREBV	100			

TABLE 2. The set of large-scale CUTE problems and their dimensions.

No.	Problem	n	No.	Problem	n	No.	Problem	n
1	ARWHEAD	5000	9	ENGVAL1	5000	17	POWER	10000
2	BDQRTIC	5000	10	EXTROSNB	1000	18	SCHMVETT	5000
3	BOX	10000	11	FLETCHCR	1000	19	SPARSQUR	10000
4	BRYBND	5000	12	LIARWHD	5000	20	SROSENBR	5000
5	COSINE	10000	13	MOREBV	5000	21	TOINTGSS	5000
6	DQDRTIC	5000	14	NONDIA	5000	22	TQUARTIC	5000
7	EDENSCH	2000	15	PENALTY1	1000	23	TRIDIA	5000
8	EG2	1000	16	POWELLSG	5000	24	WOODS	4000

Obviously, $p^D(\sigma)$ increases monotonously. If $p^D(\sigma)$ gets close to one fast, DFLS is a good algorithm. In a similar way, we define $p^A(\sigma)$ and $p^L(\sigma)$ for Algorithms ALS and LBFGS, respectively.

Using the performance profiles, we report numerical results of Algorithms DFLS, ALS and LBFGS for small-scale and large-scale CUTE test problems [1, 15]. Table 1 lists eighty small-scale problems whose dimensions are less than or equal to five hundred. All the problems use default initial points provided by the CUTE collection and find the same critical points.

FIGURE 2. The performance profiles on the number of function evaluations for small-scale CUTE problems.

FIGURE 3. The performance profiles on the number of gradient evaluations for small-scale CUTE problems.

For these small-scale problems, the performance profiles of Algorithms DFLS, ALS and LBFGS on the numbers of function and gradient evaluations and CPU times are illustrated in Figures 2, 3 and 4, respectively. From the viewpoint of function and gradient evaluations, DFLS and ALS which exploit negative curvature directions from exact Hessians are more efficient than the first-order LBFGS algorithm. However, both DFLS and ALS are slower than LBFGS as shown in

FIGURE 4. The performance profiles on the CPU times for small-scale CUTE problems.

Figure 4, since the computation of the gradient-related direction and the negative curvature direction is more expensive than computing the search direction for LBFGS. We remark that LBFGS is even faster for large-scale problems. This is illustrated in Figure 7.

Next, we focus on comparisons on Algorithms DFLS and ALS. Although more gradient evaluations are performed in dwindling filter steps, the performance of DFLS on the number of gradient evaluations is almost the same as ALS. Taking advantage of dwindling filter steps which provide more freedom to accept trial points, DFLS promotes the sequence of iterates to converge fast and saves many function evaluations.

For large-scale problems, we select twenty-four unconstrained problems from CUTE collection and list them in Table 2. To examine the robustness and efficiencies of Algorithms DFLS, ALS and LBFGS, we set the initial point of each problem as $10 \times x_0$, where x_0 is the default initial point. And these three algorithms return the same optimal function value. So we infer that the same critical point has been obtained for different algorithms. The performance profiles on the number of function and gradient evaluations and CPU times for large-scale problems are shown in Figures 5, 6 and 7. We find that DFLS saves more function and gradient evaluations than ALS. In particular, there are six problems in which DFLS outperforms ALS, while DFLS fails to do so only in one problem. So, we say that the DFLS algorithm is the best solver in these three algorithms.

When examining the remarkably successful example for the dwindling filter, we plot function values versus iterations on the EXTROSNB problem in Figure 8. DFLS solves this problem with 24% dwindling filter steps and does not clear out the filter. From Figure 8, we note that the sequence of function values does not decrease monotonously as iterations progress. Thanks to the nonmonotone, the EXTROSNB problem is rapidly solved when the dwindling filter is employed.

FIGURE 5. The performance profiles on the number of function evaluations for large-scale CUTE problems.

FIGURE 6. The performance profiles on the number of gradient evaluations for large-scale CUTE problems.

Finally, we list detailed numerical results for 80 small CUTE problems in Tables 3 and 4, and the numerical results for 24 large CUTE problems in Table 5. These numerical results show that the algorithm DFLS is significantly efficient.

FIGURE 7. The performance profiles on the CPU times for large-scale CUTE problems.

FIGURE 8. Function values versus iterations on the EXTROSNB problem.

TABLE 3. Numerical results for the set of small CUTE problems.

No	Problem	n	Objective	DFLS			ALS			L-BFGS		
				nf	ng	cpu	nf	ng	cpu	nf	ng	cpu
1	AKIVA	2	6.166e+00	145	145	0.0	145	135	0.0	23	23	0.0
2	ALLINITU	4	5.744e+00	23	16	0.0	17	11	0.0	14	14	0.0
3	ARWHEAD	100	0.000e+00	7	7	0.0	72	6	0.0	12	12	0.0
4	BARD	3	8.215e-03	16	16	0.0	15	14	0.0	24	24	0.0
5	BDQRTIC	100	3.788e+02	14	14	0.0	14	14	0.0	111	111	0.0
6	BEALE	2	1.774e-16	17	11	0.0	15	10	0.0	16	16	0.0
7	BOX	100	-1.124e+01	7	6	0.0	7	6	0.0	13	13	0.0
8	BOX3	3	1.198e-12	9	9	0.0	9	9	0.0	24	24	0.0
9	BRKMCC	2	1.690e-01	5	5	0.0	5	5	0.0	8	8	0.0
10	BROWNAL	200	4.910e-10	5	5	0.0	5	5	0.0	14	14	0.0
11	BROWNB	2	8.911e-14	41	20	0.0	206	52	0.0	26	26	0.0
12	BROWNDEN	4	8.582e+04	12	12	0.0	15	13	0.0	44	44	0.0
13	BRYBND	100	2.432e-14	20	16	0.0	18	16	0.0	129	129	0.0
14	CHNROSNB	50	1.952e-13	66	53	0.0	158	86	0.0	298	298	0.0
15	CLIFF	2	1.998e-01	29	29	0.0	29	29	0.0	42	42	0.0
16	COSINE	100	-9.900e+01	21	9	0.0	16	7	0.0	38	38	0.0
17	CUBE	2	8.920e-13	80	45	0.0	68	45	0.0	50	50	0.0
18	CURLY10	100	-1.003e+04	87	47	0.0	50	42	0.0	1029	1029	0.0
19	CURLY20	100	-1.003e+04	65	31	0.0	122	32	0.0	1132	1132	0.0
20	CURLY30	100	-1.003e+04	54	28	0.0	88	24	0.0	1294	1294	0.1
21	DENSCHNA	2	2.739e-14	7	7	0.0	7	7	0.0	11	11	0.0
22	DENSCHNB	2	1.705e-14	9	8	0.0	9	8	0.0	9	9	0.0
23	DENSCHNC	2	1.757e-18	13	13	0.0	13	13	0.0	18	18	0.0
24	DENSCHND	3	1.254e-08	4389	4205	0.0	4247	4191	0.0	65	65	0.0
25	DENSCHNE	3	2.245e-15	21	18	0.0	15	11	0.0	43	43	0.0
26	DENSCHNF	2	2.270e-17	7	7	0.0	7	7	0.0	10	10	0.0
27	DIXON3DQ	100	1.010e-09	95	95	0.0	95	95	0.0	661	661	0.0
28	DQDRTIC	100	3.365e-15	7	7	0.0	7	7	0.0	22	22	0.0
29	ENGVAL1	100	1.091e+02	11	11	0.0	11	11	0.0	20	20	0.0
30	ENGVAL2	3	1.219e-14	28	20	0.0	22	20	0.0	37	37	0.0
31	ERRINROS	50	3.990e+01	92	81	0.0	167	111	0.0	196	196	0.0
32	EXPFIT	2	2.405e-01	16	11	0.0	15	11	0.0	17	17	0.0
33	EXTROSNB	100	3.068e-07	871	794	0.1	85176	3687	1.1	13914	13914	0.3
34	FLETCHBV2	100	-5.140e-01	17	17	0.0	17	17	0.0	216	216	0.0
35	FLETCHCR	100	3.781e-14	143	141	0.0	295	197	0.0	605	605	0.0
36	FREUROTH	100	1.196e+04	49	38	0.0	127	31	0.0	50	50	0.0
37	GENROSE	500	1.000e+00	1317	530	1.0	599	336	0.6	1268	1268	0.2
38	GULF	3	1.508e-09	312	281	0.1	103	58	0.0	68	68	0.0
39	HAIRY	2	2.000e+01	5691	2330	0.0	3331	1487	0.0	92	92	0.0
40	HATFLDD	3	6.631e-08	31	27	0.0	30	28	0.0	24	24	0.0

TABLE 4. Numerical results for the set of small CUTE problems (continued).

No	Problem	n	Objective	DFLS			ALS			L-BFGS		
				nf	ng	cpu	nf	ng	cpu	nf	ng	cpu
41	HATFLDE	3	5.120e-07	33	29	0.0	34	29	0.0	43	43	0.0
42	HATFLDFL	3	6.226e-05	89	39	0.0	141	39	0.0	818	818	0.0
43	HELIX	3	4.911e-17	28	19	0.0	28	20	0.0	33	33	0.0
44	HILBERTA	2	9.208e-19	3	3	0.0	3	3	0.0	7	7	0.0
45	HILBERTB	10	2.292e-14	6	6	0.0	6	6	0.0	7	7	0.0
46	HIMMELBB	2	6.987e-14	50	19	0.0	55	38	0.0	21	21	0.0
47	HIMMELBF	4	3.186e+02	1997	1400	0.0	1606	1198	0.0	53	53	0.0
48	HIMMELBG	2	7.288e-14	99	43	0.0	389	190	0.0	13	13	0.0
49	HUMPS	2	1.339e-12	3217	1321	0.0	16961	8251	0.0	343	343	0.0
50	KOWOSB	4	3.078e-04	28	23	0.0	21	20	0.0	47	47	0.0
51	LIARWHD	100	2.769e-14	13	13	0.0	13	13	0.0	20	20	0.0
52	MARATOSB	2	-1.000e+00	3631	2866	0.0	2666	1094	0.0	1528	1528	0.0
53	MEXHAT	2	-4.001e-02	465	136	0.0	207	33	0.0	54	54	0.0
54	MOREBV	100	4.507e-08	6361	6361	2.9	6361	6361	2.8	9190	9190	0.2
55	NONDIA	100	3.795e-21	11	11	0.0	11	11	0.0	18	18	0.0
56	NONDQUAR	100	4.680e-07	152	148	0.0	5534	296	0.1	2183	2183	0.0
57	OSBORNEA	5	5.465e-05	62	41	0.0	176	133	0.0	194	194	0.0
58	OSBORNEB	11	4.014e-02	58	37	0.0	32	24	0.0	254	254	0.0
59	PALMER5C	6	2.128e+00	7	7	0.0	7	7	0.0	16	16	0.0
60	PENALTY1	100	9.025e-04	25	25	0.0	50	40	0.0	70	70	0.0
61	PENALTY2	200	4.712e+13	31	31	0.0	141	35	0.0	138	138	0.0
62	PENALTY3	200	9.987e-04	3447	337	17.9	8120	427	28.7	96	96	0.7
63	POWELLSC	100	8.281e-09	18	18	0.0	18	18	0.0	68	68	0.0
64	POWER	100	4.846e-10	24	24	0.0	24	24	0.0	53	53	0.0
65	ROSENBR	2	4.318e-14	31	27	0.0	36	27	0.0	48	48	0.0
66	S308	2	7.732e-01	12	12	0.0	12	12	0.0	14	14	0.0
67	SCHMVETT	100	-2.940e+02	22	22	0.0	22	22	0.0	49	49	0.0
68	SINEVAL	2	1.611e-16	115	80	0.0	120	74	0.0	97	97	0.0
69	SISSER	2	4.487e-10	15	15	0.0	15	15	0.0	15	15	0.0
70	SPARSINE	100	2.267e-13	119	78	0.1	17	17	0.0	407	407	0.0
71	SPARSQR	100	1.863e-09	18	18	0.0	18	18	0.0	26	26	0.0
72	TOINTGOR	50	1.374e+03	9	9	0.0	9	9	0.0	133	133	0.0
73	TOINTGSS	100	1.010e+01	22	22	0.0	22	22	0.0	33	33	0.0
74	TOINTPSP	50	2.256e+02	626	571	0.0	22	16	0.0	126	126	0.0
75	TOINTQOR	50	1.175e+03	8	8	0.0	8	8	0.0	43	43	0.0
76	TQUARTIC	100	9.349e-12	12	10	0.0	16	12	0.0	23	23	0.0
77	TRIDIA	100	4.760e-14	13	13	0.0	13	13	0.0	238	238	0.0
78	VAREIGVL	50	1.757e-15	7	7	0.0	7	7	0.0	29	29	0.0
79	WATSON	12	3.151e-07	53	53	0.0	27	15	0.0	2294	2294	0.0
80	YFITU	3	2.080e-12	150	125	0.0	123	83	0.0	98	98	0.0

TABLE 5. Numerical results for the set of large CUTE problems with $10 \times x_0$.

No	Problem	n	Objective	DFLS			ALS			L-BFGS		
				nf	ng	cpu	nf	ng	cpu	nf	ng	cpu
1	ARWHEAD	5000	0.000e+00	14	14	0.1	14	14	0.1	30	30	0.1
2	BDQRTIC	5000	2.001e+04	21	21	0.2	21	21	0.2	282	282	0.6
3	BOX	10000	-1.865e+03	8	6	0.1	8	6	0.1	12	12	0.1
4	BRYBND	5000	7.264e-15	108	108	1.6	108	108	1.5	67	67	0.2
5	COSINE	10000	-9.999e+03	31	13	0.1	15	7	0.1	9	9	0.0
6	DQDRITIC	5000	1.074e-13	7	7	0.0	7	7	0.0	26	26	0.0
7	EDENSCH	2000	1.200e+04	21	21	0.1	21	21	0.0	47	47	0.0
8	EG2	1000	-9.989e+02	4	4	0.0	4	4	0.0	25	25	0.0
9	ENGVAL1	5000	5.549e+03	17	17	0.1	85	18	0.1	30	30	0.1
10	EXTROSNB	1000	5.826e-07	1028	1028	1.0	112617	4308	12.3	13290	13290	2.4
11	FLETCHCR	1000	1.372e-12	1426	1345	3.3	8889	4452	10.1	5672	5672	1.4
12	LIARWHD	5000	1.027e-12	23	23	0.1	23	23	0.1	39	39	0.1
13	MOREBV	5000	3.032e-08	67	67	1.7	67	67	1.7	45	45	0.1
14	NONDIA	5000	3.352e-12	34	34	0.1	48	42	0.1	59	59	0.1
15	PENALTY1	1000	9.686e-03	157	74	0.0	170	61	0.0	86	86	0.0
16	POWELLSC	5000	8.722e-07	23	23	0.1	23	23	0.1	75	75	0.1
17	POWER	10000	5.339e-10	58	58	2.0	58	58	1.9	618	618	1.2
18	SCHMVETT	5000	-1.499e+04	1126	178	9.1	22125	6846	20.7	23	23	0.1
19	SPARSQR	10000	1.436e-08	28	28	0.8	28	28	0.8	49	49	0.4
20	SROSENBR	5000	5.481e-13	17	17	0.0	64	43	0.1	60	60	0.1
21	TOINTGSS	5000	1.000e+01	65	65	0.6	65	65	0.6	14	14	0.0
22	TQUARTIC	5000	0.000e+00	1	1	0.0	1	1	0.0	1	1	0.0
23	TRIDIA	5000	2.263e-13	596	596	11.4	596	596	11.1	2004	2004	1.8
24	WOODS	4000	1.255e-12	65	57	0.2	84	57	0.2	50	50	0.1

6. CONCLUSION

In this paper, a dwindling multidimensional filter technique is presented in a line search algorithm framework for solving large-scale unconstrained optimization problems. Under some mild assumptions, we prove that the new algorithm converges to the first-order and second-order critical points. Preliminary numerical experiments and numerical comparison with ALS and L-BFGS indicate that the new algorithm is robust and efficient.

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REFERENCES

- [1] I. Bongartz, A. R. Conn, Nicholas I. M. Gould, and Philippe L. Toint, *Cute: constrained and unconstrained testing environment*, ACM Transactions on Mathematical Software **21** (1995), no. 1, 123–160.
- [2] Yan Nan Chen and Wen Yu Sun, *A new nonmonotone optimization method with trust region and second-order line search* (Chinese, with English summary), Numer. Math. J. Chinese Univ. **32** (2010), no. 4, 369–386. MR2790772 (2011m:90154)
- [3] Elizabeth D. Dolan and Jorge J. Moré, *Benchmarking optimization software with performance profiles*, Math. Program. **91** (2002), no. 2, Ser. A, 201–213, DOI 10.1007/s101070100263. MR1875515 (2002j:90001)
- [4] G. Fasano, *Planar conjugate gradient algorithm for large-scale unconstrained optimization. I. Theory*, J. Optim. Theory Appl. **125** (2005), no. 3, 523–541, DOI 10.1007/s10957-005-2087-1. MR2148660 (2006a:90163)
- [5] G. Fasano, *Planar conjugate gradient algorithm for large-scale unconstrained optimization. II. Application*, J. Optim. Theory Appl. **125** (2005), no. 3, 543–558, DOI 10.1007/s10957-005-2088-0. MR2148661 (2006a:90164)
- [6] Giovanni Fasano and Massimo Roma, *Iterative computation of negative curvature directions in large scale optimization*, Comput. Optim. Appl. **38** (2007), no. 1, 81–104, DOI 10.1007/s10589-007-9034-z. MR2332402 (2008i:90084)
- [7] M. C. Ferris, S. Lucidi, and M. Roma, *Nonmonotone curvilinear line search methods for unconstrained optimization*, Comput. Optim. Appl. **6** (1996), no. 2, 117–136, DOI 10.1007/BF00249642. MR1398263 (97d:90087)
- [8] Roger Fletcher, Nicholas I. M. Gould, Sven Leyffer, Philippe L. Toint, and Andreas Wächter, *Global convergence of a trust-region SQP-filter algorithm for general nonlinear programming*, SIAM J. Optim. **13** (2002), no. 3, 635–659 (2003), DOI 10.1137/S1052623499357258. MR1972208 (2004b:90105)
- [9] Roger Fletcher and Sven Leyffer, *Nonlinear programming without a penalty function*, Math. Program. **91** (2002), no. 2, Ser. A, 239–269, DOI 10.1007/s101070100244. MR1875517 (2002j:90074)
- [10] Roger Fletcher, Sven Leyffer, and Philippe L. Toint, *On the global convergence of a filter-SQP algorithm*, SIAM J. Optim. **13** (2002), no. 1, 44–59 (electronic), DOI 10.1137/S105262340038081X. MR1922433 (2003g:90121)
- [11] Roger Fletcher, Sven Leyffer, and Philippe L. Toint, *A brief history of filter methods*, SIAG/OPT Views-and-News **18** (2007), no. 1, 2–12.
- [12] Clóvis C. Gonzaga, Elizabeth Karas, and Márcia Vanti, *A globally convergent filter method for nonlinear programming*, SIAM J. Optim. **14** (2003), no. 3, 646–669 (electronic), DOI 10.1137/S1052623401399320. MR2085935 (2005d:49049)
- [13] Nicholas I. M. Gould, Sven Leyffer, and Philippe L. Toint, *A multidimensional filter algorithm for nonlinear equations and nonlinear least-squares*, SIAM J. Optim. **15** (2004), no. 1, 17–38, DOI 10.1137/S1052623403422637. MR2112974 (2005m:90117)

- [14] N. I. M. Gould, S. Lucidi, M. Roma, and Ph. L. Toint, *Exploiting negative curvature directions in linesearch methods for unconstrained optimization*, Optim. Methods Softw. **14** (2000), no. 1-2, 75–98, DOI 10.1080/10556780008805794. International Conference on Nonlinear Programming and Variational Inequalities (Hong Kong, 1998). MR1809604 (2001m:90104)
- [15] Nicholas I. M. Gould, Dominique Orban, and Philippe L. Toint, *GALAHAD, a library of thread-safe Fortran 90 packages for large-scale nonlinear optimization*, ACM Trans. Math. Software **29** (2003), no. 4, 353–372, DOI 10.1145/962437.962438. MR2077337 (2005b:90006)
- [16] Nick I. M. Gould, Caroline Sainvitu, and Philippe L. Toint, *A filter-trust-region method for unconstrained optimization*, SIAM J. Optim. **16** (2005), no. 2, 341–357, DOI 10.1137/040603851. MR2197983 (2006i:90076)
- [17] Nataša Krejić, Zorana Lužanin, and Irena Stojkovska, *Gauss-Newton-based BFGS methods with filter for unconstrained minimization*, Appl. Math. Comput. **211** (2009), no. 2, 354–362, DOI 10.1016/j.amc.2009.01.041. MR2524164
- [18] ChengJin Li and WenYu Sun, *On filter-successive linearization methods for nonlinear semi-definite programming*, Sci. China Ser. A **52** (2009), no. 11, 2341–2361, DOI 10.1007/s11425-009-0168-6. MR2566650 (2010j:90104)
- [19] Shujun Li and Zhenhai Liu, *A new trust region filter algorithm*, Appl. Math. Comput. **204** (2008), no. 1, 485–489, DOI 10.1016/j.amc.2008.07.007. MR2458387
- [20] Dong C. Liu and Jorge Nocedal, *On the limited memory BFGS method for large scale optimization*, Math. Programming **45** (1989), no. 3, (Ser. B), 503–528, DOI 10.1007/BF01589116. MR1038245 (90m:90253)
- [21] Jun Long and Sanyun Zeng, *A new filter-Levenberg-Marquardt method with disturbance for solving nonlinear complementarity problems*, Appl. Math. Comput. **216** (2010), no. 2, 677–688, DOI 10.1016/j.amc.2010.01.098. MR2601536
- [22] Stefano Lucidi, Francesco Rochetich, and Massimo Roma, *Curvilinear stabilization techniques for truncated Newton methods in large scale unconstrained optimization*, SIAM J. Optim. **8** (1998), no. 4, 916–939 (electronic), DOI 10.1137/S1052623495295250. MR1641270 (2000d:90073)
- [23] Garth P. McCormick, *A modification of Armijo’s step-size rule for negative curvature*, Math. Programming **13** (1977), no. 1, 111–115. MR0461907 (57 #1889)
- [24] Wei Hua Miao and Wen Yu Sun, *A filter trust-region method for unconstrained optimization* (Chinese, with English summary), Numer. Math. J. Chinese Univ. **29** (2007), no. 1, 88–96. MR2337211 (2009g:90147)
- [25] Jorge J. Moré and Danny C. Sorensen, *On the use of directions of negative curvature in a modified Newton method*, Math. Programming **16** (1979), no. 1, 1–20, DOI 10.1007/BF01582091. MR517757 (80b:90119)
- [26] Pu-yan Nie, Ming-yong Lai, Shu-jin Zhu, and Pei-ai Zhang, *A line search filter approach for the system of nonlinear equations*, Comput. Math. Appl. **55** (2008), no. 9, 2134–2141, DOI 10.1016/j.camwa.2007.08.037. MR2401120 (2009b:90145)
- [27] Jorge Nocedal, *Updating quasi-Newton matrices with limited storage*, Math. Comp. **35** (1980), no. 151, 773–782, DOI 10.2307/2006193. MR572855 (81g:65077)
- [28] Jorge Nocedal and Stephen J. Wright, *Numerical optimization*, 2nd ed., Springer Series in Operations Research and Financial Engineering, Springer, New York, 2006. MR2244940 (2007a:90001)
- [29] Ademir A. Ribeiro, Elizabeth W. Karas, and Clóvis C. Gonzaga, *Global convergence of filter methods for nonlinear programming*, SIAM J. Optim. **19** (2008), no. 3, 1231–1249, DOI 10.1137/060672285. MR2460740 (2010f:90164)
- [30] Caroline Sainvitu and Philippe L. Toint, *A filter-trust-region method for simple-bound constrained optimization*, Optim. Methods Softw. **22** (2007), no. 5, 835–848, DOI 10.1080/10556780701322970. MR2354770 (2008g:90116)
- [31] Wenyu Sun, *Filter methods for optimization: motivation and development*, invited talk, International Conference on Engineering Mathematics and Computational Mathematics, Hong Kong Polytechnical University, December 2009.
- [32] Wenyu Sun and Ya xiang Yuan, *Optimization Theory and Methods: Nonlinear Programming*, Springer Optimization and Its Applications, vol. 1, Springer, New York, 2006.
- [33] Wen-yu Sun and Qun-yan Zhou, *An unconstrained optimization method using nonmonotone second order Goldstein’s line search*, Sci. China Ser. A **50** (2007), no. 10, 1389–1400, DOI 10.1007/s11425-007-0072-x. MR2390457 (2009b:90147)

- [34] Andreas Wächter and Lorenz T. Biegler, *Line search filter methods for nonlinear programming: local convergence*, SIAM J. Optim. **16** (2005), no. 1, 32–48 (electronic), DOI 10.1137/S1052623403426544. MR2177768 (2006e:90121)
- [35] Andreas Wächter and Lorenz T. Biegler, *Line search filter methods for nonlinear programming: motivation and global convergence*, SIAM J. Optim. **16** (2005), no. 1, 1–31 (electronic), DOI 10.1137/S1052623403426556. MR2177767 (2006e:90120)
- [36] Zhujun Wang and Detong Zhu, *A filter-line-search method for unconstrained optimization*, J. Appl. Math. Comput. **34** (2010), no. 1-2, 329–342, DOI 10.1007/s12190-009-0324-8. MR2718789 (2012a:90179)
- [37] Zhenghao Yang, Wenyu Sun, and Chuangyin Dang, *A filter-trust-region method for LC^1 unconstrained optimization and its global convergence*, Anal. Theory Appl. **24** (2008), no. 1, 55–66, DOI 10.1007/s10496-008-0055-y. MR2422460 (2009g:90117)
- [38] Yan Zhang, Wenyu Sun, and Liqun Qi, *A nonmonotone filter Barzilai-Borwein method for optimization*, Asia-Pac. J. Oper. Res. **27** (2010), no. 1, 55–69, DOI 10.1142/S0217595910002582. MR2646952 (2011k:90188)
- [39] Qun-yan Zhou and Wen-yu Sun, *An adaptive nonmonotonic trust region method with curvilinear searches*, J. Comput. Math. **24** (2006), no. 6, 761–770. MR2269958 (2007f:90108)
- [40] Qun-yan Zhou and Wen-yu Sun, *A nonmonotone second-order steplength method for unconstrained minimization*, J. Comput. Math. **25** (2007), no. 1, 104–112. MR2292432 (2007j:90073)

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