ON THE KAZHDAN–LUSZTIG CELLS IN TYPE E_8

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ABSTRACT. In 1979, Kazhdan and Lusztig introduced the notion of “cells” (left, right and two-sided) for a Coxeter group W , a concept with numerous applications in Lie theory and around. Here, we address algorithmic aspects of this theory for finite W which are important in applications, e.g., run explicitly through all left cells, determine the values of Lusztig’s $\mathbf{a}$ -function, identify the characters of left cell representations. The aim is to show how type E_8 (the largest group of exceptional type) can be handled systematically and efficiently, too. This allows us, for the first time, to solve some open questions in this case, including Kottwitz’ conjecture on left cells and involutions.

1. INTRODUCTION

Let W be a Coxeter group. Kazhdan and Lusztig [30] introduced certain polynomials $P_{y,w} \in \mathbb{Z}[v]$ (where $y, w \in W$ and v is an indeterminate), which have many remarkable properties and appear in a number of problems in the representation theory of Lie algebras and algebraic groups. As far as W itself is concerned, the polynomials $P_{y,w}$ give rise to partitions of W into left, right and two-sided “cells”. These play an important role, for example, in the classification of the irreducible characters of reductive groups over finite fields [33]. Since the appearance of [30], various authors have contributed to the program of determining the Kazhdan–Lusztig cells for certain types of W ; see, for example, the comments in [8, §4], [29, 7.12 and 7.15] and [42, §1.7]. The situation for finite Coxeter groups is as follows.

For type A_n (when W is a symmetric group), the cells are determined in terms of the Knuth–Robinson–Schensted correspondence (see [30], [34] and also [15]). For the classical types B_n and D_n ; see the work of Garfinkle [11] and re-interpretations of this work by Bonnafé et al. [3]. For type $I_2(m)$ (when W is a dihedral group), the cells are easily determined by explicit computation (by hand); see [29, 7.15]. For the groups of exceptional type, the determination of the cells heavily relies on computers. See Alvis [1] (types H_3 , H_4), Takahashi [44] (F_4), Tong [45] (E_6), Chen–Shi [7] (E_7) and Chen [6] (E_8).

The above results on the various types of groups raise the question if it is possible to characterise and to work with the cells in terms of some general principles. For applications and experiments, we would also like to be able to reconstruct explicitly the cell partition of W in a systematic and efficient way. Furthermore, we typically need to know more than just the partition of the set W into cells. One of the most important aspects of the theory is that every left cell gives rise to a representation of W ; so, for example, given an element $w \in W$, we would like to be able to determine

Received by the editor March 10, 2014 and, in revised form, April 14, 2014.

2010 *Mathematics Subject Classification.* Primary 20C40; Secondary 20C08, 20F55.

This work was supported by DFG Priority Programme SPP 1489.

the left cell Γ containing w and to identify the representation afforded by Γ (its dimension, its character).

The methods developed in [19] allow us to deal with questions of this kind for any finite W of rank up to around 8, with the exception of type E_8 . The group of type E_8 has 696729600 elements; it is—by far—the computationally most challenging case among the finite Coxeter groups of exceptional type. (Note that the left cells are determined in [6], but not the corresponding information on left cell representations. Note also that the results in [48] are concerned with a modification of the polynomials $P_{y,w}$ due to Lusztig–Vogan, so they do not help us here; the highly efficient methods of DuCloux [8], [9] are not sufficient either for the kind of questions that we are addressing here.)

The main purpose of this paper is to show how type E_8 can be dealt with efficiently, too. The new algorithms are designed to work with any finite W as input (not just type E_8 particularly); they are freely available in the latest version of the computer algebra package `PyCox` [22]. The original motivation for this work was a conjecture due to Kottwitz [31], concerning the characters of left cell representations and intersections of left cells with conjugacy classes of involutions. By work of Kottwitz himself, Casselman [5], Bonnafé and the first-named author [4], [19], [21], this conjecture was known to hold except possibly for type E_8 . The algorithms developed in this paper allow us to verify Kottwitz’s conjecture for type E_8 in a straightforward way (by an almost automatic procedure). Hence, this conjecture is now known to hold for any finite Coxeter group.

Systematic experiments based on the methods presented in this paper suggest a general characterisation of left cells inside a given two-sided cell, in terms of a slight variation of Vogan’s generalised τ -invariant [46, §3]; see Conjecture 6.9.

This paper is organised as follows. In Section 2, we briefly recall the basic definitions concerning the polynomials $P_{y,w}$ and the cells of W . In Section 3, we present results of Lusztig [33, Chap. 5] which show that the two-sided cells of W are in bijective correspondence with the so-called “special representations” of W . By [33, 5.27], this correspondence gives rise to the definition of a numerical function $w \mapsto \mathbf{a}(w)$ on W , which is constant on the two-sided cells. In Section 4, we discuss the problem of explicitly computing $\mathbf{a}(w)$ for any given $w \in W$. The main idea is to use Lusztig’s “leading coefficients of character values” [33], [36] and their refinements for matrix representations introduced in [13]. In Section 5, we recall some basic results about the Kazhdan–Lusztig star operations [30]; these lead to the definition of Vogan’s [46] “generalized τ -invariant” for elements of W . We then show how to determine a (relatively small) set $\mathfrak{C}_{\text{left}}^\circ(W)$ of left cells of W such that *any* left cell of W can be reached from a unique cell in $\mathfrak{C}_{\text{left}}^\circ(W)$ by a straightforward procedure (repeated applications of star operations); see Remark 5.10. In Section 6, we apply these general methods to type E_8 . Here, there are 101796 left cells in total but the set $\mathfrak{C}_{\text{left}}^\circ(W)$ contains only 106 left cells. Using the knowledge of $\mathfrak{C}_{\text{left}}^\circ(W)$, we obtain the desired algorithms for efficiently dealing with all the left cells of W . Applications, including Kottwitz’ conjecture, are discussed Section 7.

A key idea in this work is to use a relatively small subset $\check{D} \subseteq W$ with the following properties: (1) It is defined in general terms and is known to contain—by a theoretical argument (see Proposition 4.1)—representatives of all left cells of W , (2) there is a general algorithm for the determination of $\check{D}$ (see [19, §5]) and (3) this algorithm also produces additional information like values of the $\mathbf{a}$ -function

and the characters of cell representations. It can be shown that $\check{D}$ is in fact the set of “distinguished involutions” as defined by Lusztig [35], but we do not need this result here. In our approach, knowing $\check{D}$ constitutes the first step in describing the left cells of W . In type E_8 , for example, it quickly leads to a new and independent proof of the main result of Chen [6]; see Example 6.3. This is the basis for the experiments leading to the formulation of Conjecture 6.9.

2. KAZHDAN–LUSZTIG POLYNOMIALS AND CELLS

Let W be a Coxeter group with generating set S . Let $l: W \rightarrow \mathbb{Z}_{\geq 0}$ be the usual length function with respect to S . We briefly recall the definition of cells from [30]. For this purpose, let $\mathcal{H}$ be the one-parameter generic Iwahori–Hecke algebra. This is an associative algebra over the ring $A = \mathbb{Z}[v, v^{-1}]$ of Laurent polynomials in one variable v . The algebra $\mathcal{H}$ is free as an A -module with basis $\{T_w \mid w \in W\}$. The multiplication is determined by

$$T_s T_w = \begin{cases} T_{sw} & \text{if } l(sw) > l(w), \\ T_{sw} + (v - v^{-1})T_w & \text{if } l(sw) < l(w), \end{cases}$$

where $s \in S$ and $w \in W$. For basic properties of W and $\mathcal{H}$, we refer to [26], [38]. Let $\{C'_w \mid w \in W\}$ be the new basis of $\mathcal{H}$ introduced in [30, Theorem 1.1]. The change of basis to the old basis is given by equations

$$C'_w = \sum_{y \in W} v^{l(y)-l(w)} P_{y,w} T_y \quad \text{where} \quad P_{y,w} \in \mathbb{Z}[v].$$

Here are some of the properties of the polynomials $P_{y,w}$ (see also Remark 2.2 below). We have $P_{w,w} = 1$ and $P_{y,w} = 0$ unless $y \leq w$, where $\leq$ is the Bruhat–Chevalley order. If $y < w$, then $P_{y,w} \in \mathbb{Z}[v]$ has constant term 1 and degree at most $l(w) - l(y) - 1$; furthermore, only even powers of v will occur in $P_{y,w}$. If $y < w$, we will denote by $\mu(y, w)$ the coefficient of $v^{l(w)-l(y)-1}$ in $P_{y,w}$. (Thus, $\mu(y, w) = 0$ if $l(y) \equiv l(w) \pmod{2}$.) We write $y \leftarrow_L w$ if one of the following conditions holds:

- $y = w$, or
- $y < w$ and $\mu(y, w) \neq 0$, or
- there exists some $s \in S$ such that $y = sw > w$.

The Kazhdan–Lusztig left pre-order $\leq_L$ is the transitive closure of the relation $\leftarrow_L$, that is, we have $x \leq_L y$ if there exists a sequence $x = x_0, x_1, \dots, x_k = y$ of elements in W such that $x_{i-1} \leftarrow_L x_i$ for all i . The equivalence relation associated with $\leq_L$ will be denoted by $\sim_L$ and the corresponding equivalence classes are called the *left cells* of W .

We write $x \leq_R y$ if $x^{-1} \leq_L y^{-1}$. The equivalence relation associated with $\leq_R$ will be denoted by $\sim_R$ and the corresponding equivalence classes are called the *right cells* of W . Finally, we define a pre-order $\leq_{LR}$ by the condition that $x \leq_{LR} y$ if there exists a sequence $x = x_0, x_1, \dots, x_k = y$ such that, for each $i \in \{1, \dots, k\}$, we have $x_{i-1} \leq_L x_i$ or $x_{i-1} \leq_R x_i$. The equivalence relation associated with $\leq_{LR}$ will be denoted by $\sim_{LR}$ and the corresponding equivalence classes are called the *two-sided cells* of W .

Remark 2.1. Among other reasons, the partition of W into left cells is important because it gives rise to representations of W and the corresponding algebra $\mathcal{H}$ in terms of so-called “ W -graphs” (see [30, Theorem 1.3]). Indeed, let Γ be a left cell

and $[\Gamma]_A$ an A -module which is free over A with basis $\{e_x \mid x \in \Gamma\}$. Then $[\Gamma]_A$ is an $\mathcal{H}$ -module, where the action of T_s ($s \in S$) is given as follows:

$$T_s \cdot e_y = \begin{cases} -v^{-1}e_y & \text{if } sy < y, \\ ve_y + \sum_{x \in \Gamma: sx < x} \tilde{\mu}(x, y)e_x & \text{if } sy > y. \end{cases}$$

Here, we set $\tilde{\mu}(x, y) = \mu(x, y)$ if $x < y$ and $\tilde{\mu}(x, y) = \mu(y, x)$ if $y < x$; otherwise, we set $\tilde{\mu}(x, y) = 0$. (See also [30, Def. 1.2].)

Remark 2.2. By the definitions, the cells of W are determined via the knowledge of the coefficients $\mu(y, w)$ of the polynomials $P_{y, w}$. Now, there is a recursive and purely combinatorial algorithm for the computation of $P_{y, w}$. Assume that $w \neq 1$ and let $s \in S$ be such that $sw < w$. Then we have (see [30, (2.2.c) and (2.3.g)]):

- If $sy > y$, then $P_{y, w} = P_{sy, w}$.
- If $sy < y$, then $P_{y, w} = P_{sy, sw} + v^2 P_{y, sw} - \sum_{\substack{y \leq z < sw \\ sz < z}} \mu(z, sw) v^{l(w) - l(z)} P_{y, z}$.

For large W , these formulae quickly become unusable. Using the concept of “induction of cells” [12], one can improve these recursion formulae, as explained in [19, §4]. In this way, we obtain an algorithm for the computation of left cells which works quite efficiently for all finite W of rank ≤ 7 ; see [19, Table 2, p. 246]. For example, let W be of type E_6 . The left cells in this case have been determined by Tong [45]. Using the computer algebra package PyCox [22], we obtain the cells by an automatic procedure as follows:

```
>>> W=coxeter("E", 6); W.order
51840
>>> kl=klcells(W, 1, v)
#I 652 left cells (21 non-equivalent), mues:1
```

(This takes about 30 seconds; of course, this varies with the available computer. See the help menu of `klcells` for further information about the format of this command.) This also works for type E_7 ; the computation of the 6364 left cells using `klcells` takes about 4 hours in this case. (See Chen–Shi [7] where different methods were used.) However, type E_8 remains by far out of reach in this approach.

Remark 2.3. Assume that W is finite. Then we denote by $\text{Irr}(W)$ the set of simple $\mathbb{R}[W]$ -modules (up to isomorphism). Note that $\mathbb{R}$ is a splitting field for W (see [26, 6.3.8]). Let Γ be a left cell. By specializing $v \mapsto 1$, the $\mathcal{H}$ -module $[\Gamma]_A$ in Remark 2.1 becomes an $\mathbb{R}[W]$ -module which we denote by $[\Gamma]_1$. For any $E \in \text{Irr}(W)$, we denote by $m(\Gamma, E)$ the multiplicity of E as a constituent of $[\Gamma]_1$. We shall also need to address the problem of computing these multiplicities without having to work out the character values of $[\Gamma]_1$; this will be done in Proposition 4.1 below.

3. TWO-SIDED CELLS AND SPECIAL REPRESENTATIONS

We shall assume from now on that W is finite. By the above definitions, the two-sided cells of W are derived from the knowledge of the relation $\leq_L$. Thus, it would seem that the determination of the two-sided cells is at least as difficult as the determination of the left cells. There is, however, a different way to approach the two-sided cells, using the representation theory of W . This is based on the following constructions. Since the left cells form a partition of W , we have a direct

sum decomposition of left $\mathbb{R}[W]$ -modules

$$\mathbb{R}[W] \cong \bigoplus_{\Gamma \text{ left cell of } W} [\Gamma]_1.$$

Hence, given $E \in \text{Irr}(W)$, there exists a left cell Γ such that E is a constituent of $[\Gamma]_1$.

Definition 3.1. Let $E \in \text{Irr}(W)$. Then all left cells Γ such that E is a constituent of $[\Gamma]_1$ are contained in the same two-sided cell. (See [33, 5.1, 5.15] or [24, §2.2].) This two-sided cell, therefore, only depends on E and will be denoted by $\mathcal{F}_E$. Thus, we obtain a partition

$$\text{Irr}(W) = \bigsqcup_{\mathcal{F} \text{ two-sided cell}} \text{Irr}(W \mid \mathcal{F}),$$

where $\text{Irr}(W \mid \mathcal{F})$ consists of all $E \in \text{Irr}(W)$ such that $\mathcal{F}_E = \mathcal{F}$.

It is remarkable that one can prove some things about the above partition of $\text{Irr}(W)$ without first working out the two-sided cells. To state this more precisely, we need some further notation. Let $\mathcal{H}$ be the one-parameter generic Iwahori–Hecke algebra associated with W , as in the previous section. Let $K = \mathbb{R}(v)$. By extension of scalars, we obtain a K -algebra $\mathcal{H}_K = K \otimes_A \mathcal{H}$. It is known that $\mathcal{H}_K$ is split semisimple and abstractly isomorphic to $K[W]$ (see [26, 9.3.5]); furthermore, the map $v \mapsto 1$ induces a bijection between $\text{Irr}(\mathcal{H}_K)$ and $\text{Irr}(W)$ (see [26, 8.1.7]). Given $E \in \text{Irr}(W)$, we denote by E_v the corresponding irreducible representation of $\mathcal{H}_K$. We have $\text{trace}(T_w, E_v) \in \mathbb{R}[v, v^{-1}]$ for all $w \in W$ (see [26, 9.3.5]). We define

$$\mathbf{a}_E := \min\{i \in \mathbb{Z}_{\geq 0} \mid v^i \text{trace}(T_w, E_v) \in \mathbb{R}[v] \text{ for all } w \in W\}.$$

Finally, let $\mathbf{b}_E$ be the smallest integer $i \geq 0$ such that E occurs as a constituent of the i -th symmetric power of the natural reflection representation of W . Then, as in [33, 4.1], the set of *special representations* of W is defined by

$$\mathcal{S}(W) := \{E \in \text{Irr}(W) \mid \mathbf{a}_E = \mathbf{b}_E\}.$$

Now we can state:

Theorem 3.2 (Lusztig). *Let $\mathcal{F}$ be a two-sided cell of W . Then $\mathcal{F} = \mathcal{F}_{E_0}$ for a unique $E_0 \in \mathcal{S}(W)$. Furthermore, the function $E \mapsto \mathbf{a}_E$ is constant on $\text{Irr}(W \mid \mathcal{F})$.*

The functions $E \mapsto \mathbf{a}_E$, $E \mapsto \mathbf{b}_E$ and, hence, the sets $\mathcal{S}(W)$ are explicitly known in all cases; see the tables in [33, Chap. 4] (for finite Weyl groups) and [26, §6.5] (where the types $I_2(m)$, H_3 , H_4 are included in the discussion).

For example, if W is of type E_8 , we have $|\text{Irr}(W)| = 112$ and there are 46 special representations; they are listed in Table 1 (which is taken from [33, 4.13.1]). Consequently, by Theorem 3.2, we already know that there are 46 two-sided cells of W . Using [33, 5.25.2, 5.26, 12.3.7], it also follows that

$$\text{Number of left cells of } W = \sum_{E_0 \in \mathcal{S}(W)} \dim E_0 = 101796.$$

We shall not need the latter result, since we will obtain this number independently in the course of our computations; see Example 6.3.

Definition 3.3 (Lusztig [33, 5.27]). Let $w \in W$. If $\mathcal{F}$ is the two-sided cell containing w , then we set $\mathbf{a}(w) := \mathbf{a}_E$ where $E \in \text{Irr}(W \mid \mathcal{F})$. (By Theorem 3.2, this is well defined.)

TABLE 1. The 46 special representations for W of type E_8

E	$\mathbf{a}_E$	E	$\mathbf{a}_E$	E	$\mathbf{a}_E$	E	$\mathbf{a}_E$	E	$\mathbf{a}_E$
1_x	0	3240_z	9	4200_z	15	$2800'_z$	25	$560'_z$	47
8_z	1	2268_x	10	5600_z	15	$4096'_x$	26	$210'_x$	52
35_x	2	2240_x	10	4480_y	16	$2240'_x$	28	$112'_z$	63
112_z	3	4096_z	11	2100_y	20	$2268'_x$	30	$35'_x$	74
210_x	4	525_x	12	$4200'_z$	21	$3240'_x$	31	$8'_z$	91
560_z	5	4200_x	12	$5600'_z$	21	$1400'_x$	32	$1'_x$	120
567_x	6	2800_z	13	$2835'_x$	22	$525'_x$	36		
700_x	6	4536_z	13	$6075'_x$	22	$1400'_z$	37		
1400_z	7	2835_x	14	$4536'_z$	23	$700'_x$	42		
1400_x	8	6075_x	14	$4200'_x$	24	$567'_x$	46		

Comments on the proof of Theorem 3.2. Contrary to the previous results, and those discussed in Section 4 below, there does not seem to exist an elementary direct proof of Theorem 3.2. Using standard operations in the character ring of W (induction from parabolic subgroups, tensoring with the sign character), Lusztig [33, 4.2] has defined another partition of $\text{Irr}(W)$ into so-called “families”. As shown in [33, Theorem 5.25] (for finite Weyl groups, using deep results from the theory of Lie algebras and algebraic groups), we have $\mathcal{F}_E = \mathcal{F}_{E'}$ if and only if E, E' belong to the same “family”; a relatively simple direct argument for the types $I_2(m)$, H_3 , H_4 can be found in [18, Example 3.6]. (Another proof, for general W , is given in [38, Prop. 23.3], which relies on certain positivity properties for $\mathcal{H}$. There is now a general, purely algebraic proof of these positivity properties, due to Elias–Williamson [10, Cor. 1.2].) Then the statements in Theorem 3.2 follow from the analogous results for “families”, and these are contained in [33, Chap. 4] (for finite Weyl groups) and [26, §6.5] (where the types $I_2(m)$, H_3 , H_4 are included in the discussion). $\square$

Remark 3.4. By the definitions, it is clear that every two-sided cell is at the same time a union of left cells and a union of right cells. It is, however, not clear at all that two-sided cells are the smallest subsets of W with this property. This follows from the following implication, where $x, y \in W$:

$$(A) \quad x \leq_L y \quad \text{and} \quad x \sim_{LR} y \quad \Rightarrow \quad x \sim_L y.$$

The fact that (A) holds is implicit in the proof of Theorem 3.2. When W is a finite Weyl group, (A) was first proved by Lusztig; see [32, Lemma 4.1] or [34, Cor. 5.5] (using similar methods as discussed in the proof of Theorem 3.2). For type H_4 , (A) has been verified in [1, Cor. 3.3]; a similar verification also works for H_3 . For type $I_2(m)$, see [38, 8.7]. Once (A) is known to hold, it follows that the partition of W into left cells determines the partitions into right cells and into two-sided cells.

4. CELLS AND LEADING COEFFICIENTS

In the previous section, we have seen that one can attach to any element $w \in W$ a numerical value $\mathbf{a}(w)$: we have $\mathbf{a}(w) = \mathbf{a}_E$ where $E \in \text{Irr}(W)$ is such that $w \in \mathcal{F}_E$. The purpose of this section is to address the following problem:

Given $w \in W$, how can we actually find some $E \in \text{Irr}(W)$ such that $w \in \mathcal{F}_E$?

We will see that this problem can be solved using Lusztig's *leading coefficients of character values* [33], [36] and their refinements introduced in [13]. Let $E \in \text{Irr}(W)$ and consider the corresponding irreducible representation E_v of $\mathcal{H}_K$. Recall that

$$v^{\mathbf{a}_E} \text{trace}(T_w, E_v) \in \mathbb{R}[v] \quad \text{for all } w \in W.$$

Consequently, there are unique numbers $c_{w,E} \in \mathbb{R}$ ($w \in W$) such that

$$v^{\mathbf{a}_E} \text{trace}(T_w, E_v) = (-1)^{l(w)} c_{w,E} + \text{"higher terms"},$$

where "higher terms" means an $\mathbb{R}$ -linear combination of monomials v^i where $i > 0$. Given E , there is at least one $w \in W$ such that $c_{w,E} \neq 0$ (by the definition of $\mathbf{a}_E$). Since $\text{trace}(T_w, E_v) = \text{trace}(T_{w^{-1}}, E_v)$ for all $w \in W$ (see [26, 8.2.6]), we certainly have

$$c_{w,E} = c_{w^{-1},E} \quad \text{for all } w \in W.$$

These numbers are called the *leading coefficients of character values*; see [33], [36]. We now describe some general constructions involving these leading coefficients.

Let $E \in \text{Irr}(W)$. Since the sum of all $c_{w,E}^2$ ($w \in W$) is strictly positive, we can write that sum as $f_E \dim E$ where $f_E \in \mathbb{R}_{>0}$. With this notation, we can state:

Proposition 4.1 (Cf. [19, 5.6]). *We define real numbers as follows:*

$$\check{n}_w := \sum_{E \in \text{Irr}(W)} f_E^{-1} c_{w,E} \quad \text{for all } w \in W.$$

Then the following hold.

- (a) Let Γ be a left cell and $E \in \text{Irr}(W)$. Then the multiplicity $m(\Gamma, E)$ (see Remark 2.3) is given by

$$m(\Gamma, E) = \sum_{w \in \Gamma} \check{n}_w c_{w,E}.$$

- (b) Every left cell of W contains an element of the set $\check{\mathcal{D}} := \{w \in W \mid \check{n}_w \neq 0\}$. Thus, the number of left cells of W is less than or equal to $|\check{\mathcal{D}}|$.

Proof. (a) Let $E, E' \in \text{Irr}(W)$. By [33, 5.8] (for finite Weyl groups) and [20, Cor. 3.8] (in general), we have the following orthogonality relations:

$$\sum_{w \in \Gamma} c_{w,E} c_{w,E'} = \begin{cases} m(\Gamma, E) f_E & \text{if } E \cong E', \\ 0 & \text{otherwise.} \end{cases}$$

Using these relations and the defining formula for $\check{n}_w$, the expression $\sum_{w \in \Gamma} \check{n}_w c_{w,E}$ reduces to $m(\Gamma, E)$. (See also [24, Example 1.8.5].)

(b) Let Γ be a left cell. Then there is some $E \in \text{Irr}(W)$ such that $m(\Gamma, E) > 0$. So (a) shows that there exists some $w \in \Gamma$ such that $\check{n}_w \neq 0$, that is, we have $w \in \Gamma \cap \check{\mathcal{D}}$. $\square$

Corollary 4.2 (Cf. Lusztig [33, Lemma 5.2]). *Let $w \in W$ and assume that there exists some $E \in \text{Irr}(W)$ such that $c_{w,E} \neq 0$. Then $w \in \mathcal{F}_E$.*

Proof. Let Γ be the left cell such that $w \in \Gamma$. Then the orthogonality relations used in the above proof show that $\sum_{x \in \Gamma} c_{x,E}^2 = f_E m(\Gamma, E)$. Since the sum contains some strictly positive term (the one corresponding to $x = w$), we have $m(\Gamma, E) > 0$ and so $\Gamma \subseteq \mathcal{F}_E$. $\square$

The above methods allow us to solve the problem raised at the beginning of this section for those elements $w \in W$ for which there exists some $E \in \text{Irr}(W)$ such that $c_{w,E} \neq 0$. The new ingredient to deal with *all* elements of W are the leading matrix coefficients introduced in [13]. For each $E \in \text{Irr}(W)$, we consider a matrix representation

$$\rho_v^E: \mathcal{H}_K \rightarrow M_{d_E}(K) \quad (d_E = \dim E)$$

affording $E_v \in \text{Irr}(\mathcal{H}_K)$. Now recall that K is the field of fractions of $\mathbb{R}[v]$. Let us consider the discrete valuation ring

$$\mathcal{O} = \{f/g \in K \mid f, g \in \mathbb{R}[v], g(0) \neq 0\} \subseteq K.$$

In particular, we have a ring homomorphism $\mathcal{O} \rightarrow \mathbb{R}$ given by evaluation at 0. Following [16, 2.2, 4.3], we say that ρ_v^E is a *balanced representation* if there exists a symmetric matrix $\Omega^\lambda \in M_{d_E}(\mathcal{O})$ such that

$$\det(\Omega^E) \in \mathcal{O}^\times \quad \text{and} \quad \Omega^E \rho_v^E(T_{w^{-1}}) = \rho_v^E(T_w)^{\text{tr}} \Omega^E \quad \text{for all } w \in W.$$

By [16, 2.3] (see also [24, §1.4]), we can always choose ρ_v^E to be balanced; let us now assume that this is the case. Then we have

$$v^{\mathbf{a}_E} \rho_v^E(T_w) \in M_{d_E}(\mathcal{O}) \quad \text{for all } w \in W.$$

For $1 \leq i, j \leq d_E$, we denote by $c_{w,E}^{ij} \in \mathbb{R}$ the value at 0 of the (i, j) -entry of the matrix $(-1)^{l(w)} v^{\mathbf{a}_E} \rho_v^E(T_w)$. These real numbers are called *leading matrix coefficients*. Note that

$$c_{w,E} = \sum_{1 \leq i \leq d_E} c_{w,E}^{ii} \quad \text{for all } w \in W,$$

where $c_{w,E}$ are Lusztig's leading coefficients of character values as introduced earlier.

Definition 4.3 (Cf. [17, Def. 3.1], [24, §1.6]). Let $w \in W$ and $E \in \text{Irr}(W)$. Then write $E \rightsquigarrow w$ if $c_{w,E}^{ij} \neq 0$ for some $i, j \in \{1, \dots, d_E\}$. By [16, 3.9], the relation $\rightsquigarrow$ does not depend on the choice of the balanced representation ρ_v^E .

Lemma 4.4. *Let $w \in W$. Then there exists some $E \in \text{Irr}(W)$ such that $E \rightsquigarrow w$. For any such E , we have $w \in \mathcal{F}_E$. Consequently, we have $\mathbf{a}(w) = \mathbf{a}_E$ (see Definition 3.3).*

Proof. By the orthogonality relations in [16, 3.3], we have

$$1 = \sum_{E \in \text{Irr}(W)} \sum_{1 \leq i, j \leq d_E} c_{w,E}^{ij} c_{w^{-1},E}^{ji}.$$

Hence, there exists some $E \in \text{Irr}(W)$ such that $E \rightsquigarrow w$. Now let $E \in \text{Irr}(W)$ be arbitrary such that $E \rightsquigarrow w$. Let Γ be the left cell such that $w \in \Gamma$. Then, by [13, Prop. 4.7], [17, Lemma 3.2], we have $m(\Gamma, E) > 0$ and so $w \in \mathcal{F}_E$. $\square$

Remark 4.5. Let W be of exceptional type. Then Lusztig [32, §5] (type H_3), Alvis–Lusztig [2] (H_4), Naruse [41] (F_4, E_6), Howlett–Yin [28] (E_6, E_7) and Howlett [27] (E_8) explicitly determined matrix representations ρ_v^E for all $E \in \text{Irr}(W)$ (in terms of abstract W -graphs, as defined in [30]). Thus, for each $E \in \text{Irr}(W)$, we are given explicit matrices $\{\rho_v^E(T_s) \mid s \in S\}$, with entries in $\mathbb{R}[v, v^{-1}]$. In [16, Example 4.6] (types H_3, H_4) and [25, §4] (types F_4, E_6, E_7, E_8), it is shown how to construct

symmetric matrices $\Omega^E \in M_{d_E}(\mathbb{R}[v])$ such that some entry of Ω^E has a non-zero constant term and such that

$$\Omega^E \rho_v^E(T_s) = \rho_v^E(T_s)^{\text{tr}} \Omega^E \quad \text{for all } s \in S.$$

One can check that, in all cases, $\det(\Omega^E) \in \mathbb{R}[v]$ has a non-zero constant term. Hence, the above matrix representations are balanced in the sense defined above. (It is conjectured in [16, 4.6] that every W -graph representation is automatically balanced; see also [24, 1.4.14].) In Michel’s development version of the computer algebra package CHEVIE [40], these representations are obtained via the command **Representations**. Thus, given an element $w \in W$, we consider a reduced expression $w = s_1 s_2 \cdots s_l$ with $s_i \in S$ and obtain $\rho_v^E(T_w) = \rho_v^E(T_{s_1}) \cdots \rho_v^E(T_{s_l})$. We can then check if the relation $E \rightsquigarrow w$ holds. Note that, in type E_8 , we have $\max\{l(w)\} = 120$ and $\max\{\dim E\} = 7168$; in extreme cases, the computation of the matrix $\rho_v^E(T_w)$ may take 10 minutes or even more. Therefore, we will have to try to reduce as much as possible the number of elements $w \in W$ for which we need to apply this method to work out $\mathbf{a}(w)$. The techniques for achieving such a reduction will be discussed in the following section.

(The computation [25] of the symmetric matrices Ω^E for type E_8 took several months; these matrices are available upon request. Note that we do not need to know them for establishing the relation $E \rightsquigarrow w$; as far as type E_8 is concerned, they were *only* needed to prove that Howlett’s matrix representations indeed are balanced.)

5. STAR OPERATIONS AND INDUCTION OF CELLS

We now turn to the determination of the left cells of W . A first approximation is obtained by the following constructions, which already appeared in [30]. For any $w \in W$, we denote by $\mathcal{R}(w) := \{s \in S \mid ws < w\}$ the *right descent set* of w .

Proposition 5.1 ([30, Prop. 2.4]). *Let $x, y \in W$. If $x \sim_L y$, then $\mathcal{R}(x) = \mathcal{R}(y)$. Thus, for any $I \subseteq S$, the set $\{w \in W \mid \mathcal{R}(w) = I\}$ is a union of left cells of W .*

The above result can be refined considerably. Following Kazhdan–Lusztig [30, 4.1], let us consider two generators $s, t \in S$ such that st has order 3. We set

$$\mathcal{D}_R(s, t) := \{w \in W \mid \mathcal{R}(w) \cap \{s, t\} \text{ has exactly one element}\}.$$

If $w \in \mathcal{D}_R(s, t)$, then exactly one of the elements ws, wt belongs to $\mathcal{D}_R(s, t)$; we denote it w^* . The map

$$\sigma_{s,t}: \mathcal{D}_R(s, t) \rightarrow \mathcal{D}_R(s, t), \quad w \mapsto w^*,$$

is an involution, called *star operation*. Now let Γ be a left cell. By Proposition 5.1, we have $\Gamma \subseteq \mathcal{D}_R(s, t)$ or $\Gamma \cap \mathcal{D}_R(s, t) = \emptyset$.

Proposition 5.2 ([30, Cor. 4.3]). *Let $s, t \in S$ be as above and Γ be a left cell such that $\Gamma \subseteq \mathcal{D}_R(s, t)$. Then $\Gamma^* := \{w^* \mid w \in \Gamma\}$ also is a left cell and the map $[\Gamma]_A \rightarrow [\Gamma^*]_A$, $e_w \mapsto e_{w^*}$, is an isomorphism of $\mathcal{H}$ -modules. Furthermore, $w \sim_R w^*$ for all $w \in \Gamma$.*

Remark 5.3. Let $y, w \in W$. We write $y \overset{*}{\leftrightarrow} w$ if w is obtained from y via repeated applications of star operations, that is, there is a sequence of elements $y =$

$y_0, y_1, \dots, y_m = w$ in W such that, for each $i \in \{1, \dots, m\}$, we have $y_i = \sigma_{s_i, t_i}(y_{i-1})$ for some $s_i, t_i \in S$ such that $s_i t_i$ has order 3 and $y_{i-1} \in \mathcal{D}_R(s_i, t_i)$. The set

$$R^*(w) := \{y \in W \mid y \overset{*}{\leftrightarrow} w\}$$

will be called the (*right*) *star orbit* of w . By Proposition 5.2, $R^*(w)$ is contained in a right cell of W . Similarly, the *left star orbit* of w is defined by

$$L^*(w) := \{y \in W \mid y^{-1} \overset{*}{\leftrightarrow} w^{-1}\}.$$

Since $y \sim_L w \Leftrightarrow y^{-1} \sim_R w^{-1}$, the set $L^*(w)$ is contained in a left cell. In PyCox, these operations are performed by the commands `klstarorbitem` and `leftklstarorbitem`.

Remark 5.4. Let $\mathfrak{C}_{\text{left}}(W)$ be the set of all left cells of W . The above result shows that the star operations permute the elements of $\mathfrak{C}_{\text{left}}(W)$. We shall denote by $\mathfrak{C}_{\text{left}}^\circ(W) \subseteq \mathfrak{C}_{\text{left}}(W)$ a subset such that any $\Gamma \in \mathfrak{C}_{\text{left}}(W)$ can be reached from a unique element of $\mathfrak{C}_{\text{left}}^\circ(W)$ via repeated applications of star operations. From a practical point of view, this is particularly useful for storing results on a computer: Since star operations are performed quite easily and efficiently, it will only be necessary to store the cells in $\mathfrak{C}_{\text{left}}^\circ(W)$. The PyCox command `klcells` returns such a subset $\mathfrak{C}_{\text{left}}^\circ(W)$ of the set of all left cells of W .

For example, if W is of type E_6 , then the output of the `klcells` command in Remark 2.2 tells us that $|\mathfrak{C}_{\text{left}}(W)| = 652$ and $|\mathfrak{C}_{\text{left}}^\circ(W)| = 21$.

In combination, Propositions 5.1 and 5.2 show that, if Γ is a left cell, then we not only have $\mathcal{R}(y) = \mathcal{R}(w)$ but also $\mathcal{R}(y^*) = \mathcal{R}(w^*)$ for all $y, w \in \Gamma$. Iterating this process leads us to the following notion.

Definition 5.5 (Cf. Vogan [46, 3.10], simply-laced version¹). For any $n \geq 0$, we define a relation $\approx_n$ on W inductively as follows: First, let $n = 0$ and $y, w \in W$. Then $y \approx_0 w$ if $\mathcal{R}(y) = \mathcal{R}(w)$. Now let $n > 0$ and assume that the relation $\approx_{n-1}$ has been already defined on W . Let $y, w \in W$. Then $y \approx_n w$ if $y \approx_{n-1} w$ and if $\sigma_{s,t}(y) \approx_{n-1} \sigma_{s,t}(w)$ for any $s, t \in S$ such that st has order 3 and $y, w \in \mathcal{D}_R(s, t)$.

If $y \approx_n w$ for all $n \geq 0$, then we say that y, w have the same *generalized τ -invariant*. This defines an equivalence relation on W ; the corresponding equivalence classes will be called τ -cells.

Corollary 5.6. *Let Γ be a left cell of W . Then Γ is contained in a τ -cell.*

Proof. This is clear by Propositions 5.1 and 5.2. □

Thus, by computing the τ -cells of W , we obtain a first approximation to the partition of W into left cells. This is quite powerful. For example, by Kazhdan–Lusztig [30, §5], the left cells in type A_{n-1} (where W is the symmetric group $\mathfrak{S}_n$) are precisely given by the τ -cells. A similar result also holds for W of type E_6 , as shown by Tong [45, §7]. In PyCox, the τ -cells are computed by the command `gentaucells` (which just implements the procedure described in Definition 5.5;

¹The general version would also take into account generators $s, t \in S$ such that st has order strictly bigger than 3; see [46, 3.10] and Remark 6.8 below, but the relations (and the proofs) are somewhat more complicated; the “simply-laced” version is sufficient for our purposes here.

note that, when W is finite, one only needs to verify $y \approx_n w$ for a finite number of n .) For example, Tong's result is recovered as follows:

```
>>> W=coxeter("E", 6)
>>> len(gentacells(W, allwords(W)))
652      # the number of left cells
```

In general, the τ -cells will just be unions of left cells.

Finally, we show how to obtain the complete set of all left cells of W (and not just their representatives in $\tilde{D}$). This will rely on two techniques: (1) orbits under the star operations (see Remark 5.4) and (2) the concept of “induction of cells” [12]. Let us explain how (2) works. Let $I \subseteq S$ and consider the parabolic subgroup $W_I \subseteq W$. Then

$$X_I := \{w \in W \mid \mathcal{R}(w) \cap I = \emptyset\}$$

is the set of distinguished left coset representatives of W_I in W . The map $X_I \times W_I \rightarrow W$, $(x, u) \mapsto xu$, is a bijection and we have $l(xu) = l(x) + l(u)$ for all $x \in X_I$ and $u \in W_I$; see [26, §2.1]. Thus, given $w \in W$, we can write uniquely $w = xu$ where $x \in X_I$ and $u \in W_I$. In this case, we denote $\text{pr}_I(w) := u$. Let $\sim_{L,I}$ be the equivalence relation on W_I for which the equivalence classes are the left cells of W_I .

Proposition 5.7 ([12]). *Let $I \subseteq S$. If $w, w' \in W$ are such that $w \sim_L w'$, then $\text{pr}_I(w) \sim_{L,I} \text{pr}_I(w')$. In particular, if Γ is a left cell of W_I , then $X_I\Gamma$ is a union of left cells of W .*

We now show that the induction of cells is compatible with the star operations.

Remark 5.8. Let $I \subseteq S$ and $s, t \in I$ be generators such that st has order 3. Then we can define star operations with respect to W_I and s, t . As above we set

$$\mathcal{D}_R^I(s, t) := \{u \in W_I \mid \mathcal{R}(u) \cap \{s, t\} \text{ has exactly one element}\}$$

and obtain a bijection $\sigma_{s,t}^I: \mathcal{D}_R^I(s, t) \rightarrow \mathcal{D}_R^I(s, t)$, $u \mapsto u^*$. Now note that $\mathcal{D}_R^I(s, t) = \mathcal{D}_R(s, t) \cap W_I$ and $\sigma_{s,t}^I$ is the restriction of $\sigma_{s,t}$ from $\mathcal{D}_R(s, t)$ to $\mathcal{D}_R^I(s, t)$. Further note that $X_I\mathcal{D}_R^I(s, t) \subseteq \mathcal{D}_R(s, t)$ and

$$\sigma_{s,t}(xu) = x\sigma_{s,t}^I(u) \quad \text{for all } x \in X_I \text{ and } u \in \mathcal{D}_R^I(s, t).$$

Thus, more intuitively, we can also write $(xu)^* = xu^*$ for $x \in X_I$ and $u \in \mathcal{D}_R^I(s, t)$.

Corollary 5.9 (Cf. [14, 3.9, 3.10]). *In the setting of Remark 5.8, let Γ be a left cell of W_I such that $\Gamma \subseteq \mathcal{D}_R^I(s, t)$. Let $\Gamma^* := \{u^* \mid u \in \Gamma\} \subseteq \mathcal{D}_R^I(s, t)$ (where we use the star operation with respect to W_I, s, t). By Propositions 5.2 and 5.7, each of the two sets $X_I\Gamma$, $X_I\Gamma^*$ is a union of left cells of W . Now let $X_I\Gamma = \Gamma_1 \amalg \dots \amalg \Gamma_r$ be the partition of $X_I\Gamma$ into left cells of W . Then the star operation (with respect to W, s, t) is defined for each Γ_i and $X_I\Gamma^* = \Gamma_1^* \amalg \dots \amalg \Gamma_r^*$ is the partition of $X_I\Gamma^*$ into left cells of W .*

Proof. Since $X_I\Gamma \subseteq X_I\mathcal{D}_R^I(s, t) \subseteq \mathcal{D}_R(s, t)$, the star operation (with respect to W, s, t) is defined for each element of $X_I\Gamma$. Now, the bijection $\Gamma \rightarrow \Gamma^*$, $u \mapsto u^*$, induces a bijection

$$X_I\Gamma \rightarrow X_I\Gamma^*, \quad xu \mapsto xu^* \quad (x \in X_I, u \in \Gamma).$$

By Remark 5.8, we have $(xu)^* = xu^*$ for $x \in X_I$ and $u \in \mathcal{D}_R^I(s, t)$. Thus, the above bijection $X_I\Gamma \rightarrow X_I\Gamma^*$ is the restriction of $\sigma_{s,t}$ from $\mathcal{D}_R(s, t)$ to $X_I\Gamma$. Consequently, we have $\Gamma_i^* \subseteq X_I\Gamma^*$ for $1 \leq i \leq r$ which yields the desired assertion. $\square$

Remark 5.10. The practical use of Proposition 5.7 is as follows. Let $I \subsetneq S$ and assume that the left cells of W_I have already been determined; let $\mathfrak{C}_{\text{left}}(W_I) = \{\Gamma_1, \dots, \Gamma_r\}$. For each i , let $X_I \Gamma_i = Y_{i,1} \amalg \dots \amalg Y_{i,t_i}$ be the decomposition into τ -cells. Then the sets $\{Y_{i,j} \mid 1 \leq i \leq r, 1 \leq j \leq t_i\}$ form a partition of W ; by Corollary 5.6 and Proposition 5.7, each $Y_{i,j}$ is a union of left cells of W .

Using star operations, we can go one step further. Let $R \subseteq \{1, \dots, r\}$ be a subset such that $\mathfrak{C}_{\text{left}}^\circ(W_I) = \{\Gamma_i \mid i \in R\}$ is a set of left cells as in Remark 5.4. Now let Γ be any left cell of W . Then Corollary 5.9 shows that there exists a left cell Γ' of W which is contained in the set

$$\Upsilon_I(W) := \bigcup_{i \in R, 1 \leq j \leq t_i} Y_{i,j} \subseteq W$$

and such that Γ is obtained via repeated applications of star operations from Γ' (with respect to generators $s, t \in I$). Using repeated applications of star operations with respect to *all* generators in S , we obtain a subset $\mathfrak{C}_{\text{left}}^\circ(W) \subseteq \Upsilon_I(W)$ as in Remark 5.4.

6. TYPE E_8

We shall now apply the methods developed so far to the group W of type E_8 . Recall that, by Lusztig's results in Section 3, we already know that there are 46 two-sided cells of W , corresponding to the special representations in $\mathcal{S}(W)$. The first step is to consider the leading coefficients of character values from Section 4.

Example 6.1. Let W be of type E_8 . We have $|\text{Irr}(W)| = 112$. The entries of the matrix of all leading coefficients $(c_{w,E})$ have been determined by Lusztig [36, 3.14]. This shows, for example, that $|c_{w,E}| \leq 8$ for all E, w ; however, the columns of the matrix are not matched with the various elements of W . Using [19, §5, Algorithm B], we can determine the matrix $(c_{w,E})$ together with the labelling of the columns by the elements of W . (See the description of the PyCox command `distinguishedinvolutions`; note that this computation takes nearly 18 days and requires about 36 GB of main memory.) By inspection of the output, we can verify the following statements.

- (a) The set $W^* := \{w \in W \mid c_{w,E} \neq 0 \text{ for some } E \in \text{Irr}(W)\}$ contains precisely 208422 elements; furthermore, $|\check{\mathcal{D}}| = 101796$ and $\check{n}_d = 1$ for all $d \in \check{\mathcal{D}}$.
- (b) Let $\mathcal{I} := \{w \in W \mid w^2 = 1\}$ be the set of involutions in W . This set $\mathcal{I}$ is computed as explained in [19, §2]; we have $|\mathcal{I}| = 199952$ and $\check{\mathcal{D}} \subseteq \mathcal{I} \subseteq W^*$.
- (c) For any $E_0 \in \mathcal{S}(W)$, we set $\mathcal{F}_{E_0}^* := \{w \in W \mid c_{w,E_0} \neq 0\}$. Then we have

$$W^* := \{w \in W \mid c_{w,E} \neq 0 \text{ for some } E \in \text{Irr}(W)\} = \bigcup_{E_0 \in \mathcal{S}(W)} \mathcal{F}_{E_0}^*,$$

where the union on the right hand side is disjoint.

Thus, for any $w \in W^*$, there is a unique $E_0 \in \mathcal{S}(W)$ such that $c_{w,E_0} \neq 0$. Then we have $w \in \mathcal{F}_{E_0}$ and so $\mathbf{a}(w) = \mathbf{a}_{E_0}$ (see Definition 3.3).

Remark 6.2. Most of the statements in Example 6.1 can also be deduced from theoretical arguments and, hence, hold in general. (But we still need to know the sets $\mathcal{I}$, $\check{\mathcal{D}}$ explicitly in type E_8 in the subsequent discussion.) For example, if W is a finite Weyl group, then (c) holds by [33, Prop. 7.1] and [36, 3.14]. If W is of type H_3 , H_4 or $I_2(m)$, then (c) is checked by explicit computation; see [19, Rem. 5.12]. The

inclusion $\mathcal{I} \subseteq W^*$ in (b) holds by the remark just after [36, 3.5(b)]. The argument for proving this works in general, once the positivity properties already mentioned in the comments on the proof of Theorem 3.2 are known to hold. Similarly, as observed in [17, 3.7], these positivity properties together with the techniques in [36] also imply that $\check{\mathcal{D}}$ is the set of “distinguished involutions” defined by Lusztig [35]; in particular, $\check{\mathcal{D}} \subseteq \mathcal{I}$.

Example 6.3. Let W be of type E_8 . The set $\check{\mathcal{D}}$ is directly available in PyCox via the command `distinva`, which returns a pair of lists: the first one containing the elements $w \in \check{\mathcal{D}}$, the second one containing the corresponding values $\mathbf{a}(w)$. (This uses pre-stored data from the output of the original computation in Example 6.1 and, hence, just takes a few seconds.) We now decompose $\check{\mathcal{D}}$ into τ -cells:

```
>>> W=coxeter("E", 8)
>>> gt=gentaucells(W, distinva(W)[0]); len(gt)
81901
```

(This computation takes a few days.) Thus, we obtain the partition $\check{\mathcal{D}} = \check{\mathcal{D}}_1 \amalg \dots \amalg \check{\mathcal{D}}_{81901}$ where each $\check{\mathcal{D}}_i$ is a τ -cell of $\check{\mathcal{D}}$. Using the knowledge of the values $\mathbf{a}(w)$ for all $w \in \check{\mathcal{D}}$, we can now check that the following property holds:

(*) For each $i \in \{1, 2, 3, \dots, 81901\}$, the function $w \mapsto \mathbf{a}(w)$ is injective on $\check{\mathcal{D}}_i$.

Since the function $w \mapsto \mathbf{a}(w)$ is constant on left cells (see Definition 3.3) and since elements in the same left cell have the same generalized τ -invariant (see Corollary 5.6), we conclude that each left cell of W contains a unique element of $\check{\mathcal{D}}$; in particular, there are precisely 101796 left cells of W . This yields a new proof of the following result:

Theorem 6.4 (Chen [6]). *Let W be of type E_8 and $w, w' \in W$. Then $w \sim_L w'$ if and only if $\mathbf{a}(w) = \mathbf{a}(w')$ (see Definition 3.3) and w, w' have the same generalized τ -invariant (see Definition 5.5).*

Proof. Let $w, w' \in W$. If $w \sim_L w'$, then w, w' belong to the same two-sided cell and so $\mathbf{a}(w) = \mathbf{a}(w')$; furthermore, w, w' have the same generalized τ -invariant by Corollary 5.6. Conversely, assume that $\mathbf{a}(w) = \mathbf{a}(w')$ and that w, w' have the same generalized τ -invariant. By Proposition 4.1(b), there exist $d, d' \in \check{\mathcal{D}}$ such that $w \sim_L d$ and $w' \sim_L d'$. Then $\mathbf{a}(d) = \mathbf{a}(w)$ and $\mathbf{a}(w') = \mathbf{a}(d')$ (by Definition 3.3) and so $\mathbf{a}(d) = \mathbf{a}(d')$. Furthermore, by Corollary 5.6, the elements d, d' have the same generalized τ -invariant and so $d, d' \in \check{\mathcal{D}}_i$ for some i . Hence, Example 6.3(*) implies that $d = d'$ and so $w \sim_L d \sim_L w'$. $\square$

Corollary 6.5. *Let W be of type E_8 and $w, w' \in W^*$, where W^* is defined in Example 6.1. Then $w \sim_L w'$ if and only if w, w' have the same generalized τ -invariant and if there exists some $E_0 \in \mathcal{S}(W)$ such that $c_{w, E_0} \neq 0$ and $c_{w', E_0} \neq 0$.*

Proof. Let $w, w' \in W^*$. If $w \sim_L w'$, then w, w' have the same generalized τ -invariant. Furthermore, by Example 6.1, there exist $E_0, E'_0 \in \mathcal{S}(W)$ such that $c_{w, E_0} \neq 0$ and $c_{w', E'_0} \neq 0$. Then $w \in \mathcal{F}_{E_0}$ and $w' \in \mathcal{F}_{E'_0}$. Since $w \sim_L w'$, we conclude that $\mathcal{F}_{E_0} = \mathcal{F}_{E'_0}$ and so $E_0 = E'_0$, using Theorem 3.2. Conversely, assume that w, w' have the same generalized τ -invariant and that there exists some $E_0 \in \mathcal{S}(W)$ such that $c_{w, E_0} \neq 0$ and $c_{w', E_0} \neq 0$. Then $w, w' \in \mathcal{F}_{E_0}$ and so $\mathbf{a}(w) = \mathbf{a}(w')$. Hence, we have $w \sim_L w'$ by Theorem 6.4. $\square$

Example 6.6. Let W be of type E_8 . Because of the sheer size of W , it would be impossible to determine directly all values $\mathbf{a}(w)$ for $w \in W$ and the partition of W into τ -cells. Instead we apply the procedure in Remark 5.10 in order to obtain a set $\Upsilon_I(W)$ and then a set of left cells $\mathfrak{C}_{\text{left}}^\circ(W) \subseteq \Upsilon_I(W)$ as in Remark 5.4. For this purpose, let I be such that W_I is of type E_7 . The set X_I of coset representatives is obtained by the PyCox command `redleftcosetreps`; we have $|X_I| = 240$. The PyCox command `klcells` shows that $|\mathfrak{C}_{\text{left}}(W_I)| = 6364$ and $|\mathfrak{C}_{\text{left}}^\circ(W_I)| = 56$ (see [19, Table 2, p. 246]); furthermore, the largest left cell of W_I has size 1024. Hence, we will have to deal with 56 sets $X_I\Gamma_i$ each of which contains at most $240 \cdot 1024$ elements. For sets of this size, the partition into τ -cells is readily determined using the `gentaucells` command of PyCox. Explicitly, we find that $|\Upsilon_I(W)| = 4305120$ and that $\Upsilon_I(W)$ decomposes into 614 τ -cells. Now let $\mathcal{C} \subseteq \Upsilon_I(W)$ be such a τ -cell. The decomposition of $\mathcal{C}$ into left cells is determined as follows.

- By Example 6.3, $|\mathcal{C} \cap \check{\mathcal{D}}|$ is the number of left cells contained in $\mathcal{C}$. Hence, if $|\mathcal{C} \cap \check{\mathcal{D}}| = 1$, then $\mathcal{C}$ is a left cell. This applies to 522 of the τ -cells in $\Upsilon_I(W)$.

Now assume that $|\mathcal{C} \cap \check{\mathcal{D}}| > 1$. Let $A := \{\mathbf{a}(w) \mid w \in \mathcal{C}\}$ and set $\mathcal{C}_{(i)} := \{w \in \mathcal{C} \mid \mathbf{a}(w) = i\}$ for $i \in A$. By Theorem 6.4, each set $\mathcal{C}_{(i)}$ is a left cell of W . Thus, if we can determine $\mathbf{a}(w)$ for all $w \in \mathcal{C}$, then we can decompose $\mathcal{C}$ into left cells. By Remark 5.3, $\mathcal{C}$ is a union of left star orbits. So it suffices to determine $\mathbf{a}(w)$ for one element w in each such orbit. If the orbit contains an involution w , then we obtain $\mathbf{a}(w)$ from the results in Example 6.1. It turns out that, inside the 92 τ -cells $\mathcal{C}$ such that $|\mathcal{C} \cap \check{\mathcal{D}}| > 1$, there are 561 left star orbits which do not contain an involution. For representatives of these 561 orbits, we then have to rely on the techniques described in Remark 4.5 in order to determine $\mathbf{a}(w)$. (These computations take a few days.) Eventually, we find that $\Upsilon_I(W)$ decomposes into 746 left cells. Out of these 746, we further obtain a set $\mathfrak{C}_{\text{left}}^\circ(W)$ consisting of 106 left cells.

Remark 6.7. In PyCox, the left cells in $\mathfrak{C}_{\text{left}}^\circ(W)$ are available via the command `klcellreps`, which also returns additional information, e.g., the decomposition of $[\Gamma]_1$ into irreducibles and the unique $E_0 \in \mathcal{S}(W)$ such that $\Gamma \subseteq \mathcal{F}_{E_0}$, for any $\Gamma \in \mathfrak{C}_{\text{left}}^\circ(W)$. The star orbit of a left cell in $\mathfrak{C}_{\text{left}}^\circ(W)$ is computed by the PyCox command `cellrepstarorbit`. In type E_8 , it takes about 32 hours and a computer with 128 GB of main memory to produce the complete list of all 101796 left cells of W out of the 106 left cells in $\mathfrak{C}_{\text{left}}^\circ(W)$. However, just using the set $\mathfrak{C}_{\text{left}}^\circ(W)$ alone, we now obtain an efficient algorithm for determining, for *any* $w \in W$, the unique $E_0 \in \mathcal{S}(W)$ such that $w \in \mathcal{F}_{E_0}$. Indeed, given w , we compute $R^*(w)$ as defined in Remark 5.3. By the definition of $\mathfrak{C}_{\text{left}}^\circ(W)$, there exists some $\Gamma \in \mathfrak{C}_{\text{left}}^\circ(W)$ such that $\Gamma \cap R^*(w) \neq \emptyset$; let $y \in \Gamma \cap R^*(w)$. Since $R^*(w)$ is contained in a right cell, we have $w \sim_{LR} y$. The unique $E_0 \in \mathcal{S}(W)$ such that $y \in \mathcal{F}_{E_0}$ is already known (since $y \in \Gamma$ and $\Gamma \in \mathfrak{C}_{\text{left}}^\circ(W)$); hence, $w \in \mathcal{F}_{E_0}$ and $\mathbf{a}(w) = \mathbf{a}(y) = \mathbf{a}_{E_0}$. This procedure is implemented in the Python command `klcellrepelm` (and this function does not require much main memory on a computer). By a slight modification of this procedure, one can construct the left cell containing any given element $w \in W$; this is implemented in the PyCox command `leftcellem`. (See the help menus of these functions for further information.)

Remark 6.8. The results of Garfinkle [11] and the fact that an equivalence like that in Theorem 6.4 even holds in type E_8 suggest that something similar should

be true for the left cells in any finite Coxeter group W . In order to formulate a general statement, we strengthen the definition of the generalized τ -invariant in Definition 5.5 using Lusztig’s method of “strings” [34, §10]. To explain how this works, let us consider again two generators $s, t \in S$ but drop the assumption that st has order 3. Assume that st has order $m \geq 3$. Let $W' \subseteq W$ be the parabolic subgroup generated by s, t . For any $w \in W$, the coset $W'w$ can be partitioned into four subsets: one consists of the unique element x of minimal length, one consists of the unique element of maximal length, one consists of the $(m-1)$ elements $sx, tsx, stsx, \dots$ and one consists of the $(m-1)$ elements $tx, stx, tstx, \dots$. As in [34, 10.2], the last two subsets (ordered as above) are called *strings*. By [34, 10.6] (see also [47, §4] for the case $m = 4$), we can now define an involution

$$\mathcal{D}_R(s, t) \rightarrow \mathcal{D}_R(s, t), \quad w \mapsto \tilde{w},$$

as follows. Let $w \in \mathcal{D}_R(s, t)$. Then w^{-1} is contained in a unique string $\sigma_{w^{-1}}$ (with respect to s, t). Let $i \in \{1, \dots, m-1\}$ be the index such that w^{-1} is the i th element of $\sigma_{w^{-1}}$. Then $\tilde{w}$ is defined to be the element such that $\tilde{w}^{-1}$ is the $(m-i)$ th element of $\sigma_{w^{-1}}$.

Now let $\Gamma \subseteq \mathcal{D}_R(s, t)$ be a left cell. Then $\tilde{\Gamma} = \{\tilde{w} \mid w \in \Gamma\}$ also is a left cell by [34, Prop. 10.7] (the assumption on W being “crystallographic” is now superfluous, thanks to Elias and Williamson [10]). Hence, as in Section 5, we do not only have $\mathcal{R}(y) = \mathcal{R}(w)$ but also $\mathcal{R}(\tilde{y}) = \mathcal{R}(\tilde{w})$ for all $y, w \in \Gamma$. Iterating this process (exactly as in Definition 5.5 but now allowing any generators s, t such that st has order at least 3), we obtain a stronger version of the generalised τ -invariant; the corresponding equivalence classes of W are called $\tilde{\tau}$ -cells. We can now state the following variation of Vogan’s conjecture [46, 3.11].

Conjecture 6.9. *For any finite Coxeter group W , two elements $w, w' \in W$ belong to the same left cell if and only if w, w' belong to the same two-sided cell and to the same $\tilde{\tau}$ -cell.*

We have checked that Conjecture 6.9 is true for all classical types B_n, D_n where $n \leq 9$, and for all exceptional types including the non-crystallographic types H_3, H_4 ; in type F_4 , the left cells are precisely the $\tilde{\tau}$ -cells. It is even possible to formulate a version of Conjecture 6.9 for cells with respect to unequal parameters; see [23].

7. APPLICATIONS

Using the results in the previous section, it should now be possible to answer any concrete question concerning the partition into Kazhdan–Lusztig cells for W of type E_8 .

Example 7.1. This arises from the work of Lusztig [37] on rationality properties of unipotent representations. Let W be of type E_8 and C_0 be the unique conjugacy class of W whose elements have order 6 and $|C_0| = 4480$. Quite remarkably, we have $l(w) = 40$ for all $w \in C_0$ in this case; see [26, Appendix B.6].

TABLE 2. Intersections of two-sided cells with $C_{\min}$ for cuspidal classes in type E_8

C	$o(w)$	$ C_{\min} $	$ C_{\min} \cap \mathcal{F}_{E_0} $ for $E_0 \in S(W)$
$8A_1$	2	1	$1 \cdot 1'$
$2D_4(a_1)$	4	15120	$2100 \cdot 2100_y \cup 13020 \cdot 4480_y$
$D_4 + 4A_1$	6	56	$14 \cdot 567_x \cup 2 \cdot 112_z \cup 40 \cdot 1400_z$
$4A_2$	3	4480	$4480 \cdot 4480_y$
$E_8(a_8)$	6	4480	$4480 \cdot 4480_y$
$E_7(a_4) + A_1$	6	11592	$174 \cdot 2240_x \cup 2944 \cdot 4480_y \cup 128 \cdot 2835'_x \cup 1760 \cdot 4200_x \cup 1408 \cdot 6075_x \cup 80 \cdot 6075'_x \cup 780 \cdot 2800_z \cup 786 \cdot 4096_z$ $\cup 480 \cdot 4200_z \cup 590 \cdot 4200_z \cup 348 \cdot 4536_z \cup 142 \cdot 4536'_z \cup 1098 \cdot 5600_z \cup 874 \cdot 5600_z$
$2D_4$	6	4070	$42 \cdot 2100_y \cup 14 \cdot 2240_z \cup 52 \cdot 2240'_z \cup 1814 \cdot 4480_y \cup 64 \cdot 2268'_z \cup 270 \cdot 2835'_x \cup 206 \cdot 4200_y \cup 96 \cdot 4200'_x$ $\cup 418 \cdot 6075_x \cup 38 \cdot 6075'_x \cup 82 \cdot 2800_z \cup 222 \cdot 4096_z \cup 228 \cdot 4200'_z \cup 106 \cdot 4536_z \cup 12 \cdot 5600_z \cup 382 \cdot 5600'_z$
$2A_3 + 2A_1$	4	1260	$208 \cdot 2268_x \cup 464 \cdot 4096_y \cup 258 \cdot 6075_x \cup 108 \cdot 1400'_z \cup 114 \cdot 4536'_z \cup 106 \cdot 5600'_z$
$D_8(a_3)$	8	7748	$14 \cdot 210_y \cup 168 \cdot 525_z \cup 236 \cdot 567_x \cup 366 \cdot 700_z \cup 1614 \cdot 1400_x \cup 452 \cdot 2240_z \cup 404 \cdot 2268_x$ $\cup 376 \cdot 4200_y \cup 44 \cdot 6075_x \cup 92 \cdot 560_z \cup 1908 \cdot 1400_z \cup 20 \cdot 2800_y \cup 1502 \cdot 3240_z \cup 4552 \cdot 4096_z$
$D_6 + 2A_1$	10	256	$102 \cdot 4480_y \cup 4 \cdot 2268'_z \cup 26 \cdot 4200_x \cup 42 \cdot 6075_x \cup 26 \cdot 5600_z \cup 56 \cdot 5600'_z$
$2A_4$	5	7952	$38 \cdot 567_x \cup 134 \cdot 1400_x \cup 1058 \cdot 2240_z \cup 440 \cdot 4480_y \cup 272 \cdot 2268_x \cup 668 \cdot 2835_x \cup 2242 \cdot 4200_x$ $\cup 200 \cdot 6075_x \cup 480 \cdot 1400_z \cup 172 \cdot 2800_z \cup 856 \cdot 3240_z \cup 1344 \cdot 4096_z \cup 48 \cdot 4536_z$
$E_8(a_6)$	10	3370	$12 \cdot 210_y \cup 56 \cdot 567_x \cup 198 \cdot 700_y \cup 734 \cdot 1400_z \cup 348 \cdot 2240_z \cup 42 \cdot 2268_x$ $\cup 146 \cdot 4200_x \cup 122 \cdot 560_z \cup 966 \cdot 1400_z \cup 610 \cdot 3240_z \cup 136 \cdot 4096_z$
$E_6(a_2) + A_2$	6	16374	$310 \cdot 1400_x \cup 110 \cdot 2240_x \cup 1774 \cdot 4480_y \cup 510 \cdot 2268_x \cup 4536 \cdot 4200_z \cup 2116 \cdot 6075_x \cup 124 \cdot 1400_z$ $\cup 1638 \cdot 2800_z \cup 786 \cdot 3240_z \cup 1422 \cdot 4096_z \cup 382 \cdot 4200_z \cup 626 \cdot 4536_z \cup 1040 \cdot 5600_z$
$E_8(a_3)$	12	2696	$34 \cdot 210_x \cup 186 \cdot 567_x \cup 364 \cdot 700_y \cup 604 \cdot 1400_x \cup 78 \cdot 2240_z \cup 380 \cdot 560_z \cup 870 \cdot 1400_z \cup 180 \cdot 3240_z$
$A_5 + A_2 + A_1$	6	3752	$84 \cdot 2240_z \cup 1148 \cdot 4480_y \cup 356 \cdot 2835_x \cup 364 \cdot 4200_y \cup 492 \cdot 6075_x \cup 166 \cdot 2800_z$ $\cup 308 \cdot 4096_z \cup 294 \cdot 4200_z \cup 136 \cdot 4536_z \cup 54 \cdot 5600_z \cup 350 \cdot 5600'_z$
$D_8(a_1)$	12	2040	$52 \cdot 210_x \cup 40 \cdot 525_x \cup 36 \cdot 567_x \cup 82 \cdot 700_y \cup 400 \cdot 1400_z \cup 52 \cdot 2240_z \cup 158 \cdot 560_z \cup 688 \cdot 1400_z \cup 532 \cdot 3240_z$
D_8	14	852	$24 \cdot 210_x \cup 12 \cdot 525_x \cup 16 \cdot 567_x \cup 120 \cdot 700_y \cup 200 \cdot 1400_x \cup 112 \cdot 560_z \cup 272 \cdot 1400_z \cup 12 \cdot 2800_z \cup 84 \cdot 3240_z$
$A_7 + A_1$	8	2080	$202 \cdot 1400_x \cup 58 \cdot 2240_z \cup 200 \cdot 4480_y \cup 206 \cdot 2268_x \cup 208 \cdot 4200_z \cup 180 \cdot 6075_x$ $\cup 240 \cdot 2800_y \cup 332 \cdot 3240_z \cup 374 \cdot 4096_z \cup 80 \cdot 5600_z$
$E_7 + A_1$	18	192	$4 \cdot 525_x \cup 8 \cdot 567_x \cup 8 \cdot 700_y \cup 100 \cdot 1400_x \cup 8 \cdot 560_z \cup 52 \cdot 1400_z \cup 12 \cdot 2800_z$
A_8	9	2816	$6 \cdot 210_x \cup 78 \cdot 567_x \cup 172 \cdot 700_y \cup 648 \cdot 1400_z \cup 478 \cdot 2240_z \cup 62 \cdot 560_z \cup 790 \cdot 1400_z \cup 582 \cdot 3240_z$
$E_8(a_4)$	18	732	$104 \cdot 210_x \cup 24 \cdot 567_x \cup 114 \cdot 700_y \cup 80 \cdot 1400_x \cup 12 \cdot 560_z \cup 278 \cdot 560_z \cup 120 \cdot 1400_z$
$E_8(a_2)$	20	624	$12 \cdot 35_x \cup 202 \cdot 210_y \cup 16 \cdot 567_x \cup 32 \cdot 700_y \cup 16 \cdot 1400_z \cup 172 \cdot 112_z \cup 242 \cdot 560_z \cup 32 \cdot 1400_z$
$D_5(a_1) + A_3$	12	15134	$38 \cdot 525_x \cup 42 \cdot 567_x \cup 258 \cdot 1400_x \cup 1648 \cdot 2240_x \cup 1012 \cdot 4480_y \cup 620 \cdot 2268_x \cup 114 \cdot 42835_x \cup 3082 \cdot 4200_z$ $\cup 1318 \cdot 6075_x \cup 620 \cdot 1400_z \cup 1086 \cdot 2800_z \cup 410 \cdot 3240_z \cup 2388 \cdot 4096_z \cup 200 \cdot 4536_z \cup 268 \cdot 5600_z$
$E_6 + A_2$	12	840	$32 \cdot 567_x \cup 28 \cdot 700_y \cup 524 \cdot 1400_z$
$E_8(a_7)$	12	2360	$2 \cdot 567_x \cup 40 \cdot 700_y \cup 1136 \cdot 1400_x \cup 92 \cdot 2240_z \cup 188 \cdot 2268_x \cup 32 \cdot 4200_y \cup 50 \cdot 560_z \cup 536 \cdot 1400_z \cup 64 \cdot 3240_z \cup 220 \cdot 4096_z$
$E_7(a_2) + A_1$	12	1758	$6 \cdot 525_x \cup 10 \cdot 567_x \cup 4 \cdot 700_y \cup 492 \cdot 1400_z \cup 32 \cdot 4480_y \cup 84 \cdot 2268_x \cup 40 \cdot 4200_z \cup 106 \cdot 6075_x$ $\cup 10 \cdot 560_z \cup 302 \cdot 1400_z \cup 196 \cdot 2800_z \cup 140 \cdot 3240_z \cup 30 \cdot 4096_z \cup 32 \cdot 5600_z$
$E_8(a_1)$	24	320	$12 \cdot 35_x \cup 114 \cdot 210_y \cup 16 \cdot 700_z \cup 90 \cdot 112_z \cup 88 \cdot 560_z$
$D_8(a_2)$	30	4996	$24 \cdot 210_x \cup 60 \cdot 525_x \cup 32 \cdot 567_x \cup 292 \cdot 700_y \cup 840 \cdot 1400_x \cup 468 \cdot 2240_z \cup 180 \cdot 2268_x \cup 38 \cdot 4200_y \cup 34 \cdot 6075_x$ $\cup 104 \cdot 560_z \cup 976 \cdot 1400_y \cup 224 \cdot 2800_z \cup 1024 \cdot 3240_z \cup 528 \cdot 4096_z \cup 44 \cdot 4200_z \cup 128 \cdot 4536_z$
$E_8(a_5)$	15	1516	$2 \cdot 35_x \cup 174 \cdot 210_x \cup 28 \cdot 567_x \cup 368 \cdot 700_y \cup 194 \cdot 1400_z \cup 20 \cdot 2240_z \cup 44 \cdot 112_z \cup 338 \cdot 560_z \cup 336 \cdot 1400_z \cup 12 \cdot 3240_z$
E_8	30	128	$14 \cdot 35_x \cup 48 \cdot 210_y \cup 60 \cdot 112_z$

Lusztig [37, 2.17] observed that

- $|C_0| = 4480$ is equal to the number of left cells in the two-sided cell $\mathcal{F}$ of W attached to C_0 by the method described in [37, 2.17].

He remarks that “this suggests that $C_0 \subseteq \mathcal{F}$ and that any left cell in $\mathcal{F}$ contains a unique element of C_0 ”. Using PyCox, we can confirm that this suggestion is true, as follows. First, we identify our class C_0 in the list of the 112 classes returned by the command `conjugacyclasses`.

```
>>> W=coxeter("E", 8)
>>> c=conjugacyclasses(W)
>>> [i for i in range(112) if c['classlengths'][i]==4480 and
      W.permorder(W.wordtoperm(c['reps'][i]))==6]
[10]
>>> cl=conjugacyclass(W, W.wordtoperm(c['reps'][10]))
# Size of class: 4480
>>> set([W.permlength(w) for w in cl])
40
```

The last command shows that all elements in C_0 indeed have length 40. Next, we check how C_0 is partitioned into τ -cells:

```
>>> len(gentaucells(W, cl)) # This will take almost an hour.
4480
```

Thus, all the elements lie in pairwise different τ -cells and, hence, in pairwise different left cells of W . Finally, we check that all elements of C_0 lie in the same two-sided cell.

```
>>> klcellrepelm(W, cl[0])['special'] # see Remark 6.7
'4480_y'
>>> set([klcellrepelm(W, w)['special'] for w in cl])
set(['4480_y'])
```

(This takes about a quarter of an hour.) Thus, we have $C_0 \subseteq \mathcal{F}_{4480_y}$. By Table 1, we also obtain $\mathbf{a}(w) = 16$ for all $w \in C_0$.

More generally, let C be any conjugacy class of W . Let $d_C = \min\{l(w) \mid w \in C\}$ and $C_{\min} = \{w \in C \mid l(w) = d_C\}$ be the set of elements of minimal length in C ; see [26, §3.1]. Furthermore, we say that C is cuspidal if $C \cap W_I = \emptyset$ for any proper $I \subsetneq S$. We note that the class C_0 considered above is a cuspidal class such that $C_0 = C_{0, \min}$. Table 2 shows the cardinalities of the intersections $C_{\min} \cap \mathcal{F}_{E_0}$ as E_0 runs over the set $\mathcal{S}(W)$ of special representations. (The notation $n_1 * E_1 \cup n_2 * E_2 \cup \dots$ means $|C_{\min} \cap \mathcal{F}_{E_i}| = n_i$ for $i = 1, 2, \dots$. The above example C_0 corresponds to the fifth row of the table. Let C'_0 be the class corresponding to the fourth row. Lusztig pointed out that $C'_0 = C_0^2$ and $l(w^2) = 2l(w)$ for $w \in C_0$; this may be the reason why the results are the same for C_0, C'_0 .)

Next, we discuss the following conjecture.

Conjecture 7.2 (Lusztig, cf. [43], [49]). *Every left cell Γ of W is left-connected, that is, for any two elements $x, y \in \Gamma$, there is a chain of generators $s_1, s_2, \dots, s_n$ in S such that $y = s_n \cdots s_2 s_1 x$ and all intermediate elements $s_1 x, s_2 s_1 x, \dots, s_{n-1} \cdots s_2 s_1 x$ lie in Γ .*

Example 7.3. Using the PyCox commands `klcellreps` and `cellrepstarorbit` (see Remark 6.7), we have a way of running through all the left cells of W . Furthermore, it is straightforward to write a function which verifies if a given left cell is left-connected or not. In this way, we have verified that Conjecture 7.2 holds for all W of exceptional type $H_3, H_4, F_4, E_6, E_7, E_8$. (For type E_8 , this takes about 2 or 3 days; note that it is not necessary to keep all the left cells at once in the main memory of the computer.) For type A_n , the conjecture holds by [30, §5]; for $I_2(m)$, it follows easily from the description of the left cells in [38, 7.15]. For type B_n , it follows from Garfinkle [11, Theorem 3.5.9]. The question seems to be open for type D_n .

Finally, we come to Kottwitz' conjecture [31]. Let W be a finite Coxeter group and C be a conjugacy class of involutions in W . Following [31, §1], [39, 6.3], let V_C be an $\mathbb{R}$ -vector space with a basis $\{a_w \mid w \in C\}$. Then there is a linear action of W on V_C such that, for any $s \in S$ and $w \in C$, we have

$$s.a_w = \begin{cases} -a_w & \text{if } sw = ws \text{ and } \ell(sw) < \ell(w), \\ a_{sws} & \text{otherwise.} \end{cases}$$

Conjecture 7.4 (Kottwitz [31, §1]). *Let C be a conjugacy class of involutions and Γ be a left cell of W . Then $\dim \operatorname{Hom}_W(V_C, [\Gamma]_1) = |C \cap \Gamma|$.*

By work of Kottwitz himself, Casselman [5], Bonnafé and the first-named author [4], [19], [21], this conjecture is already known to hold except possibly for W of type E_8 . The verification for type E_8 is now a matter of combining various pieces of known information. The decompositions of the representations V_C into irreducibles can be computed using the known character table of W ; see [5]. In PyCox, this is done using the command `involutionmodel`. Now let Γ be a left cell. By Example 6.3, there is a unique element $d \in \Gamma \cap \tilde{D}$; we then write $\Gamma = \Gamma_d$. By Proposition 4.1 and Example 6.1(a), we have $m(\Gamma_d, E) = c_{d,E}$ for all $E \in \operatorname{Irr}(W)$. Hence, we have

$$\dim \operatorname{Hom}_W(V_C, [\Gamma_d]_1) = \sum_{E \in \operatorname{Irr}(W)} c_{d,E} \dim \operatorname{Hom}_W(V_C, E),$$

and these dimensions can be explicitly determined using the results in Example 6.1 and the output of `involutionmodel` (or the tables in [5]). Finally, by Example 6.1(c), there is a unique $E_0 \in \mathcal{S}(W)$ such that $c_{d,E_0} \neq 0$. Then, by Corollary 6.5, we have

$$\begin{aligned} C \cap \Gamma_d &= \{w \in C \mid \mathbf{a}(w) = \mathbf{a}(d) \text{ and } d, w \text{ belong to the same } \tau\text{-cell}\} \\ &= \{w \in C \mid c_{w,E_0} \neq 0 \text{ and } d, w \text{ belong to the same } \tau\text{-cell}\}. \end{aligned}$$

These intersections can be determined using the PyCox command `gentaucells` (applied to C) and the results in Example 6.1 (or the command `klcellrepelm` in Remark 6.7). In this way, we have verified that Conjecture 7.4 holds for W of type E_8 .

ACKNOWLEDGEMENTS

We thank G. Lusztig for pointing out the suggestion in [37, 2.17]. We also thank T. Pietraho who helped to clarify some points about Garfinkle's work [11].

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