MATHEMATICS OF COMPUTATION
Volume 84, Number 292, March 2015, Pages 767–801
S 0025-5718(2014)02856-1
Article electronically published on August 5, 2014

ALGORITHM FOR CONSTRUCTING SYMMETRIC DUAL FRAMELET FILTER BANKS

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Abstract. Dual wavelet frames and their associated dual framelet filter banks are often constructed using the oblique extension principle. In comparison with the construction of tight wavelet frames and tight framelet filter banks, it is indeed quite easy to obtain some particular examples of dual framelet filter banks with or without symmetry from any given pair of low-pass filters. However, such constructed dual framelet filter banks are often too particular to have some desirable properties such as balanced filter supports between primal and dual filters. From the point of view of both theory and application, it is important and interesting to have an algorithm which is capable of finding *all* possible dual framelet filter banks with symmetry and with the shortest possible filter supports from any given pair of low-pass filters with symmetry. However, to our best knowledge, this issue has not been resolved yet in the literature and one often has to solve systems of nonlinear equations to obtain nontrivial dual framelet filter banks. Given the fact that the construction of dual framelet filter banks is widely believed to be very flexible, the lack of a systematic algorithm for constructing all dual framelet filter banks in the literature is a little bit surprising to us. In this paper, by solving only small systems of linear equations, we shall completely settle this problem by introducing a step-by-step efficient algorithm to construct all possible dual framelet filter banks with or without symmetry and with the shortest possible filter supports. As a byproduct, our algorithm leads to a simple algorithm for constructing all symmetric tight framelet filter banks with two high-pass filters from a given low-pass filter with symmetry. Examples will be provided to illustrate our algorithm. To explain and to understand better our algorithm and dual framelet filter banks, we shall also discuss some properties of our algorithms and dual framelet filter banks in this paper.

1. Introduction and motivations

Wavelets and framelets with associated filter banks have many applications in areas such as image processing and scientific computing ([1, 3, 6]). On the one hand, dual framelet filter banks generalize biorthogonal wavelet filter banks by using more than one pair of high-pass filters. On the other hand, dual framelet filter banks include tight framelet filter banks as special cases by allowing the use of different sets of filters for analysis and synthesis. Therefore, dual framelet filter banks include both biorthogonal wavelet filter banks and tight framelet filter banks

Received by the editor July 23, 2013.
2010 *Mathematics Subject Classification.* Primary 42C40, 42C15.
Key words and phrases. Dual framelet filter banks, oblique extension principle, symmetry, tight framelet filter banks, complex symmetry, moment correcting filters, linear-phase moments, vanishing moments.
This research was supported in part by the Natural Sciences and Engineering Research Council of Canada (NSERC Canada) under Grant RGP 228051.

as special cases. The design of a dual framelet filter bank is quite different in nature to the construction of a biorthogonal wavelet filter bank. With the added redundancy and flexibility in a dual framelet filter bank, it is well known in the literature ([2, 4, 5, 10, 15]) that it is often much more flexible and less restrictive to construct dual framelet filter banks than biorthogonal wavelet filter banks or tight framelet filter banks.

In this paper we discuss how to systematically design all possible dual framelet filter banks with some desirable properties such as vanishing moments, symmetry, and short filter supports. To do so, let us recall some basic definitions and notation. By $l_0(\mathbb{Z})$ we denote the linear space of all sequences $u = \{u(k)\}_{k\in\mathbb{Z}} : \mathbb{Z} \to \mathbb{C}$ such that $\{k \in \mathbb{Z} \,:\, u(k) \neq 0\}$ is a finite set. For $u = \{u(k)\}_{k\in\mathbb{Z}} \in l_0(\mathbb{Z})$, the z-transform of u is a Laurent polynomial defined to be $\mathsf{u}(z) := \sum_{k\in\mathbb{Z}} u(k)z^k$. We define another associated sequence $u^\star$ by $u^\star(k) = \overline{u(-k)}$, $k \in \mathbb{Z}$. For a matrix $\mathsf{P}(z) = \sum_{k\in\mathbb{Z}} P_k z^k$ of Laurent polynomials, we define $\mathsf{P}^\star(z) := \sum_{k\in\mathbb{Z}} P_k^\star z^{-k}$, where $P_k^\star := \overline{P_k}^{\mathsf{T}}$ denotes the complex conjugate of the transpose of the matrix P_k.

Since all wavelet or framelet filter banks are often obtained via the oblique extension principle (see [2, 4, 5]), we first revisit here the oblique extension principle (OEP). For $\tilde{a}, \tilde{b}_1, \ldots, \tilde{b}_s, a, b_1, \ldots, b_s, \Theta \in l_0(\mathbb{Z})$, we say that $(\{\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\}, \{a; b_1, \ldots, b_s\})_\Theta$ is *a dual framelet filter bank* if the following perfect reconstruction condition holds:

$$\begin{bmatrix} \tilde{\mathsf{b}}_1(z) & \cdots & \tilde{\mathsf{b}}_s(z) \\ \tilde{\mathsf{b}}_1(-z) & \cdots & \tilde{\mathsf{b}}_s(-z) \end{bmatrix} \begin{bmatrix} \mathsf{b}_1(z) & \cdots & \mathsf{b}_s(z) \\ \mathsf{b}_1(-z) & \cdots & \mathsf{b}_s(-z) \end{bmatrix}^\star = \mathcal{M}_{a,\tilde{a},\Theta}(z), \tag{1.1}$$

where

$$\mathcal{M}_{a,\tilde{a},\Theta}(z) := \begin{bmatrix} \Theta(z) - \Theta(z^2)\tilde{\mathsf{a}}(z)\mathsf{a}^\star(z) & -\Theta(z^2)\tilde{\mathsf{a}}(z)\mathsf{a}^\star(-z) \\ -\Theta(z^2)\tilde{\mathsf{a}}(-z)\mathsf{a}^\star(z) & \Theta(-z) - \Theta(z^2)\tilde{\mathsf{a}}(-z)\mathsf{a}^\star(-z) \end{bmatrix}. \tag{1.2}$$

It is trivial to observe that $(\{\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\}, \{a; b_1, \ldots, b_s\})_\Theta$ is a dual framelet filter bank if and only if $(\{a; b_1, \ldots, b_s\}, \{\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\})_{\Theta^\star}$ is a dual framelet filter bank. $\{a; b_1, \ldots, b_s\}_\Theta$ is called *a tight framelet filter bank* if $(\{a; b_1, \ldots, b_s\}, \{a; b_1, \ldots, b_s\})_\Theta$ is a dual framelet filter bank.

The low-pass filters a and $\tilde{a}$ are often given in advance. As we shall see in Section 2, one often designs a moment correcting filter Θ with some desirable properties first. Then the matrix $\mathcal{M}_{a,\tilde{a},\Theta}$ is given and the construction of high-pass filters $b_1, \ldots, b_s, \tilde{b}_1, \ldots, \tilde{b}_s$ now becomes how to factorize a given matrix $\mathcal{M}_{a,\tilde{a},\Theta}$ of Laurent polynomials in (1.2) so that (1.1) holds.

Under the natural assumption $\mathsf{a}(1) = \tilde{\mathsf{a}}(1) = 1$, we can define functions $\boldsymbol{\varphi}, \tilde{\boldsymbol{\varphi}}, \tilde{\boldsymbol{\eta}}$, $\boldsymbol{\psi}^{[\ell]}$, $\tilde{\boldsymbol{\psi}}^{[\ell]}$, $\ell = 1, \ldots, s$ on the real line $\mathbb{R}$ by

$$\boldsymbol{\varphi}(\xi) := \prod_{j=1}^{\infty} \mathsf{a}(e^{-i2^{-j}\xi}),\ \tilde{\boldsymbol{\varphi}}(\xi) := \prod_{j=1}^{\infty} \tilde{\mathsf{a}}(e^{-i2^{-j}\xi}),\ \tilde{\boldsymbol{\eta}}(\xi) := \Theta(e^{-i\xi})\tilde{\boldsymbol{\varphi}}(\xi),\ \xi \in \mathbb{R} \tag{1.3}$$

and

$$\boldsymbol{\psi}^{[\ell]}(2\xi) := \mathsf{b}_\ell(e^{-i\xi})\boldsymbol{\varphi}(\xi), \quad \tilde{\boldsymbol{\psi}}^{[\ell]}(2\xi) := \tilde{\mathsf{b}}_\ell(e^{-i\xi})\tilde{\boldsymbol{\varphi}}(\xi), \quad \ell = 1, \ldots, s,\ \xi \in \mathbb{R}. \tag{1.4}$$

Then all the above functions are well defined ([1, 3]). Under the assumption $\mathsf{a}(1) = \tilde{\mathsf{a}}(1) = \Theta(1) = 1$, it has been shown in [11, Theorem 2] and [12, Theorem 17] that $(\{\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\}, \{a; b_1, \ldots, b_s\})_\Theta$ is a dual framelet filter bank if and only if

$(\{\tilde{\boldsymbol{\eta}}; \tilde{\boldsymbol{\psi}}^{[1]}, \ldots, \tilde{\boldsymbol{\psi}}^{[s]}\}, \{\boldsymbol{\varphi}; \boldsymbol{\psi}^{[1]}, \ldots, \boldsymbol{\psi}^{[s]}\})$ forms a frequency-based dual framelet, that is,

$$\sum_{k\in\mathbb{Z}} \langle \mathbf{f}, \tilde{\boldsymbol{\eta}}_{1;0,k}\rangle \langle \boldsymbol{\varphi}_{1;0,k}, \mathbf{g}\rangle + \sum_{j=0}^{\infty}\sum_{\ell=1}^{s}\sum_{k\in\mathbb{Z}} \langle \mathbf{f}, \tilde{\boldsymbol{\psi}}^{[\ell]}_{2^{-j};0,k}\rangle \langle \boldsymbol{\psi}^{[\ell]}_{2^{-j};0,k}, \mathbf{g}\rangle = 2\pi \langle \mathbf{f}, \mathbf{g}\rangle$$

for all compactly supported functions $\mathbf{f}, \mathbf{g} \in C^\infty(\mathbb{R})$, where the infinite series converges in an appropriate sense as described in [11] and

$$\boldsymbol{\psi}_{\lambda;n,k}(x) := |\lambda|^{1/2} e^{-ik\lambda x} f(\lambda x - n) \quad \text{for} \quad \lambda, k, n, x \in \mathbb{R}.$$

Due to this natural link between a dual framelet filter bank and a frequency-based dual framelet, in this paper we only deal with filter banks without discussing wavelets and framelets on the real line.

For a dual framelet filter bank $(\{\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\}, \{a; b_1, \ldots, b_s\})_\Theta$, it is often desirable for the high-pass filters to possess certain numbers of vanishing moments:

$$\mathsf{b}_\ell(z) = (1 - z^{-1})^{n_b} \mathring{\mathsf{b}}_\ell(z), \qquad \tilde{\mathsf{b}}_\ell(z) = (1 - z^{-1})^{n_{\tilde{b}}} \mathring{\tilde{\mathsf{b}}}_\ell(z), \qquad \ell = 1, \ldots, s, \tag{1.5}$$

where n_b and $n_{\tilde{b}}$ are nonnegative integers. Moreover, n_b and $n_{\tilde{b}}$ are called the orders of vanishing moments of the primal high-pass filters $b_1, \ldots, b_s$ and the dual high-pass filters $\tilde{b}_1, \ldots, \tilde{b}_s$, respectively, if all $\mathring{\mathsf{b}}_\ell$ and $\mathring{\tilde{\mathsf{b}}}_\ell$ are Laurent polynomials such that both $\sum_{\ell=1}^{s} |\mathring{\mathsf{b}}_\ell(1)|$ and $\sum_{\ell=1}^{s} |\mathring{\tilde{\mathsf{b}}}_\ell(1)|$ are nonzero.

Now the perfect reconstruction condition for a dual framelet filter bank $(\{\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\}, \{a; b_1, \ldots, b_s\})_\Theta$ in (1.1) can be equivalently expressed as

$$\begin{bmatrix} \mathring{\tilde{\mathsf{b}}}_1(z) & \cdots & \mathring{\tilde{\mathsf{b}}}_s(z) \\ \mathring{\tilde{\mathsf{b}}}_1(-z) & \cdots & \mathring{\tilde{\mathsf{b}}}_s(-z) \end{bmatrix} \begin{bmatrix} \mathring{\mathsf{b}}_1(z) & \cdots & \mathring{\mathsf{b}}_s(z) \\ \mathring{\mathsf{b}}_1(-z) & \cdots & \mathring{\mathsf{b}}_s(-z) \end{bmatrix}^\star = \mathcal{M}_{a,\tilde{a},\Theta|n_b,n_{\tilde{b}}}(z), \tag{1.6}$$

where

$$\mathcal{M}_{a,\tilde{a},\Theta|n_b,n_{\tilde{b}}}(z) := \begin{bmatrix} \mathsf{A}(z) & \mathsf{B}(z) \\ \mathsf{B}(-z) & \mathsf{A}(-z) \end{bmatrix} \tag{1.7}$$

with

$$\mathsf{A}(z) := \frac{\Theta(z) - \Theta(z^2)\tilde{\mathsf{a}}(z)\mathsf{a}^\star(z)}{(1-z)^{n_b}(1-z^{-1})^{n_{\tilde{b}}}}, \quad \mathsf{B}(z) := -\Theta(z^2)\frac{\tilde{\mathsf{a}}(z)}{(1+z)^{n_b}}\frac{\mathsf{a}^\star(-z)}{(1-z^{-1})^{n_{\tilde{b}}}}. \tag{1.8}$$

Now the construction of a dual framelet filter bank with preassigned orders of vanishing moments is simply to factorize the matrix $\mathcal{M}_{a,\tilde{a},\Theta|n_b,n_{\tilde{b}}}$ in (1.7) so that (1.6) is satisfied.

To reduce computational complexity in the implementation of a dual framelet filter bank, we prefer a small number s of high-pass filters. As shown in Theorem 7, it is often necessary that $s > 1$. Hence, in this paper we shall consider the case $s = 2$ for a dual framelet filter bank. For the case $s = 2$, (1.6) takes the following equivalent form:

$$\begin{bmatrix} \mathring{\mathsf{b}}_1(z) & \mathring{\mathsf{b}}_2(z) \\ \mathring{\mathsf{b}}_1(-z) & \mathring{\mathsf{b}}_2(-z) \end{bmatrix} \begin{bmatrix} \mathring{\tilde{\mathsf{b}}}_1(z) & \mathring{\tilde{\mathsf{b}}}_2(z) \\ \mathring{\tilde{\mathsf{b}}}_1(-z) & \mathring{\tilde{\mathsf{b}}}_2(-z) \end{bmatrix}^\star = \begin{bmatrix} \mathsf{A}^\star(z) & \mathsf{B}^\star(-z) \\ \mathsf{B}^\star(z) & \mathsf{A}^\star(-z) \end{bmatrix}. \tag{1.9}$$

It is easy to find particular solutions to (1.9) by choosing $\mathring{\mathsf{b}}_1$ and $\mathring{\mathsf{b}}_2$ in such a way that the determinant of the first 2×2 matrix on the left side of (1.9) is a nonzero

monomial. We present here two particular constructions known in the literature, for example, see [4,5]. The first construction is

$$\mathring{\mathsf{b}}_1(z) = 1, \qquad \mathring{\mathsf{b}}_2(z) = z. \tag{1.10}$$

Then it follows from (1.9) that we must have

$$\mathring{\tilde{\mathsf{b}}}_1(z) = [\mathsf{A}(z) + \mathsf{B}(z)]/2, \qquad \mathring{\tilde{\mathsf{b}}}_2(z) = z[\mathsf{A}(z) - \mathsf{B}(z)]/2. \tag{1.11}$$

The second construction is

$$\mathring{\mathsf{b}}_1(z) = (1+z)/2, \qquad \mathring{\mathsf{b}}_2(z) = (1-z)/2. \tag{1.12}$$

Then the perfect reconstruction condition in (1.9) will force

$$\mathring{\tilde{\mathsf{b}}}_1(z) = [(1+z)\mathsf{A}(z)+(1-z)\mathsf{B}(z)]/2, \ \mathring{\tilde{\mathsf{b}}}_2(z) = [(1-z)\mathsf{A}(z)+(1+z)\mathsf{B}(z)]/2. \tag{1.13}$$

We now discuss filter support and symmetry property of a dual framelet filter bank. For a filter $u = \{u(k)\}_{k\in\mathbb{Z}} \in l_0(\mathbb{Z})$, if $u(m)u(n) \neq 0$ and $u(k) = 0$ for all $k \in \mathbb{Z}\backslash[m,n]$, then we define

$$\operatorname{fsupp}(u) := [m,n], \qquad \operatorname{len}(u) := |\operatorname{fsupp}(u)| := n - m. \tag{1.14}$$

The filter support $\operatorname{fsupp}(u)$ is simply the shortest interval containing all the positions of the nonzero coefficients of u. We say that u has *symmetry* if

$$u(c-k) = \epsilon u(k), \qquad \forall\, k \in \mathbb{Z} \tag{1.15}$$

for some $c \in \mathbb{Z}$ and $\epsilon \in \{-1,1\}$. A filter u is *symmetric* about the point $\frac{c}{2}$ if (1.15) holds with $\epsilon = 1$, and *antisymmetric* about the point $\frac{c}{2}$ if (1.15) holds with $\epsilon = -1$. We call $\frac{c}{2}$ the *symmetry center* of the filter u, which is simply the center of its filter support $\operatorname{fsupp}(u)$. It is often convenient to use a *symmetry operator* S to record the symmetry type of a filter having symmetry. For this purpose, we define

$$\mathsf{S}\mathsf{u}(z) := \frac{\mathsf{u}(z)}{\mathsf{u}(z^{-1})}, \qquad z \in \mathbb{C}\backslash\{0\}. \tag{1.16}$$

Now it is straightforward to see that (1.15) holds if and only if $\mathsf{S}\mathsf{u}(z) = \epsilon z^c$.

Note that the perfect reconstruction condition in (1.1) with $s = 2$ implies

$$\tilde{\mathsf{b}}_1(z)\mathsf{b}_1^\star(-z) + \tilde{\mathsf{b}}_2(z)\mathsf{b}_2^\star(-z) = -\Theta(z^2)\tilde{\mathsf{a}}(z)\mathsf{a}^\star(-z). \tag{1.17}$$

If all the filters $\tilde{a}, \tilde{b}_1, \tilde{b}_2, a, b_1, b_2, \Theta$ are required to have symmetry, as we shall see from Lemma 8, it is natural to have $\mathsf{S}(\tilde{\mathsf{b}}_1(z)\mathsf{b}_1^\star(-z)) = \mathsf{S}(\tilde{\mathsf{b}}_2(z)\mathsf{b}_2^\star(-z))$, and from (1.17) we must have the following relation on the lengths of filter supports:

$$\begin{aligned} &\max(\operatorname{len}(b_1) + \operatorname{len}(\tilde{b}_1), \operatorname{len}(b_2) + \operatorname{len}(\tilde{b}_2)) \\ &\qquad = \operatorname{len}(a) + \operatorname{len}(\tilde{a}) + 2\operatorname{len}(\Theta) + 2\epsilon_{\text{len}}, \qquad \epsilon_{\text{len}} \in \mathbb{N} \cup \{0\}. \end{aligned} \tag{1.18}$$

Therefore, from any given filters $a, \tilde{a}, \Theta$ with symmetry, it is natural and important to construct all dual framelet filter banks $(\{\tilde{a}; \tilde{b}_1, \tilde{b}_2\}, \{a; b_1, b_2\})_\Theta$ having symmetry and the shortest possible filter supports:

$$\max(\operatorname{len}(b_1) + \operatorname{len}(\tilde{b}_1), \operatorname{len}(b_2) + \operatorname{len}(\tilde{b}_2)) = \operatorname{len}(a) + \operatorname{len}(\tilde{a}) + 2\operatorname{len}(\Theta). \tag{1.19}$$

We now examine the filter length and symmetry property of the two particular constructions in (1.10)–(1.13). Assume that all the filters $a, \tilde{a}, \Theta$ have the following symmetry:

$$\mathsf{S}\Theta(z) = \epsilon_\Theta z^{c_\Theta}, \qquad \mathsf{S}\mathsf{a}(z) = \epsilon z^c, \qquad \mathsf{S}\tilde{\mathsf{a}}(z) = \tilde{\epsilon} z^{\tilde{c}} \tag{1.20}$$

and both A and B in (1.8) are Laurent polynomials. In order for the Laurent polynomial A to have symmetry, by Lemma 8, it is natural to require that

$$\tilde{c} = c - c_\Theta \qquad \text{and} \qquad \tilde{\epsilon} = \epsilon. \tag{1.21}$$

Then A and B have the following symmetry:

$$\mathsf{SA}(z) = (-1)^{n_{\tilde{b}}+n_b}\epsilon_\Theta z^{c_\Theta+n_{\tilde{b}}-n_b}, \qquad \mathsf{SB}(z) = (-1)^{c+n_{\tilde{b}}}\epsilon_\Theta z^{c_\Theta+n_{\tilde{b}}-n_b}. \tag{1.22}$$

We consider two cases according to either $\mathsf{SA}(z) = \mathsf{SB}(z)$ or $\mathsf{SA}(z) = -\mathsf{SB}(z)$.

Case 1: If $c + n_b$ is an even integer, then $\mathsf{SA}(z) = \mathsf{SB}(z)$. The first particular construction of a dual framelet filter bank $(\{\tilde{a}; \tilde{b}_1, \tilde{b}_2\}, \{a; b_1, b_2\})_\Theta$ in (1.10) and (1.11) indeed has symmetry and the shortest possible filter support satisfying (1.19) with

$$\operatorname{len}(b_1) = \operatorname{len}(b_2) = n_b, \quad \max(\operatorname{len}(\tilde{b}_1), \operatorname{len}(\tilde{b}_2)) = \operatorname{len}(a) + \operatorname{len}(\tilde{a}) + 2\operatorname{len}(\Theta) - n_b.$$

Since $\mathsf{S}((1+z)\mathsf{A}(z)) = -\mathsf{S}((1-z)\mathsf{B}(z))$, the dual filters $\tilde{b}_1$ and $\tilde{b}_2$ in (1.13) usually do not possess any symmetry, even though all other filters $b_1, b_2, a, \tilde{a}, \Theta$ have symmetry. Hence, the second particular construction of a dual framelet filter bank $(\{\tilde{a}; \tilde{b}_1, \tilde{b}_2\}, \{a; b_1, b_2\})_\Theta$ in (1.12) and (1.13) lacks symmetry.

Case 2: If $c + n_b$ is an odd integer, then $\mathsf{SA}(z) = -\mathsf{SB}(z)$. The dual filters $\tilde{b}_1$ and $\tilde{b}_2$ in (1.11) usually do not possess any symmetry, even though all other filters $b_1, b_2, a, \tilde{a}, \Theta$ have symmetry. Therefore, the first particular construction in (1.10) and (1.11) lacks symmetry. But the second particular construction of a dual framelet filter bank $(\{\tilde{a}; \tilde{b}_1, \tilde{b}_2\}, \{a; b_1, b_2\})_\Theta$ in (1.12) and (1.13) indeed has symmetry and the shortest possible filter support satisfying (1.19), since by $\mathsf{SA}(z) = -\mathsf{SB}(z)$, we deduce from (1.13) that

$$\operatorname{len}(b_1) = \operatorname{len}(b_2) = n_b+1, \quad \max(\operatorname{len}(\tilde{b}_1), \operatorname{len}(\tilde{b}_2)) = \operatorname{len}(a)+\operatorname{len}(\tilde{a})+2\operatorname{len}(\Theta)-n_b-1.$$

Though the two particular constructions in (1.10)–(1.13) are explicit and very simple, a shortcoming of the two particular constructions is that their filter supports are quite unbalanced: the filter supports of b_1 and b_2 are very short while the filter supports of $\tilde{b}_1$ and $\tilde{b}_2$ are usually very long. For the purpose of implementation and performance of a discrete framelet transform employing a dual framelet filter bank, it is often desirable for all high-pass filters to have more or less balanced filter supports. For certain applications such as signal and image denoising, it is of interest to have a dual framelet filter bank $(\{\tilde{a}; \tilde{b}_1, \tilde{b}_2\}, \{a; b_1, b_2\})_\Theta$ which is close to a tight framelet filter bank, that is, $\tilde{a} \approx a, \tilde{b}_1 \approx b_1$, and $\tilde{b}_2 \approx b_2$. Furthermore, if $\tilde{a} = a$ and if the necessary and sufficient condition in [13, 14] is satisfied, then it is very much desired that such an algorithm is able to obtain all the tight framelet filter banks with symmetry and with two high-pass filters as special cases. Undoubtedly, the particular constructions in (1.10)–(1.13) cannot achieve this goal. In fact, to our best knowledge, so far there is no systematic algorithm available in the literature to achieve such a purpose. This difficulty is probably caused by the fact that

the overwhelming flexibility and freedom in using four high-pass filters in a dual framelet filter bank make the task of finding all dual framelet filter banks much harder.

The above discussions motivate us to develop a systematic algorithm to construct *all* possible dual framelet filter banks with symmetry and with short filter supports. Though many particular constructions of various dual framelet filter banks with or without symmetry appeared in the literature (see [2, 4, 5, 7, 8, 10, 16, 17, 21, 22] and references therein), to our best knowledge, so far there is no systematic algorithm available in the literature to construct *all* possible dual framelet filter banks $(\{\tilde{a}; \tilde{b}_1, \tilde{b}_2\}, \{a; b_1, b_2\})_\Theta$ with symmetry and with the shortest possible filter supports derived from any given filters $a, \tilde{a}, \Theta$ with symmetry. The only method that we know so far is [17, Appendix] where a system of nonlinear equations has to be solved in order to obtain some nontrivial examples of dual framelet filter banks other than the two particular constructions in (1.10)–(1.13). Given the fact that the construction of dual framelet filter banks is widely believed to be very flexible, the lack of a systematic algorithm for constructing dual framelet filter banks in the literature is a little bit surprising to us. The main goal of this paper is to fill this gap by developing a systematic and satisfactory algorithm to construct all dual framelet filter banks $(\{\tilde{a}; \tilde{b}_1, \tilde{b}_2\}, \{a; b_1, b_2\})_\Theta$ with symmetry and with the shortest possible filter supports derived from any given filters $a, \tilde{a}, \Theta$ with symmetry.

The structure of the paper is as follows. In Section 2 we shall present a step-by-step algorithm for constructing all dual framelet filter banks having symmetry and the shortest possible filter supports satisfying (1.18) with $\epsilon_{\text{len}} \in \{0, 1\}$. Therefore, our algorithm not only finds all symmetric dual framelet filter banks $(\{\tilde{a}; \tilde{b}_1, \tilde{b}_2\}, \{a; b_1, b_2\})_\Theta$ with the shortest possible filter supports satisfying (1.19) but also all those having slightly longer filter supports satisfying (1.18) with $\epsilon_{\text{len}} = 1$. Our algorithm only involves solving small systems of linear equations and is able to efficiently find all possible dual framelet filter banks having symmetry and the shortest possible filter support from any given filters $a, \tilde{a}, \Theta$ with symmetry. Moreover, if $\tilde{a} = a$ and if the necessary and sufficient condition in [13, 14] is satisfied, then our algorithm is able to obtain all the tight framelet filter banks with symmetry and with two high-pass filters. Our algorithm for constructing dual framelet filter banks naturally leads to a simple algorithm, stated in detail in Section 3, for constructing all tight framelet filter banks with symmetry and two high-pass filters. In fact, our algorithm in Section 3 for constructing all symmetric tight framelet filter banks $(\{\tilde{a}; \tilde{b}_1, \tilde{b}_2\}, \{a; b_1, b_2\})_\Theta$ is not only slightly more general but also much simpler than those algorithms developed in [13, 14, 19]. In Section 4 we shall present several examples to illustrate our algorithm. To better understand our algorithms and dual framelet filter banks, since the presentation of our algorithm appears to be somewhat complicated at the first glance, we shall discuss in Section 5 some basic properties of dual framelet filter banks and provide some explanations for our algorithm.

2. Algorithm for constructing symmetric dual framelet filter banks

Comparing with the design of tight framelet filter banks with symmetry, since a dual framelet filter bank employs four high-pass filters, the construction of dual framelet filter banks with symmetry has much more flexibility and freedom. However, such overwhelming flexibility and freedom also make it much more difficult in

finding all possible dual framelet filter banks. In fact, to our best knowledge, there is no systematic algorithm available so far in the literature for constructing all symmetric dual framelet filter banks $(\{\tilde{a};\tilde{b}_1,\tilde{b}_2\},\{a;b_1,b_2\})_\Theta$ with the shortest possible filter supports satisfying (1.19). In this section, we completely settle this problem by presenting a step-by-step systematic algorithm to construct all dual framelet filter banks $(\{\tilde{a};\tilde{b}_1,\tilde{b}_2\},\{a;b_1,b_2\})_\Theta$ having symmetry and the shortest possible filter supports satisfying (1.18) with $\epsilon_{\rm len}\in\{0,1\}$ from any given filters $a,\tilde{a},\Theta$ with symmetry.

In this paper we deal with both real-valued and complex-valued dual framelet filter banks. For complex-valued filters, there is another closely related notion of symmetry similar to (1.15). We say that a filter $u=\{u(k)\}_{k\in\mathbb{Z}}:\mathbb{Z}\to\mathbb{C}$ has *complex symmetry* if

$$u(c-k)=\epsilon\overline{u(k)},\qquad \forall\, k\in\mathbb{Z} \tag{2.1}$$

for some $c\in\mathbb{Z}$ and $\epsilon\in\{-1,1\}$. That is, $u^\star(k)=\epsilon u(c+k)$ for all $k\in\mathbb{Z}$, where $u^\star(k):=\overline{u(-k)}, k\in\mathbb{Z}$. Define a *complex symmetry operator* $\mathbb{S}$ by

$$\mathbb{S}\mathsf{u}(z):=\frac{\mathsf{u}(z)}{\mathsf{u}^\star(z)},\qquad z\in\mathbb{C}\backslash\{0\}. \tag{2.2}$$

Then a filter u has complex symmetry in (2.1) if and only if $\mathbb{S}\mathsf{u}(z)=\epsilon z^c$. If u is identically zero, then $\mathbb{S}\mathsf{u}$ and Su can be assigned any types of [complex] symmetry. It is trivial to see that a filter u has real-valued coefficients if and only if $\mathsf{u}^\star(z)=\mathsf{u}(z^{-1})$. Therefore, for a real-valued filter u, there is no difference between symmetry and complex symmetry since $\mathsf{Su}=\mathbb{S}\mathsf{u}$. If a filter u has symmetry and $\lambda\in\mathbb{C}\backslash\{0\}$, then λu also has symmetry. However, if u has complex symmetry, it is not necessary that λu has complex symmetry and for a nontrivial filter u, in fact, λu also has complex symmetry if and only if $\lambda\in\mathbb{R}$ or $i\lambda\in\mathbb{R}$.

For a filter u and a nonnegative integer m, we define the order of *sum rules* of u to be $\mathrm{sr}(u):=m$, where m is the largest integer such that $\mathsf{u}(z)=\mathcal{O}(|1+z|^m), z\to-1$. Similarly, we define the order of *vanishing moments* of the filter u to be $\mathrm{vm}(u):=n$, where n is the largest integer such that $\mathsf{u}(z)=\mathcal{O}(|1-z|^n), z\to 1$. To guarantee that both A and B in (1.8) are Laurent polynomials, under the natural assumption $\mathsf{a}(1)\tilde{\mathsf{a}}(1)\Theta(1)\neq 0$, it is necessary and sufficient to require that

$$\begin{aligned}&0\leqslant n_b\leqslant \mathrm{sr}(\tilde{a}),\qquad 0\leqslant n_{\tilde{b}}\leqslant \mathrm{sr}(a),\\ &\Theta(z)-\Theta(z^2)\tilde{\mathsf{a}}(z)\mathsf{a}^\star(z)=\mathcal{O}(|1-z|^{n_b+n_{\tilde{b}}}),\qquad z\to 1.\end{aligned} \tag{2.3}$$

For an integer j, we define $\mathrm{odd}(j):=1$ if j is odd, and $\mathrm{odd}(j):=0$ if j is even, that is, $\mathrm{odd}(j):=\frac{1-(-1)^j}{2}$. By $\mathrm{coeff}(\mathsf{p},z,j)$ we denote the coefficient of z^j in a Laurent polynomial p.

We now present an algorithm to construct all dual framelet filter banks having symmetry or complex symmetry and having real coefficients or complex coefficients with filter supports satisfying (1.18) with $\epsilon_{\rm len}\in\{0,1\}$. The presentation and statement of the following algorithm appear to be somewhat complicated at first glance. To understand better our algorithm and dual framelet filter banks, we shall provide some explanations for the following algorithm in Section 5. Here we only present the algorithm and its proof without explaining its various relations and assumptions appearing in the algorithm.

Algorithm 1. *Let $a, \tilde{a}, \Theta \in l_0(\mathbb{Z})$ be given filters having [complex] symmetry (and real coefficients) satisfying* (1.20), (1.21), *and* (2.3) *for some $c, \tilde{c}, c_\Theta \in \mathbb{Z}$, $\epsilon, \tilde{\epsilon}, \epsilon_\Theta \in \{-1, 1\}$, and $n_b, n_{\tilde{b}} \in \mathbb{N} \cup \{0\}$. Assume that $\mathrm{len}(a) + \mathrm{len}(\tilde{a}) + \mathrm{len}(\Theta) > 0$, that is, $\mathsf{a}, \tilde{\mathsf{a}}, \Theta$ cannot be simultaneously monomials. For the case of complex symmetry, replace the symmetry operator S by the complex symmetry operator $\mathbb{S}$ throughout.*

(S1) *Define Laurent polynomials A and B as in* (1.8), *and*

$$\mathsf{p}(z^2) := \gcd(\mathsf{A}(z), \mathsf{A}(-z), \mathsf{B}(z), \mathsf{B}(-z)), \quad \mathring{\mathsf{A}}(z) := \frac{\mathsf{A}(z)}{\mathsf{p}(z^2)}, \quad \mathring{\mathsf{B}}(z) := \frac{\mathsf{B}(z)}{\mathsf{p}(z^2)}. \tag{2.4}$$

Then $\mathsf{p}, \mathsf{A}, \mathsf{B}, \mathring{\mathsf{A}}, \mathring{\mathsf{B}}$ have [complex] symmetry (and real coefficients). Define ϵ_0, c_0, n_0 by

$$\epsilon_0 z^{c_0} := \mathsf{S}\mathring{\mathsf{A}}(z) \quad \text{and} \quad [c_0 - n_0, n_0] := \mathrm{fsupp}(\mathring{\mathsf{A}}). \tag{2.5}$$

(S2) *Select $\mathsf{d}, c_1, \epsilon_1, n_1, n_2, \epsilon_{\mathrm{len}}$ as follows:*

(1) *Select a Laurent polynomial d with [complex] symmetry (and real coefficients) such that*

$$\mathsf{d}(z) \mid \mathsf{D}(z) \quad \text{with} \quad \mathsf{D}(z^2) := [\mathring{\mathsf{A}}(z)\mathring{\mathsf{A}}(-z) - \mathring{\mathsf{B}}(z)\mathring{\mathsf{B}}(-z)]^\star. \tag{2.6}$$

Define $\epsilon_{\mathsf{d}}, c_{\mathsf{d}}, n_{\mathsf{d}}$ by $\epsilon_{\mathsf{d}} z^{c_{\mathsf{d}}} := \mathsf{S}\mathsf{d}(z)$ and $[c_{\mathsf{d}} - n_{\mathsf{d}}, n_{\mathsf{d}}] := \mathrm{fsupp}(\mathsf{d})$. We often restrict $c_{\mathsf{d}} \in \{0, 1\}$;

(2) *Select $c_1 \in \{\mathrm{odd}(c + n_b), \mathrm{odd}(c + n_b) + 2\}$. Define $c_2 := 2c_{\mathsf{d}} + 2 - c_1$.*

(3) *Select $\epsilon_1 = 1$ if $(-1)^{c_1}\epsilon_{\mathsf{d}} = -1$, otherwise, select $\epsilon_1 \in \{-1, 1\}$. Define $\epsilon_2 := (-1)^{c_1}\epsilon_{\mathsf{d}}\epsilon_1$.*

(4) *Select $\epsilon_{\mathrm{len}} = 0$ for the shortest filter support satisfying* (1.19), *otherwise, select $\epsilon_{\mathrm{len}} = 1$.*

(5) *Select $n_1 \in \mathbb{Z}$ satisfying $\frac{c_1}{2} \leqslant n_1 \leqslant \frac{c_1 - c_0}{2} + n_0 + \epsilon_{\mathrm{len}}$.*

(6) *Select $n_2 \in \mathbb{Z}$ satisfying $\max(\frac{c_2}{2}, 2n_{\mathsf{d}} + 1 - n_1) \leqslant n_2 \leqslant \frac{c_2 - c_0}{2} + n_0 + \epsilon_{\mathrm{len}}$.*

(S3) *Parameterize a filter $\mathring{\mathsf{b}}_1$ such that $\mathsf{S}\mathring{\mathsf{b}}_1(z) = \epsilon_1 z^{c_1}$ and $\mathrm{fsupp}(\mathring{\mathsf{b}}_1) = [c_1 - n_1, n_1]$. Find the unknown coefficients of $\mathring{\mathsf{b}}_1$ by solving a system X_1 of linear equations induced by $\mathcal{R}_1(z) = 0$ and*

$$\mathit{coeff}(\mathring{\tilde{\mathsf{b}}}_2^\star, z, j) = 0, \quad j = n_0 - n_2 - c_0 + 1 + \epsilon_{\mathrm{len}}, \ldots, n_0 + n_1 - c_0 - 2n_{\mathsf{d}} - 1, \tag{2.7}$$

where $\mathcal{R}_1$ and $\mathring{\tilde{\mathsf{b}}}_2^\star$ are Laurent polynomials uniquely determined, through long division using the divisor $\mathsf{d}(z^2)$, by $\mathrm{fsupp}(\mathcal{R}_1) \subseteq [2(c_{\mathsf{d}} - n_{\mathsf{d}}), 2n_{\mathsf{d}} - 1]$ and

$$\mathring{\mathsf{B}}^\star(z)\mathring{\mathsf{b}}_1(z) - \mathring{\mathsf{A}}^\star(z)\mathring{\mathsf{b}}_1(-z) = \mathsf{d}(z^2)z\mathring{\tilde{\mathsf{b}}}_2^\star(z) + \mathcal{R}_1(z). \tag{2.8}$$

If X_1 has no nontrivial solution, restart the algorithm from (S2) by selecting other choices of $\mathsf{d}, c_1, \epsilon_1, n_1, n_2, \epsilon_{\mathrm{len}}$.

(S4) *Parameterize a filter $\mathring{\mathsf{b}}_2$ such that $\mathsf{S}\mathring{\mathsf{b}}_2(z) = \epsilon_2 z^{c_2}$ and $\mathrm{fsupp}(\mathring{\mathsf{b}}_2) = [c_2 - n_2, n_2]$. Find the unknown coefficients of the filter $\mathring{\mathsf{b}}_2$ by solving a system X_2 of linear equations induced by $\mathcal{R}_2(z) = 0$ and*

$$\mathit{coeff}(\mathring{\tilde{\mathsf{b}}}_1^\star, z, j) = 0, \quad j = n_0 - n_1 - c_0 + 1 + \epsilon_{\mathrm{len}}, \ldots, n_0 + n_2 - c_0 - 2n_{\mathsf{d}} - 1, \tag{2.9}$$

where $\mathcal{R}_2$ and $\mathring{\tilde{\mathsf{b}}}_1^\star$ are Laurent polynomials uniquely determined, through long division using the divisor $\mathsf{d}(z^2)$, by $\mathrm{fsupp}(\mathcal{R}_2) \subseteq [2(c_{\mathsf{d}} - n_{\mathsf{d}}), 2n_{\mathsf{d}} - 1]$ and

$$\mathring{\mathsf{B}}^\star(z)\mathring{\mathsf{b}}_2(z) - \mathring{\mathsf{A}}^\star(z)\mathring{\mathsf{b}}_2(-z) = -\mathsf{d}(z^2)z\mathring{\tilde{\mathsf{b}}}_1^\star(z) + \mathcal{R}_2(z). \tag{2.10}$$

If X_2 has no nontrivial solution, restart the algorithm from (S2) by selecting other choices of $\mathsf{d}, c_1, \epsilon_1, n_1, n_2, \epsilon_{\text{len}}$.

(S5) *There must exist a complex number $\lambda \in \mathbb{C}$ such that*

$$\lambda \mathsf{d}(z^2) = z^{-1}[\mathring{\mathsf{b}}_1(z)\mathring{\mathsf{b}}_2(-z) - \mathring{\mathsf{b}}_1(-z)\mathring{\mathsf{b}}_2(z)]. \tag{2.11}$$

If $\lambda = 0$, then restart the algorithm from (S2) by selecting other choices of d, c_1, ϵ_1, $n_1, n_2, \epsilon_{\text{len}}$. *Otherwise, replace* $\mathring{\tilde{\mathsf{b}}}_1, \mathring{\tilde{\mathsf{b}}}_2$ *by* $\bar{\lambda}^{-1}\mathring{\tilde{\mathsf{b}}}_1, \bar{\lambda}^{-1}\mathring{\tilde{\mathsf{b}}}_2$, *respectively. Moreover,*

$$\mathsf{S}\mathring{\tilde{\mathsf{b}}}_1(z) = \epsilon_0\epsilon_1 z^{c_0+c_1}, \qquad \mathsf{S}\mathring{\tilde{\mathsf{b}}}_2(z) = \epsilon_0\epsilon_2 z^{c_0+c_2}. \tag{2.12}$$

(S6) *Find Laurent polynomials* q *and* $\tilde{\mathsf{q}}$ *having [complex] symmetry (and real coefficients) such that* $\mathsf{p}(z) = \tilde{\mathsf{q}}(z)\mathsf{q}^\star(z)$. *Define*

$$\mathsf{b}_1(z) := (1-z^{-1})^{n_b}\mathring{\mathsf{b}}_1(z)\mathsf{q}(z^2), \quad \mathsf{b}_2(z) := (1-z^{-1})^{n_b}\mathring{\mathsf{b}}_2(z)\mathsf{q}(z^2) \tag{2.13}$$

and

$$\tilde{\mathsf{b}}_1(z) := (1-z^{-1})^{n_{\tilde{b}}}\mathring{\tilde{\mathsf{b}}}_1(z)\tilde{\mathsf{q}}(z^2), \quad \tilde{\mathsf{b}}_2(z) := (1-z^{-1})^{n_{\tilde{b}}}\mathring{\tilde{\mathsf{b}}}_2(z)\tilde{\mathsf{q}}(z^2). \tag{2.14}$$

Then $(\{\tilde{a}; \tilde{b}_1, \tilde{b}_2\}, \{a; b_1, b_2\})_\Theta$ *is a dual framelet filter bank having [complex] symmetry (and real coefficients) such that* $\text{vm}(b_1) \geqslant n_b, \text{vm}(b_2) \geqslant n_b$, $\text{vm}(\tilde{b}_1) \geqslant n_{\tilde{b}}$, $\text{vm}(\tilde{b}_2) \geqslant n_{\tilde{b}}$, *and*

$$\max(\text{len}(b_1) + \text{len}(\tilde{b}_1), \text{len}(b_2) + \text{len}(\tilde{b}_2)) \leqslant \text{len}(a) + \text{len}(\tilde{a}) + 2\,\text{len}(\Theta) + 2\epsilon_{\text{len}}. \tag{2.15}$$

Proof. We first look at the symmetry property and filter supports of $\mathring{\mathsf{A}}$ and $\mathring{\mathsf{B}}$. By our assumption in (1.21), we have

$$\mathsf{S}(\Theta(z^2)\tilde{\mathsf{a}}(z)\mathsf{a}^\star(z)) = \epsilon_\Theta\epsilon\tilde{\epsilon}z^{2c_\Theta+\tilde{c}-c} = \epsilon_\Theta z^{c_\Theta} = \mathsf{S}\Theta(z). \tag{2.16}$$

Hence, both A and B have symmetry. Since $\text{len}(a)+\text{len}(\tilde{a})+\text{len}(\Theta) > 0$, it is trivial to see that

$$\text{len}(\Theta) < \text{len}(a) + \text{len}(\tilde{a}) + 2\,\text{len}(\Theta) = \text{len}(\Theta(z^2)\tilde{\mathsf{a}}(z)\mathsf{a}^\star(z)).$$

From the definition of A and B in (1.8), it follows from the above relation and (2.16) that $\text{fsupp}(\mathsf{A}) = \text{fsupp}(\mathsf{B})$. Since both A and B have symmetry, p has symmetry too. Define $\epsilon_{\mathsf{p}} z^{c_{\mathsf{p}}} := \mathsf{S}\mathsf{p}(z)$ to be the symmetry type of the Laurent polynomial p. By the definition of $\mathring{\mathsf{A}}$ and $\mathring{\mathsf{B}}$ in (2.4), we conclude that

$$\text{fsupp}(\mathring{\mathsf{A}}) = \text{fsupp}(\mathring{\mathsf{B}}) = [c_0 - n_0, n_0], \qquad \mathsf{S}\mathring{\mathsf{A}}(z) = \epsilon_0 z^{c_0}, \qquad \mathsf{S}\mathring{\mathsf{B}}(z) = \epsilon_{\mathring{\mathsf{B}}} z^{c_0} \tag{2.17}$$

with

$$\epsilon_0 = (-1)^{n_{\tilde{b}}+n_b}\epsilon_\Theta\epsilon_{\mathsf{p}}, \qquad \epsilon_{\mathring{\mathsf{B}}} = \epsilon_0(-1)^{c+n_b}, \qquad c_0 = c_\Theta + n_{\tilde{b}} - n_b - 2c_{\mathsf{p}}. \tag{2.18}$$

By (2.18) and $\mathsf{S}\mathring{\mathsf{b}}_1(z) = \epsilon_1 z^{c_1}$, we have

$$\mathsf{S}(\mathring{\mathsf{B}}^\star(z)\mathring{\mathsf{b}}_1(z)) = \epsilon_{\mathring{\mathsf{B}}}\epsilon_1 z^{-c_0}z^{c_1} = (-1)^{c+n_b}\epsilon_0\epsilon_1 z^{c_1-c_0}$$

and

$$\mathsf{S}(\mathring{\mathsf{A}}^\star(z)\mathring{\mathsf{b}}_1(-z)) = \epsilon_0\epsilon_1 z^{-c_0}(-z)^{c_1} = (-1)^{c_1}\epsilon_0\epsilon_1 z^{c_1-c_0}.$$

By item (2) of (S2), we have $(-1)^{c_1} = (-1)^{c+n_b}$. Therefore,

$$\mathsf{S}(\mathring{\mathsf{B}}^\star(z)\mathring{\mathsf{b}}_1(z)) = (-1)^{c+n_b}\epsilon_0\epsilon_1 z^{c_1-c_0} = (-1)^{c_1}\epsilon_0\epsilon_1 z^{c_1-c_0} = \mathsf{S}(\mathring{\mathsf{A}}^\star(z)\mathring{\mathsf{b}}_1(-z)).$$

Consequently, it follows from (2.8) and $\mathcal{R}_1 = 0$ that

$$\mathsf{S}(\mathsf{d}(z^2)z\mathring{\tilde{\mathsf{b}}}_2^\star(z)) = \mathsf{S}(\mathring{\mathsf{A}}^\star(z)\mathring{\mathsf{b}}_1(-z)) = (-1)^{c_1}\epsilon_0\epsilon_1 z^{c_1-c_0},$$

from which we conclude that $\mathring{\tilde{\mathsf{b}}}_2$ has symmetry such that

$$\begin{aligned}\mathsf{S}\mathring{\tilde{\mathsf{b}}}_2(z) &= \frac{\mathsf{S}(\mathsf{d}(z^2)z)}{\mathsf{S}(\mathring{\mathsf{A}}^\star(z)\mathring{\mathsf{b}}_1(-z))} = \frac{\epsilon_{\mathsf{d}} z^{2c_{\mathsf{d}}+2}}{(-1)^{c_1}\epsilon_0\epsilon_1 z^{c_1-c_0}} \\ &= (-1)^{c_1}\epsilon_{\mathsf{d}}\epsilon_1\epsilon_0 z^{2c_{\mathsf{d}}+2-c_1+c_0} = \epsilon_0\epsilon_2 z^{c_0+c_2},\end{aligned}$$

where we used the definition of c_2 and ϵ_2 in (S2). Hence, the second identity in (2.12) holds.

On the other hand, by (2.17) and (2.18), since $\mathrm{fsupp}(\mathring{\mathsf{A}}) = \mathrm{fsupp}(\mathring{\mathsf{B}})$, we obtain

$$\begin{aligned}\mathrm{fsupp}(\mathring{\mathsf{B}}^\star(z)\mathring{\mathsf{b}}_1(z)) &= \mathrm{fsupp}(\mathring{\mathsf{b}}_1) - \mathrm{fsupp}(\mathring{\mathsf{B}}) = \mathrm{fsupp}(\mathring{\mathsf{b}}_1) - \mathrm{fsupp}(\mathring{\mathsf{A}}) \\ &= \mathrm{fsupp}(\mathring{\mathsf{A}}^\star(z)\mathring{\mathsf{b}}_1(-z)).\end{aligned}$$

Hence, by $\mathrm{fsupp}(\mathsf{d}) = [c_{\mathsf{d}} - n_{\mathsf{d}}, n_{\mathsf{d}}]$ and $\mathrm{fsupp}(\mathsf{b}_1) = [c_1 - n_1, n_1]$, we deduce from (2.8) and $\mathcal{R}_1 = 0$ that

$$\mathrm{fsupp}(\mathring{\tilde{\mathsf{b}}}_2^\star) \subseteq [c_1 - n_0 - n_1 + 2n_{\mathsf{d}} - 2c_{\mathsf{d}} - 1, n_0 + n_1 - c_0 - 2n_{\mathsf{d}} - 1]. \tag{2.19}$$

By the proved symmetry property $\mathsf{S}\mathring{\tilde{\mathsf{b}}}_2(z) = \epsilon_0\epsilon_2 z^{c_0+c_2}$ and (2.7), using the definition $c_2 = 2c_{\mathsf{d}} + 2 - c_1$, we obtain

$$\mathrm{fsupp}(\mathring{\tilde{\mathsf{b}}}_2) \subseteq [c_0 - n_0 + n_2 - \epsilon_{\mathrm{len}}, c_2 + n_0 - n_2 + \epsilon_{\mathrm{len}}]. \tag{2.20}$$

By a similar argument and using (S4) instead of (S3), we can check that the first identity in (2.12) holds and

$$\mathrm{fsupp}(\mathring{\tilde{\mathsf{b}}}_1) \subseteq [c_0 - n_0 + n_1 - \epsilon_{\mathrm{len}}, c_1 + n_0 - n_1 + \epsilon_{\mathrm{len}}]. \tag{2.21}$$

Since $\mathcal{R}_1 = \mathcal{R}_2 = 0$, (2.8) and (2.10) together imply

$$\mathsf{d}(z^2)\begin{bmatrix} z\mathring{\tilde{\mathsf{b}}}_1^\star(z) \\ z\mathring{\tilde{\mathsf{b}}}_2^\star(z)\end{bmatrix} = \begin{bmatrix} \mathring{\mathsf{b}}_2(-z) & -\mathring{\mathsf{b}}_2(z) \\ -\mathring{\mathsf{b}}_1(-z) & \mathring{\mathsf{b}}_1(z)\end{bmatrix}\begin{bmatrix}\mathring{\mathsf{A}}^\star(z) \\ \mathring{\mathsf{B}}^\star(z)\end{bmatrix}. \tag{2.22}$$

Therefore, multiplying $\begin{bmatrix} \mathring{\mathsf{b}}_1(z) & \mathring{\mathsf{b}}_2(z) \\ \mathring{\mathsf{b}}_1(-z) & \mathring{\mathsf{b}}_2(-z)\end{bmatrix}$ from the left on both sides of (2.22), we have

$$\mathsf{d}(z^2)\begin{bmatrix} \mathring{\mathsf{b}}_1(z) & \mathring{\mathsf{b}}_2(z) \\ \mathring{\mathsf{b}}_1(-z) & \mathring{\mathsf{b}}_2(-z)\end{bmatrix}\begin{bmatrix} \mathring{\tilde{\mathsf{b}}}_1^\star(z) \\ \mathring{\tilde{\mathsf{b}}}_2^\star(z)\end{bmatrix} = \mathsf{D}_{\mathring{\mathsf{b}}}(z^2)\begin{bmatrix}\mathring{\mathsf{A}}^\star(z) \\ \mathring{\mathsf{B}}^\star(z)\end{bmatrix},$$

where $\mathsf{D}_{\mathring{\mathsf{b}}}(z^2) := z^{-1}[\mathring{\mathsf{b}}_1(z)\mathring{\mathsf{b}}_2(-z) - \mathring{\mathsf{b}}_1(-z)\mathring{\mathsf{b}}_2(z)]$. From the above identity we further deduce that

$$\begin{bmatrix} \mathring{\mathsf{b}}_1(z) & \mathring{\mathsf{b}}_2(z) \\ \mathring{\mathsf{b}}_1(-z) & \mathring{\mathsf{b}}_2(-z)\end{bmatrix}\begin{bmatrix} \mathring{\tilde{\mathsf{b}}}_1(z) & \mathring{\tilde{\mathsf{b}}}_2(z) \\ \mathring{\tilde{\mathsf{b}}}_1(-z) & \mathring{\tilde{\mathsf{b}}}_2(-z)\end{bmatrix}^\star = \frac{\mathsf{D}_{\mathring{\mathsf{b}}}(z^2)}{\mathsf{d}(z^2)}\begin{bmatrix} \mathring{\mathsf{A}}^\star(z) & \mathring{\mathsf{B}}^\star(-z) \\ \mathring{\mathsf{B}}^\star(z) & \mathring{\mathsf{A}}^\star(-z)\end{bmatrix}. \tag{2.23}$$

Since $\gcd(\mathring{\mathsf{A}}(z), \mathring{\mathsf{A}}(-z), \mathring{\mathsf{B}}(z), \mathring{\mathsf{B}}(-z)) = 1$ by (S1), we obtain $\gcd(\mathring{\mathsf{A}}^\star(z),\ \mathring{\mathsf{A}}^\star(-z),\ \mathring{\mathsf{B}}^\star(z),\ \mathring{\mathsf{B}}^\star(-z)) = 1$. Therefore, we must have $\mathsf{d}(z^2) \mid \mathsf{D}_{\mathring{\mathsf{b}}}(z^2)$. Thus, the above relation in (2.23) particularly implies

$$\mathring{\mathsf{b}}_1(z)\mathring{\tilde{\mathsf{b}}}_1^\star(z) + \mathring{\mathsf{b}}_2(z)\mathring{\tilde{\mathsf{b}}}_2^\star(z) = \frac{\mathsf{D}_{\mathring{\mathsf{b}}}(z^2)}{\mathsf{d}(z^2)}\mathring{\mathsf{A}}^\star(z). \tag{2.24}$$

Since $\mathrm{fsupp}(\mathring{b}_1) \subseteq [c_1 - n_1, n_1]$ and $\mathrm{fsupp}(\mathring{b}_2) \subseteq [c_2 - n_2, n_2]$, by (2.20) and (2.21), we must have

$$\begin{aligned} &\mathrm{fsupp}(\mathring{\mathsf{b}}_1(z)\overset{\circ}{\tilde{\mathsf{b}}}{}_1^\star(z)) \subseteq [-n_0 - \epsilon_{\mathrm{len}}, n_0 - c_0 + \epsilon_{\mathrm{len}}], \\ &\mathrm{fsupp}(\mathring{\mathsf{b}}_2(z)\overset{\circ}{\tilde{\mathsf{b}}}{}_2^\star(z)) \subseteq [-n_0 - \epsilon_{\mathrm{len}}, n_0 - c_0 + \epsilon_{\mathrm{len}}]. \end{aligned} \tag{2.25}$$

Consequently, it follows from (2.24) and $\mathrm{fsupp}(\mathring{\mathsf{A}}^\star) = [-n_0, n_0 - c_0]$ that

$$\mathrm{fsupp}(\mathsf{D}_{\mathring{\mathsf{b}}}(z^2)/\mathsf{d}(z^2)) \subseteq [-\epsilon_{\mathrm{len}}, \epsilon_{\mathrm{len}}].$$

Since $\epsilon_{\mathrm{len}} \in \{0, 1\}$, this forces $\mathrm{fsupp}(\mathsf{D}_{\mathring{\mathsf{b}}}(z^2)/\mathsf{d}(z^2)) \subseteq \{0\}$. That is, $\lambda := \mathsf{D}_{\mathring{\mathsf{b}}}(z^2)/\mathsf{d}(z^2)$ must be a constant. In other words, $\mathsf{D}_{\mathring{\mathsf{b}}}(z) = \lambda \mathsf{d}(z)$. By our assumption $\lambda \neq 0$, after replacing $\overset{\circ}{\tilde{\mathsf{b}}}{}_1, \overset{\circ}{\tilde{\mathsf{b}}}{}_2$ by $\bar{\lambda}^{-1}\overset{\circ}{\tilde{\mathsf{b}}}{}_1, \bar{\lambda}^{-1}\overset{\circ}{\tilde{\mathsf{b}}}{}_2$, respectively, we must have

$$\begin{bmatrix} \overset{\circ}{\tilde{\mathsf{b}}}{}_1(z) & \overset{\circ}{\tilde{\mathsf{b}}}{}_2(z) \\ \overset{\circ}{\tilde{\mathsf{b}}}{}_1(-z) & \overset{\circ}{\tilde{\mathsf{b}}}{}_2(-z) \end{bmatrix} \begin{bmatrix} \mathring{\mathsf{b}}_1(z) & \mathring{\mathsf{b}}_2(z) \\ \mathring{\mathsf{b}}_1(-z) & \mathring{\mathsf{b}}_2(-z) \end{bmatrix}^\star = \begin{bmatrix} \mathring{\mathsf{A}}(z) & \mathring{\mathsf{B}}(z) \\ \mathring{\mathsf{B}}(-z) & \mathring{\mathsf{A}}(-z) \end{bmatrix}. \tag{2.26}$$

For the case of complex symmetry, since $\lambda = \mathsf{D}_{\mathring{\mathsf{b}}}(z^2)/\mathsf{d}(z^2)$ and since both $\mathsf{D}_{\mathring{\mathsf{b}}}$ and d have complex symmetry, the constant λ must have complex symmetry. This is only possible for $\lambda \in \mathbb{R}$ or $i\lambda \in \mathbb{R}$. Hence, complex symmetry will be preserved after replacing $\overset{\circ}{\tilde{\mathsf{b}}}{}_1, \overset{\circ}{\tilde{\mathsf{b}}}{}_2$ by $\bar{\lambda}^{-1}\overset{\circ}{\tilde{\mathsf{b}}}{}_1, \bar{\lambda}^{-1}\overset{\circ}{\tilde{\mathsf{b}}}{}_2$, respectively. If all filters have real coefficients, then λ must be a real number and therefore, all the constructed high-pass filters must have real coefficients too.

Now it is straightforward to check that $(\{\tilde{a}; \tilde{b}_1, \tilde{b}_2\}, \{a; b_1, b_2\})_\Theta$ is a dual framelet filter bank with [complex] symmetry (and real coefficients).

By (2.17) and(2.18), we have

$$\mathrm{len}(\mathring{\mathsf{A}}) = \mathrm{len}(\mathring{\mathsf{B}}) = 2n_0 - c_0 = 2n_0 + n_b - n_{\tilde{b}} + 2c_{\mathsf{p}} - c_\Theta.$$

We now check the inequality in (2.15). It follows from (2.13), (2.14), and (2.25) that

$$\begin{aligned} \mathrm{len}(b_1) + \mathrm{len}(\tilde{b}_1) &= n_b + n_{\tilde{b}} + 2\,\mathrm{len}(\mathsf{p}) + \mathrm{len}(\mathring{b}_1) + \mathrm{len}(\overset{\circ}{\tilde{b}}_1) \\ &\leqslant n_b + n_{\tilde{b}} + 2\,\mathrm{len}(\mathsf{p}) + 2n_0 - c_0 + 2\epsilon_{\mathrm{len}}. \end{aligned}$$

By the definition of B in (1.8), we have

$$2\,\mathrm{len}(\mathsf{p}) + 2n_0 - c_0 = 2\,\mathrm{len}(\mathsf{p}) + \mathrm{len}(\mathring{\mathsf{B}}) = \mathrm{len}(\mathsf{B}) = \mathrm{len}(a) + \mathrm{len}(\tilde{a}) + 2\,\mathrm{len}(\Theta) - n_b - n_{\tilde{b}}.$$

We deduce from the above inequalities that

$$\mathrm{len}(b_1) + \mathrm{len}(\tilde{b}_1) \leqslant \mathrm{len}(a) + \mathrm{len}(\tilde{a}) + 2\,\mathrm{len}(\Theta) + 2\epsilon_{\mathrm{len}}.$$

Similarly, we can verify

$$\mathrm{len}(b_2) + \mathrm{len}(\tilde{b}_2) \leqslant \mathrm{len}(a) + \mathrm{len}(\tilde{a}) + 2\,\mathrm{len}(\Theta) + 2\epsilon_{\mathrm{len}}.$$

Hence, (2.15) holds. This completes the proof. □

We now make some remarks on Algorithm 1. Since a Laurent polynomial d is selected in advance in (S2), the unique Laurent polynomials $\overset{\circ}{\tilde{\mathsf{b}}}{}_2$ and $\mathcal{R}_1$ in (2.8) and $\overset{\circ}{\tilde{\mathsf{b}}}{}_1$ and $\mathcal{R}_2$ in (2.10) can be easily obtained by long division. Moreover, since $\mathring{\mathsf{A}}$ and $\mathring{\mathsf{B}}$ are given, all the coefficients in $\overset{\circ}{\tilde{\mathsf{b}}}{}_2$ are linear combinations of the unknown coefficients in $\mathring{b}_1$. Therefore, the system X_1 consists of linear equations involving only the unknown coefficients from $\mathring{b}_1$. Similarly, the system X_2 consists of linear

equations involving only the unknown coefficients from $\mathring{b}_2$. Therefore, we only have to solve two small systems X_1 and X_2 of linear equations in Algorithm 1. Hence, whether there exists a nontrivial solution to X_1 or X_2 can be completely determined. For the case $\Theta = 1$, it is easy to check that we must have $\mathsf{p}(z) = 1$ in (2.4). Since we often take n_b to be the largest possible integer, without any loss of generality, we can always take $\epsilon_1 = 1$ in (S2) since $|\mathring{\mathsf{b}}_1(1)| + |\mathring{\mathsf{b}}_2(1)| \neq 0$. It is pretty trivial to see that the first particular construction in (1.10) and (1.11) is covered by Algorithm 1 by selecting $\mathsf{d} = 1, c_1 = 0, \epsilon_1 = 1, n_1 = 0, n_2 = 1, \epsilon_{\rm len} = 0$ and $\mathsf{q} = \tilde{\mathsf{q}} = 1$, while the second particular construction in (1.12) and (1.13) is covered by Algorithm 1 by selecting $\mathsf{d} = 1, c_1 = 1, \epsilon_1 = 1, n_1 = 1, n_2 = 1, \epsilon_{\rm len} = 0$ and $\mathsf{q} = \tilde{\mathsf{q}} = 1$.

Note that if two Laurent polynomials d in (2.6) differ only by a multiplication of a monomial while other choices of $\epsilon_1, c_1, n_1, n_2, \epsilon_{\rm len}$ are the same, then the corresponding constructed dual framelet filter banks are essentially the same. Since all the possible choices of $\mathsf{d}, c_1, \epsilon_1, n_1, n_2, \epsilon_{\rm len}$ in (S2) are finite, there are essentially only finitely many cases involved in Algorithm 1. We shall explain the seemingly complicated relations and constraints on parameters in (S2) in Section 5. We shall see then that Algorithm 1 is capable of finding all possible dual framelet filter banks $(\{\tilde{a}; \tilde{b}_1, \tilde{b}_2\}, \{a; b_1, b_2\})_\Theta$ having [complex] symmetry (and real coefficients) and having the shortest possible filter support satisfying (1.18) with $\epsilon_{\rm len} \in \{0, 1\}$ from any given filters $a, \tilde{a}, \Theta$ with [complex] symmetry.

In the rest of this section, we address the issue on the construction of a desirable moment correcting filter Θ from a given pair of low-pass filters. The following result (also cf. [4, 5]) guarantees the existence of a desired moment correcting filter Θ having [complex] symmetry and the shortest possible filter support.

Lemma 1. *Let u be a filter having symmetry $\mathsf{S}\mathsf{u}(z) = z^c$ (or complex symmetry $\mathbb{S}\mathsf{u}(z) = z^c$) and $\mathsf{u}(1) = 1$. For any nonnegative integer n, there exists a filter $\Theta \in l_0(\mathbb{Z})$ such that*

$$\Theta(1) = 1 \qquad \text{and} \qquad \Theta(z) - \Theta(z^2)\mathsf{u}(z) = \mathcal{O}(|1-z|^n), \qquad z \to 1, \tag{2.27}$$

$\mathsf{S}\Theta(z) = z^{-c}$ (or $\mathbb{S}\Theta(z) = z^{-c}$), and $\operatorname{fsupp}(\Theta) \subseteq [-c-m, m]$ with $m := \lceil \frac{n-c-1}{2} \rceil$. Moreover, if u has real coefficients, then Θ has real coefficients.

Proof. By [9, Lemma 2.2], there always exists a unique filter Θ with $\operatorname{fsupp}(\Theta) \subseteq [m-n+1, m]$ such that (2.27) holds. For the case of symmetry, we replace Θ by $[\Theta(z) + z^{-c}\Theta(z^{-1})]/2$; for the case of complex symmetry, we replace Θ by $[\Theta(z) + z^{-c}\Theta^\star(z)]/2$. Noting that $-c-m \leqslant m-n+1$, we see that Θ is a desired moment correcting filter satisfying all the requirements. □

Since (2.27) induces a system of linear equations, a desired moment correcting filter Θ in Lemma 1 can be easily obtained by solving a linear system which is guaranteed to have a solution.

When a moment correcting filter Θ is given in advance, we can also design a filter $\tilde{a}$ derived from a given low-pass filter a such that (2.27) holds. Since a general moment correcting filter does not introduce any additional difficulty, here we only discuss the commonly used case $\Theta = 1$. By [9, Lemma 2.2], the following result can be proved in the same way as in Lemma 1.

Proposition 2. *Let M, N be positive integers and $a \in l_0(\mathbb{Z})$ be a filter with $\mathsf{a}(1) = 1$. For any subset Λ of $\mathbb{Z}$ such that the cardinality of Λ is N, there exists a unique*

solution $\{t_k\}_{k\in\Lambda}$ *to the system of linear equations induced by*

$$\tilde{\mathsf{a}}(z)\mathsf{a}^\star(z) = 1 + \mathcal{O}(|z-1|^N), \qquad z \to 1, \quad \textit{with} \quad \tilde{\mathsf{a}}(z) := (1+z)^M \sum_{k\in\Lambda} t_k z^k. \tag{2.28}$$

If the filter a *is real-valued, then so is the filter* $\tilde{a}$. *If in addition* a *has symmetry* $\mathsf{S}\mathsf{a}(z) = z^c$ *(or complex symmetry* $\mathbb{S}\mathsf{a}(z) = z^c$*) for some* $c \in \mathbb{Z}$, *take* $\Lambda = \{\lceil \frac{c-M+1-N}{2} \rceil, \ldots, \lfloor \frac{c-M-1+N}{2} \rfloor\}$ *provided that* $c+N+M$ *is an odd integer (this requirement can be dropped if* N *is even and either* a *is real-valued or* a *has symmetry), then* $\mathsf{S}\tilde{\mathsf{a}}(z) = z^c$ *(or* $\mathbb{S}\tilde{\mathsf{a}}(z) = z^c$*).*

Using Lemma 1 or Proposition 2, we shall present several examples of dual framelet filter banks in Section 4 to illustrate Algorithm 1.

If $\tilde{a} = a$ and if the necessary and sufficient condition in [13, 14] is satisfied, then Algorithm 1 is able to obtain all the tight framelet filter banks with symmetry and with two high-pass filters as special cases. We shall discuss this issue in detail in Section 3. Algorithm 1 can be also straightforwardly modified to handle dual framelet filter banks without symmetry but with the shortest possible filter support. For the convenience of the reader, a detailed algorithm is provided as follows.

Algorithm 2. *Let* $a, \tilde{a}, \Theta \in l_0(\mathbb{Z})$ *and* $n_b, n_{\tilde{b}} \in \mathbb{N} \cup \{0\}$ *satisfying* (2.3).

(S1) *Define* A *and* B *as in* (1.8) *and* $\mathsf{p}, \mathring{\mathsf{A}}, \mathring{\mathsf{B}}$ *as in* (2.4). *Define* $[m_0, n_0] := \mathrm{fsupp}(\mathring{\mathsf{B}}^\star)$ *and assume that* $\mathrm{fsupp}(\mathring{\mathsf{A}}) = \mathrm{fsupp}(\mathring{\mathsf{B}})$.

(S2) *Select* $\epsilon_{\mathrm{len}}, s_1, s_2 \in \{0,1\}$ *and* $\ell_1, \ell_2 \in \mathbb{N} \cup \{0\}$ *such that* $\max(\ell_1, \ell_2) \leqslant n_0 - m_0 + 2\epsilon_{\mathrm{len}}$. *Select a polynomial* d *satisfying* (2.6) *and* $\lceil \frac{s_1+s_2-1}{2} \rceil \leqslant m_{\mathsf{d}} \leqslant n_{\mathsf{d}} \leqslant \lfloor \frac{s_1+s_2+\ell_1+\ell_2-1}{2} \rfloor$, *where* $[m_{\mathsf{d}}, n_{\mathsf{d}}] := \mathrm{fsupp}(\mathsf{d})$.

(S3) *Parameterize a filter* $\mathring{\mathsf{b}}_1$ *by* $\mathring{\mathsf{b}}_1(z) := z^{s_1} \sum_{j=0}^{\ell_1} t_j z^j$. *Find the unknown coefficients* $\{t_0, \ldots, t_{\ell_1}\}$ *by solving a system* X_1 *of linear equations induced by* $\mathcal{R}_1(z) = 0$ *and*

$$\mathit{coeff}(\mathring{\tilde{\mathsf{b}}}_2^\star, z, j) = 0, \quad \begin{aligned} & j = m_0 + s_1 - 2m_{\mathsf{d}} - 1, \ldots, m_0 - s_2 - \epsilon_{\mathrm{len}} - 1 \quad \textit{and} \\ & j = n_0 - s_2 - \ell_2 + \epsilon_{\mathrm{len}} + 1, \ldots, n_0 + s_1 + \ell_1 - 2n_{\mathsf{d}} - 1, \end{aligned}$$

where $\mathcal{R}_1$ *and* $\mathring{\tilde{\mathsf{b}}}_2^\star$ *are Laurent polynomials uniquely determined by* $\mathrm{fsupp}(\mathcal{R}_1) \subseteq [2m_{\mathsf{d}}, 2n_{\mathsf{d}} - 1]$ *and* (2.8).

(S4) *Parameterize a filter* $\mathring{\mathsf{b}}_2$ *by* $\mathring{\mathsf{b}}_2(z) := z^{s_2} \sum_{j=0}^{\ell_2} \tilde{t}_j z^j$. *Find the unknown coefficients* $\{\tilde{t}_0, \ldots, \tilde{t}_{\ell_2}\}$ *by solving a system* X_2 *of linear equations induced by* $\mathcal{R}_2(z) = 0$ *and*

$$\mathit{coeff}(\mathring{\tilde{\mathsf{b}}}_1^\star, z, j) = 0, \quad \begin{aligned} & j = m_0 + s_2 - 2m_{\mathsf{d}} - 1, \ldots, m_0 - s_1 - \epsilon_{\mathrm{len}} - 1 \quad \textit{and} \\ & j = n_0 - s_1 - \ell_1 + \epsilon_{\mathrm{len}} + 1, \ldots, n_0 + s_2 + \ell_2 - 2n_{\mathsf{d}} - 1, \end{aligned}$$

where $\mathcal{R}_2$ *and* $\mathring{\tilde{\mathsf{b}}}_1^\star$ *are Laurent polynomials uniquely determined by* $\mathrm{fsupp}(\mathcal{R}_2) \subseteq [2m_{\mathsf{d}}, 2n_{\mathsf{d}} - 1]$ *and* (2.10). *If both* X_1 *and* X_2 *have only trivial solution, restart the algorithm from (S2) by selecting other choices of* $\mathsf{d}, \epsilon_{\mathrm{len}}, s_1, s_2, \ell_1, \ell_2$.

(S5) *There must exist* $\lambda \in \mathbb{C}$ *such that* (2.11) *holds. If* $\lambda = 0$, *then restart the algorithm from (S2) by selecting other choices of* d, ϵ_{len}, s_1, s_2, ℓ_1, ℓ_2. *Otherwise, replace* $\mathring{\tilde{\mathsf{b}}}_1, \mathring{\tilde{\mathsf{b}}}_2$ *by* $\bar{\lambda}^{-1}\mathring{\tilde{\mathsf{b}}}_1, \bar{\lambda}^{-1}\mathring{\tilde{\mathsf{b}}}_2$, *respectively.*

(S6) *Find Laurent polynomials* q *and* $\tilde{\mathsf{q}}$ *such that* $\mathsf{p}(z) = \tilde{\mathsf{q}}(z)\mathsf{q}^\star(z)$. *Define* $\mathsf{b}_1, \mathsf{b}_2$ *as in* (2.13) *and* $\tilde{\mathsf{b}}_1, \tilde{\mathsf{b}}_2$ *as in* (2.14).

Then $(\{\tilde{a}; \tilde{b}_1, \tilde{b}_2\}, \{a; b_1, b_2\})_\Theta$ *is a dual framelet filter bank satisfying* (2.15).

Proof. Note that $\mathrm{fsupp}(\mathring{\mathsf{b}}_1) \subseteq [s_1, s_1 + \ell_1]$ and $\mathrm{fsupp}(\mathring{\mathsf{b}}_2) \subseteq [s_2, s_2 + \ell_2]$. (2.10) implies that $\mathrm{fsupp}(\mathring{\tilde{\mathsf{b}}}_1^\star) \subseteq [m_0 + s_2 - 2m_{\mathsf{d}} - 1, n_0 + s_2 + \ell_2 - 2n_{\mathsf{d}} - 1]$ and (2.8) implies that $\mathrm{fsupp}(\mathring{\tilde{\mathsf{b}}}_2^\star) \subseteq [m_0 + s_1 - 2m_{\mathsf{d}} - 1, n_0 + s_1 + \ell_1 - 2n_{\mathsf{d}} - 1]$. The constraint on the coefficients of $\mathring{\tilde{\mathsf{b}}}_1^\star$ in (S4) implies $\mathrm{fsupp}(\mathring{\tilde{\mathsf{b}}}_1^\star) \subseteq [m_0 - s_1 - \epsilon_{\mathrm{len}}, n_0 - s_1 - \ell_1 + \epsilon_{\mathrm{len}}]$, and the constraint on the coefficients of $\mathring{\tilde{\mathsf{b}}}_2^\star$ in (S3) implies $\mathrm{fsupp}(\mathring{\tilde{\mathsf{b}}}_2^\star) \subseteq [m_0 - s_2 - \epsilon_{\mathrm{len}}, n_0 - s_2 - \ell_2 + \epsilon_{\mathrm{len}}]$. Therefore, $\mathrm{fsupp}(\mathring{\mathsf{b}}_1(z)\mathring{\tilde{\mathsf{b}}}_1^\star(z)) \subseteq [m_0 - \epsilon_{\mathrm{len}}, n_0 + \epsilon_{\mathrm{len}}]$ and $\mathrm{fsupp}(\mathring{\mathsf{b}}_2(z)\mathring{\tilde{\mathsf{b}}}_2^\star(z)) \subseteq [m_0 - \epsilon_{\mathrm{len}}, n_0 + \epsilon_{\mathrm{len}}]$. By the same proof of Algorithm 1, $(\{\tilde{a}; \tilde{b}_1, \tilde{b}_2\}, \{a; b_1, b_2\})_\Theta$ is a dual framelet filter bank satisfying (2.15). $\square$

3. Algorithm for constructing symmetric tight framelet filter banks

Since a tight framelet filter bank is a special case of dual framelet filter banks by using the same set of filters for both analysis and synthesis, Algorithm 1 developed in Section 2 allows us to obtain all symmetric tight framelet filter banks as special cases. To reduce the complexity of Algorithm 1 for constructing symmetric tight framelet filter banks, in this section, we derive from Algorithm 1 a simple algorithm for constructing all tight framelet filter banks having [complex] symmetry (and real coefficients) from any given filters a and Θ with [complex] symmetry.

For $x \in \mathbb{R}$, its sign function is defined to be $\mathrm{sgn}(x) := 1$ if $x > 0$, $\mathrm{sgn}(0) = 0$, and $\mathrm{sgn}(x) = -1$ if $x < 0$. Recall that $\{a; b_1, b_2\}_\Theta$ is a tight framelet filter bank if

$$\Theta(z^2)\mathsf{a}(z)\mathsf{a}^\star(z) + \mathsf{b}_1(z)\mathsf{b}_1^\star(z) + \mathsf{b}_2(z)\mathsf{b}_2^\star(z) = \Theta(z) \tag{3.1}$$

and

$$\Theta(z^2)\mathsf{a}(z)\mathsf{a}^\star(-z) + \mathsf{b}_1(z)\mathsf{b}_1^\star(-z) + \mathsf{b}_2(z)\mathsf{b}_2^\star(-z) = 0. \tag{3.2}$$

We first show that the symmetry types of the high-pass filters b_1 and b_2 in a tight framelet filter bank $\{a; b_1, b_2\}_\Theta$ with [complex] symmetry are uniquely determined by the filters a and Θ.

Theorem 3. *Let* $\{a; b_1, b_2\}_\Theta$ *be a tight framelet filter bank such that all the filters* $a, b_1, b_2, \Theta \in l_0(\mathbb{Z})$ *are not identically zero and have symmetry:*

$$\mathsf{S}\Theta(z) = 1, \qquad \mathsf{S}\mathsf{a}(z) = \epsilon z^c, \qquad \mathsf{S}\mathsf{b}_1(z) = \epsilon_1 z^{c_1}, \qquad \mathsf{S}\mathsf{b}_2(z) = \epsilon_2 z^{c_2} \tag{3.3}$$

[or (3.3) *holds with* S *being replaced by the complex symmetry operator* $\mathbb{S}$*]. If*

$$\mathrm{len}(b_2) \leqslant \mathrm{len}(b_1) \leqslant \mathrm{len}(a) + \mathrm{len}(\Theta) \neq 0, \tag{3.4}$$

then up to a trivial switch of b_1 *and* b_2 *for the case* $\mathrm{len}(b_1) = \mathrm{len}(b_2)$*, the symmetry centers* c_1 *and* c_2 *are essentially uniquely determined by*

$$\tfrac{c_1}{2} - (\tfrac{c}{2} - n_\Theta) \in 2\mathbb{Z}, \qquad \tfrac{c_2}{2} = n_{\mathcal{M}} + 1 - \tfrac{c_1}{2}, \tag{3.5}$$

and for the case of symmetry (or real coefficients), ϵ_1 *and* ϵ_2 *are uniquely determined by*

$$\epsilon_1 = -\epsilon sgn(\Theta(n_\Theta)), \qquad \epsilon_2 = (-1)^c \epsilon_1 sgn(\lambda), \tag{3.6}$$

where $\Theta(n_\Theta)$ is the leading coefficient of Θ (that is, $\Theta(n_\Theta) \neq 0$ and $\Theta(k) = 0$ for all $k > n_\Theta$), $\lambda z^{2n_{\mathcal{M}}}$ is the leading term of the Laurent polynomial $\det(\mathcal{M}_{a,a,\Theta}(z))$. *Moreover, if $c + n_{\mathcal{M}}$ is an even integer, then* $\operatorname{len}(b_2) < \operatorname{len}(b_1) = \operatorname{len}(a) + \operatorname{len}(\Theta)$.

Proof. To verify the claims in (3.5) and (3.6), we simply compare the leading coefficients in the two equations in (3.1) and (3.2). Since $\Theta(n_\Theta)$ is the leading coefficients of Θ and $\mathsf{S}\Theta(z) = 1$, we have $\operatorname{fsupp}(\Theta) = [-n_\Theta, n_\Theta]$. Define $n, n_1, n_2 \in \mathbb{Z}$ by

$$[c - n, n] := \operatorname{fsupp}(a), \qquad [c_1 - n_1, n_1] := \operatorname{fsupp}(b_1), \qquad [c_2 - n_2, n_2] := \operatorname{fsupp}(b_2).$$

For the case of complex symmetry, we define

$$\lambda_0 := \epsilon\Theta(n_\Theta)(a(n))^2, \qquad \lambda_1 := \epsilon_1(b_1(n_1))^2, \qquad \lambda_2 := \epsilon_2(b_2(n_2))^2.$$

For the case of symmetry or real coefficients, we define

$$\lambda_0 := \epsilon\Theta(n_\Theta)|a(n)|^2, \qquad \lambda_1 := \epsilon_1|b_1(n_1)|^2, \qquad \lambda_2 := \epsilon_2|b_2(n_2)|^2. \tag{3.7}$$

The leading terms of each addent in (3.1) are

$$\lambda_0 z^{2n_\Theta+2n-c}, \qquad \lambda_1 z^{2n_1-c_1}, \qquad \lambda_2 z^{2n_2-c_2}, \qquad \Theta(n_\Theta)z^{n_\Theta}, \tag{3.8}$$

and the leading terms of each addent in (3.2) are

$$(-1)^{n-c}\lambda_0 z^{2n_\Theta+2n-c}, \qquad (-1)^{n_1-c_1}\lambda_1 z^{2n_1-c_1}, \qquad (-1)^{n_2-c_2}\lambda_2 z^{2n_2-c_2}, \tag{3.9}$$

respectively. Note that all $\lambda_0, \lambda_1, \lambda_2$ are nonzero. Our assumption in (3.4) becomes $2n_2 - c_2 \leqslant 2n_1 - c_1 \leqslant 2n - c + 2n_\Theta \neq 0$, from the last relation we must have $n_\Theta < 2n-c+2n_\Theta$ (otherwise, $\operatorname{len}(a) = \operatorname{len}(\Theta) = 0$). Since the perfect reconstruction condition in (3.1) and (3.2) must hold, the above inequalities imply that we have two cases to consider. By a simple argument (see [13, 14]) on symmetry types of addents in (3.2), we can easily deduce from (3.2) that

$$c_1 - c \in 2\mathbb{Z} \qquad \text{and} \qquad c_2 - c \in 2\mathbb{Z}. \tag{3.10}$$

Case 1: $2n_2 - c_2 = 2n_1 - c_1 = 2n - c + 2n_\Theta$. By (3.10), the following two equations must hold:

$$\lambda_0 + \lambda_1 + \lambda_2 = 0, \qquad \lambda_0 + (-1)^{n_1-n}\lambda_1 + (-1)^{n_2-n}\lambda_2 = 0, \tag{3.11}$$

from which we deduce that $(-1)^{n_1-n} = (-1)^{n_2-n} = 1$; otherwise, the above two equations will force at least one of $\lambda_0, \lambda_1, \lambda_2$ to be zero. Hence, $n_1 - n \in 2\mathbb{Z}$ and $n_2 - n \in 2\mathbb{Z}$. Now we deduce from $2n_1 - c_1 = 2n - c + 2n_\Theta$ that

$$c_1 = c + 2n_1 - 2n - 2n_\Theta = c - 2n_\Theta + 4k$$

with $k := (n_1 - n)/2 \in \mathbb{Z}$. Hence, $c_1 - (c - 2n_\Theta) \in 4\mathbb{Z}$. Similarly, we also have $c_2 - (c - 2n_\Theta) \in 4\mathbb{Z}$. Since $\{a; b_1, b_2\}_\Theta$ is a tight framelet filter bank, we must have $\Theta(z) \geqslant 0$ for all $z \in \mathbb{T} := \{\zeta \in \mathbb{C} \ : \ |\zeta| = 1\}$. Consequently, $\mathbb{S}\Theta(z) = 1$. If all filters have real coefficients or have symmetry, by $\mathsf{S}\Theta(z) = \mathbb{S}\Theta(z) = 1$, the moment correcting filter Θ must have real coefficients. Therefore, by (3.7), all $\lambda_0, \lambda_1, \lambda_2$ are real numbers. Now it follows from the first identity of (3.11) that at least one of the signs of λ_1 and λ_2 must be different to that of λ_0. Without loss of generality, we assume $\lambda_0\lambda_1 < 0$. That is, we must have $\epsilon_1 = -\operatorname{sgn}(\lambda_0) = -\epsilon\operatorname{sgn}(\Theta(n_\Theta))$.

Case 2: $2n_2 - c_2 < 2n_1 - c_1 = 2n - c + 2n_\Theta$. By (3.10), the following two equations must hold:

$$\lambda_0 + \lambda_1 = 0, \qquad \lambda_0 + (-1)^{n_1-n}\lambda_1 = 0, \tag{3.12}$$

from which we must have $(-1)^{n_1-n} = 1$, that is, $n_1 - n \in 2\mathbb{Z}$. We deduce from $2n_1 - c_1 = 2n - c + 2n_\Theta$ that

$$c_1 = c + 2n_1 - 2n - 2n_\Theta = c - 2n_\Theta + 4k,$$

where $k := (n_1 - n)/2 \in \mathbb{Z}$. Hence, we also have $c_1 - (c - 2n_\Theta) \in 4\mathbb{Z}$. If all the filters have symmetry or real coefficients, then it follows from the first equation of (3.12) that $\lambda_1 = -\lambda_0$, that is, we must have $\epsilon_1 = -\epsilon \mathrm{sgn}(\Theta(n_\Theta))$.

We now investigate the property of c_2 and ϵ_2. By the perfect reconstruction condition in (1.1) with $s = 2$, we have $\mathsf{D}(z^2)\mathsf{D}^\star(z^2) = \det(\mathcal{M}_{a,a,\Theta}(z))$, where $\mathsf{D}(z^2) := z^{-1}[\mathsf{b}_1(z)\mathsf{b}_2(-z) - \mathsf{b}_1(-z)\mathsf{b}_2(z)]$. By (3.10), we have $\mathsf{S}(\mathsf{b}_1(z)\mathsf{b}_2(-z)) = \mathsf{S}(\mathsf{b}_1(-z)\mathsf{b}_2(z))$. Hence, it follows from the definition of D that $\mathsf{SD}(z) = \epsilon_1\epsilon_2(-1)^c z^{\frac{c_1+c_2}{2}-1}$, where we used (3.10). From the relation $\mathsf{D}(z^2)\mathsf{D}^\star(z^2) = \det(\mathcal{M}_{a,a,\Theta}(z))$ and comparing their leading coefficients, we must have $c_1 + c_2 - 2 = 2n_{\mathcal{M}}$ and $\epsilon_1\epsilon_2(-1)^c = \mathrm{sgn}(\lambda)$. Hence, the second relation in (3.5) holds and $\epsilon_2 = (-1)^c\epsilon_1\mathrm{sgn}(\lambda)$.

If $c + n_{\mathcal{M}}$ is an even integer, by (3.10), we conclude that $\frac{c_2}{2} - (\frac{c}{2} - n_\Theta) \notin 2\mathbb{Z}$. Therefore, Case 1 cannot happen. As a consequence, we must have Case 2, that is, $\mathrm{len}(b_2) < \mathrm{len}(b_1) = \mathrm{len}(a) + \mathrm{len}(\Theta)$. □

If $\{a; b_1, b_2\}_\Theta$ is a tight framelet filter bank and $\lambda_1, \lambda_2 \in \mathbb{T}$, then it is trivial to check that $\{a; \lambda_1 b_1(\cdot - k_1), \lambda_2 b_2(\cdot - k_2)\}_\Theta$ is also a tight framelet filter bank provided that k_1 and k_2 are even integers. However, if all filters are not identically zero in a tight framelet filter bank $\{a; b_1, b_2\}_\Theta$, then $\{a; \lambda_1 b_1(\cdot - k_1), \lambda_2 b_2(\cdot - k_2)\}_\Theta$ cannot be a tight framelet filter bank if at least one of k_1 and k_2 is an odd integer. Therefore, Theorem 3 tells us that up to even integer shifts the symmetry types of the high-pass filters b_1 and b_2 in a tight framelet filter bank $\{a; b_1, b_2\}_\Theta$ with [complex] symmetry and the shortest possible filter supports are uniquely determined by the filters a and Θ, that is, $c_1 = c - 2n_\Theta$ and $c_2 = 2n_{\mathcal{M}} + 2 - c_1$. For the case of complex symmetry, since $\mathbb{S}(i\mathsf{b}_1) = -\mathbb{S}\mathsf{b}_1$, we can always assume $\epsilon_1 = \epsilon_2 = 1$. That is, for the case of complex symmetry, there is no essential difference for a filter being complex symmetric or complex antisymmetric.

We now derive from Algorithm 1 a simple algorithm for constructing all symmetric tight framelet filter banks with the shortest possible filter supports.

Algorithm 3. *Let $a, \Theta \in l_0(\mathbb{Z})$ be filters having [complex] symmetry (and real coefficients) such that $\mathsf{S}a(z) = \epsilon z^c$ with $\epsilon \in \{-1, 1\}$ and $c \in \mathbb{Z}$, $\mathsf{S}\Theta(z) = 1$, $\Theta(z) \geqslant 0$ for all $z \in \mathbb{T}$, and*

$$0 \leqslant n_b \leqslant \mathrm{sr}(a), \quad \Theta(z) - \Theta(z^2)\mathsf{a}(z)\mathsf{a}^\star(z) = \mathcal{O}(|1 - z|^{2n_b}), \qquad z \to 1. \tag{3.13}$$

Assume that $\mathrm{len}(a) + \mathrm{len}(\Theta) > 0$. For the case of complex symmetry, replace S by $\mathbb{S}$ throughout.

(S1) *Define Laurent polynomials A and B as in (1.8) and $\mathsf{p}, \mathring{\mathsf{A}}, \mathring{\mathsf{B}}$ as in (2.4) with $\tilde{a} = a$ and $n_{\tilde{b}} = n_b$ such that $\mathsf{p}(z) \geqslant 0$ for all $z \in \mathbb{T}$.*

(S2) *Select a Laurent polynomial d with [complex] symmetry (and real coefficients) such that $\mathsf{d}(z)\mathsf{d}^\star(z) = \mathsf{D}(z)$, where D is defined in (2.6). Define $\epsilon_{\mathsf{d}} z^{c_{\mathsf{d}}} := \mathsf{S}\mathsf{d}(z)$, $[c_{\mathsf{d}} - n_{\mathsf{d}}, n_{\mathsf{d}}] := \mathrm{fsupp}(\mathsf{d})$, and*

$$\begin{aligned} &c_1 := c - n_b - 2n_\Theta, \quad \epsilon_1 := \epsilon(-1)^{n_b+1} sgn(\Theta(n_\Theta)), \\ &[-n_0, n_0] := \mathrm{fsupp}(\mathring{\mathsf{A}}), \quad n_1 := \tfrac{n_0+c_1}{2}, \end{aligned} \tag{3.14}$$

where n_Θ is defined in Theorem 3. If $\Theta(n_\Theta)$ is not a real number, define $\epsilon_1 := 1$.

(S3) *Parameterize a filter $\mathring{\mathsf{b}}_1$ such that $\mathsf{S}\mathring{\mathsf{b}}_1(z) = \epsilon_1 z^{c_1}$ and $\mathrm{fsupp}(\mathring{\mathsf{b}}_1) = [c_1 - n_1, n_1]$. Find the unknown coefficients of $\mathring{\mathsf{b}}_1$ by solving a system X of linear equations induced by $\mathcal{R}(z) = 0$ and*

$$\mathit{coeff}(\mathring{\mathsf{b}}_2^\star, z, j) = 0, \quad j = n_1 - c_{\mathsf{d}}, \ldots, n_0 + n_1 - 2n_{\mathsf{d}} - 1, \tag{3.15}$$

where $\mathcal{R}$ and $\mathring{\mathsf{b}}_2^\star$ are uniquely determined by $\mathrm{fsupp}(\mathcal{R}) \subseteq [2(c_{\mathsf{d}} - n_{\mathsf{d}}), 2n_{\mathsf{d}} - 1]$ and

$$\mathring{\mathsf{B}}^\star(z)\mathring{\mathsf{b}}_1(z) - \mathring{\mathsf{A}}^\star(z)\mathring{\mathsf{b}}_1(-z) = \mathsf{d}(z^2)z\mathring{\mathsf{b}}_2^\star(z) + \mathcal{R}(z). \tag{3.16}$$

(S4) *For any nontrivial solution to the homogeneous system X in (S3), there must exist $\lambda > 0$ such that (2.11) holds. Replace $\mathring{\mathsf{b}}_1, \mathring{\mathsf{b}}_2$ by $\lambda^{-1/2}\mathring{\mathsf{b}}_1, \lambda^{-1/2}\mathring{\mathsf{b}}_2$, respectively.*

(S5) *Find two Laurent polynomials $\mathsf{q}_1, \mathsf{q}_2$ with [complex] symmetry (and real coefficients) such that*

$$\mathsf{q}_1(z)\mathsf{q}_1^\star(z) + \mathsf{q}_2(z)\mathsf{q}_2^\star(z) = \mathsf{p}(z) \quad \text{and} \quad \frac{\mathsf{S}\mathsf{q}_1(z)}{\mathsf{S}\mathsf{q}_2(z)} = (-1)^{c_1}\epsilon_{\mathsf{d}} z^{c_{\mathsf{d}} - c_1 + 1}. \tag{3.17}$$

Define

$$\begin{aligned} \mathsf{b}_1(z) &:= (1 - z^{-1})^{n_b}[\mathring{\mathsf{b}}_1(z)\mathsf{q}_1(z^2) + \mathring{\mathsf{b}}_2(z)\mathsf{q}_2(z^2)], \\ \mathsf{b}_2(z) &:= (1 - z^{-1})^{n_b}[\mathring{\mathsf{b}}_2(z)\mathsf{q}_1^\star(z^2) - \mathring{\mathsf{b}}_1(z)\mathsf{q}_2^\star(z^2)]. \end{aligned} \tag{3.18}$$

Then $\{a; b_1, b_2\}_\Theta$ is a tight framelet filter bank having [complex] symmetry (and real coefficients) such that $\mathrm{vm}(b_1) \geqslant n_b, \mathrm{vm}(b_2) \geqslant n_b$, and $\max(\mathrm{len}(b_1), \mathrm{len}(b_2)) \leqslant \mathrm{len}(a) + \mathrm{len}(\Theta)$.

Proof. By the definition of $\mathring{\mathsf{A}}$, we see that $n_1 := \frac{n_0 + c_1}{2}$ must be an integer. By the same proof of (2.19) in Algorithm 1 with $c_0 = 0$, we have

$$\mathrm{fsupp}(\mathring{\mathsf{b}}_2^\star) \subseteq [c_1 - n_0 - n_1 + 2n_{\mathsf{d}} - 2c_{\mathsf{d}} - 1, n_0 + n_1 - 2n_{\mathsf{d}} - 1]$$

and $\mathsf{S}\mathring{\mathsf{b}}_2(z) = \epsilon_2 z^{c_2}$ with $c_2 := 2c_{\mathsf{d}} + 2 - c_1$ and $\epsilon_2 := (-1)^{c_1}\epsilon_{\mathsf{d}}\epsilon_1$. It follows from (3.14) and (3.15) that

$$\mathrm{fsupp}(\mathring{\mathsf{b}}_2^\star) \subseteq [-c_2 - n_1 + c_{\mathsf{d}} + 1, n_1 - c_{\mathsf{d}} - 1] = [c_1 - n_1 - c_{\mathsf{d}} - 1, n_1 - c_{\mathsf{d}} - 1].$$

Hence, $\mathrm{len}(\mathring{\mathsf{b}}_2) \leqslant 2n_1 - c_1 = n_0$. By the definition of n_1, we also have $\mathrm{len}(\mathring{\mathsf{b}}_1) \leqslant 2n_1 - c_1 = n_0$.

Since $\mathcal{R} = 0$, we deduce from (3.16) that $\mathring{\mathsf{b}}_2(z)\mathsf{d}^\star(z^2)z^{-1} = \mathring{\mathsf{B}}(z)\mathring{\mathsf{b}}_1^\star(z) - \mathring{\mathsf{A}}(z)\mathring{\mathsf{b}}_1^\star(-z)$, from which we see that

$$\begin{aligned} &[\mathring{\mathsf{B}}^\star(z)\mathring{\mathsf{b}}_2(z) - \mathring{\mathsf{A}}^\star(z)\mathring{\mathsf{b}}_2(-z)]\mathsf{d}^\star(z^2)z^{-1} \\ &\quad = \mathring{\mathsf{B}}^\star(z)[\mathring{\mathsf{b}}_2(z)\mathsf{d}^\star(z^2)z^{-1}] + \mathring{\mathsf{A}}^\star(z)[\mathring{\mathsf{b}}_2(-z)\mathsf{d}^\star(z^2)(-z)^{-1}] \\ &\quad = [\mathring{\mathsf{B}}^\star(z)\mathring{\mathsf{B}}(z) - \mathring{\mathsf{A}}^\star(z)\mathring{\mathsf{A}}(-z)]\mathring{\mathsf{b}}_1^\star(z) + [\mathring{\mathsf{A}}^\star(z)\mathring{\mathsf{B}}(-z) - \mathring{\mathsf{B}}^\star(z)\mathring{\mathsf{A}}(z)]\mathring{\mathsf{b}}_1^\star(-z). \end{aligned}$$

Since $\Theta(z) \geqslant 0$ and $\mathsf{D}(z) \geqslant 0$ for all $z \in \mathbb{T}$, we shall see in Lemma 6 that $\mathcal{M}_{a,a,\Theta}(z) \geqslant 0$ for all $z \in \mathbb{T}$. Therefore,

$$\begin{bmatrix} \mathring{\mathsf{A}}(z) & \mathring{\mathsf{B}}(z) \\ \mathring{\mathsf{B}}(-z) & \mathring{\mathsf{A}}(-z) \end{bmatrix} \geqslant 0, \qquad \forall\, z \in \mathbb{T}. \tag{3.19}$$

Since $\mathring{\mathsf{A}}^\star(z) = \mathring{\mathsf{A}}(z)$ and $\mathring{\mathsf{B}}^\star(z) = \mathring{\mathsf{B}}(-z)$, by $\mathsf{d}(z^2)\mathsf{d}^\star(z^2) = \mathsf{D}^\star(z^2) = \mathring{\mathsf{A}}(z)\mathring{\mathsf{A}}(-z) - \mathring{\mathsf{B}}(z)\mathring{\mathsf{B}}(-z)$, we deduce from the above identities that

$$[\mathring{\mathsf{B}}^\star(z)\mathring{\mathsf{b}}_2(z) - \mathring{\mathsf{A}}^\star(z)\mathring{\mathsf{b}}_2(-z)]\mathsf{d}^\star(z^2)z^{-1} = -\mathsf{d}(z^2)\mathsf{d}^\star(z^2)\mathring{\mathsf{b}}_1^\star(z).$$

Since d is not identically zero, the above identity implies

$$\mathring{\mathsf{B}}^\star(z)\mathring{\mathsf{b}}_2(z) - \mathring{\mathsf{A}}^\star(z)\mathring{\mathsf{b}}_2(-z) = -\mathsf{d}(z^2)z\mathring{\mathsf{b}}_1^\star(z). \tag{3.20}$$

Now the same proof to (2.23) in Algorithm 1 shows that (2.23) is satisfied with $\mathring{\tilde{\mathsf{b}}}_1 = \mathring{\mathsf{b}}_1$ and $\mathring{\tilde{\mathsf{b}}}_2 = \mathring{\mathsf{b}}_2$ and $\lambda := \mathsf{D}_{\mathring{\mathsf{b}}}(z^2)/\mathsf{d}(z^2)$ is a complex number. By (3.19), it follows from (2.23) that λ must be a nonnegative real number. If $\lambda = 0$, then (2.23) implies $\mathring{\mathsf{b}}_1(z)\mathring{\mathsf{b}}_1^\star(z) + \mathring{\mathsf{b}}_2(z)\mathring{\mathsf{b}}_2^\star(z) = 0$, which is only possible when $\mathring{\mathsf{b}}_1 = \mathring{\mathsf{b}}_2 = 0$, a contradiction to our assumption. Hence, $\lambda > 0$. After replacing $\mathring{\mathsf{b}}_1, \mathring{\mathsf{b}}_2$ by $\lambda^{-1/2}\mathring{\mathsf{b}}_1, \lambda^{-1/2}\mathring{\mathsf{b}}_2$, respectively, we see that (2.26) is satisfied with $\mathring{\tilde{\mathsf{b}}}_1 = \mathring{\mathsf{b}}_1$ and $\mathring{\tilde{\mathsf{b}}}_2 = \mathring{\mathsf{b}}_2$. Note that

$$\begin{bmatrix} \mathsf{b}_1(z) & \mathsf{b}_2(z) \\ \mathsf{b}_1(-z) & \mathsf{b}_2(-z) \end{bmatrix} = \begin{bmatrix} (1-z^{-1})^{n_b} & \\ & (1+z^{-1})^{n_b} \end{bmatrix} \begin{bmatrix} \mathring{\mathsf{b}}_1(z) & \mathring{\mathsf{b}}_2(z) \\ \mathring{\mathsf{b}}_1(-z) & \mathring{\mathsf{b}}_2(-z) \end{bmatrix} \begin{bmatrix} \mathsf{q}_1(z^2) & -\mathsf{q}_2^\star(z^2) \\ \mathsf{q}_2(z^2) & \mathsf{q}_1^\star(z^2) \end{bmatrix}.$$

Using (3.17), we can directly check that $\{a; b_1, b_2\}_\Theta$ is a tight framelet filter bank having [complex] symmetry (and real coefficients). □

We make some remarks here about Algorithm 3. From [13, Theorem 4.2] (also see [14, 19]), there exists a tight framelet filter bank $\{a; b_1, b_2\}_\Theta$ with [complex] symmetry (and real coefficients) if and only if

(i) $\Theta(z) \geqslant 0$ for all $z \in \mathbb{T}$;
(ii) there exists a Laurent polynomial d with [complex] symmetry (and real coefficients) such that $\mathsf{d}(z)\mathsf{d}^\star(z) = \mathsf{D}(z)$;
(iii) there exist Laurent polynomials $\mathsf{q}_1, \mathsf{q}_2$ with [complex] symmetry (and real coefficients) such that (3.17) is satisfied.

For the case of complex symmetry (and real coefficients), a Laurent polynomial d in (S2) satisfying (ii) is essentially unique and can be easily derived from D (see [13, Theorem 2.8] and [14]). For the case of symmetry, there are essentially finitely many such choices of d satisfying (ii) (see [13, Theorem 2.9]). Laurent polynomials $\mathsf{q}_1, \mathsf{q}_2$ with [complex] symmetry satisfying (iii) can be found by [13, Theorems 2.6 and 2.7]. Under the conditions in (i)–(iii), an algorithm, based on matrix factorization with symmetry, has been developed in [13, 14, 19] to construct a tight framelet filter bank $\{a; b_1, b_2\}_\Theta$ with [complex] symmetry (and real coefficients) derived from any given filters a and Θ with symmetry. If the conditions in (i)–(iii) are satisfied, it is guaranteed by [13] that the overdetermined system X in (S3) of Algorithm 3 must have a solution. Comparing with the algorithms proposed in [13, 14, 19], our algorithm in Algorithm 3 is much simpler and more efficient.

Without the symmetry constraint, from any given filters a and Θ, Algorithm 3 can be easily modified to construct all possible tight framelet filter banks $\{a; b_1, b_2\}_\Theta$ having the shortest filter support. By a similar argument as in Algorithms 1 and 3, for the convenience of the reader, we provide an algorithm here.

Algorithm 4. *Let $a, \Theta \in l_0(\mathbb{Z})$ satisfying $\Theta(z) \geqslant 0$ and $\Theta(z) - \Theta(z^2)\mathsf{a}(z)\mathsf{a}^\star(z) \geqslant 0$ for all $z \in \mathbb{T}$. Let $n_b \in \mathbb{N} \cup \{0\}$ satisfying* (3.13). *Assume that* $\operatorname{len}(a) + \operatorname{len}(\Theta) > 0$.

(S1) *Define* A *and* B *as in* (1.8) *and* $\mathsf{p}, \mathring{\mathsf{A}}, \mathring{\mathsf{B}}$ *as in* (2.4) *with* $\tilde{a} = a$ *and* $n_{\tilde{b}} = n_b$ *such that* $\mathsf{p}(z) \geqslant 0$ *for all* $z \in \mathbb{T}$. *Define* $[-n_0, n_0] := \operatorname{fsupp}(\mathring{\mathsf{A}})$.

(S2) *Select* $\epsilon_{\rm len}, s_1, s_2 \in \{0, 1\}$ *and a polynomial* d *such that* $\mathsf{d}(z)\mathsf{d}^\star(z) = \mathsf{D}(z)$ *and* $\lceil \frac{s_1+s_2-1}{2} \rceil \leqslant m_{\mathsf{d}} \leqslant n_{\mathsf{d}} \leqslant \lfloor \frac{s_1+s_2-1}{2} \rfloor + n_0 + \epsilon_{\rm len}$, *where* D *is defined in* (2.6) *and* $[m_{\mathsf{d}}, n_{\mathsf{d}}] := \operatorname{fsupp}(\mathsf{d})$.

(S3) *Parameterize a filter* $\mathring{\mathsf{b}}_1$ *by* $\mathring{\mathsf{b}}_1(z) = z^{s_1} \sum_{j=0}^{n_0+\epsilon_{\rm len}} t_j z^j$. *Find the unknown coefficients* $\{t_0, \ldots, t_{n_0+\epsilon_{\rm len}}\}$ *by solving a system* X *of linear equations induced by* $\mathcal{R}(z) = 0$ *and*

$$\begin{aligned} \mathit{coeff}(\mathring{\mathsf{b}}_2^\star, z, j) = 0, \quad & j = s_1 - n_0 - 2m_{\mathsf{d}} - 1, \ldots, s_2 - 1 \quad \text{and} \\ & j = s_2 + n_0 + \epsilon_{\rm len} + 1, \ldots, s_1 + 2n_0 - 2n_{\mathsf{d}} + \epsilon_{\rm len} - 1, \end{aligned}$$

where $\mathcal{R}$ *and* $\mathring{\mathsf{b}}_2^\star$ *are uniquely determined by* $\operatorname{fsupp}(\mathcal{R}) \subseteq [2m_{\mathsf{d}}, 2n_{\mathsf{d}} - 1]$ *and* (3.16).

(S4) *For any nontrivial solution to the homogeneous system* X *in (S3), there must exist* $\lambda > 0$ *such that* (2.11) *holds. Replace* $\mathring{\mathsf{b}}_1, \mathring{\mathsf{b}}_2$ *by* $\lambda^{-1/2}\mathring{\mathsf{b}}_1, \lambda^{-1/2}\mathring{\mathsf{b}}_2$, *respectively.*

(S5) *Find Laurent polynomials* $\mathsf{q}_1, \mathsf{q}_2$ *such that* $\mathsf{q}_1(z)\mathsf{q}_1^\star(z) + \mathsf{q}_2(z)\mathsf{q}_2^\star(z) = \mathsf{p}(z)$.

Define b_1 *and* b_2 *as in* (3.18). *Then* $\{a; b_1, b_2\}_\Theta$ *is a tight framelet filter bank satisfying* $\max(\operatorname{len}(b_1), \operatorname{len}(b_2)) \leqslant \operatorname{len}(a) + \operatorname{len}(\Theta) + \epsilon_{\rm len}$.

4. Examples of symmetric dual framelet filter banks

In this section, we present a few examples to illustrate Algorithm 1. Though many different dual framelet filter banks with short filter supports can be derived by Algorithm 1 from given filters $a, \tilde{a}$ and Θ, for simplicity of presentation, we only provide examples of real-valued symmetric dual framelet filter banks $(\{\tilde{a}; \tilde{b}_1, \tilde{b}_2\}, \{a; b_1, b_2\})_\Theta$ such that all the four high-pass filters have more or less the same length of filter supports. We always take $\epsilon_{\rm len} = 0$ in this section so that the constructed symmetric dual framelet filter banks have the shortest possible filter supports satisfying (1.19). Since Algorithm 3 is a special case of Algorithm 1, we do not apply Algorithm 3 explicitly for constructing symmetric tight framelet filter banks.

Before we present several examples, let us recall some basic definitions. Let $u \in l_0(\mathbb{Z})$ such that $\operatorname{fsupp}(u) = [m, n]$. It is handy to list the filter u as $u = \{u(m), \ldots, \underline{\mathbf{u(0)}}, \ldots, u(n)\}_{[m,n]}$, where we underlined and boldfaced the number $u(0)$ to indicate its position at the origin. For a filter $a \in l_0(\mathbb{Z})$ with $\mathsf{a}(1) = 1$, we define $\boldsymbol{\varphi}$ as in (1.3). If $\boldsymbol{\varphi} \in L_2(\mathbb{R})$, then we can define a function $\phi^a(x) := \lim_{n\to\infty} \frac{1}{2\pi} \int_{-n}^{n} \boldsymbol{\varphi}(\xi) e^{i\xi x} d\xi$ in $L_2(\mathbb{R})$. More generally, $\widehat{\phi^a} = \boldsymbol{\varphi}$ in the sense of tempered distributions. For $b \in l_0(\mathbb{Z})$, we define $\psi^{a,b} := 2\sum_{k\in\mathbb{Z}} b(k)\phi^a(2\cdot - k)$ or, equivalently, $\widehat{\psi^{a,b}}(2\xi) = \mathsf{b}(e^{-i\xi})\boldsymbol{\varphi}(\xi)$. Write $\mathsf{a}(z) = (1+z)^{\operatorname{sr}(a)}\mathsf{v}(z)$ for some $v \in l_0(\mathbb{Z})$ and define $\sum_{k=-K}^{K} w(k)z^k := \mathsf{v}(z)\mathsf{v}^\star(z)$. The *smoothness exponent* $\operatorname{sm}(a)$ of a filter a ([9]) is defined to be

$$\operatorname{sm}(a) := -1/2 - \log_2 \sqrt{\rho(a)}, \tag{4.1}$$

where $\rho(a)$ denotes the spectral radius, the largest modulus of all the eigenvalues, of the square matrix $(w(2j-k))_{-K\leqslant j,k\leqslant K}$. The smoothness exponent $\mathrm{sm}(a)$ is closely related to the smoothness of the refinable function ϕ^a and the stability of affine wavelet systems; see [3, 9, 11, 15] and references therein.

Example 1. Let $a = \{\frac{1}{8}, \underline{\mathbf{3}}, \frac{3}{8}, \frac{1}{8}\}_{[-1,2]}, \tilde{a} = \{\underline{\mathbf{1}}, \frac{1}{2}\}_{[0,1]}$. If $\Theta = \delta$ (that is, $\Theta = \{\underline{\mathbf{1}}\}_{[0,0]}$) and $n_b = n_{\tilde{b}} = 1$, then $\mathsf{p}(z) = 1$ and $\mathsf{D}(z) = \frac{1}{8}$. Taking $\mathsf{d} = 1$, $c_1 = 0, \epsilon_1 = 1, n_1 = 0, n_2 = 2$, we have

$$\tilde{a} = \tfrac{1}{2}\{\underline{\mathbf{1}}, 1\}_{[0,1]}, \quad \tilde{b}_1 = \tfrac{1}{4}\{2\bar{t}-1, -2\bar{t}-1, \underline{\mathbf{2\bar{t}+1}}, -2\bar{t}+1\}_{[-2,1]}, \quad \tilde{b}_2 = \tfrac{1}{2}\{\underline{\mathbf{-1}}, 1\}_{[0,1]}$$

and

$$a = \{\tfrac{1}{8}, \underline{\mathbf{3}}, \tfrac{3}{8}, \tfrac{1}{8}\}_{[-1,2]}, \quad b_1 = \tfrac{1}{4}\{-1, \underline{\mathbf{1}}\}_{[-1,0]}, \quad b_2 = \tfrac{1}{4}\{-t, \underline{\mathbf{t-2}}, 2-t, t\}_{[-1,2]},$$

When $t = 1/2$, this is simply the first particular construction given in (1.10) and (1.11). When $t = 0$, multiplying $\tilde{b}_1$ by $1/2$ and b_1 by 2, the above dual framelet filter bank becomes

$$\tilde{a} = \tfrac{1}{2}\{1, \underline{\mathbf{1}}\}_{[0,1]}, \quad \tilde{b}_1 = \tfrac{1}{8}\{-1, -1, \underline{\mathbf{1}}, 1\}_{[-2,1]}, \tilde{b}_2 = \tfrac{1}{2}\{\underline{\mathbf{-1}}, 1\}_{[0,1]}$$

and

$$a = \tfrac{1}{8}\{1, \underline{\mathbf{3}}, 3, 1\}_{[-1,2]}, \quad b_1 = \tfrac{1}{2}\{-1, \underline{\mathbf{1}}\}_{[-1,0]}, \quad b_2 = \tfrac{1}{2}\{\underline{\mathbf{-1}}, 1\}_{[0,1]}, \tag{4.2}$$

which is essentially the first particular construction in (1.10) and (1.11) by switching a with $\tilde{a}$.

Example 2. Let $a = \tilde{a} = \{\frac{1}{8}, \underline{\mathbf{3}}, \frac{3}{8}, \frac{1}{8}\}_{[-1,2]}$. If $\Theta = \delta$ and $n_b = n_{\tilde{b}} = 1$, then $\mathsf{p}(z) = 1$ and $\mathsf{D}(z) = \frac{3}{16}$. Taking $\mathsf{d} = 1, c_1 = 0, \epsilon_1 = 1, n_1 = 1, n_2 = 2$, we have

$$\tilde{a} = \tfrac{1}{8}\{1, \underline{\mathbf{3}}, 3, 1\}_{[-1,2]}, \quad \tilde{b}_1 = \tfrac{3}{8}\{-1, \underline{\mathbf{1}}\}_{[-1,0]}, \quad \tilde{b}_2 = \tfrac{1}{8}\{-1, \underline{\mathbf{-3}}, 3, 1\}_{[-1,2]}$$

and

$$a = \tfrac{1}{8}\{1, \underline{\mathbf{3}}, 3, 1\}_{[-1,2]}, \quad b_1 = \tfrac{1}{2}\{-1, \underline{\mathbf{1}}\}_{[-1,0]}, \quad b_2 = \tfrac{1}{8}\{-1, \underline{\mathbf{-3}}, 3, 1\}_{[-1,2]}, \tag{4.3}$$

which is essentially a tight framelet filter bank and can be also easily obtained by Algorithm 3.

Taking $\mathsf{d} = 1, c_1 = 0, \epsilon_1 = 1, n_1 = 0, n_2 = 1$, we have $\{a; b_1, b_2\}$ as in (4.2) and

$$\tilde{a} = \tfrac{1}{8}\{1, \underline{\mathbf{3}}, 3, 1\}_{[-1,2]}, \quad \tilde{b}_1 = \tfrac{1}{32}\{-1, -3, -10, \underline{\mathbf{10}}, 3, 1\}_{[-3,2]},$$
$$\tilde{b}_2 = \tfrac{1}{8}\{-1, \underline{\mathbf{-3}}, 3, 1\}_{[-1,2]}.$$

This example shows that for a given $\{a; b_1, b_2\}$, there are many choices of $\{\tilde{a}; \tilde{b}_1, \tilde{b}_2\}$ so that $(\{\tilde{a}; \tilde{b}_1, \tilde{b}_2\}, \{a; b_1, b_2\})_\Theta$ forms a dual framelet filter bank.

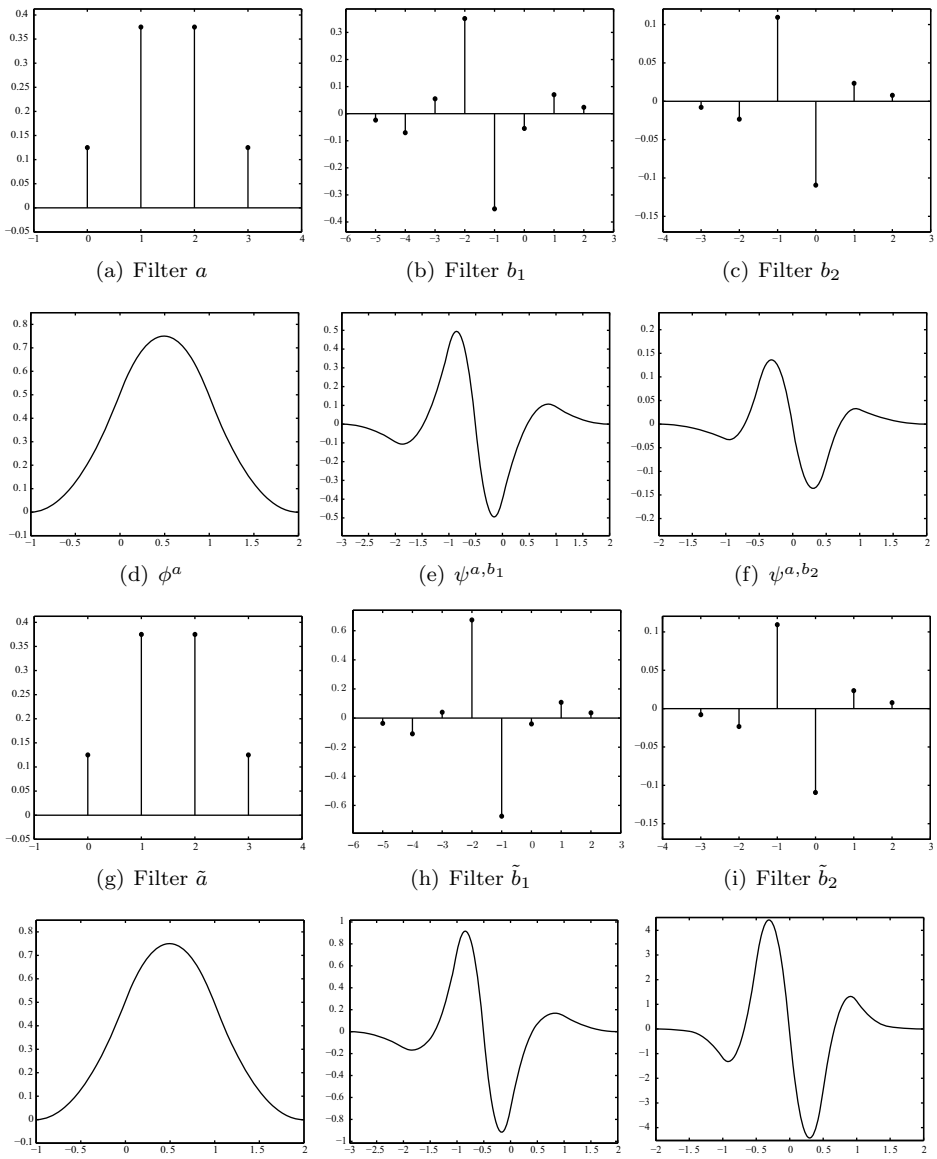
By Lemma 1 we can take $\Theta = \{\frac{13}{240}, -\frac{7}{15}, \underline{\mathbf{\frac{73}{40}}}, -\frac{7}{15}, \frac{13}{240}\}_{[-2,2]}$ with $n_b = n_{\tilde{b}} = 3$. Then $\mathsf{p}(z) = 1$ and $\mathsf{D}(x) = \frac{2719}{92160} + \frac{247}{184320}(z + z^{-1})$. Taking $\mathsf{d}(z) = 1, c_1 = 0, \epsilon_1 = 1, n_1 = 2, n_2 = 3$, we have

$$\tilde{b}_1 = \tfrac{1}{720}\{-26, -78, 29, 485, -485, \underline{\mathbf{-29}}, 78, 26\}_{[-5,2]},$$
$$\tilde{b}_2 = \tfrac{1}{144}\{-13, -153, 524, \underline{\mathbf{-524}}, 153, 13\}_{[-3,2]}$$

and

$$b_1 = \tfrac{1}{128}\{-3, -9, 7, 45, -45\underline{\mathbf{-7}}, 9, 3\}_{[-5,2]}, \quad b_2 = \tfrac{1}{128}\{-1, -3, 14, \underline{\mathbf{-14}}, 3, 1\}_{[-3,2]}$$

with $\mathrm{vm}(b_1) = \mathrm{vm}(b_2) = \mathrm{vm}(\tilde{b}_1) = \mathrm{vm}(\tilde{b}_2) = 3$. Since D has two real roots, using $\mathsf{d}(z) = z - \frac{4\sqrt{458247}-2719}{247}$, we can also easily obtain a dual framelet filter bank which involves radicals.



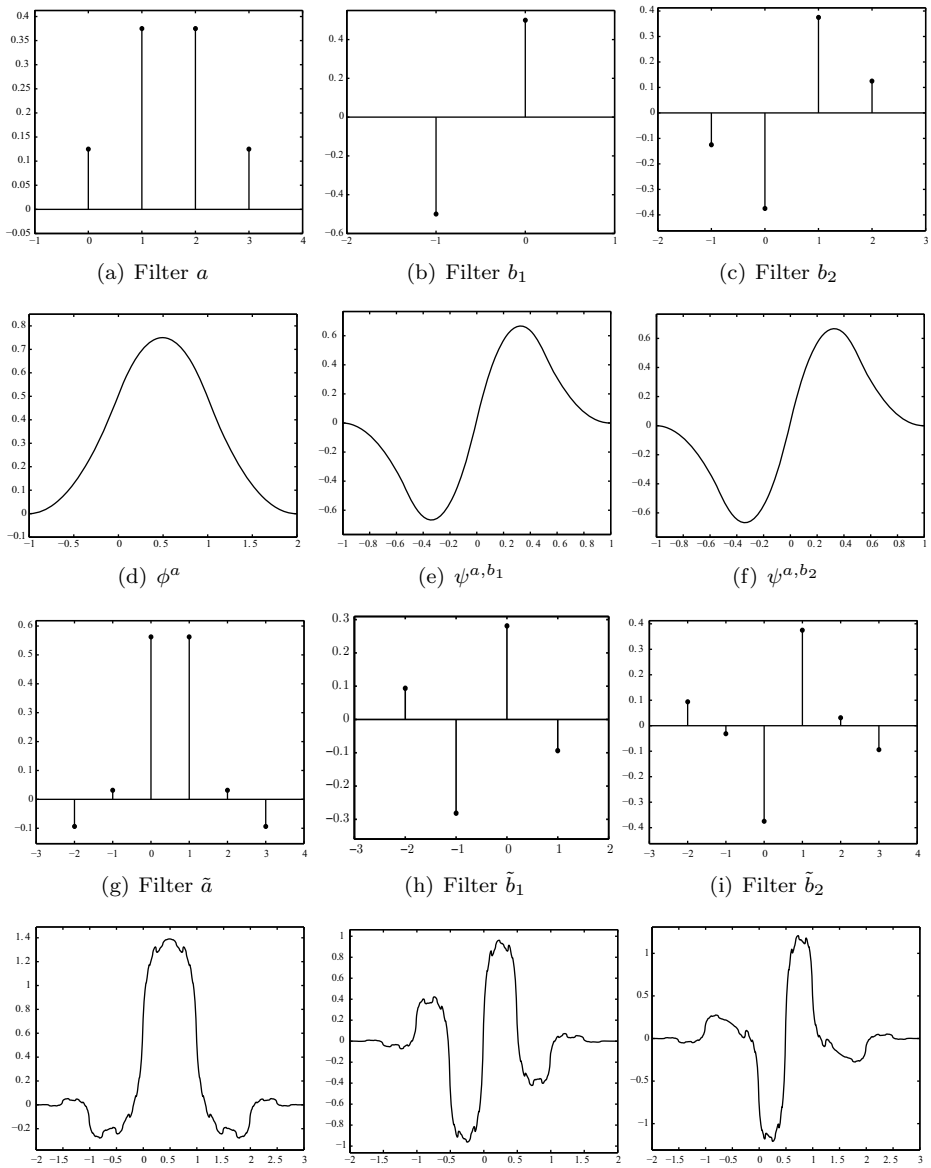
(a) Filter a (b) Filter b_1 (c) Filter b_2

(d) ϕ^a (e) ψ^{a,b_1} (f) ψ^{a,b_2}

(g) Filter $\tilde{a}$ (h) Filter $\tilde{b}_1$ (i) Filter $\tilde{b}_2$

(j) $\phi^{\tilde{a}}$ (k) $\psi^{\tilde{a},\tilde{b}_1}$ (l) $\psi^{\tilde{a},\tilde{b}_2}$

FIGURE 4.1. Filters $a, b_1, b_2, \tilde{a}, \tilde{b}_1, \tilde{b}_2$ and functions $\phi^a, \psi^{a,b_1}, \psi^{a,b_2}$, $\phi^{\tilde{a}}, \psi^{\tilde{a},\tilde{b}_1}, \psi^{\tilde{a},\tilde{b}_2}$ of the dual framelet filter bank $(\{\tilde{a}; \tilde{b}_1, \tilde{b}_2\}, \{a; b_1, b_2\})_\Theta$ given in Example 2 with $\Theta = \{\frac{13}{240}, -\frac{7}{15}, \underline{\mathbf{\frac{73}{40}}}, -\frac{7}{15}, \frac{13}{240}\}_{[-2,2]}$.



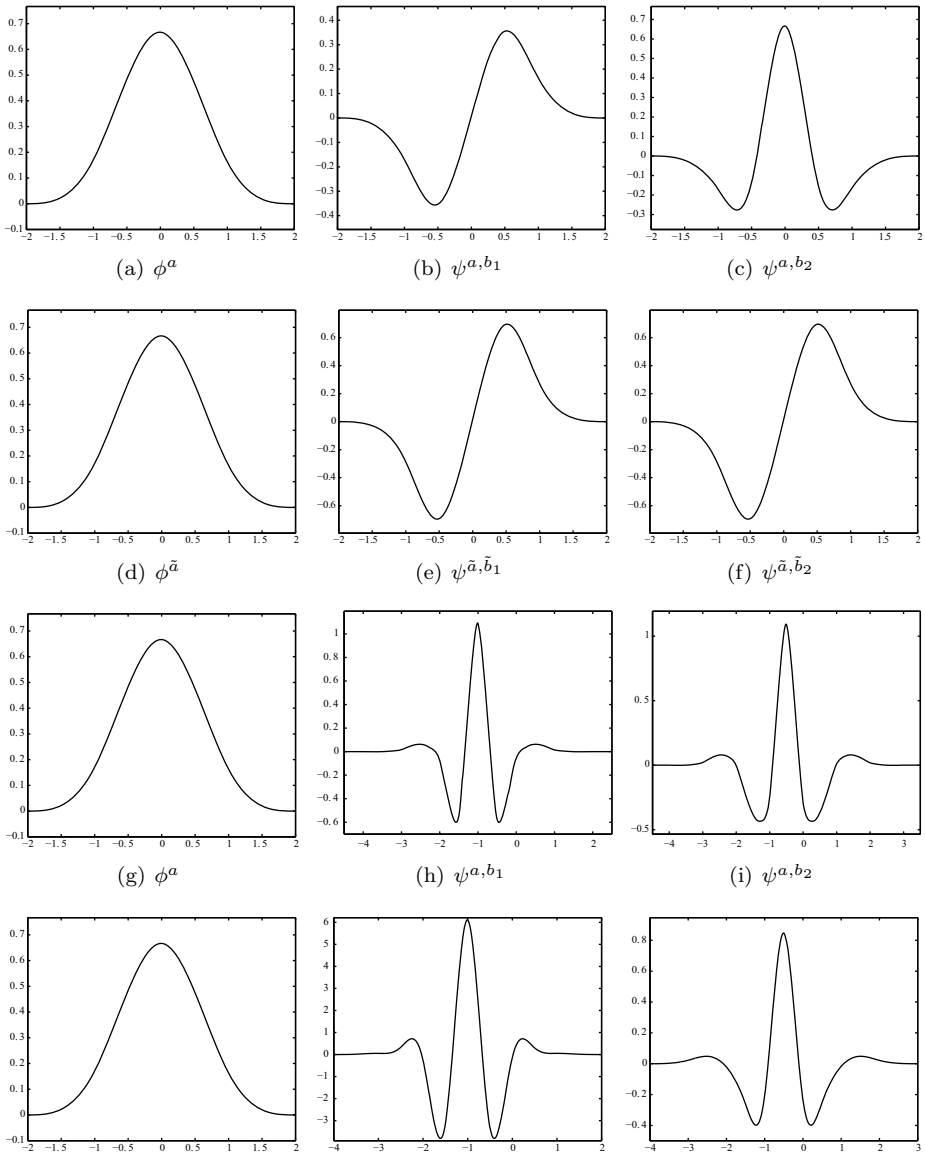
(a) Filter a (b) Filter b_1 (c) Filter b_2

(d) ϕ^a (e) ψ^{a,b_1} (f) ψ^{a,b_2}

(g) Filter $\tilde{a}$ (h) Filter $\tilde{b}_1$ (i) Filter $\tilde{b}_2$

(j) $\phi^{\tilde{a}}$ (k) $\psi^{\tilde{a},\tilde{b}_1}$ (l) $\psi^{\tilde{a},\tilde{b}_2}$

FIGURE 4.2. Filters $a, b_1, b_2, \tilde{a}, \tilde{b}_1, \tilde{b}_2$ and functions ϕ^a, ψ^{a,b_1}, ψ^{a,b_2}, $\phi^{\tilde{a}}$, $\psi^{\tilde{a},\tilde{b}_1}$, $\psi^{\tilde{a},\tilde{b}_2}$ of the dual framelet filter bank $(\{\tilde{a};\tilde{b}_1,\tilde{b}_2\}, \{a;b_1,b_2\})_\Theta$ given in Example 3.



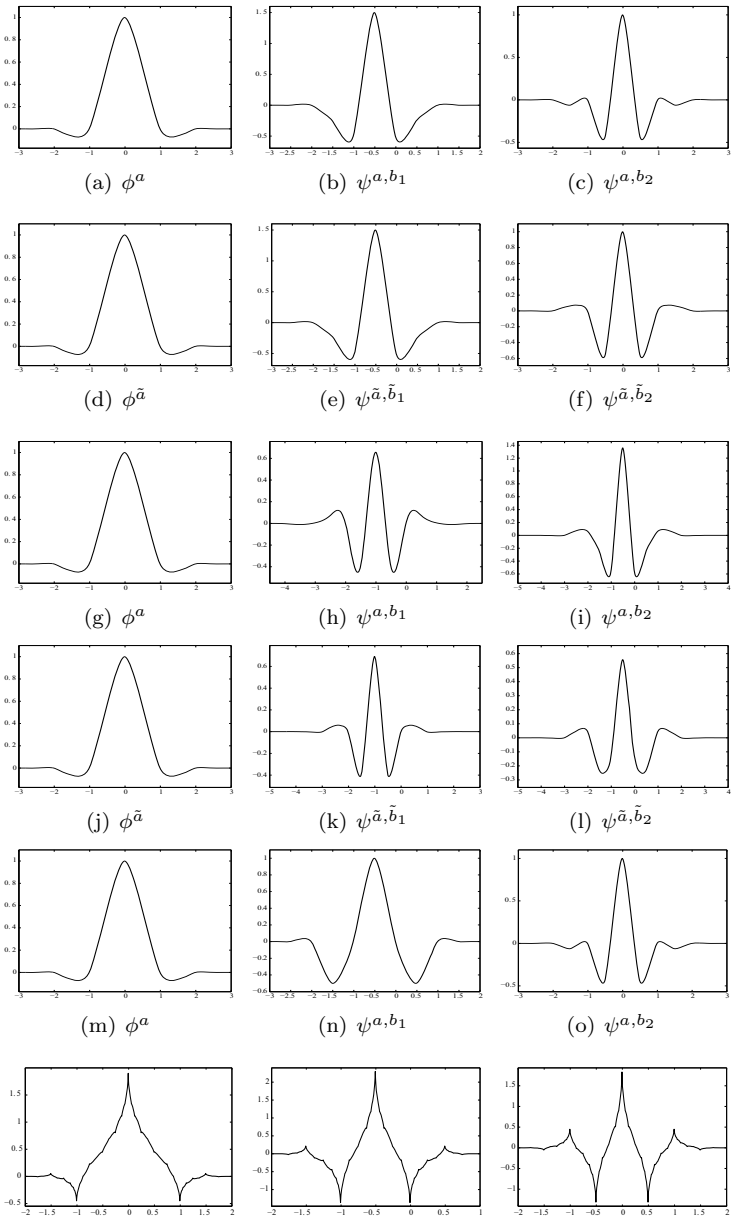
(a) ϕ^{a} (b) ψ^{a,b_1} (c) ψ^{a,b_2}

(d) $\phi^{\tilde{a}}$ (e) $\psi^{\tilde{a},\tilde{b}_1}$ (f) $\psi^{\tilde{a},\tilde{b}_2}$

(g) ϕ^{a} (h) ψ^{a,b_1} (i) ψ^{a,b_2}

(j) $\phi^{\tilde{a}}$ (k) $\psi^{\tilde{a},\tilde{b}_1}$ (l) $\psi^{\tilde{a},\tilde{b}_2}$

FIGURE 4.3. Functions $\phi^{a}, \psi^{a,b_1}, \psi^{a,b_2}, \phi^{\tilde{a}}, \psi^{\tilde{a},\tilde{b}_1}, \psi^{\tilde{a},\tilde{b}_2}$ of the dual framelet filter bank $(\{\tilde{a};\tilde{b}_1,\tilde{b}_2\},\{a;b_1,b_2\})_{\Theta}$ given in Example 4. (a)–(f) are for $\Theta=\delta$. (g)-(l) are for $\Theta=\{-\frac{311}{15120},\ \frac{22}{105},\ -\frac{1657}{1680},\ \underline{\mathbf{\frac{2452}{945}}},\ -\frac{1657}{1680},\ \frac{22}{105},\ -\frac{311}{15120}\}_{[-3,3]}$.



(a) ϕ^a (b) ψ^{a,b_1} (c) ψ^{a,b_2}

(d) $\phi^{\tilde{a}}$ (e) $\psi^{\tilde{a},\tilde{b}_1}$ (f) $\psi^{\tilde{a},\tilde{b}_2}$

(g) ϕ^a (h) ψ^{a,b_1} (i) ψ^{a,b_2}

(j) $\phi^{\tilde{a}}$ (k) $\psi^{\tilde{a},\tilde{b}_1}$ (l) $\psi^{\tilde{a},\tilde{b}_2}$

(m) ϕ^a (n) ψ^{a,b_1} (o) ψ^{a,b_2}

(p) $\phi^{\tilde{a}}$ (q) $\psi^{\tilde{a},\tilde{b}_1}$ (r) $\psi^{\tilde{a},\tilde{b}_2}$

FIGURE 4.4. Functions $\phi^a, \psi^{a,b_1}, \psi^{a,b_2}, \phi^{\tilde{a}}, \psi^{\tilde{a},\tilde{b}_1}, \psi^{\tilde{a},\tilde{b}_2}$ of the dual framelet filter bank $(\{\tilde{a};\tilde{b}_1,\tilde{b}_2\},\{a;b_1,b_2\})_\Theta$ given in Example 5. (a)–(f) are for $\Theta = \delta$. (g)–(l) are for $\Theta = \{-\frac{11}{5040}, \frac{4}{105}, -\frac{223}{1680}, \underline{\frac{\mathbf{376}}{\mathbf{315}}}, -\frac{223}{1680}, \frac{4}{105}, -\frac{11}{5040}\}_{[-3,3]}$. (m)–(r) are for $\Theta = \delta$ and $\tilde{a}$ in (4.4).

Example 3. Let $a = \{\frac{1}{8}, \underline{\mathbf{\frac{3}{8}}}, \frac{3}{8}, \frac{1}{8}\}_{[-1,2]}$ and $\Theta = \delta$. Setting $M = 3$ and $N = 4$ in Proposition 2, we have

$$\tilde{a} = \{-\tfrac{3}{32}, \tfrac{1}{32}, \underline{\mathbf{\tfrac{9}{16}}}, \tfrac{9}{16}, \tfrac{1}{32}, -\tfrac{3}{32}\}_{[-2,3]}$$

with $\mathrm{sm}(\tilde{a}) \approx 1.0981905$ and $\mathrm{sr}(\tilde{a}) = 3$. If $n_b = 1$ and $n_{\tilde{b}} = 3$, then $\mathsf{p}(z) = 1$, $\mathsf{D}(z) = -\frac{3}{128}z^{-1}$ and $n_b = 1, n_{\tilde{b}} = 3$. Taking $\mathsf{d} = 1, c_1 = 0, \epsilon_1 = 1, n_1 = 0, n_2 = 2$, we have

$$\tilde{b}_1 = \tfrac{3}{32}\{1, -3, \underline{\mathbf{3}}, -1\}_{[-2,1]}, \qquad \tilde{b}_2 = \tfrac{1}{32}\{3, -1, \underline{\mathbf{-12}}, 12, 1, -3\}_{[-2,3]}$$

and $\{a; b_1, b_2\}$ is given in (4.3). Then $\mathrm{vm}(b_1) = \mathrm{vm}(b_2) = 1$ and $\mathrm{vm}(\tilde{b}_1) = \mathrm{vm}(\tilde{b}_2) = 3$.

Example 4. Let $a = \tilde{a} = \{\frac{1}{16}, \frac{1}{4}, \underline{\mathbf{\frac{3}{8}}}, \frac{1}{4}, \frac{1}{16}\}_{[-2,2]}$. If $\Theta = \delta$ and $n_b = n_{\tilde{b}} = 1$, then $\mathsf{p}(z) = 1$ and $\mathsf{D}(z) = \frac{15}{64} + \frac{1}{128}(z + z^{-1})$. Taking $\mathsf{d}(z) = 1, c_1 = 1, \epsilon_1 = 1, n_1 = 2, n_2 = 2$, we have

$$\tilde{b}_1 = \tfrac{1}{12}\{-1, -6, \underline{\mathbf{0}}, 6, 1\}_{[-2,2]}, \quad \tilde{b}_2 = \tfrac{1}{48}\{-1, -12, \underline{\mathbf{26}}, -12, -1\}_{[-2,2]}$$

and

$$b_1 = \tfrac{1}{16}\{-1, -4, \underline{\mathbf{0}}, 4, -1\}_{[-2,2]}, \qquad b_2 = \tfrac{1}{16}\{-1, -4, \underline{\mathbf{10}}, -4, -1\}_{[-2,2]}$$

with $\mathrm{vm}(b_1) = \mathrm{vm}(\tilde{b}_1) = 1$ and $\mathrm{vm}(b_2) = \mathrm{vm}(\tilde{b}_2) = 2$. Moreover, $\tilde{\mathsf{b}}_1(-1) = \mathsf{b}(-1) = 0$. Though we can also take $\mathsf{d}(z) = z + 15 \pm 4\sqrt{14}$, no solutions can be found by Algorithm 1.

By Lemma 1 we can also take

$$\Theta = \{-\tfrac{311}{15120}, \tfrac{22}{105}, -\tfrac{1657}{1680}, \underline{\mathbf{\tfrac{2452}{945}}}, -\tfrac{1657}{1680}, \tfrac{22}{105}, -\tfrac{311}{15120}\}_{[-3,3]}$$

with $n_b = n_{\tilde{b}} = 4$. Taking $\mathsf{d}(z) = 1, c_1 = 0, \epsilon_1 = 1, n_1 = 3, n_2 = 4$, we have

$$\begin{aligned}\tilde{b}_1 &= \tfrac{1}{86400}\{933, -308, 20504, -94172, 146086, -94172, \underline{\mathbf{20504}}, -308, 933\}_{[-6,2]},\\ \tilde{b}_2 &= \tfrac{1}{151200}\{1244, 4976, 701, -16420, -51345, 121688, \underline{\mathbf{-51345}}, -16420,\\ &\qquad\qquad 701, 4976, 1244\}_{[-6,4]},\end{aligned}$$

and

$$\begin{aligned}b_1 &= \tfrac{1}{128}\{1, 4, -4, -36, 70, -36, \underline{\mathbf{-4}}, 4, 1\}_{[-6,2]},\\ b_2 &= \tfrac{1}{512}\{5, 20, 1, -96, -70, 280, \underline{\mathbf{-70}}, -96, 1, 20, 5\}_{[-6,4]}\end{aligned}$$

with $\mathrm{vm}(b_1) = \mathrm{vm}(b_2) = \mathrm{vm}(\tilde{b}_1) = \mathrm{vm}(\tilde{b}_2) = 4$.

Let $\Theta = \delta$. Setting $M = 2$ and $N = 4$ in Proposition 2, we obtain

$$\tilde{a} = \{-\tfrac{3}{16}, \tfrac{1}{4}, \underline{\mathbf{\tfrac{7}{8}}}, \tfrac{1}{4}, -\tfrac{3}{16}\}_{[-2,2]}$$

with $\mathrm{sm}(\tilde{a}) \approx 0.098191$ and $\mathrm{sr}(\tilde{a}) = 2$. Let $n_b = 1$ and $n_{\tilde{b}} = 4$. Then $\mathsf{p} = 1$ and $\mathsf{D} = -\frac{3}{128}z^{-1}$. Taking $\mathsf{d} = 1, c_1 = 1, \epsilon_1 = 1, n_1 = n_2 = 2$, we have

$$\tilde{b}_1 = \tfrac{1}{8}\{1, -2, \underline{\mathbf{0}}, 2, -1\}_{[-2,2]}, \qquad \tilde{b}_2 = \tfrac{1}{16}\{1, -4, \underline{\mathbf{6}}, -4, 1\}_{[-2,2]}$$

and

$$b_1 = \tfrac{1}{8}\{-1, -4, \underline{\mathbf{0}}, 4, 1\}_{[-2,2]}, \qquad b_2 = \tfrac{1}{16}\{-1, -4, \underline{\mathbf{10}}, -4, -1\}_{[-2,2]}$$

with $\mathrm{vm}(b_1) = 1, \mathrm{vm}(b_2) = 2$ and $\mathrm{vm}(\tilde{b}_1) = 3, \mathrm{vm}(\tilde{b}_2) = 4$. Moreover, $\tilde{\mathsf{b}}_1(-1) = \mathsf{b}_1(-1) = 0$.

If $n_b = n_{\tilde{b}} = 2$, then $\mathsf{p} = 1$ and $\mathsf{D} = \frac{3}{128}$. Taking $\mathsf{d} = 1, c_1 = 0, \epsilon_1 = 1$, $n_1 = 1, n_2 = 2$, we have

$$\tilde{b}_1 = \tfrac{1}{2}\{-1, 2, \underline{\mathbf{-1}}\}_{[-2,0]}, \qquad \tilde{b}_2 = \tfrac{1}{16}\{-3, 4, \underline{\mathbf{-2}}, 4, -3\}_{[-2,2]}$$

and

$$b_1 = \tfrac{1}{4}\{-1, 2, \underline{\mathbf{-1}}\}_{[-2,0]}, \qquad b_2 = \tfrac{1}{16}\{-1, -4, \underline{\mathbf{10}}, -4, -1\}_{[-2,2]}$$

with $\mathrm{vm}(\tilde{b}_1) = \mathrm{vm}(\tilde{b}_2) = \mathrm{vm}(b_1) = \mathrm{vm}(b_2) = 2$.

Example 5. Let $a = \tilde{a} = \{-\frac{1}{32}, 0, \frac{9}{32}, \underline{\mathbf{\frac{1}{2}}}, \frac{9}{32}, 0, -\frac{1}{32}\}_{[-3,3]}$. If $\Theta = \delta$ and $n_b = n_{\tilde{b}} = 2$, then $\mathsf{p}(z) = 1$ and $\mathsf{D}(z) = \frac{7}{256} - \frac{1}{512}(z + z^{-1})$. Taking $\mathsf{d} = 1, c_1 = 0, \epsilon_1 = 1, n_1 = 2$, $n_2 = 3$, we have

$$\tilde{b}_1 = \tfrac{1}{8}\{-1, -2, 6, \underline{\mathbf{-2}}, -1\}_{[-3,1]}, \qquad \tilde{b}_2 = \tfrac{1}{32}\{1, 0, -9, \underline{\mathbf{16}}, -9, 0, 1\}_{[-3,3]}$$

and

$$b_1 = \tfrac{1}{8}\{-1, 0, 2, \underline{\mathbf{0}}, -1\}_{[-3,1]}, \qquad b_2 = \tfrac{1}{32}\{-1, 0, -7, \underline{\mathbf{16}}, -7, 0, -1\}_{[-3,3]}$$

with $\mathrm{vm}(b_1) = \mathrm{vm}(b_2) = \mathrm{vm}(\tilde{b}_1) = 2$ and $\mathrm{vm}(\tilde{b}_2) = 4$.

By Lemma 1 we can take $\Theta = \{-\frac{11}{5040}, \frac{4}{105}, -\frac{223}{1680}, \underline{\mathbf{\frac{376}{315}}}, -\frac{223}{1680}, \frac{4}{105}, -\frac{11}{5040}\}_{[-3,3]}$ with $n_b = n_{\tilde{b}} = 4$. Then $\mathsf{p} = 1$. Take $\mathsf{d} = 1, c_1 = 0, \epsilon_1 = 1, n_1 = 4, n_2 = 5$, we have

$$\begin{aligned}\tilde{b}_1 = \tfrac{1}{13440}\{&-55, -64, 1489, -112, -10554, 18592, -10554,\\ &\underline{\mathbf{-112}}, 1489, -64, -55\}_{[-7,3]},\\ \tilde{b}_2 = \tfrac{1}{40320}\{&-44, 0, 933, 896, -4668, -2736, 11238, \underline{\mathbf{-2736}},\\ &-4668, 896, 933, 0, -44\}_{[-7,5]},\end{aligned}$$

and

$$\begin{aligned}b_1 &= \tfrac{1}{512}\{-1, 0, 3, 16, -66, 96, -66, \underline{\mathbf{16}}, 3, 0, -1\}_{[-7,3]},\\ b_2 &= \tfrac{1}{512}\{-1, 0, 18, 16, -63, -144, 348, \underline{\mathbf{-144}}, -63, 16, 18, 0, -1\}_{[-7,5]}\end{aligned}$$

with $\mathrm{vm}(b_1) = \mathrm{vm}(b_2) = \mathrm{vm}(\tilde{b}_1) = \mathrm{vm}(\tilde{b}_2) = 4$.

If $\Theta = \delta$ and $n_b = n_{\tilde{b}} = 2$, setting $M = 2$ and $N = 4$ in Proposition 2, we obtain

$$\tilde{a} = \{-\tfrac{1}{16}, \tfrac{1}{4}, \underline{\mathbf{\tfrac{5}{8}}}, \tfrac{1}{4}, -\tfrac{1}{16}\}_{[-2,2]} \tag{4.4}$$

with $\mathrm{sm}(\tilde{a}) \approx 0.885296$ and $\mathrm{sr}(\tilde{a}) = 2$. Then $\mathsf{p} = 1$ and $\mathsf{D} = \frac{1}{64}$. Taking $\mathsf{d} = 1, c_1 = 0, \epsilon_1 = 1, n_1 = 2, n_2 = 3$, we have

$$\tilde{b}_1 = \tfrac{1}{4}\{-1, 2, \underline{\mathbf{-1}}\}_{[-2,0]}, \qquad \tilde{b}_2 = \tfrac{1}{16}\{1, -4, \underline{\mathbf{6}}, -4, 1\}_{[-2,2]},$$

and

$$b_1 = \tfrac{1}{4}\{-1, 0, 2, \underline{\mathbf{0}}, -1\}_{[-3,1]}, \qquad b_2 = \tfrac{1}{32}\{-1, 0, -7, \underline{\mathbf{16}}, -7, 0, -1\}_{[-3,3]}$$

with $\mathrm{vm}(b_1) = \mathrm{vm}(b_2) = \mathrm{vm}(\tilde{b}_1) = 2$ and $\mathrm{vm}(\tilde{b}_2) = 4$.

5. Properties of dual framelet filter banks and explanations for algorithm 1

To understand better our algorithms and dual framelet filter banks, in this section we investigate several basic properties of dual framelet filter banks. Then we shall provide some explanations and discussions on Algorithm 1.

Let us first recall the multilevel discrete framelet transform associated with a dual framelet filter bank from [5]. Recall that $l(\mathbb{Z})$ denotes the set of all sequences on $\mathbb{Z}$. For a filter $u = \{u(k)\}_{k\in\mathbb{Z}} \in l_0(\mathbb{Z})$, the *subdivision operator* $\mathcal{S}_u : l(\mathbb{Z}) \to l(\mathbb{Z})$ and the *transition operator* $\mathcal{T}_u : l(\mathbb{Z}) \to l(\mathbb{Z})$ are defined to be

$$[\mathcal{S}_u v](n) := 2\sum_{k\in\mathbb{Z}} v(k)u(n-2k), \qquad n \in \mathbb{Z}, \tag{5.1}$$

$$[\mathcal{T}_u v](n) := 2\sum_{k\in\mathbb{Z}} v(k)\overline{u(k-2n)}, \qquad n \in \mathbb{Z} \tag{5.2}$$

for $v \in l(\mathbb{Z})$. In terms of the z-transform, we have

$$[\mathcal{S}_{\mathsf{u}}\mathsf{v}](z) = 2\mathsf{u}(z)\mathsf{v}(z^2), \qquad [\mathcal{T}_{\mathsf{u}}\mathsf{v}](z^2) = \mathsf{u}^\star(z)\mathsf{v}(z) + \mathsf{u}^\star(-z)\mathsf{v}(-z), \tag{5.3}$$

where $u^\star(k) := \overline{u(-k)}, k \in \mathbb{Z}$ and $\mathsf{u}^\star(z) = \sum_{k\in\mathbb{Z}} u^\star(k)z^k = \sum_{k\in\mathbb{Z}} \overline{u(k)}z^{-k}$.

Let $(\{\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\}, \{a; b_1, \ldots, b_s\})_\Theta$ be a dual framelet filter bank. For $J \in \mathbb{N}$, *a J-level discrete framelet decomposition* is given by

$$v_{j+1} := \tfrac{\sqrt{2}}{2}\mathcal{T}_{\tilde{a}} v_j, \quad w_{j+1;\ell} := \tfrac{\sqrt{2}}{2}\mathcal{T}_{\tilde{b}_\ell} v_j, \qquad \ell = 1, \ldots, s, \quad j = 0, \ldots, J-1, \tag{5.4}$$

where $v_0 : \mathbb{Z} \to \mathbb{C}$ is an input signal. After a J-level discrete framelet decomposition, the original input signal v_0 is decomposed into one sequence v_J of low-pass framelet coefficients and sJ sequences $w_{j;\ell}$ of high-pass framelet coefficients for $\ell = 1, \ldots, s$ and $j = 1, \ldots, J$. Such framelet coefficients are often processed by thresholding or quantization. *A J-level discrete framelet reconstruction* ([5]) is

$$\breve{v}_J := \Theta^\star * \mathring{v}_J, \tag{5.5}$$

$$\breve{v}_j := \frac{\sqrt{2}}{2}\mathcal{S}_a \breve{v}_{j+1} + \frac{\sqrt{2}}{2}\sum_{\ell=1}^{s} \mathcal{S}_{b_\ell} \mathring{w}_{j+1;\ell}, \qquad j = J-1, \ldots, 0, \tag{5.6}$$

$$\text{recover } \mathring{v}_0 \text{ from } \breve{v}_0 \text{ via the relation } \mathring{v}_0 = \Theta^\star * \breve{v}_0. \tag{5.7}$$

When nothing is performed on the framelet coefficients, that is, $\mathring{v}_J = v_J$ and $\mathring{w}_{j;\ell} = w_{j,\ell}$ for all $\ell = 1, \ldots, s$ and $j = 1, \ldots, J$, we say that the above J-level discrete framelet transform has the perfect reconstruction property if $\mathring{v}_0 = v_0$.

The following result shows that the condition (1.1) in the definition of a dual framelet filter bank $(\{\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\}, \{a; b_1, \ldots, b_s\})_\Theta$ corresponds to the perfect reconstruction property of its associated discrete framelet transform.

Theorem 4. *Let $\tilde{a}, \tilde{b}_1, \ldots, \tilde{b}_s, a, b_1, \ldots, b_s, \Theta \in l_0(\mathbb{Z})$. The following statements are equivalent:*

(i) *The perfect reconstruction property holds: for all $v \in l(\mathbb{Z})$,*

$$\frac{1}{2}\mathcal{S}_a(\Theta^\star * \mathcal{T}_{\tilde{a}} v) + \frac{1}{2}\sum_{\ell=1}^{s} \mathcal{S}_{b_\ell}\mathcal{T}_{\tilde{b}_\ell} v = \Theta^\star * v. \tag{5.8}$$

(ii) *The identity in* (5.8) *holds for all $v \in l_0(\mathbb{Z})$.*

(iii) *The identity in* (5.8) *holds for the particular sequences* $v = \delta$ *and* $\delta(\cdot - 1)$, *where* δ *is the Dirac sequence such that* $\delta(0) = 1$ *and* $\delta(k) = 0$ *for all* $k \neq 0$.

(iv) $(\{\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_s\}, \{a; b_1, \ldots, b_s\})_\Theta$ *is a dual framelet filter bank.*

Proof. (i)$\Longrightarrow$(ii)$\Longrightarrow$(iii) is trivial, since $\{\delta, \delta(\cdot - 1)\} \subseteq l_0(\mathbb{Z}) \subseteq l(\mathbb{Z})$. By (5.3), (5.8) can be equivalently rewritten using the z-transform as follows:

$$\begin{aligned} &\mathsf{v}(z)\left[\Theta^\star(z^2)\tilde{\mathsf{a}}^\star(z)\mathsf{a}(z) + \sum_{\ell=1}^{s} \tilde{\mathsf{b}}_\ell^\star(z)\mathsf{b}_\ell(z)\right] \\ &\quad + \mathsf{v}(-z)\left[\Theta^\star(z^2)\tilde{\mathsf{a}}^\star(-z)\mathsf{a}(z) + \sum_{\ell=1}^{s} \tilde{\mathsf{b}}_\ell^\star(-z)\mathsf{b}_\ell(z)\right] = \mathsf{v}(z)\Theta^\star(z). \end{aligned} \tag{5.9}$$

To prove (iii)$\Longrightarrow$(iv), plugging $v = \delta$ into (5.9) and noting $\mathsf{v}(z) = 1$, we see that

$$\begin{aligned} \Theta^\star(z) = &\left[\Theta^\star(z^2)\tilde{\mathsf{a}}^\star(z)\mathsf{a}(z) + \sum_{\ell=1}^{s} \tilde{\mathsf{b}}_\ell^\star(z)\mathsf{b}_\ell(z)\right] \\ &+ \left[\Theta^\star(z^2)\tilde{\mathsf{a}}^\star(-z)\mathsf{a}(z) + \sum_{\ell=1}^{s} \tilde{\mathsf{b}}_\ell^\star(-z)\mathsf{b}_\ell(z)\right]. \end{aligned}$$

Plugging $v = \delta(\cdot - 1)$ into (5.9) and noting $\mathsf{v}(z) = z$, we deduce from (5.9) that

$$\begin{aligned} \Theta^\star(z) = &\left[\Theta^\star(z^2)\tilde{\mathsf{a}}^\star(z)\mathsf{a}(z) + \sum_{\ell=1}^{s} \tilde{\mathsf{b}}_\ell^\star(z)\mathsf{b}_\ell(z)\right] \\ &- \left[\Theta^\star(z^2)\tilde{\mathsf{a}}^\star(-z)\mathsf{a}(z) + \sum_{\ell=1}^{s} \tilde{\mathsf{b}}_\ell^\star(-z)\mathsf{b}_\ell(z)\right]. \end{aligned}$$

From these two identities, it is straightforward to see that (1.1) must hold. Therefore, (iii)$\Longrightarrow$(iv).

If (1.1) is satisfied, then it is straightforward to see that (5.9) holds for all $v \in l_0(\mathbb{Z})$. That is, we proved (iv)$\Longrightarrow$(ii). The claim (ii)$\Longrightarrow$(i) follows from the locality of the subdivision operator and transition operator. □

For a tight framelet filter bank, we have the following result.

Proposition 5. *Let* $\theta, a, b_1, \ldots, b_s \in l_0(\mathbb{Z})$ *be finitely supported sequences on* $\mathbb{Z}$. *Then*

$$\|\theta * \mathcal{T}_a v\|_{l_2(\mathbb{Z})}^2 + \|\mathcal{T}_{b_1} v\|_{l_2(\mathbb{Z})}^2 + \cdots + \|\mathcal{T}_{b_s} v\|_{l_2(\mathbb{Z})}^2 = 2\|\theta * v\|_{l_2(\mathbb{Z})}^2, \qquad \forall\, v \in l_2(\mathbb{Z}), \tag{5.10}$$

if and only if $\{a; b_1, \ldots, b_s\}_\Theta$ *is a tight framelet filter bank with* $\Theta := \theta * \theta^\star$.

Proof. Note that $\Theta^\star = \Theta$ and $\|\theta * v\|_{l_2(\mathbb{Z})}^2 = \langle \Theta^\star * v, v\rangle$. By the duality between $\mathcal{S}_a$ and $\mathcal{T}_a$, we have

$$\|\theta * \mathcal{T}_a v\|_{l_2(\mathbb{Z})}^2 = \langle \theta * \mathcal{T}_a v, \theta * \mathcal{T}_a v\rangle = \langle \Theta^\star * \mathcal{T}_a v, \mathcal{T}_a v\rangle = \langle \mathcal{S}_a(\Theta^\star * \mathcal{T}_a v), v\rangle.$$

Now all the claims follow from Theorem 4 by using the polarization identity. □

We now show that the moment correcting filter Θ in every tight framelet filter bank must take the special form in Proposition 5.

Lemma 6. *Let $\mathcal{M}_{a,a,\Theta}$ be defined in* (1.2) *with $\tilde{a} = a$. Then*

$$\mathcal{M}_{a,a,\Theta}(z) \geqslant 0 \qquad \forall\, z \in \mathbb{T} := \{\zeta \in \mathbb{C} \,:\, |\zeta| = 1\}, \tag{5.11}$$

if and only if

$$\Theta(z) \geqslant 0, \qquad \forall\, z \in \mathbb{T} \tag{5.12}$$

and

$$\begin{aligned} &\det(\mathcal{M}_{a,a,\Theta}(z)) \\ &\quad = \Theta(z)\Theta(-z) - \Theta(z^2)\big[\Theta(-z)\mathsf{a}(z)\mathsf{a}^\star(z) + \Theta(z)\mathsf{a}(-z)\mathsf{a}^\star(-z)\big] \geqslant 0 \end{aligned} \tag{5.13}$$

for all $z \in \mathbb{T}$. Consequently, if $\{a; b_1, \ldots, b_s\}_\Theta$ is a tight framelet filter bank, then there exists $\theta \in l_0(\mathbb{Z})$ such that $\Theta = \theta * \theta^\star$.

Proof. Suppose that (5.11) holds. Then it is trivial that (5.13) holds, $\mathcal{M}^\star_{a,a,\Theta}(z) = \mathcal{M}_{a,a,\Theta}(z)$, and the $(1,1)$-entry of $\mathcal{M}_{a,a,\Theta}$ must be nonnegative, that is,

$$\Theta(z) - \Theta(z^2)\mathsf{a}(z)\mathsf{a}^\star(z) \geqslant 0 \qquad \forall\, z \in \mathbb{T}. \tag{5.14}$$

Since $\mathcal{M}^\star_{a,a,\Theta}(z) = \mathcal{M}_{a,a,\Theta}(z)$, we have $[\mathcal{M}_{a,a,\Theta}(z)]^\star_{1,2} = [\mathcal{M}_{a,a,\Theta}(z)]_{2,1}$, from which we conclude that $\Theta^\star = \Theta$ if a is not identically zeros. If a is identically zero, then $[\mathcal{M}_{a,a,\Theta}(z)]^\star_{1,1} = [\mathcal{M}_{a,a,\Theta}(z)]_{1,1}$ implies that $\Theta^\star = \Theta$. Therefore, $\Theta(z) \in \mathbb{R}$ for all $z \in \mathbb{T}$.

We use proof by contradiction to show that (5.12) must hold. Suppose that $\Theta(e^{-i\xi_0}) < 0$ for some $\xi_0 \in \mathbb{R}$. Since Θ is continuous on $\mathbb{T}$, there exists a nonempty open interval (c, d) such that $\Theta(e^{-i\xi}) < 0$ for all $\xi \in (c, d)$. However, if $\Theta(e^{-i\xi}) < 0$, then (5.14) will force $\Theta(e^{-i2\xi}) < 0$. Consequently, $\Theta(e^{-i\xi}) < 0$ for all $\xi \in (2^n c, 2^n d)$ and $n \in \mathbb{N}$. Since $\mathbb{T}$ is compact and $c < d$, we must have $\Theta(e^{-i\xi}) < 0$ for all $\xi \in \mathbb{R}$. Hence, $\Theta(z^2)\Theta(z) > 0$ for all $z \in \mathbb{T}$ and by (5.13) we have

$$\begin{aligned} &\big[\Theta(z) - \Theta(z^2)\mathsf{a}(z)\mathsf{a}^\star(z)\big]\Theta(-z) \\ &\qquad \geqslant \Theta(z)\Theta(-z) - \Theta(z^2)\big[\Theta(-z)\mathsf{a}(z)\mathsf{a}^\star(z) + \Theta(z)\mathsf{a}(-z)\mathsf{a}^\star(-z)\big] \geqslant 0 \end{aligned} \tag{5.15}$$

for all $z \in \mathbb{T}$. Since $\Theta(z) < 0$ for all $z \in \mathbb{T}$, the above inequality and (5.14) imply

$$\Theta(z) - \Theta(z^2)\mathsf{a}(z)\mathsf{a}^\star(z) = 0 \qquad \forall\, z \in \mathbb{T}. \tag{5.16}$$

From (5.15) and (5.16), we see that $\Theta(z^2)\Theta(z)\mathsf{a}(-z)\mathsf{a}^\star(-z) = 0$ which forces a to be identically zero. By (5.16) again, we conclude that Θ is identically zero, which is a contradiction to $\Theta(z) < 0$ for all $z \in \mathbb{T}$. Therefore, $\Theta(z) \geqslant 0$ for all $z \in \mathbb{T}$ must hold.

Conversely, suppose that (5.12) and (5.13) hold. Then (5.15) holds. Now by (5.12), we see that (5.14) holds. That is, the $(1,1)$-entry of $\mathcal{M}_{a,a,\Theta}$ must be nonnegative. Since (5.13) holds, by results from linear algebra, we conclude that (5.11) must hold.

By (1.1) with $\tilde{a} = a, \tilde{b}_1 = b_1, \ldots, \tilde{b}_s = b_s$, we must have $\mathcal{M}_{a,a,\Theta}(z) \geqslant 0$ for all $z \in \mathbb{T}$. Therefore, $\Theta(z) \geqslant 0$ for all $z \in \mathbb{T}$. By Fejér-Riesz lemma, we see that $\Theta = \theta * \theta^\star$ for some $\theta \in l_0(\mathbb{Z})$. □

We now explain the choice $s = 2$ in Algorithm 1. The number of high-pass filters is preferred to be as small as possible in applications. As demonstrated by the following result, if the number of high-pass filters in a dual framelet filter bank is $s = 1$, then it is essentially a biorthogonal wavelet filter bank.

Theorem 7. *Let $(\{\tilde{a};\tilde{b}\},\{a;b\})_\Theta$ be a dual framelet filter bank such that Θ is not identically zero. Then there exists a nonzero number $\lambda\in\mathbb{C}$ such that*

$$\Theta(z^2)=\lambda\Theta(z)\Theta(-z),\qquad \forall\, z\in\mathbb{T} \tag{5.17}$$

and $(\{\mathring{a};\mathring{b}\},\{a;b\})$ is a biorthogonal wavelet filter bank, that is,

$$\begin{bmatrix}\mathring{\mathsf{a}}(z) & \mathring{\mathsf{b}}(z)\\ \mathring{\mathsf{a}}(-z) & \mathring{\mathsf{b}}(-z)\end{bmatrix}\begin{bmatrix}\mathsf{a}(z) & \mathsf{b}(z)\\ \mathsf{a}(-z) & \mathsf{b}(-z)\end{bmatrix}^\star=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \tag{5.18}$$

where all the above filters are finitely supported and are given by

$$\mathring{\mathsf{a}}(z):=\tilde{\mathsf{a}}(z)\lambda\Theta(-z)\qquad \textit{and}\qquad \mathring{\mathsf{b}}(z):=\tilde{\mathsf{b}}(z)/\Theta(z). \tag{5.19}$$

Moreover, (5.18) *implies*

$$\begin{aligned}&\mathring{\mathsf{b}}(\xi)=\overline{c^{-1}}z^{2n-1}\mathsf{a}^\star(-z),\\ &\mathsf{b}(z)=cz^{2n-1}\mathring{\mathsf{a}}^\star(-z),\qquad \textit{for some } c\in\mathbb{C}\backslash\{0\},\ n\in\mathbb{Z}.\end{aligned} \tag{5.20}$$

If, in addition, $\tilde{a}=a$ and $\tilde{b}=b$, that is, $\{a;b\}_\Theta$ is a tight framelet filter bank, then $\Theta=\theta*\theta^\star$ for some $\theta\in l_0(\mathbb{Z})$ and $\{\breve{a};\breve{b}\}$ is an orthogonal wavelet filter bank, where $\breve{a},\breve{b}\in l_0(\mathbb{Z})$ are given by

$$\breve{\mathsf{a}}(z):=\mathsf{a}(z)\sqrt{\lambda}\boldsymbol{\theta}(-z),\qquad \breve{\mathsf{b}}(z):=\mathsf{b}(z)/\boldsymbol{\theta}(z). \tag{5.21}$$

Proof. Since $s=1$, (1.1) can be rewritten as the following equivalent matrix form:

$$\begin{bmatrix}\tilde{\mathsf{a}}(z) & \tilde{\mathsf{b}}(z)\\ \tilde{\mathsf{a}}(-z) & \tilde{\mathsf{b}}(-z)\end{bmatrix}\begin{bmatrix}\Theta(z^2) & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\mathsf{a}(z) & \mathsf{b}(z)\\ \mathsf{a}(-z) & \mathsf{b}(-z)\end{bmatrix}^\star=\begin{bmatrix}\Theta(z) & 0\\ 0 & \Theta(-z)\end{bmatrix}. \tag{5.22}$$

Taking determinant on both sides of (5.22), we have

$$\Theta(z^2)\tilde{\boldsymbol{\eta}}(z)\boldsymbol{\eta}^\star(z)=\Theta(z)\Theta(-z), \tag{5.23}$$

where

$$\tilde{\boldsymbol{\eta}}(z):=z\big(\tilde{\mathsf{a}}(z)\tilde{\mathsf{b}}(-z)-\tilde{\mathsf{a}}(-z)\tilde{\mathsf{b}}(z)\big)\ \text{and}\ \boldsymbol{\eta}(z):=z\big(\mathsf{a}(z)\mathsf{b}(-z)-\mathsf{a}(-z)\mathsf{b}(z)\big). \tag{5.24}$$

Note that the filter support of $\Theta(z^2)$ is the same as the filter support of $\Theta(z)\Theta(-z)$. Since (5.23) implies that $\Theta(z^2)$ is a factor of $\Theta(z)\Theta(-z)$ and Θ is not identically zero, we see that (5.17) must hold for some $\lambda\in\mathbb{C}\backslash\{0\}$. Consequently, we have $\lambda\tilde{\boldsymbol{\eta}}(z)\boldsymbol{\eta}^\star(z)=1$. By the definition of $\boldsymbol{\eta}$ and $\tilde{\boldsymbol{\eta}}$, we must have $\boldsymbol{\eta}(z)=-cz^{2n}$ for some nonzero $c\in\mathbb{C}$ and $n\in\mathbb{Z}$. By a direct calculation, it follows from (5.17) and (5.22) that (5.18) holds with $\mathring{\mathsf{a}}$ and $\mathring{\mathsf{b}}$ being defined in (5.19). By (5.18), we deduce that

$$\begin{bmatrix}\mathring{\mathsf{a}}(z) & \mathring{\mathsf{b}}(z)\\ \mathring{\mathsf{a}}(-z) & \mathring{\mathsf{b}}(-z)\end{bmatrix}=\begin{bmatrix}\mathsf{a}^\star(z) & \mathsf{a}^\star(-z)\\ \mathsf{b}^\star(z) & \mathsf{b}^\star(-z)\end{bmatrix}^{-1}=\frac{1}{z\boldsymbol{\eta}^\star(z)}\begin{bmatrix}\mathsf{b}^\star(-z) & -\mathsf{a}^\star(-z)\\ -\mathsf{b}^\star(z) & \mathsf{a}^\star(z)\end{bmatrix}. \tag{5.25}$$

Plugging $\boldsymbol{\eta}(z)=-cz^{2n}$ into the above identity and comparing the entries of the matrices on both sides, we conclude that (5.20) holds. Consequently, $\mathring{b}$ must be a finitely supported sequence.

When $\tilde{a}=a$ and $\tilde{b}=b$, by (5.19), $\breve{a}$ and $\breve{b}$ are finitely supported filters. Since $\Theta(z)\geqslant 0$ for all $z\in\mathbb{T}$ by Lemma 6, we see that $\lambda>0$. Using (5.17) and (5.22), we can directly check that $\{\breve{a};\breve{b}\}$ is an orthogonal wavelet filter bank. $\square$

We make a remark here on discrete framelet transform. There is a de-convolution in (5.7) to recover $\mathring{v}_0$ from $\breve{v}_0$ if Θ is not a nonzero monomial. We can easily avoid this troubling deconvolution by the following argument. Let $(\{\tilde{a}_m; \tilde{b}_{m,1}, \ldots, \tilde{b}_{m,s_m}\}, \{a_m; b_{m,1}, \ldots, b_{m,s_m}\})_{\Theta_m}$, $m = 1, \ldots, n$ be a family of dual framelet filter banks. For any input signal v, let $\breve{v}_{0,m}$ be the reconstructed signal *before deconvolution* using the dual framelet filter bank $(\{\tilde{a}_m; \tilde{b}_{m,1}, \ldots, \tilde{b}_{m,s_m}\}, \{a_m; b_{m,1}, \ldots, b_{m,s_m}\})_{\Theta_m}$. Suppose that there exist $\tilde{\Theta}_1, \ldots, \tilde{\Theta}_n \in l_0(\mathbb{Z})$ such that

$$\tilde{\Theta}_1(z)\Theta_1^\star(z) + \cdots + \tilde{\Theta}_n(z)\Theta_n^\star(z) = 1. \tag{5.26}$$

Avoiding deconvolution in (5.7), we can recover $\mathring{v}_0$ via $\mathring{v}_0 = \tilde{\Theta}_1 * \breve{v}_{0,1} + \cdots + \tilde{\Theta}_n * \breve{v}_{0,n}$.

In the rest of this section, we provide some explanations and discussions on Algorithm 1. Let us first discuss symmetry property for a linear combination of filters with symmetry. Let u and v be filters having [complex] symmetry. If both u and v have the same symmetry types, then it is trivial to see that $u \pm v$ also has [complex] symmetry. However, if the symmetry types of u and v are different, then $u \pm v$ usually does not have any symmetry. More precisely, we have the following result.

Lemma 8. *Let $u, v \in l_0(\mathbb{Z})$ be nontrivial filters having symmetry $\mathsf{Su}(z) = \epsilon_u z^{c_u}$ and $\mathsf{Sv}(z) = \epsilon_v z^{c_v}$ (or complex symmetry $\mathbb{S}\mathsf{u}(z) = \epsilon_u z^{c_u}$ and $\mathbb{S}\mathsf{v}(z) = \epsilon_v z^{c_v}$) for some $\epsilon_u, \epsilon_v \in \{-1, 1\}$ and $c_u, c_v \in \mathbb{Z}$. If $u + v$ also has symmetry $\mathsf{S}(\mathsf{u}+\mathsf{v}) = \epsilon z^c$ (or complex symmetry $\mathbb{S}(\mathsf{u}+\mathsf{v}) = \epsilon z^c$), then one of the following two cases must hold:*

(i) *$\epsilon_u = \epsilon_v$ and $c_u = c_v$, that is, u and v have the same [complex] symmetry type.*

(ii) *$\epsilon\epsilon_u z^{c-c_u} \neq 1$ and $\epsilon\epsilon_v z^{c-c_v} \neq 1$, that is, $\mathsf{Su} \neq \mathsf{S}(\mathsf{u}+\mathsf{v}) \neq \mathsf{Sv}$ (or $\mathbb{S}\mathsf{u} \neq \mathbb{S}(\mathsf{u}+\mathsf{v}) \neq \mathbb{S}\mathsf{v}$ for complex symmetry). Moreover, u and v must take the particular form: $\mathsf{u}(z) = \mathsf{w}(z)\mathring{\mathsf{u}}(z), \mathsf{v}(z) = \mathsf{w}(z)\mathring{\mathsf{v}}(z)$ with $\mathsf{w} := \gcd(\mathsf{u}, \mathsf{v})$ and*

$$\mathring{\mathsf{u}}(z) := \frac{1 - \epsilon\epsilon_v z^{c-c_v}}{\mathsf{q}(z)}, \qquad \mathring{\mathsf{v}}(z) := \frac{\epsilon\epsilon_u z^{c-c_u} - 1}{\mathsf{q}(z)}$$

where q is a Laurent polynomial with [complex] symmetry such that

$$\mathsf{q}(z) \mid \gcd(1 - \epsilon\epsilon_u z^{c-c_u}, 1 - \epsilon\epsilon_v z^{c-c_v}).$$

Proof. We only prove the case of symmetry while the case of complex symmetry can be proved in the same way. Since $\mathsf{u}(z^{-1}) = \epsilon_u z^{-c_u}\mathsf{u}(z)$ and $\mathsf{v}(z^{-1}) = \epsilon_v z^{-c_v}\mathsf{v}(z)$, it follows from $\mathsf{S}(\mathsf{u}+\mathsf{v}) = \epsilon z^c$ that

$$\mathsf{u}(z) + \mathsf{v}(z) = \epsilon z^c(\mathsf{u}(z^{-1}) + \mathsf{v}(z^{-1})) = \epsilon\epsilon_u z^{c-c_u}\mathsf{u}(z) + \epsilon\epsilon_v z^{c-c_v}\mathsf{v}(z)$$

from which we deduce that

$$\mathsf{u}(z)(\epsilon\epsilon_u z^{c-c_u} - 1) = \mathsf{v}(z)(1 - \epsilon\epsilon_v z^{c-c_v}). \tag{5.27}$$

If either $\epsilon\epsilon_u z^{c-c_u} - 1 = 0$ or $1 - \epsilon\epsilon_v z^{c-c_v} = 0$, since both u and v are not identically zero, we conclude from (5.27) that both $\epsilon\epsilon_u z^{c-c_u} - 1 = 0$ and $1 - \epsilon\epsilon_v z^{c-c_v} = 0$ must hold. That is, Case (i) must hold.

Otherwise, both $\epsilon\epsilon_u z^{c-c_u} - 1 \neq 0$ and $1 - \epsilon\epsilon_v z^{c-c_v} \neq 0$. That is, we must have Case (ii) with $\epsilon\epsilon_u z^{c-c_u} \neq 1$ and $\epsilon\epsilon_v z^{c-c_v} \neq 1$. For this case, we now show that u and v must take the special form. Since $\mathsf{w} := \gcd(\mathsf{u}, \mathsf{v})$ and both u and v have [complex] symmetry, w must have [complex] symmetry (see [13]). Therefore, (5.27) implies

$$\mathring{\mathsf{u}}(z)(\epsilon\epsilon_u z^{c-c_u} - 1) = \mathring{\mathsf{v}}(z)(1 - \epsilon\epsilon_v z^{c-c_v}). \tag{5.28}$$

Note that $\gcd(\mathring{\mathsf{u}}, \mathring{\mathsf{v}}) = 1$. From (5.28), we must have $\mathring{\mathsf{u}}(z) \mid (1 - \epsilon\epsilon_v z^{c-c_v})$. Define a Laurent polynomial $\mathsf{q}(z) := (1 - \epsilon\epsilon_v z^{c-c_v})/\mathring{\mathsf{u}}(z)$. Then q has [complex] symmetry. Now it follows from (5.28) again that

$$\mathring{\mathsf{u}}(z)\mathsf{q}(z) = 1 - \epsilon\epsilon_v z^{c-c_v} \qquad \text{and} \qquad \mathring{\mathsf{v}}(z)\mathsf{q}(z) = \epsilon\epsilon_u z^{c-c_u} - 1.$$

Hence, u and v must take the special form in Case (ii). □

By Lemma 8, if $\mathsf{u}+\mathsf{v} = \mathsf{w}$ and all $\mathsf{u}, \mathsf{v}, \mathsf{w}$ have [complex] symmetry, then it is very natural to assume that $\mathsf{S}\mathsf{u} = \mathsf{S}\mathsf{v} = \mathsf{S}\mathsf{w}$ (or $\mathbb{S}\mathsf{u} = \mathbb{S}\mathsf{v} = \mathbb{S}\mathsf{w}$ for the case of complex symmetry).

Since we discuss Algorithm 1 in this section, without further mention, we always assume that $(\{\tilde{a}; \tilde{b}_1, \tilde{b}_2\}, \{a; b_1, b_2\})_\Theta$ is a dual framelet filter bank such that

(1) $\min(\mathrm{vm}(b_1), \mathrm{vm}(b_2)) \geqslant n_b$ and $\min(\mathrm{vm}(\tilde{b}_1), \mathrm{vm}(\tilde{b}_2)) \geqslant n_{\tilde{b}}$ so that $\mathring{\mathsf{b}}_1, \mathring{\mathsf{b}}_2, \mathring{\tilde{\mathsf{b}}}_1, \mathring{\tilde{\mathsf{b}}}_2$ in (1.5) are well defined;

(2) all filters have symmetry such that (1.20) holds and

$$\mathsf{S}\mathring{\mathsf{b}}_1(z) = \epsilon_1 z^{c_1}, \quad \mathsf{S}\mathring{\mathsf{b}}_2(z) = \epsilon_2 z^{c_2}, \quad \mathsf{S}\mathring{\tilde{\mathsf{b}}}_1(z) = \tilde{\epsilon}_1 z^{\tilde{c}_1}, \quad \mathsf{S}\mathring{\tilde{\mathsf{b}}}_2(z) = \tilde{\epsilon}_2 z^{\tilde{c}_2}. \tag{5.29}$$

For the case of complex symmetry, replace S by the complex symmetry operator $\mathbb{S}$ throughout;

(3) all filters $\mathsf{b}_1, \mathsf{b}_2, \tilde{\mathsf{b}}_1, \tilde{\mathsf{b}}_2, \mathsf{a}, \tilde{\mathsf{a}}, \Theta$ are not identically zero.

Note that the perfect reconstruction condition (1.1) with $s = 2$ in the definition of a dual framelet filter bank can be rewritten as (1.17) plus

$$\tilde{\mathsf{b}}_1(z)\mathsf{b}_1^\star(z) + \tilde{\mathsf{b}}_2(z)\mathsf{b}_2^\star(z) = \Theta(z) - \Theta(z^2)\tilde{\mathsf{a}}(z)\mathsf{a}^\star(z). \tag{5.30}$$

According to Lemma 8, since all filters have symmetry, by (1.17) it is natural to assume that

$$\mathsf{S}(\tilde{\mathsf{b}}_1(z)\mathsf{b}_1^\star(-z)) = \mathsf{S}(\tilde{\mathsf{b}}_2(z)\mathsf{b}_2^\star(-z)) = \mathsf{S}(\Theta(z^2)\tilde{\mathsf{a}}(z)\mathsf{a}^\star(-z)). \tag{5.31}$$

Note that $\mathsf{S}\mathsf{u}(-z) = \mathsf{S}\mathsf{u}(z)$ or $\mathsf{S}\mathsf{u}(-z) = -\mathsf{S}\mathsf{u}(z)$ for any filter u having symmetry. By (5.31), we have two cases to consider.

Case 1: $\mathsf{S}(\tilde{\mathsf{b}}_1(z)\mathsf{b}_1^\star(z)) = \mathsf{S}(\tilde{\mathsf{b}}_2(z)\mathsf{b}_2^\star(z))$. By (5.30) and Lemma 8, it is very natural to assume that

$$\mathsf{S}(\tilde{\mathsf{b}}_1(z)\mathsf{b}_1^\star(z)) = \mathsf{S}(\tilde{\mathsf{b}}_2(z)\mathsf{b}_2^\star(z)) = \mathsf{S}\Theta(z) = \mathsf{S}(\Theta(z^2)\tilde{\mathsf{a}}(z)\mathsf{a}^\star(z)). \tag{5.32}$$

Case 2: $\mathsf{S}(\tilde{\mathsf{b}}_1(z)\mathsf{b}_1^\star(z)) = -\mathsf{S}(\tilde{\mathsf{b}}_2(z)\mathsf{b}_2^\star(z))$. By (5.31) and $\mathsf{S}(\Theta(z^2)\tilde{\mathsf{a}}(z)\mathsf{a}^\star(-z)) = \pm\mathsf{S}(\Theta(z^2)\tilde{\mathsf{a}}(z)\mathsf{a}^\star(z))$, we have either

$$\mathsf{S}(\Theta(z^2)\tilde{\mathsf{a}}(z)\mathsf{a}^\star(z)) = \mathsf{S}(\tilde{\mathsf{b}}_1(z)\mathsf{b}_1^\star(z)) \quad \text{or} \quad \mathsf{S}(\Theta(z^2)\tilde{\mathsf{a}}(z)\mathsf{a}^\star(z)) = \mathsf{S}(\tilde{\mathsf{b}}_2(z)\mathsf{b}_2^\star(z)).$$

Therefore, by (5.30) and Lemma 8, it is natural to have $\mathsf{S}(\tilde{\mathsf{b}}_1(z)\mathsf{b}_1^\star(z)) = \mathsf{S}\Theta(z) = \mathsf{S}(\tilde{\mathsf{b}}_2(z)\mathsf{b}_2^\star(z))$, which contradicts our assumption $\mathsf{S}(\tilde{\mathsf{b}}_1(z)\mathsf{b}_1^\star(z)) = -\mathsf{S}(\tilde{\mathsf{b}}_2(z)\mathsf{b}_2^\star(z))$. Hence, this case can rarely happen.

It follows from the last identity in (5.32) that (1.21) must be satisfied. Consequently, both A and $\mathring{\mathsf{A}}$ have symmetry. For simplicity, in the following discussion we assume $\mathsf{p} = 1$ and therefore, $\mathring{\mathsf{A}} = \mathsf{A}$ and $\mathring{\mathsf{B}} = \mathsf{B}$. The general case is similar. Note that $\mathsf{S}\mathring{\mathsf{B}}(z) = (-1)^{c+n_b}\mathsf{S}\mathring{\mathsf{A}}(z)$. Since $\mathrm{len}(a) + \mathrm{len}(\tilde{a}) + \mathrm{len}(\Theta) \neq 0$, we already proved that (2.17) and (2.18) hold.

By (1.5) with $s = 2$, using (1.9), we can equivalently express the conditions in (5.31) and (5.32) as

$$(-1)^{c_1} = (-1)^{c_2} = (-1)^{c+n_b}, \quad \epsilon_1\tilde{\epsilon}_1 = \epsilon_2\tilde{\epsilon}_2 = \epsilon_0, \quad \tilde{c}_1 - c_1 = \tilde{c}_2 - c_2 = c_0. \tag{5.33}$$

Our choice of c_1 in (2) of (S2) is fully justified by the first relation in (5.33).

We now show that the following well-defined Laurent polynomial d has symmetry, where

$$\begin{aligned} \mathsf{d}(z^2) &:= z^{-1}[\mathring{\mathsf{b}}_1(z)\mathring{\mathsf{b}}_2(-z) - \mathring{\mathsf{b}}_1(-z)\mathring{\mathsf{b}}_2(z)], \\ \tilde{\mathsf{d}}(z^2) &:= z^{-1}[\mathring{\tilde{\mathsf{b}}}_1(z)\mathring{\tilde{\mathsf{b}}}_2(-z) - \mathring{\tilde{\mathsf{b}}}_1(-z)\mathring{\tilde{\mathsf{b}}}_2(z)]. \end{aligned} \tag{5.34}$$

By $(-1)^{c_1} = (-1)^{c_2}$ in (5.33), we can easily deduce that $c_1 + c_2$ is an even integer and

$$\mathsf{S}(\mathring{\mathsf{b}}_1(z)\mathring{\mathsf{b}}_2(-z)) = (-1)^{c_2}\epsilon_1\epsilon_2 z^{c_1+c_2} = (-1)^{c_1}\epsilon_1\epsilon_2 z^{c_1+c_2} = \mathsf{S}(\mathring{\mathsf{b}}_1(-z)\mathring{\mathsf{b}}_2(z)).$$

Hence, d has symmetry $\epsilon_{\mathsf{d}} z^{c_{\mathsf{d}}} := \mathsf{S}\mathsf{d}(z) = (-1)^{c_1}\epsilon_1\epsilon_2 z^{\frac{c_1+c_2}{2}-1}$, that is,

$$\epsilon_{\mathsf{d}} = (-1)^{c_1}\epsilon_1\epsilon_2 \qquad \text{and} \qquad c_{\mathsf{d}} = \tfrac{c_1+c_2}{2} - 1. \tag{5.35}$$

Equation (5.35) directly leads to the relations $c_2 = 2c_{\mathsf{d}} + 2 - c_1$ and $\epsilon_2 = (-1)^{c_1}\epsilon_{\mathsf{d}}\epsilon_1$. If $(-1)^{c_1}\epsilon_{\mathsf{d}} = -1$, then $\epsilon_1\epsilon_2 = (-1)^{c_1}\epsilon_{\mathsf{d}} = -1$ and hence either ϵ_1 or ϵ_2 must be one. Without loss of any generality, we can assume $\epsilon_1 = 1$, otherwise, we simply switch $\mathring{\mathsf{b}}_1$ with $\mathring{\mathsf{b}}_2$. This justifies items (2) and (3) of (S2).

Note that $(\{\tilde{a}; \tilde{b}_1(\cdot - 2k), \tilde{b}_2\}, \{a; b_1(\cdot - 2k), b_2\})_\Theta$ is a dual framelet filter bank for every $k \in \mathbb{Z}$ and its corresponding new d in (5.34) gains a factor z^k. This explains our restriction $c_{\mathsf{d}} \in \{0, 1\}$ in item (1) of (S2), since up to an integer shift there are only two symmetry centers. It follows directly from (1.9) that $\mathsf{d}(z)\tilde{\mathsf{d}}^\star(z) = \mathsf{D}(z)$. Hence, d is a factor of D and has symmetry. That is, all cases of possible d are covered by item (1) of (S2). Note that $\mathrm{fsupp}(\mathring{\mathsf{A}}) = [c_0 - n_0, n_0]$ and

$$2n_0 - c_0 = \mathrm{len}(\mathring{\mathsf{A}}) = \mathrm{len}(\mathring{\mathsf{B}}) = \mathrm{len}(a) + \mathrm{len}(\tilde{a}) + 2\,\mathrm{len}(\Theta) - n_b - n_{\tilde{b}}. \tag{5.36}$$

Since $\mathrm{fsupp}(\mathring{\mathsf{b}}_1) = [c_1 - n_1, n_1]$ with symmetry center $\frac{c_1}{2}$, we must have $\frac{c_1}{2} \leqslant n_1$. Since we assumed $\mathsf{p} = 1$, by our assumption on filter supports in (2.15), we must impose the following constraint:

$$2n_1 - c_1 = \mathrm{len}(\mathring{\mathsf{b}}_1) \leqslant \mathrm{len}(a) + \mathrm{len}(\tilde{a}) + 2\,\mathrm{len}(\Theta) - n_b - n_{\tilde{b}} + 2\epsilon_{\mathrm{len}} = 2n_0 - c_0 + 2\epsilon_{\mathrm{len}}.$$

Hence, $\frac{c_1}{2} \leqslant n_1 \leqslant \frac{c_1 - c_0}{2} + n_0 + \epsilon_{\mathrm{len}}$. This justifies our choice of n_1 in (5) of (S2).

By the same argument, we have $\frac{c_2}{2} \leqslant n_2 \leqslant \frac{c_2 - c_0}{2} + n_0 + \epsilon_{\mathrm{len}}$. Also, by the definition of d in (5.34), we must have $2\,\mathrm{len}(\mathsf{d}) \leqslant \mathrm{len}(\mathring{\mathsf{b}}_1) + \mathrm{len}(\mathring{\mathsf{b}}_2)$, that is, $2(2n_{\mathsf{d}} - c_{\mathsf{d}}) \leqslant 2n_1 - c_1 + 2n_2 - c_2$. Using this inequality and the justified relation $c_1 + c_2 = 2c_{\mathsf{d}} + 2$ in item (2) of (S2), we conclude that $2n_{\mathsf{d}} + 1 - n_1 \leqslant n_2$. Hence, our choice of n_2 in (6) of (S2) is justified.

From (1.9) and (5.34), (2.22) holds, that is, (2.8) and (2.10) must be satisfied with $\mathcal{R}_1 = \mathcal{R}_2 = 0$. The conditions in (2.9) and (2.7) follow trivially from the requirement for the filter support in (2.15).

Though it is more complicated, the general case of p can be checked similarly. When p is not a constant, Algorithm 1 indeed may miss some very special rare solutions, solely due to our simple splitting of p in (S6). However, this can be easily avoided by constructing Laurent polynomials $\mathsf{q}_1, \ldots, \mathsf{q}_4, \tilde{\mathsf{q}}_1, \ldots, \tilde{\mathsf{q}}_4$ with symmetry such that

$$\begin{bmatrix} \mathsf{q}_1(z) & \mathsf{q}_3(z) \\ \mathsf{q}_2(z) & \mathsf{q}_4(z) \end{bmatrix} \begin{bmatrix} \tilde{\mathsf{q}}_1(z) & \tilde{\mathsf{q}}_3(z) \\ \tilde{\mathsf{q}}_2(z) & \tilde{\mathsf{q}}_4(z) \end{bmatrix}^\star = \mathsf{p}(z) I_2, \tag{5.37}$$

$$\frac{\mathsf{S}\mathsf{q}_1(z)}{\mathsf{S}\mathsf{q}_2(z)} = \frac{\mathsf{S}\mathsf{q}_3(z)}{\mathsf{S}\mathsf{q}_4(z)} = \frac{\mathsf{S}\tilde{\mathsf{q}}_1(z)}{\mathsf{S}\tilde{\mathsf{q}}_2(z)} = \frac{\mathsf{S}\tilde{\mathsf{q}}_3(z)}{\mathsf{S}\tilde{\mathsf{q}}_4(z)} = \epsilon_1\epsilon_2 z^{\frac{c_2-c_1}{2}}. \tag{5.38}$$

Note that $c_2 - c_1$ is an even integer by (5.33). Then one can directly check that $(\{\tilde{a};\tilde{b}_1,\tilde{b}_2\},\{a;b_1,b_2\})_\Theta$ is a dual framelet filter bank having [complex] symmetry, where

$$\begin{aligned}
\mathsf{b}_1(z) &:= (1-z^{-1})^{n_b}[\mathring{\mathsf{b}}_1(z)\mathsf{q}_1(z^2) + \mathring{\mathsf{b}}_2(z)\mathsf{q}_2(z^2)],\\
\mathsf{b}_2(z) &:= (1-z^{-1})^{n_b}[\mathring{\mathsf{b}}_1(z)\mathsf{q}_3(z^2) + \mathring{\mathsf{b}}_2(z)\mathsf{q}_4(z^2)],\\
\tilde{\mathsf{b}}_1(z) &:= (1-z^{-1})^{n_{\tilde{b}}}[\mathring{\tilde{\mathsf{b}}}_1(z)\tilde{\mathsf{q}}_1(z^2) + \mathring{\tilde{\mathsf{b}}}_2(z)\tilde{\mathsf{q}}_2(z^2)],\\
\tilde{\mathsf{b}}_2(z) &:= (1-z^{-1})^{n_{\tilde{b}}}[\mathring{\tilde{\mathsf{b}}}_1(z)\tilde{\mathsf{q}}_3(z^2) + \mathring{\tilde{\mathsf{b}}}_2(z)\tilde{\mathsf{q}}_4(z^2)].
\end{aligned}$$

For simplicity of presentation, here we do not address the issue about how to construct $\mathsf{q}_1,\ldots,\mathsf{q}_4,\tilde{\mathsf{q}}_1,\ldots,\tilde{\mathsf{q}}_4$ satisfying (5.37) and (5.38). Interested readers are referred to [13, 14] for solving (5.37) and (5.38) for the particular case of tight framelet filter banks as discussed in Section 3.

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