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DIRECTIONAL CHEBYSHEV-TYPE METHODS FOR SOLVING EQUATIONS

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ABSTRACT. A semi-local convergence analysis for directional Chebyshev-type methods in m-variables is presented in this study. Our convergence analysis uses recurrent relations and Newton–Kantorovich-type hypotheses. Numerical examples are also provided to show the effectiveness of the proposed method.

1. INTRODUCTION

In this study we are concerned with the problem of approximating a solution $x^\star$ of equation

$$F(x) = 0, \tag{1.1}$$

where F is a nonlinear Fréchet-differentiable mapping defined in an open convex nonempty subset $\mathcal{D}$ of $\mathbb{R}^m$ (m a natural number) with values in $\mathbb{R}$.

Computational sciences have received substantial and significant attention of researchers in recent years in several areas such as engineering sciences, economic equilibrium theory and mathematics. These sciences can solve various problems by passing first through mathematical modelling and then later looking for the solution iteratively [4, 6]. For example, finding a local minimum of function is connected to solving a set of nonlinear equations. So, numerical methods are crucial and necessary for solving nonlinear equations.

In computer graphics, the intersection of two surfaces is also modeled by nonlinear equations and can be complicated in general, because of some closed loops and singularities. This requires finding efficient algorithms for solving this intersection. We usually compute the intersection $\mathcal{C} = \mathcal{A} \cap \mathcal{B}$ of two surfaces $\mathcal{A}$ and $\mathcal{B}$ in $\mathbb{R}^3$ [8, 11]. If the two surfaces are explicitly given by

$$\mathcal{A} = \{(u, v, w)^T \,:\, w = F_1(u, v)\} \quad \text{and} \quad \mathcal{B} = \{(u, v, w)^T \,:\, w = F_2(u, v)\},$$

then the solution $x^\star = (u^\star, v^\star, w^\star)^T \in \mathcal{C}$ must satisfy the nonlinear equation

$$F_1(u^\star, v^\star) = F_2(u^\star, v^\star) \quad \text{and} \quad w^\star = F_1(u^\star, v^\star).$$

Hence, we must solve a nonlinear equation in two variables $x = (u, v)^T$ of the form

$$F(x) = F_1(x) - F_2(x) = 0,$$

which is a special case of equation (1.1).

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In mathematical programming [13], for an equality-constraint optimization problem, e.g.,

$$\min\ \psi(x)\ s.t.\quad F(x)=0$$

where $\psi,\ F:\mathcal{D}\subseteq\mathbb{R}^m\longrightarrow\mathbb{R}$ are nonlinear mappings, we usually seek a feasible point to start a numerical algorithm, which again requires the determination of $x^\star$. Other areas of applications can be found in [4]–[16].

The study about convergence matter of iterative procedures is usually centered on two types: semi-local and local convergence analysis. The semi-local convergence matter is based on the information around an initial point, to give criteria ensuring the convergence of iterative procedures; while the local one is based on the information around a solution, to find estimates of the radii of convergence balls. In the present paper we are interested only in the semi-local convergence.

An and Bai [1] used the directional Secant method (DSM)

$$\begin{aligned}
&x_{k+1}=x_k+h_k,\\
&h_k=-\frac{\theta_k\,F(x_k)}{F(x_k+\theta_k\,d_k)-F(x_k)}\,d_k,\\
&(k\geq 0,\quad x_0\in\mathbb{R}^n,\quad d_k\in\mathbb{R}^n,\quad \parallel d_k\parallel=1,\quad \theta_k\geq 0)
\end{aligned}$$

to generate a sequence $\{x_k\}$ converging to $x^\star$.

(DSM) is a usefull alternative to (DNM) [5, 8, 10]:

$$x_{k+1}=x_k-\frac{F(x_k)}{\nabla F(x_k)\,\cdot\,d_k}\,d_k\quad (k\geq 0),$$

where

$$\nabla F(x_k)=\left(\frac{\partial F(x_k)}{\partial x_1},\frac{\partial F(x_k)}{\partial x_2},\cdots,\frac{\partial F(x_k)}{\partial x_m}\right)$$

is the gradient of F and d_k is a direction at x_k.

(DNM) converges quadratically to $x^\star$, if x_0 is close enough to $x^\star$ [10]. However, as already noted in [10], the computation of the gradient $\nabla F(x_k)$ may be very expensive as it is the case when the number n of unknowns is large. In some applications, the mapping F may not be differentiable, or the gradient is impossible to compute. The (DSM) avoids these obstacles. Note that if $m=1$, (DSM) reduces to the classical Secant method, and (DNM) to Newton's method [3]. The quadratic convergence of (DSM) [1] and (DNM) [10] was established for directions d_k sufficiently close to the gradients $\nabla F(x_k)$ and under standard Newton–Kantorovich-type hypotheses.

In the present paper, we introduce the directional Chebyshev-type method (DCTM):

$$\begin{aligned}
&x_0\in\mathcal{D},\\
&y_k=x_k-A_k\,F(x_k),\quad v_k=x_k+\theta_k\,d_k,\quad A_k=\frac{\theta_k}{F(v_k)-F(x_k)}\,\cdot\,d_k\\
&z_k=x_k+a\ (y_k-x_k),\quad a\in[0,1],\\
&x_{k+1}=x_k-A_k\,(b\,F(x_k)+c\,F(z_k))\\
&(k\geq 0,\ d_k\in\mathbb{R}^m,\ \parallel d_k\parallel=1,\ \theta_k\geq 0,\ b\in[0,1],\ c\geq 0,\ c\,(1-a)=1-b)
\end{aligned}$$

to generate a sequence $\{x_k\}$ approximating the zero $x^\star$. Notice that if $a=b=1$ and $c=0$, we obtain the Secant method. However, c can be chosen to be positive in this case. Clearly, in general additional hypotheses on a, b and c are needed to obtain convergence for (DCTM). Such conditions are given later in Theorem 2.8.

(DCTM) is a Secant-type analog of the third order Chebyshev-type method (CTM) with efficiency close to Newton's and the same region of accessibility defined on Banach spaces by Ezquerro and Hernández [9]:

$$\begin{aligned}
&x_0 \in \mathcal{D},\\
&y_k = x_k - F'(x_k)^{-1} F(x_k),\\
&z_k = x_k + d\,(y_k - x_k), \quad d \in [0,1],\\
&x_{k+1} = x_k - \frac{1}{d^2}\, F'(x_k)^{-1}\,((d^2+d-1)\,F(x_k) + F(z_k)), \quad k \geq 0.
\end{aligned}$$

The paper is organized as follows: Section 2 contains the semi-local convergence of (DCTM). The numerical examples are given in Section 3.

2. Semi-local convergence analysis of (DCTM) using recurrent relations

In this section, we use the Euclidean norms for both vector and matrix. The unit direction d_k is chosen such that $d_k \approx \nabla F(x_k)/\| \nabla F(x_k) \|$. The angle between two vectors x, y in a Hilbert space H denoted by $\angle(x,y)$ is given by

$$\angle(x,y) = \arccos \frac{x \cdot y}{\| x \| \cdot \| y \|}, \quad x \neq 0, \quad y \neq 0.$$

We need the definition of the first order divided difference for a mapping.

Definition 2.1. A mapping $[.,.;F]$ belonging to $\mathcal{L}(H,\mathbb{R})$ is called the first order divided difference of F at the points x and y in H $(x \neq y)$ if the following holds:

$$[x,y;F]\,(y-x) = F(y) - F(x).$$

If F is Fréchet differentiable at x, then $[x,x;F] = \nabla F(x)$.

For $x \in \mathbb{R}^m$ and $r > 0$, we denote by $U(x,r)$ and $\overline{U}(x,r)$ the open and closed balls at x and of radius r, respectively. Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces. We shall use the following measure of invertibility in $\mathcal{L}(\mathcal{X},\mathcal{Y})$, $d(Q) = \inf_{\|x\|=1} \| Q(x) \|$ for $Q \in \mathcal{L}(\mathcal{X},\mathcal{Y})$. If Q is invertible and $Q^{-1} \in \mathcal{L}(\mathcal{Y},\mathcal{X})$, then $d(Q) = \| Q^{-1} \|^{-1}$. We also need the following Banach-type lemma on invertible operators [4, 12]

Lemma 2.2. *If Q and T belong in $\mathcal{L}(\mathcal{X},\mathcal{Y})$ such that Q is boundedly invertible and $d(Q) > \| Q - T \|$, then T is also invertible and $d(T) \geq d(Q) - \| Q - T \|$.*

We shall use the following conditions:

($\mathcal{C}_1$) $|F(x_0)| \leq \lambda$, $\| \nabla F(x_0) \| \geq \beta > 0$ and $|\nabla F(x_0) \cdot d_0| \geq \alpha \| \nabla F(x_0) \|$ with $\alpha \in [0,1]$;

($\mathcal{C}_2$) d_k $(k \geq 0)$ satisfies

- ($\mathcal{C}_{21}$) $\varrho\,|\nabla F(x_0) \cdot d_0| \leq |\nabla F(x) \cdot d_0|$, $x \in (x_0,\, x_0 + \theta_0\, d_0)$, $\varrho \in (0,1)$,
- ($\mathcal{C}_{22}$) $\angle(d_{k+1}, \nabla F(x_{k+1})) \leq \angle(d_k, \nabla F(x_k))$;

($\mathcal{C}_3$) θ_k satisfies $\theta_{k+1} \leq q\,\theta_k \| x_{k+1} - x_k \|$, $q \in (0,1)$;

($\mathcal{C}_4$) Mapping ∇F is Lipschitz with constant M on $\mathcal{D}$:

$$\| \nabla F(x) - \nabla F(y) \| \leq M \| x - y \| \quad \text{for all} \quad x, y \in \mathcal{D};$$

($\mathcal{C}_5$) Mapping F has a divided difference of order one $[x,y;F]$ at the points $(x,y) \in \mathcal{D}^2$ satisfying Definition 2.1 and

$$\| [x,z;F] - [y,z;F] \| \leq N \| x - y \| \quad \text{for all} \quad x, y, z \in \mathcal{D}.$$

We need some auxiliary lemmas to establish semi-local convergence of (DCTM).

Lemma 2.3 ([12, 3.2.2]). *Let $\mathcal{D} \subseteq \mathbb{R}^m$ be an open convex nonempty subset and $F : \mathcal{D} \subseteq \mathbb{R}^m \longrightarrow \mathbb{R}$ a differentiable mapping. Then, for any $x, y \in \mathcal{D}$, there exists a vector $\mu \in (x, y)$, such that*

$$F(y) - F(x) = \nabla F(\mu)\ (y - x),$$

where

$$(x, y) = \{z\ :\ z = x + \theta\, y, \quad 0 < \theta < 1\}$$

represents the open straight line between the points x and y.

Remark 2.4. Note that $(\mathcal{C}_{\mathbf{22}})$ implies

$$\frac{|\nabla F(x_{k+1}) \cdot d_{k+1}|}{\| \nabla F(x_{k+1}) \|} \geq \frac{|\nabla F(x_k) \cdot d_k|}{\| \nabla F(x_k) \|} \qquad (k \geq 0), \tag{2.1}$$

since $\| d_k \| = 1$ for all k. These conditions state that the direction d_k does not have to be exactly along the gradient $\nabla F(x_k)$ (which is the most common choice). Small perturbations in the angle $\langle d_k, \nabla F(x_k) \rangle$ are allowed, if they do not increase with k.

Lemma 2.5. *Let F be a nonlinear Fréchet-differentiable mapping $F : \mathcal{D} \subseteq \mathbb{R}^m \longrightarrow \mathbb{R}$ under the previous conditions $(\mathcal{C}_{\mathbf{1}})$–$(\mathcal{C}_{\mathbf{5}})$ and $\{x_k\}$ the sequence in $\mathcal{D}$ generated by (DCTM). Then the following is satisfied:*

$$|\nabla F(x_k) \cdot d_k| \geq \alpha\ \| \nabla F(x_k) \| \qquad (k \geq 0). \tag{2.2}$$

Proof. Estimate (2.2) holds for $k = 0$ by $(\mathcal{C}_{\mathbf{1}})$. Then, by a simple induction argument and (2.1) we get

$$\alpha\, \frac{\| \nabla F(x_{k+1}) \|}{|\nabla F(x_{k+1}) \cdot d_{k+1}|} \leq \alpha\, \frac{\| \nabla F(x_k) \|}{|\nabla F(x_k) \cdot d_k|} \leq 1.$$

That completes the proof of Lemma 2.5. □

We need an Ostrowski-type approximation for (DCTM) [4,6,12].

Lemma 2.6. *Let F be a nonlinear Fréchet-differentiable mapping $F : \mathcal{D} \subseteq \mathbb{R}^m \longrightarrow \mathbb{R}$. Suppose that sequence $\{x_k\}$ generated by (DCTM) is well defined. Then the following assertions hold for all $k \geq 0$:*

$$F(z_k) \;=\; (1 - a)\, F(x_k) + a \Big([z_k, x_k; F] - [v_k, x_k; F] \Big) (y_k - x_k), \tag{2.3}$$

$$x_{k+1} - y_k \;=\; -a\, c\, A_k \Big([v_k, x_k; F] - [z_k, x_k; F] \Big) (y_k - x_k) \tag{2.4}$$

and

$$\begin{aligned} F(x_{k+1}) \;=\;& \Big([x_{k+1}, x_k; F] - [v_k, x_k; F] \Big) (x_{k+1} - x_k) \\ & - a\, c \Big([z_k, x_k; F] - [v_k, x_k; F] \Big) (y_k - x_k). \end{aligned} \tag{2.5}$$

Proof. Using (DCTM), we obtain in turn that

$$\begin{aligned} F(z_k) \;&=\; (1 - a)\, F(x_k) + F(z_k) - F(x_k) + a\, [v_k, x_k; F]\, \frac{\theta_k\, F(x_k)}{F(v_k) - F(x_k)} \cdot d_k \\ &=\; (1 - a)\, F(x_k) + F(z_k) - F(x_k) + a\, \frac{F(v_k) - F(x_k)}{F(v_k) - F(x_k)}\, F(x_k) \end{aligned}$$

showing (2.3). We have for $a \neq 1$ that

$$\begin{aligned} x_{k+1} - y_k &= -A_k \, (b\, F(x_k) + \frac{1-b}{1-a}\, F(z_k)) + A_k\, F(x_k) \\ &= -A_k \, ((b-1)\, F(x_k) + \frac{1-b}{1-a}\, F(z_k)) \\ &= -A_k \, ((b-1)\, F(x_k) + (1-b)\, F(x_k)) \\ &\quad + \frac{a\,(1-b)}{1-a}\, ([z_k, x_k; F] - [v_k, x_k; F])\, (y_k - x_k), \end{aligned}$$

which is (2.4). Finally, we also have that

$$\begin{aligned} F(x_{k+1}) &= F(x_{k+1}) - F(x_k) - [v_k, x_k; F]\ (y_k - x_k) \\ &= [x_{k+1}, x_k; F]\, (x_{k+1} - x_k) - [v_k, x_k; F]\, (y_k - x_k) \\ &= \Big([x_{k+1}, x_k; F] - [v_k, x_k; F] \Big)\, (x_{k+1} - x_k) \\ &\quad + [v_k, x_k; F]\, (x_{k+1} - x_k) - [v_k, x_k; F]\, (y_k - x_k) \\ &= \Big([x_{k+1}, x_k; F] - [v_k, x_k; F] \Big)\, (x_{k+1} - x_k) + [v_k, x_k; F]\, (x_{k+1} - y_k) \\ &= \Big([x_{k+1}, x_k; F] - [v_k, x_k; F] \Big)\, (x_{k+1} - x_k) \\ &\quad + [v_k, x_k; F] \left(\frac{-a\, c\, ([v_k, x_k; F] - [z_k, x_k; F])\, (y_k - x_k)\, \theta_k \ \cdot d_k}{F(v_k) - F(x_k)} \right) \\ &= \Big([x_{k+1}, x_k; F] - [v_k, x_k; F] \Big)\, (x_{k+1} - x_k) \\ &\quad - a\, c \Big([z_k, x_k; F] - [v_k, x_k; F] \Big)\, (y_k - x_k), \end{aligned}$$

which is (2.5). That completes the proof of Lemma 2.6. □

It is convenient for us to introduce some notation and initial conditions:

$$a_0 = \frac{1}{\rho\, \alpha\, \beta}, \quad r_0 = N\, a_0\, \lambda,$$

$$t_0 = N\, \theta_0, \quad s_0 = (r_0 + t_0)\, a_0, \quad c_0 = a_0\, \lambda, \quad b_0 = \frac{M\, \lambda}{\rho\, \beta^2\, \alpha}.$$

Then, we have the following relations:

$$\begin{aligned} \| A_0 \| \leq \left| \frac{\theta_0}{F(v_0) - F(x_0)} \right| \leq \frac{1}{|\nabla F(\mu_0) \cdot d_0|} \leq \frac{1}{\rho\, \alpha\, \beta} = a_0, \\ N \ \| y_0 - x_0 \| \leq N \ \| A_0 \| \ |F(x_0)| \leq N\, a_0\, \lambda = r_0, \\ N \ \| z_0 - v_0 \| \leq r_0 + t_0, \\ N \ \| A_0 \| \, \| z_0 - v_0 \| \leq (r_0 + t_0)\, a_0 = s_0, \\ M \ \| \nabla F(x_0) \|^{-1} \| y_0 - x_0 \| \leq \frac{M\, \lambda}{\beta^2\, \rho\, \alpha} = b_0. \end{aligned}$$

Hence, we get that

$$\begin{array}{rcl} \| x_1 - y_0 \| & \leq & a\,c \, \| A_0 \| \, \| [v_0, x_0; F] - [z_0, x_0; F] \| \, \| y_0 - x_0 \| \\ & \leq & a\ c\, a_0\, N \ \| z_0 - v_0 \| \, \| y_0 - x_0 \| \leq a\,c\,s_0 \ \| y_0 - x_0 \|, \\ \| x_1 - x_0 \| & \leq & (1 + a\,c\,s_0) \ \| y_0 - x_0 \| < R, \end{array}$$

for some $R > 0$ to be determined later,

$$\begin{array}{rcl} N \ \| x_1 - v_0 \| & \leq & N \ \| x_1 - x_0 \| + N\,\theta_0 \\ & \leq & N\,((1 + a\,c\,s_0) \ \| y_0 - x_0 \| + \theta_0) \\ & \leq & (1 + a\,c\,s_0)\,(r_0 + t_0), \end{array}$$

and

$$\begin{array}{rcl} |F(x_1)| & \leq & N \ \| x_1 - v_0 \| \, \| x_1 - x_0 \| + a\,c\,N \ \| z_0 - v_0 \| \, \| y_0 - x_0 \| \\ & \leq & ((1 + a\,c\,s_0)^2 + a\,c)\,(r_0 + t_0) \ \| y_0 - x_0 \| \,. \end{array}$$

We must define auxiliary real functions

$$f(x, y) = \frac{1}{1 - (1 + a\,c\,x)\,y},$$

$$g(x) = ((1 + a\,c\,x)^2 + a\,c)\,x,$$

scalar sequences ($k \geq 1$)

$$r_k = r_{k-1}\, f(s_{k-1}, b_{k-1})\, g(s_{k-1}), \quad c_k = c_{k-1}\, f(s_{k-1}, b_{k-1})\, g(s_{k-1}),$$

$$t_k = t_{k-1}\, q\,(1 + a\,c\,s_{k-1})\, c_{k-1}, \quad a_k = a_{k-1}\, f(s_{k-1}, b_{k-1}), \quad s_k = (r_k + t_k)\, a_k$$

and

$$b_k = b_{k-1}\, f^2(s_{k-1}, b_{k-1})\, g(s_{k-1}).$$

Then, we shall show, using induction, the following recurrence relations.

Lemma 2.7. *Let us suppose that x_0, v_0, y_0, $z_0 \in \mathcal{D}$ and x_k, v_k, y_k, $z_k \in \mathcal{D}$ for $k \in \mathbb{N}^\star$. Moreover, suppose that*

$$f(s_0, b_0)^2\, g(s_0) < 1 \tag{2.6}$$

and

$$q\,(1 + a\,c\,s_0)\, f(s_0, b_0)\, c_0 < 1. \tag{2.7}$$

Then, the following relations are satisfied for $k \geq 0$:

(I_k) $\dfrac{1}{\rho\,|\nabla F(x_k)\,\cdot\, d_k|} \leq a_k$ *and* $\| A_k \| \leq a_k$,

(II_k) $\| y_k - x_k \| \leq f(s_{k-1}, b_{k-1})\, g(s_{k-1}) \ \| y_{k-1} - x_{k-1} \| \leq c_k$,

(III_k) $N \ \| A_k \| \, \| z_k - v_k \| \leq s_k$,

(IV_k) $M \ \| \nabla F(x_k) \|^{-1} \, \| y_k - x_k \| \leq b_k$,

(V_k) $\| x_{k+1} - y_k \| \leq a\,c\,s_k \ \| y_k - x_k \|$,

(VI_k) $\| x_{k+1} - x_k \| \leq (1 + a\,c\,s_k) \ \| y_k - x_k \|$,

(VII_k) $\| x_{k+1} - x_0 \| \leq (1 + a\,c\,s_0)\,(1 + q\,\theta_0)\, \dfrac{1 - (f(s_0, b_0)\, g(s_0))^{k+1}}{1 - f(s_0, b_0)\, g(s_0)} \ \| y_0 - x_0 \| \,.$

Proof. We shall first show conditions (I_k)–(VII_k) are satisfied for $k = 1$. We have in turn that

$$\begin{aligned} \| v_1 - x_0 \| &\leq \| x_1 - x_0 \| + \theta_1 \leq (1 + q\,\theta_0) \ \| x_1 - x_0 \| \\ &\leq (1 + a\,c\,s_0)\,(1 + q\,\theta_0) \ \| y_0 - x_0 \| \end{aligned}$$

and

$$\begin{array}{rcl}
\| \nabla F(x_1) \| & \geq & \| \nabla F(x_0) \| - \| \nabla F(x_1) - \nabla F(x_0) \| \\
& \geq & \| \nabla F(x_0) \| - M \ \| x_1 - x_0 \| \\
& \geq & \| \nabla F(x_0) \| - M\,(1 + a\,c\,s_0) \ \| y_0 - x_0 \| \\
& = & \| \nabla F(x_0) \| \ (1 - M\,(1 + a\,c\,s_0) \ \| \nabla F(x_0) \|^{-1} \| y_0 - x_0 \|) \\
& \geq & \| \nabla F(x_0) \| \ (1 - (1 + a\,c\,s_0)\,b_0).
\end{array}$$

Hence, we have by Lemma 2.2 that

$$\| \nabla F(x_1) \|^{-1} \leq \| \nabla F(x_0) \|^{-1} \ f(s_0, b_0),$$

(I_1)

$$\begin{aligned}
\| A_1 \| &\leq \left| \frac{\theta_1}{F(v_1) - F(x_1)} \right| \leq \left| \frac{\theta_1}{\nabla F(\mu_1)\theta_1 \cdot d_1} \right| \\
&\leq \frac{1}{\rho\,|\nabla F(x_1) \cdot d_1|} \leq \frac{\| \nabla F(x_0) \|}{\rho\,|\nabla F(x_0) \cdot d_0| \ \| \nabla F(x_1) \|} \\
&\leq \frac{f(s_0, b_0)}{\rho\,|\nabla F(x_0) \cdot d_0|} \leq a_0\,f(s_0, b_0) = a_1,
\end{aligned}$$

(II_2)

$$\begin{array}{rcl}
\| y_1 - x_1 \| & \leq & \| A_1 \| \ |F(x_1)| \\
& \leq & a_0\,f(s_0, b_0)\,(r_0 + t_0)\,((1 + a\,c\,s_0)^2 + a\,c) \ \| y_0 - x_0 \| \\
& = & f(s_0, b_0)\,g(s_0) \ \| y_0 - x_0 \|,
\end{array}$$

$$\begin{array}{rcl}
\| y_1 - x_0 \| & \leq & \| y_1 - x_1 \| + \| x_1 - x_0 \| \\
& \leq & f(s_0, b_0)\,g(s_0) \ \| y_0 - x_0 \| + (1 + a\,c\,s_0) \ \| y_0 - x_0 \| \\
& \leq & (f(s_0, b_0)\,g(s_0) + 1 + a\,c\,s_0) \ \| y_0 - x_0 \|, \\
\| z_1 - x_0 \| & \leq & a \ \| y_1 - x_1 \| + \| x_1 - x_0 \| \\
& \leq & (a\,f(s_0, b_0)\,g(s_0) + 1 + a\,c\,s_0) \ \| y_0 - x_0 \|, \\
N \ \| y_1 - x_1 \| & \leq & r_0\,f(s_0, b_0)\,g(s_0) = r_1,
\end{array}$$

$$N\,\theta_1 \leq N\,q\,\theta_0 \ \| x_1 - x_0 \| \leq N\,q\,\theta_0\,(1 + a\,c\,s_0) \ \| y_0 - x_0 \| \leq t_0\,q\,(1 + a_0\,c\,s_0)\,a_0\,\lambda = t_1,$$

(III_1)

$$N \ \| A_1 \| \, \| z_1 - v_1 \| \leq a_1\,(N \ \| y_1 - x_1 \| + N\,\theta_1) \leq a_1\,(r_1 + t_1) = s_1,$$

(IV_1)

$$\begin{array}{rcl}
M \ \| \nabla F(x_1) \|^{-1} \| y_1 - x_1 \| & \leq & M \ \| \nabla F(x_0) \|^{-1} \ f(s_0, b_0)^2\,g(s_0) \ \| y_0 - x_0 \| \\
& \leq & b_0\,f(s_0, b_0)^2\,g(s_0) = b_1,
\end{array}$$

(V_1)

$$\begin{array}{rcl}
\| x_2 - y_1 \| & \leq & a\,c \ \| A_1 \| \, \| [v_1, x_1; F] - [z_1, x_1; F] \| \, \| y_1 - x_1 \| \\
& \leq & a\,c\,N \ \| A_1 \| \, \| v_1 - z_1 \| \, \| y_1 - x_1 \| \leq a\,c\,s_1 \ \| y_1 - x_1 \|,
\end{array}$$

(VI_1)

$$\| x_2 - x_1 \| \leq \| x_2 - y_1 \| + \| y_1 - x_1 \| \leq (1 + a\,c\,s_1) \ \| y_1 - x_1 \|,$$

(VII_1)

$$\begin{aligned}\| x_2 - x_0 \| &\leq (1+a\,c\,s_1)\ \| y_1 - x_1 \| + (1+a\,c\,s_0)\ \| y_0 - x_0 \| \\ &\leq ((1+a\,c\,s_1)\, f(s_0,b_0)\, g(s_0) + 1 + a\,c\,s_0)\ \| y_0 - x_0 \| \\ &\leq (1+a\,c\,s_0)\,(1+f(s_0,b_0)\,g(s_0))\ \| y_0 - x_0 \| \\ &\leq (1+a\,c\,s_0)\,(1+q\,\theta_0)\,\frac{1-(f(s_0,b_0)\,g(s_0))^2}{1-f(s_0,b_0)\,g(s_0)}\ \| y_0 - x_0 \| \,.\end{aligned}$$

The rest follows by a simple induction argument. This completes the proof of Lemma 2.7. □

We can show the following semi-local convergence result for (DCTM).

Theorem 2.8. *Let F be a nonlinear Fréchet-differentiable mapping under conditions $(\mathcal{C}_\mathbf{1})$–$(\mathcal{C}_\mathbf{5})$. We also suppose that* (2.6) *and* (2.7) *hold. Then, if $\overline{U}(x_0,R) \subseteq \mathcal{D}$, where*

$$R = \frac{(1+a\,c\,s_0)\,(1+q\,\theta_0)\,a_0\,\lambda}{1-f(s_0,b_0)\,g(s_0)},$$

the sequence $\{x_k\}$ generated by (DCTM) starting in x_0, is well defined, remains in $\overline{U}(x_0,R)$ for all $k \geq 0$ and converges to a solution $x^\star \in \overline{U}(x_0,R)$ of equation $F(x) = 0$.

Proof. First, we have that

$$\| v_0 - x_0 \| \leq \theta_0 < R,\ \| y_0 - x_0 \| < R,\ \| z_0 - x_0 \| \leq a\ \| y_0 - x_0 \| < R$$

and

$$\| x_1 - x_0 \| \leq (1+a\,c\,s_0)\ \| y_0 - x_0 \| < R.$$

Thus, v_0, y_0, z_0, $x_1 \in \mathcal{D}$. Similarly, we get v_1, y_1, z_1, $x_2 \in \mathcal{D}$. Assume v_i, y_i, z_i, $x_{i+1} \in \mathcal{D}$, $i = 1, \ldots, k$. Then, using Lemma 2.7, we prove by induction v_{k+1}, y_{k+1}, z_{k+1}, $x_{k+2} \in \mathcal{D}$. We have in turn by the recurrence relations that

$$\begin{aligned}\| v_{k+1} - x_k \| &\leq (1+q\,\theta_k)\ \| x_{k+1} - x_k \| \\ &\leq (1+a\,c\,s_k)\,(1+q\,\theta_k)\ \| y_k - x_k \| \\ &\leq (1+a\,c\,s_0)\,(1+q\,\theta_0)\,(f(s_0,b_0)\,g(b_0))^k\ \| y_0 - x_0 \|, \\ \| v_{k+1} - x_0 \| &\leq (1+a\,c\,s_0)\,(f(s_0,b_0)\,g(s_0))^k\ (1+q\,\theta_0)\ \| y_0 - x_0 \| \\ &\quad + (1+a\,c\,s_0)\,\frac{1-(f(s_0,b_0)\,g(s_0))^k}{1-f(s_0,b_0)\,g(s_0)}\,(1+q\,\theta_0)\ \| y_0 - x_0 \| \\ &\leq (1+a\,c\,s_0)\,(1+q\,\theta_0)\,\frac{1-(f(s_0,b_0)\,g(s_0))^{k+1}}{1-f(s_0,b_0)\,g(s_0)}\ \| y_0 - x_0 \| < R,\end{aligned}$$

$$\begin{aligned}\| y_{k+1} - x_0 \| &\leq \| y_{k+1} - x_{k+1} \| + \| x_{k+1} - x_0 \| \\ &\leq f(s_k,b_k)\,g(s_k)\ \| y_k - x_k \| \\ &\quad + (1+a\,c\,s_0)\,(1+q\,\theta_0)\,\frac{1-(f(s_0,b_0)\,g(s_0))^{k+1}}{1-f(s_0,b_0)\,g(s_0)}\ \| y_0 - x_0 \| \\ &< (1+a\,c\,s_0)\,(1+q\,\theta_0)\,\frac{1-(f(s_0,b_0)\,g(s_0))^{k+2}}{1-f(s_0,b_0)\,g(s_0)}\ \| y_0 - x_0 \| < R, \\ \| z_{k+1} - x_0 \| &\leq \| x_{k+1} - x_0 \| + a\ \| y_{k+1} - x_{k+1} \| < R,\end{aligned}$$

$$
\begin{aligned}
\| x_{k+2}-x_0 \| &\leq (1+a\,c\,s_{k+1})\ \| y_{k+1}-x_{k+1} \| \\
&\quad + (1+a\,c\,s_0)\,(1+q\,\theta_0)\,\frac{1-(f(s_0,b_0)\,g(s_0))^{k+1}}{1-f(s_0,b_0)\,g(s_0)}\ \| y_0-x_0 \| \\
&\leq (1+a\,c\,s_0)\,(f(s_0,b_0)\,g(s_0))^{k+1}\ \| y_0-x_0 \| \\
&\quad + (1+a\,c\,s_0)\,(1+q\,\theta_0)\,\frac{1-(f(s_0,b_0)\,g(s_0))^{k+1}}{1-f(s_0,b_0)\,g(s_0)}\ \| y_0-x_0 \| \\
&\leq (1+a\,c\,s_0)\,(1+q\,\theta_0)\,\frac{1-(f(s_0,b_0)\,g(s_0))^{k+2}}{1-f(s_0,b_0)\,g(s_0)}\ \| y_0-x_0 \| < R.
\end{aligned}
$$

Hence, we deduce $v_{k+1},\, y_{k+1},\, z_{k+1},\, x_{k+2} \in \mathcal{D}$. Next, we shall show the convergence of the sequence $\{x_k\}$ by using recurrence relations:

$$
\| x_{k+1}-x_k \| \leq (1+a\,c\,s_k)\ \| y_k-x_k \| \leq (1+a\,c\,s_0)\,(f(s_0,b_0)\,g(s_0))^k\ \| y_0-x_0 \|\,.
$$

Then, we have that

$$
\begin{aligned}
&\| x_{k+j}-x_k \| \leq \sum_{i=k}^{i=k+j-1} \| x_{i+1}-x_i \| \\
&\leq (1+a\,c\,s_0) \sum_{i=k}^{i=k+j-1} (f(s_0,b_0)\,g(s_0))^i\ \| y_0-x_0 \| \\
&\leq (1+a\,c\,s_0)\,(f(s_0,b_0)\,g(s_0))^k\,\frac{1-(f(s_0,b_0)\,g(s_0))^j}{1-f(b_0)\,g(s_0)}\ \| y_0-x_0 \|\,.
\end{aligned}
\tag{2.8}
$$

It is obvious that $\{x_k\}$ is a complete sequence in $\mathbb{R}^m$ and as such it converges to some $x^\star \in \overline{U}(x_0,R)$ (since $\overline{U}(x_0,R)$ is a closed set).

We shall show that $F(x^\star)=0$. Indeed, we have that

$$
\begin{aligned}
|F(x_{k+1})| &\leq ((1+a\,c\,s_k)^2+a\,c)\,(r_k+t_k)\ \| y_k-x_k \| \\
&\leq ((1+a\,c\,s_0)^2+a\,c)(r_0+t_0)\,(f(s_0,b_0)\,g(s_0))^k\ \| y_0-x_0 \|\,.
\end{aligned}
$$

Thus, by letting $k\to\infty$ it follows that $|F(x_k)|\to 0$, since $f(s_0,b_0)\,g(s_0)<1$ and $((1+a\,c\,s_0)^2+a\,c)(r_0+t_0)\ \| y_0-x_0 \|$ is bounded. Hence, we deduce that $|F(x^\star)|=0$. This completes the proof of Theorem 2.8. ☐

It turns out that in an analogous way a second semi-local convergence result can be given for (DCTM). We first need the lemma.

Lemma 2.9. *Let $q\in(0,1)$, $M>0$ and $k\geq 0$. If $\theta_k\in(0,\Upsilon_k]$, where*

$$
\Upsilon_k=\frac{\sqrt{\| \nabla F(x_k) \|^2+2\,M\,q\,|F(x_k)|}-\| \nabla F(x_k) \|}{M}.
$$

Then, the following holds for all $k\geq 0$:

$$
\theta_k \leq q\ \| y_k-x_k \|\,.
$$

Proof. We have in turn that

$$\begin{aligned}
|F(v_k)-F(x_k)| &= \left|\int_{x_k}^{v_k} \nabla F(y)\,dy\right| \\
&= \left|\int_0^1 (\nabla F(x_k+\tau\,\theta_k\,d_k)\,\theta_k\,d_k-\nabla F(x_k)\,\theta_k\,d_k+\nabla F(x_k)\,\theta_k\,d_k)\,d\tau\right| \\
&\leq \theta_k\left(M\int_0^1 \tau\,\theta_k\,d\tau+\parallel \nabla F(x_k)\parallel\right) \\
&= \theta_k\left(\frac{M}{2}\,\theta_k+\parallel \nabla F(x_k)\parallel\right)=\frac{M}{2}\,\theta_k^2+\parallel \nabla F(x_k)\parallel\ \theta_k.
\end{aligned}$$

Therefore,

$$\frac{M}{2}\,\theta_k^2+\parallel \nabla F(x_k)\parallel\ \theta_k\leq q\,|F(x_k)|$$

provided that $\theta_k\in(0,\Upsilon_k]$. Then we get that

$$\parallel y_k-x_k\parallel=\frac{\theta_k\,|F(x_k)|}{|F(v_k)-F(x_k)|}\geq\frac{\theta_k\,|F(x_k)|}{q\,|F(x_k)|}=\frac{\theta_k}{q}.$$

This completes the proof of Lemma 2.9. □

We shall assume conditions $(\mathcal{C}_1)$, $(\mathcal{C}_2)$, $(\mathcal{C}_4)$, $(\mathcal{C}_5)$ and

$(\mathcal{C}_3^\star)$ $\theta_0\in(0,\Upsilon_0]$ and $0<\theta_k\leq\min\{\Upsilon_k,\ q\ \parallel y_{k-1}-x_{k-1}\parallel\}$ for $k\geq 1$.

We need to define auxiliary real functions

$$f(x,y)=\frac{1}{1-(1+a\,c\,(a+q)\,x)\,y},$$

$$g(x)=(1+q+a\,c\,(a+q)\,x)\,(1+a\,c\,(a+q)\,x)+a\,c\,(a+q)$$

and scalar sequences

$$s_k=f(s_{k-1},t_{k-1})^2\,g(s_{k-1})\,s_{k-1}^2,\quad s_0=\frac{N\,\gamma^2\,\lambda}{\beta^2},\quad \gamma=\frac{1}{\rho\,\alpha},$$

$$t_k=f(s_{k-1},t_{k-1})^2\,g(s_{k-1})\,s_{k-1}\,t_{k-1},\quad t_0=\frac{M\,\gamma\,\lambda}{\beta^2}.$$

Then, we obtain exactly as Lemma 2.7 the recurrence relations:

$(\widetilde{I_k})$ $\parallel A_k\parallel\leq\gamma\ \parallel\nabla F(x_k)\parallel^{-1}\leq\gamma\ \parallel\nabla F(x_{k-1})\parallel^{-1}\ f(s_{k-1},t_{k-1})$,

$(\widetilde{II_k})$ $\parallel y_k-x_k\parallel\leq f(s_{k-1},t_{k-1})\,g(s_{k-1})\,s_{k-1}\ \parallel y_{k-1}-x_{k-1}\parallel$,

$(\widetilde{III_k})$ $N\ \parallel A_k\parallel\parallel y_k-x_k\parallel\leq s_k$,

$(\widetilde{IV_k})$ $M\ \parallel\nabla F(x_k)\parallel^{-1}\parallel y_k-x_k\parallel\leq t_k$,

$(\widetilde{V_k})$ $\parallel x_{k+1}-y_k\parallel\leq a\,c\,(a+q)\,s_k\ \parallel y_k-x_k\parallel$,

$(\widetilde{VI_k})$ $\parallel x_{k+1}-x_k\parallel\leq(1+a\,c\,(a+q)\,s_k)\ \parallel y_k-x_k\parallel$,

$(\widetilde{VII_k})$ $\parallel x_{k+1}-x_0\parallel\leq(1+q+a\,c\,(a+q)\,s_0)\,\dfrac{1-(f(s_0,t_0)\,g(s_0)\,s_0)^k}{1-f(s_0,t_0)\,g(s_0)\,s_0}\ \parallel y_0-x_0\parallel$

$(\widetilde{VIII_k})$ $|F(x_k)|\leq N\,g(s_{k-1})\ \parallel y_{k-1}-x_{k-1}\parallel^2$.

Hence, we arrive at the second convergence result for (DCTM).

Theorem 2.10. *Let F a nonlinear Fréchet-differentiable mapping under conditions $(\mathcal{C}_1)$, $(\mathcal{C}_2)$, $(\mathcal{C}_3^\star)$, $(\mathcal{C}_4)$ and $(\mathcal{C}_5)$. We also suppose that $f(s_0,t_0)^2\,g(s_0)\,s_0<1$. Then, if $\overline{U}(x_0,R)\subseteq\mathcal{D}$, where*

$$R=\frac{1+q+a\,c\,(a+q)\,s_0}{1-f(s_0,t_0)\,g(s_0)\,s_0}\ \parallel y_0-x_0\parallel$$

the sequence $\{x_k\}$ generated by (DCTM) is well defined, remains in $\overline{U}(x_0, R)$ for all $k \geq 0$ and converges to a solution $x^\star \in \overline{U}(x_0, R)$ of equation $F(x) = 0$.

Remark 2.11. (a) Condition $(\mathcal{C}_5)$ can be replaced by the stronger, but more popular

$$\| [x, y; F] - [u, v; F] \| \leq N_1 (\| x - u \| + \| y - v \|) \quad \text{for all} \quad x, y, u, v \in \mathcal{D}.$$

In this case, we can set $M = 2\ N_1$.

(b) If directions d_k are given by $d_k = \nabla F(x_k)/\| \nabla F(x_k) \|$, the condition $(\mathcal{C}_{21})$ holds for $\alpha = 1$. A possible choice for α can also be

$$\alpha = \frac{|\nabla F(x_0) \cdot d_0|}{\| \nabla F(x_0) \|} \leq 1.$$

(c) Let $d_k = e^{m(k)}$, where $m(k)$ is the index of component of $\nabla F(x_k)$ of maximal modulus:

$$|\nabla F(x_k)\,[m(k)]| := \max_{1 \leq j \leq n} |\nabla F(x_k)\,[j]|.$$

For this choice of d_k, the results obtained here hold, if simply the Euclidean norm is replaced by the infinity norm $\| \cdot \|_\infty$.

(d) Condition $(\mathcal{C}_{22})$ may be too difficult to verify. In this case it can be replaced by the weaker (see [5])

$$\angle(d_k, \nabla F(x_k)) \leq \angle(d_0, \nabla F(x_0)).$$

Similar results can then be obtained using a different technique from recurrence relations using recurrent functions. Such a technique has been given in a Banach space setting [5, 7]. We leave the details to the motivated reader.

3. Numerical tests

We provide two numerical examples where we show the efficiency of the directional Chebyshev-type methods (DCTM) and we apply the convergence results previously obtained. For this, we compare some directional Chebyshev-type method (DCTM) with the directional Newton method (DNM). In particular, in the first numerical test we compare the iteration number, the computational order of convergence (see [17])

$$\rho \approx \frac{\ln (\|x_n - x^*\|/\|x_{n-1} - x^*\|)}{\ln (\|x_{n-1} - x^*\|/\|x_{n-2} - x^*\|)}$$

and the computational efficiency defined by $\rho^{1/(\mathrm{OC}*IN)}$, being (OC) the operational cost per iteration and (IN) the iteration number of the used method. In this case, if the operator F is such that $F : \mathcal{D} \subseteq \mathbb{R}^m \to \mathbb{R}$, then the operational cost is $4m + 3$ for methods (DCTM) and for method (DNM) it is $2m + 1$. Although for particular cases of the parameters a, b, c the operational cost of methods (DCTM) can be improved. For instance, if $a = b = c = 1$, then the operational cost of methods (DCTM) is $3m + 1$.

In the second numerical test, we consider a cubically polynomial equation with $m = 2$. We check that all conditions $(\mathcal{H})$ are satisfied and our Theorem 2.8 is applied to solve this equation using (DCTM).

Example 3.1. First, we take the following nonlinear problem considered in [1]:

$$F(x) = \sum_{i=1}^{p} (\sin x_i)^2 + \sum_{i=p+1}^{m} (\tan x_i)^2, \quad p \text{ is a given integer.}$$

TABLE 1. Iteration number (IN) for the directional methods (DCTM) and (DNM).

m	p	(DCTM) IN	(DNM) IN
	5	15	19
20	10	15	19
	15	15	19
	15	16	20
50	25	16	20
	35	16	20
	20	16	20
80	40	17	20
	60	16	20

TABLE 2. The computational order of convergence and the computational efficiency for (DCTM) and (DNM) methods.

m	p	(DCTM)		(DNM)	
		COC	CE	COC	CE
20	15	3.47348×10^{-7}	3.79615×10^{-10}	5.89895×10^{-11}	9.59233×10^{-14}
50	15	1.54433×10^{-6}	6.3921×10^{-10}	7.01974×10^{-11}	4.35207×10^{-14}
80	40	1.30575×10^{-6}	3.1871×10^{-10}	1.11893×10^{-11}	3.99680×10^{-15}

From the starting point $x_0 = (0.1, 0.1, \ldots, 0.1)$, we have obtained the iteration number (IN) given in Table 1, where the stopping criterion $|F(x_k)| < 10^{-12}$ is used.

We have considered the directional Chebyshev-type method (DCTM) with $a = b = c = 1$. The direction d_k is chosen such that it is sufficiently close to the gradient $\nabla F(x_k)$ of F in each iteration x_k. Notice that if $F(x_k) \neq 0$ and $F(x_k) \approx 0$, then the vector

$$p_k := \left(\frac{F(x_k + F(x_k)e_1)}{F(x_k)} - 1, \frac{F(x_k + F(x_k)e_2)}{F(x_k)} - 1, \ldots, \frac{F(x_k + F(x_k)e_m)}{F(x_k)} - 1 \right)$$

where e_k is the kth unit vector of $\mathbb{R}^m$ is near to $\nabla F(x_k)$. Thus, we have chosen $d_k := p_k/\|p_k\|$ in the implementation.

Observe in Table 1 that the iteration number obtained by the directional Chebyshev-type method (DCTM) with $a = b = c = 1$ is competitive if we compare it with the usual directional Newton method (DNM). On the other hand, in Table 2 we show the computational order of convergence and the computational efficiency for (DCTM) and (DNM) methods using the logarithmic scale and denoted by (COC) and (CE), respectively. We can see that both (COC) and (CE) for methods (DCTM) are better than the one obtained for (DNM).

In addition, we observe in both tables that the value of the parameter p does not have much influence in the iteration number and neither in the computational order for both directional methods.

Finally, in Figure 1 we show the computational order of convergence and the computational efficiency for the directional Chebyshev-type method with $a = b = c = 1$ (the top curve) and for the directional Newton method (the bottom curve) with $m = 80$ and $p = 40$ and using the logarithmic scale.

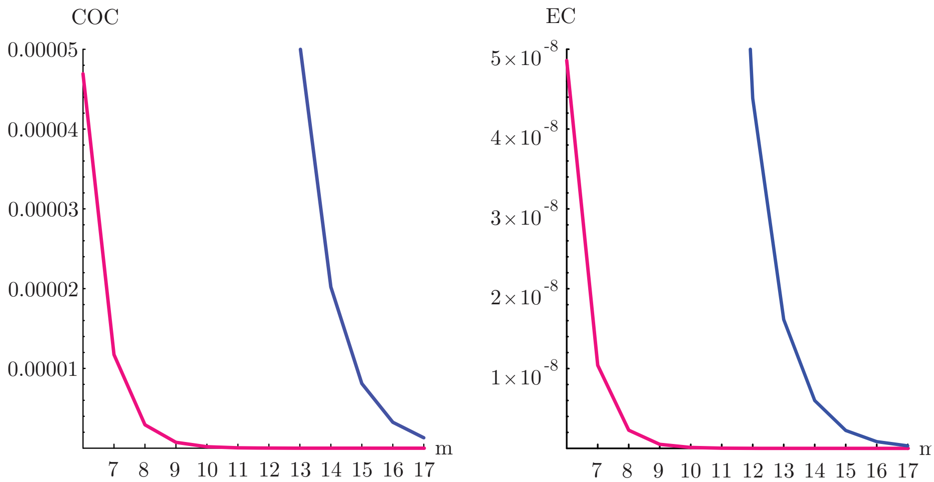

FIGURE 1. Left: The computational order of convergence for (DCTM) and (DNM) methods; Right: The computational efficiency for (DCTM) and (DNM) methods

Example 3.2. Let $m = 2$. Choose

$$x_0 = (1,1)^T, \quad \mathcal{D} = \{x : \| x - x_0 \| \leq 1 - r\} \quad \text{for} \quad r \in [0, 1/2)$$

and define function F on $\mathcal{D}$ by

$$F(x) = \frac{\varsigma_1^3 + \varsigma_2^3}{2} - 2\, r, \quad x = (\varsigma_1, \varsigma_2)^T. \tag{3.1}$$

Then, the gradient ∇F of mapping F is given by

$$\nabla F(x) = \frac{3}{2}\,(\varsigma_1^2, \varsigma_2^2)^T. \tag{3.2}$$

Let $x = (\varsigma_1, \varsigma_2)^T$, $y = (\kappa_1, \kappa_2)^T$, $z = (\tau_1, \tau_2)^T$ in $\mathcal{D}$. Using (3.2) and Definition 2.1, we have that

$$\nabla F(x) - \nabla F(y) = \frac{3}{2}\,(\varsigma_1^2 - \kappa_1^2, \varsigma_2^2 - \kappa_2^2)^T$$

and

$$\begin{aligned}
[x, z; F] - [y, z; F] &= \frac{3}{2}\,\frac{1}{3}\,(\tau_1\,\varsigma_1 + \varsigma_1^2 - \tau_1\,\kappa_1 - \kappa_1^2, \tau_2\,\varsigma_2 + \varsigma_2^2 - \tau_2\,\kappa_2 - \kappa_2^2)^T \\
&= \frac{1}{2}\,((\tau_1 + \varsigma_1 + \kappa_1)\,(\varsigma_1 - \kappa_1), (\tau_2 + \varsigma_2 + \kappa_2)\,(\varsigma_2 - \kappa_2))^T.
\end{aligned}$$

Consequently,

$$\begin{aligned}\| \nabla F(x) - \nabla F(y) \| &\leq \frac{3}{2} \| x + y \| \| x - y \| \\ &\leq \frac{3}{2} (1 - r + 1 - r) \ \| x - y \| \\ &\leq 3\,(2 - r)\,\sqrt{2} \ \| x - y \|\end{aligned}$$

and

$$\begin{aligned}\| [x, z; F] - [y, z; F] \| &\leq \frac{1}{2} \| x + y + z \| \| x - y \| \\ &\leq \frac{1}{2} (4 - 3\,r) \ \| x - y \| \\ &\leq \frac{3}{2} (2 - r)\,\sqrt{2} \ \| x - y \| \,.\end{aligned}$$

Using the above, we obtain that

$$M = 3\ (2 - r)\,\sqrt{2}, \quad \lambda = 1 - 2\ r, \quad N = \frac{3}{2}\ (2 - r)\,\sqrt{2}, \quad \beta = \frac{3\,\sqrt{2}}{2}$$

and for $d_k = \nabla F(x_k)/\| \nabla F(x_k) \|$, we can choose $\alpha = 1$, so that $(\mathcal{H}_{21})$ is satisfied. Set

$$a = b = .5, \quad c = 1, \quad \theta_0 = .5 \quad \text{and} \quad r = .495.$$

Then, in turn we get that

$$\lambda = .01, \quad M = 4.515, \quad N = 2.2575, \quad q = .435,$$

$$a_0 = .2357022605, \quad t_0 = 1.12875, \quad r_0 = .005320978531, \quad s_0 = .2673030933$$

and

$$b_0 = .005016666667, \quad c_0 = .002357022605.$$

Conditions (2.6) and (2.7) hold since

$$f(s_0, b_0)^2\, g(s_0) = .4826545981 < 1$$

and

$$q\,(1 + a\,c\,s_0)\,f(s_0, b_0)\,c_0 = .001168986615 < 1.$$

All conditions $(\mathcal{H})$ are satisfied. That is, Theorem 2.8 applies to solve equation $F(x) = 0$ and (DCTM) starting at x_0 converges to $x^\star \in U(x_0, R)$ with $R = .006255089406$. For example $x^\star = (.99999999,\ .9932883985)^T$ is a solution of (3.1).

Example 3.3. Consider the following nonlinear integral equation of mixed Hammerstein type

$$x(s) = 1 + \frac{1}{2} \int_0^1 G(s, t)\, x(t)^2\, dt, \quad s \in [0, 1],$$

where $x \in C[0, 1]$, $t \in [0, 1]$ and the kernel G is $G(s, t) = \begin{cases} (1 - s)t, & t \leq s, \\ s(1 - t), & s \leq t. \end{cases}$

Now, we consider the following quadratic integral operator:

$$F(x)(s) = x(s) - 1 - \frac{1}{2}\int_0^1 G(s,t)x(t)^2\,dt, \quad s \in [0,1], \tag{3.3}$$

where $x \in C[0,1]$, $s,t \in [0,1]$, and the kernel G is the Green function given previously.

To solve (3.3), we transform it into a finite dimensional problem by using a process of discretization. For this, we approximate the integral that appears in (3.3) by the Gauss-Legendre formula

$$\int_0^1 h(t)\,dt \simeq \sum_{i=1}^{3} w_i h(t_i),$$

where the nodes t_i and the weights w_i are known.

If we denote the approximation of $x(t_i)$ by x_i $(i = 1,2,3)$, then (3.3) is equivalent to the following nonlinear system of equations:

$$x_i - 1 - \frac{1}{2}\sum_{j=1}^{3} a_{ij}\, x_j^2 = 0, \quad i = 1,2,3, \tag{3.4}$$

where

$$a_{ij} = \begin{cases} w_j t_j(1-t_i) & \text{if } j \le i, \\ w_j t_i(1-t_j) & \text{if } j > i. \end{cases}$$

System (3.4) is now written as

$$F(\mathbf{x}) \equiv \mathbf{x} - \mathbf{1} - A\mathbf{v}_{\mathbf{x}} = 0, \qquad F : \mathbb{R}^3 \longrightarrow \mathbb{R}^3,$$

where

$$\mathbf{x} = (x_1, x_2, x_3)^T,\ \mathbf{1} = (1,1,\ldots,1)^T,\ A = (a_{ij})_{i,j=1}^3,\ \mathbf{v}_{\mathbf{x}} = (\frac{x_1^2}{2}, \frac{x_2^2}{2}, \frac{x_3^2}{2})^T.$$

If we choose $\bar{x}_0 = (1,1,1)^t$, after applying 12 iterations of method (DCTM), we obtain the numerical solution $\bar{x}^* = (1.0313\ldots, 1.0816\ldots, 1.0313\ldots)$ of system (3.4). On the other hand if we use (DNM) with the same starting point, the method converges to another solution of system (3.4), $\bar{x}^* = (4.7751\ldots, 16.5217\ldots, 4.7751\ldots)$, and using 95 iterations instead of 12 used by method (DCTM).

4. Conclusion

We presented a semi-local convergence analysis of directional Chebyshev-type methods to solve nonlinear equations under Lipschitz-type conditions on Fréchet–derivative and divided difference mapping of order one. Numerical examples demonstrating the effectiveness of the method are also presented in this study.

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