CORRIGENDUM TO: “ACCELERATION OF A TWO-GRID METHOD FOR EIGENVALUE PROBLEMS”

XIAOZHE HU AND XIAOLIANG CHENG

ABSTRACT. There is a gap in the proof of Theorem 3.3 in our paper named above. A careful estimate of the constant $C_h(\mu)$ in Lemma 3.1 is needed for the error estimates of the two-grid method to hold. In this erratum, we provide a corrected version of the proof.

In this erratum, we follow the notation in our paper *Acceleration of a two-grid method for eigenvalue problems*. The constant C_i in Theorem 3.3 in [1] depends on the constant $C_h(\mu)$ in Lemma 3.1. However, $C_h(\mu)$ behaves like $\min_i |\lambda_{i,h} - \mu| / \lambda_{i,h}$. When μ approaches $\lambda_{i,h}$, $C_h(\mu)$ degenerates. Therefore, it is necessary to estimate it more carefully than we did in our original paper. In fact, if we restrict the estimates in the orthogonal complement of $M_h(\lambda_i)$ in S_h , this constant can be bounded independently with respect to μ as shown in the following lemma. We assume the eigenvalue λ_i has multiplicity q_i and let $\Lambda = \{\lambda_{i+k,h}, k = 0, \dots, q_i - 1\}$ be the approximate eigenvalues of λ_i .

Lemma 1. *Assume that μ is not an eigenvalue, i.e., $\mu \neq \lambda_{i,h}$, $i = 1, 2, \dots, N_h$, and h is small enough, we have,*

$$(1) \quad \sup_{v \in M_h(\lambda_i)^\perp} \frac{a_\mu(u, v)}{\|v\|_a} \geq \left(\min_{\lambda_{j,h} \notin \Lambda} \frac{|\lambda_{j,h} - \mu|}{\lambda_{j,h}} \right) \|u\|_a, \quad \forall u \in M_h(\lambda_i)^\perp.$$

Moreover, if $|\mu - \lambda_i| \leq \frac{1}{2}\rho$, $\rho := \min_{j: \lambda_j \neq \lambda_i} |\lambda_j - \lambda_i|$, then

$$(2) \quad \min_{\lambda_{j,h} \notin \Lambda} \frac{|\lambda_{j,h} - \mu|}{\lambda_{j,h}} \geq C_{i,h}(\rho) > 0,$$

where the constant $C_{i,h}(\rho)$ only depends on λ_i and ρ .

Proof. Here we use the same $\nu = 1/\mu$ and T_h defined in the proof of Lemma 3.1 in [1]. It is easy to see that $(\nu I - T_h)^{-1} : M_h(\lambda_i)^\perp \rightarrow M_h(\lambda_i)^\perp$ is self-adjoint. Therefore, we have

$$\|(\nu I - T_h)^{-1}\|_{M_h(\lambda_i)^\perp \rightarrow M_h(\lambda_i)^\perp} = \left(\min_{\lambda_{j,h} \notin \Lambda} |\lambda_{j,h}^{-1} - \nu| \right)^{-1}.$$

Received by the editor December 30, 2013 and, in revised form, February 23, 2014 and May 4, 2014.

2010 *Mathematics Subject Classification.* Primary 65N15, 65N25, 65N30, 65N55.

Hence, for $u \in M_h(\lambda_i)^\perp$, we have

$$\begin{aligned} \|u\|_a &= \|(\nu I - T_h)^{-1}(\nu I - T_h)u\|_a \leq \|(\nu I - T_h)^{-1}\|_{M_h(\lambda_i)^\perp \rightarrow M_h(\lambda_i)^\perp} \|(\nu I - T_h)u\|_a \\ &= \left(\min_{\lambda_{j,h} \notin \Lambda} |\lambda_{j,h}^{-1} - \nu| \right)^{-1} \|(\nu I - T_h)u\|_a = \left(\min_{\lambda_{j,h} \notin \Lambda} \frac{|\lambda_{j,h} - \mu|}{\lambda_{j,h}} \right)^{-1} \|(I - \mu T_h)u\|_a \\ &= \left(\min_{\lambda_{j,h} \notin \Lambda} \frac{|\lambda_{j,h} - \mu|}{\lambda_{j,h}} \right)^{-1} \sup_{v \in M_h(\lambda_i)^\perp} \frac{a((I - \mu T_h)u, v)}{\|v\|_a} \\ &= \left(\min_{\lambda_{j,h} \notin \Lambda} \frac{|\lambda_{j,h} - \mu|}{\lambda_{j,h}} \right)^{-1} \sup_{v \in M_h(\lambda_i)^\perp} \frac{a_\mu(u, v)}{\|v\|_a}. \end{aligned}$$

Then (1) follows. Note that $|\lambda_{j,h} - \mu| \geq |\lambda_j - \lambda_i| - |\mu - \lambda_i| - |\lambda_j - \lambda_{j,h}|$. When h is small enough, we have $|\lambda_j - \lambda_{j,h}| \leq \frac{1}{4}\rho$ and $|\lambda_{j,h} - \mu| \geq \frac{1}{4}|\lambda_j - \lambda_i| \geq \frac{1}{4}\rho$. Hence,

$$\min_{\lambda_{j,h} \notin \Lambda} \frac{|\lambda_{j,h} - \mu|}{\lambda_{j,h}} \geq \frac{\rho}{4\mu + \rho} \geq \frac{\rho}{4\lambda_i + 3\rho} > 0.$$

Therefore, (2) follows with $C_{i,h}(\rho) = \frac{\rho}{4\lambda_i + 3\rho}$. $\square$

Based on the refined estimate of the constant, we have the following theorem, which gives the error estimates for the two-grid algorithm (Algorithm 1 in [1]).

Theorem 2. *Let (λ_i^h, u_i^h) , $(i = 1, 2, \dots, N_H)$ be calculated via Algorithm 1 in [1]. Assume that $\lambda_{i,H}$ is not an eigenvalue and H is small enough, there exists an eigenvector $u_i \in M(\lambda_i)$ such that*

$$(3) \quad \min_{\alpha \in \mathbb{R}} \|u_i - \alpha u_i^h\|_a \leq C_i(\delta_h(\lambda) + \eta(h)\delta_h^3(\lambda_i) + \eta(H)\delta_h^2(\lambda_i)\delta_H(\lambda_i) + \eta(h)\delta_H^2(\lambda_i)\delta_h(\lambda_i) + \eta(H)\delta_H^3(\lambda_i)).$$

For the eigenvalue, we have

$$(4) \quad |\lambda_i - \lambda_i^h| \leq C_i(\delta_h^2(\lambda) + \eta^2(h)\delta_h^6(\lambda_i) + \eta^2(H)\delta_h^4(\lambda_i)\delta_H^2(\lambda_i) + \eta^2(h)\delta_H^4(\lambda_i)\delta_h^2(\lambda_i) + \eta^2(H)\delta_H^6(\lambda_i)),$$

where the constants C_i are independent of the mesh size H and h .

Proof. Consider an equivalent linear system on the fine grid as follows:

$$(5) \quad a_{\lambda_{i,H}}(\tilde{u}_i^h, v_h) = (\lambda_{i,h} - \lambda_{i,H})b(u_{i,H}, v_h), \quad \forall v_h \in S_h.$$

Choose an approximate eigenfunction $u_{i,h} \in M_h(\lambda_i)$, we have

$$a(u_{i,h}, v_h) = \lambda_{i,h}b(u_{i,h}, v_h), \quad \forall v_h \in S_h.$$

Therefore,

$$(6) \quad a_{\lambda_{i,H}}(u_{i,h}, v_h) = (\lambda_{i,h} - \lambda_{i,H})b(u_{i,h}, v_h), \quad \forall v_h \in S_h.$$

Subtract (5) from (6), and we have

$$(7) \quad a_{\lambda_{i,H}}(u_{i,h} - \tilde{u}_i^h, v_h) = (\lambda_{i,h} - \lambda_{i,H})b(u_{i,h} - u_{i,H}, v_h), \quad \forall v_h \in S_h,$$

or in the following operator form,

$$(8) \quad (A_h - \lambda_{i,H}I)(u_{i,h} - \tilde{u}_i^h) = (\lambda_{i,h} - \lambda_{i,H})(u_{i,h} - u_{i,H}),$$

where the operator $A_h : S_h \mapsto S_h$ is defined by $b(A_h w_h, v_h) = a(w_h, v_h)$ for all $w_h, v_h \in S_h$. We decompose $(u_{i,h} - u_{i,H})$ on the right-hand side of (8) into

$$u_{i,h} - u_{i,H} = u_{i,h} - E_h^i u_{i,H} + E_h^i u_{i,H} - u_{i,H},$$

where E_h^i is the orthogonal spectral projection of S_h onto $M_h(\lambda_i)$. Therefore, $(u_{i,h} - E_h^i u_{i,H}) \in M_h(\lambda_i)$ and $(E_h^i u_{i,H} - u_{i,H}) \in M_h(\lambda_i)^\perp$. Then, from (8), we have

$$(9) \quad (A_h - \lambda_{i,H} I)(\hat{u}_{i,h} - \tilde{u}_i^h) = (\lambda_{i,h} - \lambda_{i,H})(E_h^i u_{i,H} - u_{i,H}),$$

where $\hat{u}_{i,h} = u_{i,h} - (\lambda_{i,h} - \lambda_{i,H})(A_h - \lambda_{i,H} I)^{-1}(u_{i,h} - E_h^i u_{i,H})$. Note that $A_h - \lambda_{i,H} I : M_h(\lambda_i) \mapsto M_h(\lambda_i)$ and $A_h - \lambda_{i,H} I : M_h(\lambda_i)^\perp \mapsto M_h(\lambda_i)^\perp$ are isomorphisms because $M_h(\lambda_i)$ is an eigenspace of operator A_h and $\lambda_{i,H}$ is not an eigenvalue of A_h by assumption. Since $(u_{i,h} - E_h^i u_{i,H}) \in M_h(\lambda_i)$ and $u_{i,h} \in M_h(\lambda_i)$, we have $\hat{u}_{i,h} \in M_h(\lambda_i)$. Similarly, $(E_h^i u_{i,H} - u_{i,H}) \in M_h(\lambda_i)^\perp$ implies $(\hat{u}_{i,h} - \tilde{u}_i^h) \in M_h(\lambda_i)^\perp$ since (9). Therefore, from the operator form (9), we have,

$$(10) \quad a_{\lambda_{i,H}}((\hat{u}_{i,h} - \tilde{u}_i^h), v_h^\perp) = (\lambda_{i,h} - \lambda_{i,H})b((E_h^i u_{i,H} - u_{i,H}), v_h^\perp), \quad \forall v_h^\perp \in M_h(\lambda_i)^\perp.$$

For $u_{i,H} \in M_H(\lambda_i)$, there exists an eigenvector $\bar{u}_i \in M(\lambda_i)$ such that $\|\bar{u}_i - u_{i,H}\|_{-a} \leq C_i \eta(H) \|\bar{u}_i - u_{i,H}\|_a$ and $\|\bar{u}_i - u_{i,H}\|_a \leq C_i \delta_H(\lambda_i)$. Moreover, for $\bar{u}_i \in M(\lambda_i)$, there exists an approximate eigenvector $\bar{u}_{i,h} \in M_h(\lambda_i)$ such that $\|\bar{u}_i - \bar{u}_{i,h}\|_{-a} \leq C_i \eta(h) \|\bar{u}_i - \bar{u}_{i,h}\|_a$ and $\|\bar{u}_i - \bar{u}_{i,h}\|_a \leq C_i \delta_h(\lambda_i)$. In general, here the two constants C_i depend on $\|\bar{u}_i\|_a$. However, note that $\|\bar{u}_i\|_a \leq \|u_{i,H}\|_a + \|\bar{u}_i - u_{i,H}\|_a \leq 1 + C_i \delta_H(\lambda_i)$, under the assumption that H is small enough, these constants are bounded from above.

Note that $(E_h^i u_{i,H} - \bar{u}_{i,h}) \in M_h(\lambda_i)$ and from (10), we have

$$a_{\lambda_{i,H}}((\hat{u}_{i,h} - \tilde{u}_i^h), v_h^\perp) = (\lambda_{i,h} - \lambda_{i,H})b((\bar{u}_{i,h} - u_{i,H}), v_h^\perp), \quad \forall v_h^\perp \in M_h(\lambda_i)^\perp.$$

According to Lemma 1 and (1), we have

$$\begin{aligned} \|\hat{u}_{i,h} - \tilde{u}_i^h\|_a &\leq C_i |\lambda_{i,h} - \lambda_{i,H}| \|\bar{u}_{i,h} - u_{i,H}\|_{-a} \\ &= C_i |(\lambda_{i,h} - \lambda_i) + (\lambda_i - \lambda_{i,H})| \|(\bar{u}_{i,h} - \bar{u}_i) + (\bar{u}_i - u_{i,H})\|_{-a} \\ &\leq C_i (|\lambda_{i,h} - \lambda_i| \|\bar{u}_{i,h} - \bar{u}_i\|_{-a} + |\lambda_{i,h} - \lambda_i| \|\bar{u}_i - u_{i,H}\|_{-a} \\ &\quad + |\lambda_i - \lambda_{i,H}| \|\bar{u}_{i,h} - \bar{u}_i\|_{-a} + |\lambda_i - \lambda_{i,H}| \|\bar{u}_i - u_{i,H}\|_{-a}) \\ &\leq C_i (\eta(h) \delta_h^3(\lambda_i) + \eta(H) \delta_h^2(\lambda_i) \delta_H(\lambda_i) \\ &\quad + \eta(h) \delta_H^2(\lambda_i) \delta_h(\lambda_i) + \eta(H) \delta_H^3(\lambda_i)). \end{aligned}$$

Then, since $\hat{u}_{i,h} \in M_h(\lambda_i)$, there exists $u_i \in M(\lambda_i)$, such that

$$\begin{aligned} \|u_i - \tilde{u}_i^h\|_a &\leq \|u_i - \hat{u}_{i,h}\|_a + \|\hat{u}_{i,h} - \tilde{u}_i^h\|_a \\ &\leq C_i (\delta_h(\lambda) + \eta(h) \delta_h^3(\lambda_i) + \eta(H) \delta_h^2(\lambda_i) \delta_H(\lambda_i) \\ &\quad + \eta(h) \delta_H^2(\lambda_i) \delta_h(\lambda_i) + \eta(H) \delta_H^3(\lambda_i)). \end{aligned}$$

Because $\tilde{u}_i^h = (\lambda_{i,h} - \lambda_{i,H})u_i^h$ according to (5), (3) follows. By Proposition 2.3 in [1], (4) follows. $\square$

MAIN RESULTS

Let $\Omega \subset \mathbb{R}^d$ ($d = 1, 2, \dots$) be a bounded polygonal convex domain and $\Gamma^h(\Omega)$ be a mesh with mesh size h . Consider

$$a(u, v) = \int_{\Omega} \sum_{i,j=1}^d a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}, \quad a_{ij} \in W_{1,\infty}(\Omega), \quad \text{and } b(u, v) = (u, v)_{L^2},$$

where $A = (a_{ij})$ is uniformly symmetric positive definite on Ω . Introduce the following spaces $\mathcal{H}_a = H_0^1(\Omega)$, $\mathcal{H}_b = L^2(\Omega)$ and the finite element space $S_h(\Omega) =$

$\{v \in C(\overline{\Omega}) \cap H_0^1(\Omega) : v|_\tau \in P_\tau^r, \forall \tau \in \Gamma_h(\Omega)\}$ where P_τ^r is the space of polynomials whose degree is not greater than the positive integer r . Furthermore, assume that $M(\lambda_i) \subset H^{r+1}(\Omega)$, and then

$$\eta(h) = \mathcal{O}(h), \quad \delta_h(\lambda_i) \leq C_i h^r.$$

Hence,

$$\min_{\alpha \in \mathbb{R}} \|u_i - \alpha u_i^h\|_a \leq C_i(h^r + H^{3r+1}) \text{ and } |\lambda_i^h - \lambda_i| \leq C_i(h^{2r} + H^{6r+2}).$$

If we use the piecewise linear finite element, namely $r = 1$, we have

$$\min_{\alpha \in \mathbb{R}} \|u_i - \alpha u_i^h\|_a \leq C_i(h + H^4) \text{ and } |\lambda_i^h - \lambda_i| \leq C_i(h^2 + H^8).$$

This means that the asymptotically optimal accuracy is obtained by taking $H = h^{\frac{1}{4}}$.

ACKNOWLEDGEMENTS

The authors thank Professor Yidu Yang and Hai Bi for pointing out the gap in our analysis as first published. They also showed similar error estimates for the two-grid method in Theorem 4.1, [2]. The authors would also like to thank the referee for helpful comments and suggestions, especially on the proof for the multiple eigenvalues case.

REFERENCES

- [1] X. Hu and X. Cheng, *Acceleration of a two-grid method for eigenvalue problems*, Math. Comp. **80** (2011), no. 275, 1287–1301, DOI 10.1090/S0025-5718-2011-02458-0. MR2785459 (2012h:65264)
- [2] Y. Yang and H. Bi, *Two-grid finite element discretization schemes based on shifted-inverse power method for elliptic eigenvalue problems*, SIAM J. Numer. Anal. **49** (2011), no. 4, 1602–1624, DOI 10.1137/100810241. MR2831063

DEPARTMENT OF MATHEMATICS, PENNSYLVANIA STATE UNIVERSITY, UNIVERSITY PARK, PENNSYLVANIA 16803

Current address: Department of Mathematics, Tufts University, Medford, Massachusetts 02155

E-mail address: xiaozhe.hu@tufts.edu

DEPARTMENT OF MATHEMATICS, ZHEJIANG UNIVERSITY, YUQUAN CAMPUS, HANGZHOU, 310027, PEOPLE'S REPUBLIC OF CHINA

E-mail address: xiaoliangcheng@zju.edu.cn