A C^1 -FINITE ELEMENT METHOD FOR THE WILLMORE FLOW OF TWO-DIMENSIONAL GRAPHS

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ABSTRACT. We consider the Willmore flow of two-dimensional graphs subject to Dirichlet boundary conditions. The corresponding evolution is described by a highly nonlinear parabolic PDE of fourth order for the height function. Based on a suitable weak form of the equation we derive a semidiscrete scheme which uses C^1 -finite elements and interpolates the Dirichlet boundary conditions. We prove quasioptimal error bounds in Sobolev norms for the solution and its time derivative and present results of test calculations.

1. INTRODUCTION

For a smooth immersed surface $f : \Sigma \rightarrow \mathbb{R}^3$ we consider the Willmore functional

$$(1.1) \quad W(f) = \frac{1}{2} \int_{\Sigma} H^2 d\mu_g.$$

Here, Σ is a two-dimensional reference manifold, g is the Riemannian metric induced by the standard Euclidean inner product in $\mathbb{R}^3$ along the pullback of f and $d\mu_g = \sqrt{\det(g)} dx$ is the induced surface element. Furthermore, $H = \kappa_1 + \kappa_2$ is the mean curvature, where κ_1, κ_2 are the principal curvatures of f . The sign of H is chosen in such a way that $H > 0$ for a sphere with outward pointing unit normal.

Apart from being of geometric interest the Willmore functional and generalizations occur in areas like biomembrane physics ([16], [25]) and image processing, where one is frequently interested in shapes corresponding to local or global minima of W . A natural way to obtain these minima is to relax an initial surface $\mathcal{M}_0 = f_0(\Sigma)$ via the L^2 -gradient flow of W , the Willmore flow, leading to the following geometric evolution equation:

$$(1.2) \quad V = \Delta_{\Gamma} H + \frac{1}{2} H^3 - 2HK \quad \text{on } \mathcal{M}(t).$$

Here, V denotes the normal velocity of the evolving surfaces $\mathcal{M}(t) = f(\Sigma, t)$, Δ_{Γ} is the Laplace Beltrami operator on f and $K = \kappa_1 \kappa_2$ is the Gauss curvature.

In this paper we assume that the surfaces $\mathcal{M}(t)$ are graphs over some bounded domain $\Omega \subset \mathbb{R}^2$ with boundary $\Gamma := \partial\Omega$ of class C^∞ , so that $\mathcal{M}(t) = f(\Omega, t)$, where

$$f(x, t) = (x, u(x, t)), \quad x \in \Omega, 0 \leq t \leq T.$$

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In this case the evolution equation (1.2) translates into the following highly non-linear fourth order parabolic PDE for the height function u (see (1.10) in [8]),

$$(1.3) \quad u_t + Q[u] \nabla \cdot \left(\frac{1}{Q[u]} \left(I - \frac{\nabla u \otimes \nabla u}{Q[u]^2} \right) \nabla (Q[u] H[u]) \right) - \frac{1}{2} Q[u] \nabla \cdot \left(\frac{H[u]^2}{Q[u]} \nabla u \right) = 0$$

in $\Omega \times (0, T]$. In the above

$$Q[u] = \sqrt{1 + |\nabla u|^2} \quad \text{and} \quad H[u] = \nabla \cdot \left(\frac{\nabla u}{Q[u]} \right)$$

denote the area element and the mean curvature respectively, while the surface is oriented by the unit normal $N = \frac{(\nabla u, -1)}{Q[u]}$. We shall impose Dirichlet boundary conditions

$$(1.4) \quad u(x, t) = g(x), \quad \frac{\partial u}{\partial \nu}(x, t) = \tilde{g}(x), \quad x \in \Gamma, 0 < t \leq T,$$

where $g, \tilde{g} : \Gamma \rightarrow \mathbb{R}$ are given smooth functions and ν denotes the unit outer normal to Γ . Finally, we prescribe the initial condition

$$(1.5) \quad u(\cdot, 0) = u_0 \quad \text{in } \Omega.$$

Let us briefly review the literature on numerical methods for Willmore flow. In [17], a discrete version of (1.1) was minimized with the help of Brakke's surface evolver ([5]). Mayer and Simonett ([21]) use a finite difference scheme in order to approximate axisymmetric solutions of Willmore flow. Finite element discretizations for the evolution of parametric surfaces by Willmore flow and related geometric evolution laws have been developed in [24], [2], [12], [4] and [23]. These methods have in common that they split the fourth order problem into two second order problems for the position and either the scalar mean curvature or the mean curvature vector. These variables are then approximated by piecewise linear or piecewise quadratic finite elements. While the above-mentioned papers treat closed surfaces, the evolution of surfaces with boundaries has been less investigated. In [7], the parametric approach from [24] is extended to surfaces with boundaries and applied to problems in surface restoration. A level set approach to Willmore flow into which boundary conditions can be incorporated is presented in [11]. Again a splitting technique is applied using the level set function and a weighted mean curvature as variables. In view of its formal similarity, this approach can also be employed to discretize the Willmore flow of graphs and an error analysis for the case of Navier boundary conditions is carried out in [8]. Similar approaches using finite differences and a discontinuous Galerkin method can be found in [22] and [29] respectively.

If one wants to avoid the above splitting approach one is naturally led to C^1 -finite elements. While their implementation is more challenging, one benefits from the necessary use of higher order polynomials in that one can work with relatively coarse meshes in order to achieve good accuracy. In [9], two of the authors carried out a corresponding error analysis for the Willmore flow of axisymmetric surfaces subject to Dirichlet boundary conditions. The aim of the current paper is to introduce and analyze a C^1 method for the initial boundary value problem (1.3)–(1.5). Contrary to the axisymmetric case, which is one-dimensional in space, the spatial variable is now defined on a domain in $\mathbb{R}^2$ with a curved boundary, so that the discretization of the Dirichlet boundary conditions (1.4) becomes an issue. We shall employ the approach from [10], which uses an exact decomposition of Ω and an interpolation of the boundary conditions (see [3, 14] for other possibilities). The underlying

variational formulation of (1.3) is derived in Section 2 and then used in Section 3 in order to discretize in space by C^1 elements that are piecewise polynomials of degree less than or equal to 3. Our main result is the following quasioptimal error estimate for the resulting semidiscrete scheme with solution u_h :

$$\left(\int_0^T \|u_t - u_{ht}\|_\Omega^2 dt \right)^{\frac{1}{2}} + \max_{0 \leq t \leq T} \sum_{|\alpha|=0}^2 h^{|\alpha|} \|D^\alpha(u - u_h)(\cdot, t)\|_\Omega \leq Ch^4 \sqrt{|\log h|}.$$

The proof of this result will be given in Section 4. The estimates are slightly worse than the bounds obtained in [9] for which the axisymmetric case do not contain the logarithmic factor. That factor arises when estimating certain nonlinear terms and is essentially due to the lack of a continuous embedding from $H^1(\Omega)$ into $L^\infty(\Omega)$ in two space dimensions (cf. Lemma 4.3). In order to obtain a practical algorithm one still needs to discretize the time variable. Details of our time discretization are given in Section 5, while Section 6 reports on implementation issues and on results of numerical tests.

Notation. We denote by $\|\cdot\|_{s,p,\Omega}$ ($s \in \mathbb{R}, 1 \leq p \leq \infty$) the norm of $W^{s,p}(\Omega)$. For $s > 0$, $W^{-s,p}(\Omega)$ is the dual space of $W^{s,p}(\Omega)$. Furthermore, if $p = 2$ we simply write $\|\cdot\|_{m,2,\Omega} = \|\cdot\|_{m,\Omega}$ and $\|\cdot\|_{0,\Omega} = \|\cdot\|_\Omega$. A corresponding notation is used for the spaces $W^{s,p}(\Gamma)$. Finally, if $m \in \mathbb{N}$, $|\cdot|_{m,p,\Omega}$ denotes the seminorm in $W^{m,p}(\Omega)$.

2. VARIATIONAL FORMULATION AND DISCRETIZATION

Let us begin by writing the Willmore functional for the graph of a function $v \in W^{2,\infty}(\Omega)$ in the form

$$(2.1) \quad W(v) = \frac{1}{2} \int_\Omega H[v]^2 Q[v] dx = \frac{1}{2} \int_\Omega [E(\nabla v) : D^2 v]^2 dx,$$

where the matrix function $E : \mathbb{R}^2 \rightarrow \text{Sym}(2)$ is defined by

$$E_{ij}(p) := \frac{1}{(1 + |p|^2)^{\frac{1}{4}}} \left(\delta_{ij} - \frac{p_i p_j}{1 + |p|^2} \right), \quad i, j = 1, 2, \quad p \in \mathbb{R}^2$$

and $\text{Sym}(n)$ denotes the space of symmetric $n \times n$ matrices. Furthermore, $A : B = \text{trace}(AB)$ for $A, B \in \text{Sym}(2)$. Note that

$$(2.2) \quad E(p) \xi \cdot \xi \geq \frac{|\xi|^2}{(1 + |p|^2)^{\frac{5}{4}}}, \quad \text{for all } \xi, p \in \mathbb{R}^2.$$

Using (2.1) we obtain for the first variation of W in direction $\varphi \in H^2(\Omega)$,

$$(2.3) \quad \langle W'(v), \varphi \rangle = \int_\Omega [E(\nabla v) : D^2 v] \left\{ [E(\nabla v) : D^2 \varphi] + \sum_{k=1}^2 \varphi_{x_k} [E_{p_k}(\nabla v) : D^2 v] \right\} dx.$$

Let us write this expression in the form

$$(2.4) \quad \langle W'(v), \varphi \rangle = \sum_{1 \leq |\alpha| \leq 2} \int_\Omega b_\alpha(\nabla v, D^2 v) D^\alpha \varphi dx,$$

where $b_\alpha : \mathbb{R}^2 \times \text{Sym}(2) \rightarrow \mathbb{R}$ is defined by

$$b_\alpha(p, S) = \begin{cases} (E(p) : S)(E_{p_k}(p) : S), & \alpha = e_k, \\ (E(p) : S)E_{kl}(p), & \alpha = e_k + e_l, \end{cases}$$

and $k, l \in \{1, 2\}$. For $\alpha, \beta \in \mathbb{N}^2, 1 \leq |\alpha|, |\beta| \leq 2$ we let

$$a_{\alpha\beta}(p, S) := \begin{cases} \frac{\partial b_\alpha}{\partial p_k}(p, S), & \beta = e_k, \\ \frac{\partial b_\alpha}{\partial S_{kl}}(p, S), & \beta = e_k + e_l. \end{cases}$$

Note that

$$(2.5) \quad a_{\alpha\beta}(p, S) = E_{ij}(p)E_{kl}(p), \quad \text{if } \alpha = e_k + e_l, \beta = e_i + e_j \text{ and } i, j, k, l \in \{1, 2\}.$$

In what follows we shall assume that $v \in C^2(\bar{\Omega})$ and that

$$(2.6) \quad \|v\|_{2,\infty,\Omega} \leq M.$$

We start with the following Gårding-type inequality.

Lemma 2.1. *Suppose that v satisfies (2.6). Then there exist $\delta > 0, \gamma, \mu \geq 0$ which only depend on M and Ω such that for all $\varphi \in H^2(\Omega)$,*

$$\sum_{1 \leq |\alpha|, |\beta| \leq 2} \int_{\Omega} a_{\alpha\beta}(\nabla v, D^2 v) D^\alpha \varphi D^\beta \varphi \, dx \geq 2\delta \|\varphi\|_{2,\Omega}^2 - \mu \|\varphi\|_{\frac{3}{2},\Gamma}^2 - \gamma \|\varphi\|_{\Omega}^2.$$

Proof. Let us abbreviate $a_{\alpha\beta}(x) := a_{\alpha\beta}(\nabla v(x), D^2 v(x))$. In view of (2.2) and (2.5) we have for all $\xi \in \mathbb{R}^2, x \in \Omega$,

$$\begin{aligned} \sum_{|\alpha|, |\beta|=2} \xi^\alpha a_{\alpha\beta}(x) \xi^\beta &= \sum_{i,j,k,l=1}^2 \xi_i \xi_j E_{ij}(\nabla v(x)) E_{kl}(\nabla v(x)) \xi_k \xi_l \\ &= (E(\nabla v(x)) \xi \cdot \xi)^2 \geq \frac{1}{2(1 + |\nabla v(x)|^2)^{\frac{5}{2}}} |\xi|^4 \geq \theta |\xi|^4, \end{aligned}$$

for some $\theta > 0$ which only depend on M . Furthermore, all coefficients $a_{\alpha\beta}$ and the modulus of continuity of the leading coefficients $a_{\alpha\beta}, |\alpha| = |\beta| = 2$ are uniformly bounded by a constant only depending on M . Hence, by Gårding's inequality there exist $\delta > 0, \tilde{\gamma} \geq 0$ which only depend on M and Ω such that

$$\sum_{1 \leq |\alpha|, |\beta| \leq 2} \int_{\Omega} a_{\alpha\beta} D^\alpha \varphi D^\beta \varphi \, dx \geq 3\delta \|\varphi\|_{2,\Omega}^2 - \tilde{\gamma} \|\varphi\|_{\Omega}^2, \quad \forall \varphi \in H_0^2(\Omega).$$

Applying this estimate to $\varphi - Z\varphi$, where $Z : H^{\frac{3}{2}}(\Gamma) \rightarrow H^2(\Omega)$ is a continuous extension operator with $Z\varphi = \varphi, \frac{\partial(Z\varphi)}{\partial\nu} = \frac{\partial\varphi}{\partial\nu}$ on Γ , we obtain the result of the lemma. $\square$

Let us next define the bilinear form $B(v; \cdot, \cdot) : H^2(\Omega) \times H^2(\Omega) \rightarrow \mathbb{R}$ by

$$B(v; \varphi, \psi) := \sum_{1 \leq |\alpha|, |\beta| \leq 2} \int_{\Omega} a_{\alpha\beta}(\nabla v, D^2 v) D^\alpha \varphi D^\beta \psi \, dx + \gamma \int_{\Omega} \varphi \psi \, dx.$$

Clearly,

$$(2.7) \quad |B(v; \varphi, \psi)| \leq C \|\varphi\|_{2,\Omega} \|\psi\|_{2,\Omega}, \quad \text{for all } \varphi, \psi \in H^2(\Omega).$$

Furthermore, Lemma 2.1 implies that

$$(2.8) \quad B(v; \varphi, \varphi) \geq 2\delta \|\varphi\|_{2,\Omega}^2 - \mu \|\varphi\|_{\frac{3}{2},\Gamma}^2, \quad \text{for all } \varphi \in H^2(\Omega),$$

where the constants in (2.7) and (2.8) again depend on M and Ω . In particular, $B(v; \cdot, \cdot)$ is coercive on $H_0^2(\Omega)$, so that we obtain together with the results in Section 15 in [1]:

Lemma 2.2. *Suppose that $v \in C^4(\bar{\Omega})$ with $\|v\|_{4,\infty,\Omega} \leq M$. Then, for every $f \in L^2(\Omega)$, the boundary value problem*

$$(2.9) \quad \sum_{1 \leq |\alpha|, |\beta| \leq 2} (-1)^{|\beta|} D^\beta (a_{\alpha\beta} D^\alpha \Phi) + \gamma \Phi = f \quad \text{in } \Omega$$

$$(2.10) \quad \Phi = \frac{\partial \Phi}{\partial \nu} = 0 \quad \text{on } \Gamma$$

has a unique solution $\Phi \in H^4(\Omega) \cap H_0^2(\Omega)$ and

$$(2.11) \quad \|\Phi\|_{4,\Omega} \leq C \|f\|_{0,\Omega}.$$

Here, $a_{\alpha\beta}(x) := a_{\alpha\beta}(\nabla v(x), D^2 v(x))$ and the constant C depends on M and Ω .

For later purposes it is convenient to derive a variational formulation of (2.9) involving test functions in $H^2(\Omega)$. Let us abbreviate

$$L_i = \sum_{|\alpha|=1}^2 a_{\alpha e_i} D^\alpha \Phi, \quad L_{ij} = \sum_{|\alpha|=1}^2 a_{\alpha, e_i + e_j} D^\alpha \Phi, \quad i, j = 1, 2.$$

Multiplying (2.9) by $\varphi \in H^2(\Omega)$ and integrating by parts we see that Φ satisfies

$$\begin{aligned} \int_{\Omega} f \varphi \, dx &= - \sum_{i=1}^2 \int_{\Omega} L_{i,x_i} \varphi \, dx + \sum_{i,j=1}^2 \int_{\Omega} L_{ij,x_i x_j} \varphi \, dx + \gamma \int_{\Omega} \Phi \varphi \, dx \\ &= B(v; \Phi, \varphi) - \sum_{i=1}^2 \int_{\Gamma} L_i \varphi \nu_i \, dS + \sum_{i,j=1}^2 \int_{\Gamma} (L_{ij,x_j} \varphi \nu_i - L_{ij} \varphi x_i \nu_j) \, dS. \end{aligned}$$

Writing $\nabla \varphi = \nabla_{\Gamma} \varphi + \frac{\partial \varphi}{\partial \nu} \nu$, where $\nabla_{\Gamma} = (\nabla_1, \nabla_2)$ is the tangential gradient on Γ we obtain with the help of integration by parts on Γ that

$$\begin{aligned} \int_{\Gamma} L_{ij} \varphi x_i \nu_j \, dS &= \int_{\Gamma} L_{ij} \nabla_i \varphi \nu_j \, dS + \int_{\Gamma} L_{ij} \frac{\partial \varphi}{\partial \nu} \nu_i \nu_j \, dS \\ &= - \int_{\Gamma} \nabla_i L_{ij} \varphi \nu_j \, dS - \int_{\Gamma} L_{ij} \varphi \nabla_i \nu_j \, dS + \int_{\Gamma} \kappa L_{ij} \varphi \nu_i \nu_j \, dS + \int_{\Gamma} L_{ij} \frac{\partial \varphi}{\partial \nu} \nu_i \nu_j \, dS. \end{aligned}$$

Here, κ is the curvature of Γ . Collecting terms we conclude that the solution Φ of (2.9), (2.10) satisfies

$$(2.12) \quad \int_{\Omega} f \varphi \, dx = B(v; \Phi, \varphi) + B_{\Gamma}(v; \Phi, \varphi), \quad \text{for all } \varphi \in H^2(\Omega),$$

where

$$\begin{aligned} (2.13) \quad B_{\Gamma}(v; \Phi, \varphi) &= - \sum_{i,j=1}^2 \int_{\Gamma} L_{ij} \nu_i \nu_j \frac{\partial \varphi}{\partial \nu} \, dS \\ &+ \int_{\Gamma} \left(- \sum_{i=1}^2 L_i \nu_i + \sum_{i,j=1}^2 (L_{ij,x_j} \nu_i + \nabla_i L_{ij} \nu_j + L_{ij} \nabla_i \nu_j - \kappa L_{ij} \nu_i \nu_j) \right) \varphi \, dS. \end{aligned}$$

In view of the continuous embedding $H^1(\Omega) \hookrightarrow L^2(\Gamma)$ we deduce that

$$\begin{aligned} |B_\Gamma(v; \Phi, \varphi)| &\leq C \sum_{|\alpha| \leq 2} \|D^\alpha \Phi\|_{1,\Gamma} \left\| \frac{\partial \varphi}{\partial \nu} \right\|_{-1,\Gamma} + C \sum_{|\alpha| \leq 3} \|D^\alpha \Phi\|_\Gamma \|\varphi\|_\Gamma \\ (2.14) \qquad &\leq C \|\Phi\|_{4,\Omega} (\|\varphi\|_\Gamma + \left\| \frac{\partial \varphi}{\partial \nu} \right\|_{-1,\Gamma}). \end{aligned}$$

Here, C depends again on Ω and M .

Let us return to the initial-boundary value problem (1.3)–(1.5) and derive a variational formulation of the underlying PDE. To begin, note that (1.3) can be written as

$$(2.15) \qquad \frac{u_t}{Q[u]} + \nabla \cdot F = 0 \quad \text{in } \Omega \times (0, T],$$

where the vector field F is given by

$$F = \frac{1}{Q[u]} \left(I - \frac{\nabla u \otimes \nabla u}{Q[u]^2} \right) \nabla (Q[u] H[u]) - \frac{1}{2} \frac{H[u]^2}{Q[u]} \nabla u.$$

A calculation shows that, for $i = 1, 2$,

$$F_i = \sum_{j=1}^2 \frac{\partial}{\partial x_j} \left([E(\nabla u) : D^2 u] E_{ij}(\nabla u) \right) - [E(\nabla u) : D^2 u] \sum_{j,k=1}^2 \frac{\partial E_{jk}}{\partial p_i}(\nabla u) u_{x_j x_k}.$$

Multiplying (2.15) by $\varphi \in H^2(\Omega)$, integrating over Ω and using integration by parts we infer

$$\begin{aligned} 0 &= \int_\Omega \left(\frac{u_t}{Q[u]} + \nabla \cdot F \right) \varphi \, dx \\ &= \int_\Omega \frac{u_t \varphi}{Q[u]} \, dx - \int_\Omega F \cdot \nabla \varphi \, dx + \int_\Gamma F \cdot \nu \varphi \, dS \\ &= \int_\Omega \frac{u_t \varphi}{Q[u]} \, dx + \langle W'(u), \varphi \rangle - \int_\Gamma [E(\nabla u) : D^2 u] E(\nabla u) \nu \cdot \nabla \varphi \, dS + \int_\Gamma F \cdot \nu \varphi \, dS, \end{aligned}$$

where we used (2.3) for the last equality. Abbreviating $\Psi := [E(\nabla u) : D^2 u] E(\nabla u) \nu$ we rewrite the third integral with the help of integration by parts on Γ ,

$$\begin{aligned} \int_\Gamma \Psi \cdot \nabla \varphi \, dS &= \int_\Gamma \Psi \cdot \nabla_\Gamma \varphi \, dS + \int_\Gamma \Psi \cdot \nu \frac{\partial \varphi}{\partial \nu} \, dS \\ &= - \int_\Gamma \nabla_\Gamma \cdot \Psi \varphi \, dS + \int_\Gamma \kappa \Psi \cdot \nu \varphi \, dS + \int_\Gamma \Psi \cdot \nu \frac{\partial \varphi}{\partial \nu} \, dS, \end{aligned}$$

so that

$$(2.16) \qquad \int_\Omega \frac{u_t \varphi}{Q[u]} \, dx + \langle W'(u), \varphi \rangle = \langle G(u), \varphi \rangle, \quad \varphi \in H^2(\Omega),$$

where

$$(2.17) \qquad \langle G(u), \varphi \rangle = \int_\Gamma (r_1 \varphi + r_2 \frac{\partial \varphi}{\partial \nu}) \, dS$$

and $r_i = r_i(u)$, $i = 1, 2$ are given in terms of u by

$$(2.18) \qquad r_1 := -\nabla_\Gamma \cdot \Psi + \kappa \Psi \cdot \nu - F \cdot \nu, \quad r_2 := \Psi \cdot \nu.$$

In particular,

$$(2.19) \qquad \int_\Omega \frac{u_t \varphi}{Q[u]} \, dx + \langle W'(u), \varphi \rangle = 0, \quad \text{for all } \varphi \in H_0^2(\Omega), 0 < t \leq T.$$

Remark 2.3. Not much is known about the existence and uniqueness of solutions for the initial-boundary value problem (1.3)–(1.5). Let us sketch a possible way to derive a corresponding result: Using (2.4) and (2.5) one can write (1.3) in the form

$$u_t + \sum_{i,j,k,l=1}^2 a_{ijkl}(\nabla u) u_{x_i x_j x_k x_l} = b(\nabla u, D^2 u, D^3 u) \quad \text{in } \Omega \times (0, T],$$

where $a_{ijkl}(p) = E_{ij}(p)E_{kl}(p)$, $p \in \mathbb{R}^2$, from which one deduces, in particular, that the problem is strictly parabolic. A local result can be obtained by linearizing a_{ijkl} around the initial data u_0 and by combining the Schauder theory for linear fourth order parabolic equations with a fixed point argument. This would give the existence of a unique solution $u \in C^{4+\alpha}(\bar{\Omega} \times [0, T])$, ($\alpha \in (0, 1)$) for some (possibly small) $T > 0$ provided that $u_0 \in C^{4+\alpha}(\bar{\Omega})$ satisfies the compatibility conditions:

$$u_0 = g, \quad \frac{\partial u_0}{\partial \nu} = \tilde{g}, \quad \sum_{i,j,k,l=1}^2 a_{ijkl}(\nabla u_0) u_{0,x_i x_j x_k x_l} = b(\nabla u_0, D^2 u_0, D^3 u_0) \quad \text{on } \partial\Omega.$$

The definition of the parabolic Hölder spaces $C^s(\bar{\Omega} \times [0, T])$ and a description of the compatibility conditions can be found in [13], II.3.1 and VI.3.2 respectively. By further strenghtening the hypotheses on u_0 we may assume that there exists $M \geq 0$ such that

$$(2.20) \quad \max_{t \in [0, T]} (\|u(\cdot, t)\|_{4, \infty, \Omega} + \|u_t(\cdot, t)\|_{4, \infty, \Omega}) \leq M.$$

Global existence results for Willmore flow have up to now only been obtained for surfaces without boundary. We refer to [20] in the parametric case and to [19] in the case of entire graphs.

3. DISCRETIZATION AND MAIN RESULT

Let $(\mathcal{T}_h)_{0 < h \leq h_0}$ be a family of subdivisions of Ω into convex quadrilaterals such that $\bar{\Omega} = \bigcup_{K \in \mathcal{T}_h} K$ and distinct quadrilaterals intersect at most in a common vertex or a common edge. Each quadrilateral is subdivided into four triangles $T_1, \dots, T_4$ by drawing its diagonals. If two vertices lie on Γ , then the corresponding edge is also contained in Γ (see Figure 1), i.e., we allow quadrilaterals having one curved edge at the boundary. We let $h = \max_{K \in \mathcal{T}_h} \text{diam}(K)$ and suppose that the family $(\mathcal{T}_h)_{0 < h \leq h_0}$ is quasiuniform in the sense that there exists a number $\rho > 0$ such that each subtriangle T_i of a quadrilateral K is starlike with respect to a disk whose diameter is ρh .

The finite element we use belongs to a family of C^1 -finite elements which were introduced and analyzed in [10]. For a mesh cell $K \in \mathcal{T}_h$, the related local finite element space is defined as

$$S_3(K) := \{p \in C^1(K) \mid p|_{T_i} \in P_3(T_i), 1 \leq i \leq 4\},$$

where we emphasize that $p|_{T_i}$ is also for boundary elements a cubic polynomial of the class P_3 in the original coordinates contrary to the isoparametric finite element concept. In order to determine a local function $p \in S_3(K)$ by means of some nodal values we introduce for a given mesh cell K the notation a_i , $i = 1, \dots, 4$, for the vertices of K , b_i for the midpoints of its edges (which are taken with respect to

FIGURE 1. Quadrilateral mesh cell K at curved boundary Γ with vertices a_i , edge midpoints b_i and subtriangles T_i , $i = 1, \dots, 4$.

arclength for boundary edges) and n_i for the edge normal vector at point b_i . Then, a function $p \in S_3(K)$ is uniquely determined by the following 16 nodal functionals:

$$N_{a_i, \alpha}(p) := D^\alpha p(a_i), \quad N_{b_i}(p) := \frac{\partial p}{\partial n_i}(b_i),$$

where $i \in \{1, \dots, 4\}$ and α varies over all multi-indices with $|\alpha| \leq 1$. We now introduce the global finite element space $X_h \subset H^2(\Omega)$ as

$$X_h := \{\varphi_h \in C^1(\bar{\Omega}) \mid \varphi_h|_K \in S_3(K), K \in \mathcal{T}_h\}.$$

Given $\psi \in H^4(\Omega)$ one defines an interpolant $\Pi_h \psi \in X_h$ elementwise on each cell K as $\Pi_h \psi|_K := p$ where $p \in S_3(K)$ is determined by the conditions

$$N_{a_i, \alpha}(p) := D^\alpha \psi(a_i), \quad N_{b_i}(p) := \frac{\partial \psi}{\partial n_i}(b_i), \quad \forall i = 1, \dots, 4, \quad |\alpha| \leq 1.$$

One can prove for the globally composed function that $\Pi_h \psi \in C^1(\bar{\Omega})$ and (see Theorem 2 in [10])

$$(3.1) \quad \|\psi - \Pi_h \psi\|_{m, \Omega} \leq Ch^{4-m} \|\psi\|_{4, \Omega}, \quad m = 0, 1, 2.$$

Furthermore, (see Theorem 5 in [10])

$$(3.2) \quad \|\psi - \Pi_h \psi\|_{\Gamma} \leq Ch^4 (\|\psi\|_{4, \Omega} + \|\psi\|_{4, \Gamma}), \quad \text{if } \psi|_{\Gamma} \in H^4(\Gamma),$$

$$(3.3) \quad \left\| \frac{\partial}{\partial \nu} (\psi - \Pi_h \psi) \right\|_{-1, \Gamma} \leq Ch^4 (\|\psi\|_{4, \Omega} + \left\| \frac{\partial \psi}{\partial \nu} \right\|_{4, \Gamma}), \quad \text{if } \frac{\partial \psi}{\partial \nu}|_{\Gamma} \in H^4(\Gamma).$$

The constants C in (3.1), (3.2) and (3.3) only depend on ρ and Ω .

We enforce the homogeneous boundary conditions on the test space X_{h0} in the following weak discrete sense [6, 10]: for a boundary edge $e \subset \Gamma$, we denote by a_1^e, a_2^e its vertices and by b^e its midpoint. Let

$$X_{h0} := \{\varphi_h \in X_h \mid \varphi_h(a_i^e) = 0, \nabla \varphi_h(a_i^e) = 0, i = 1, 2, \frac{\partial \varphi_h}{\partial \nu}(b^e) = 0 \text{ for all } e \subset \Gamma\}.$$

Note that in general $X_{h0} \not\subset H_0^2(\Omega)$. However, we have for all $\varphi_h \in X_{h0}$ (see Theorem 6 in [10])

$$(3.4) \quad \|\varphi_h\|_{s, \Gamma} \leq Ch^{\frac{5}{2}-s} \|\varphi_h\|_{2, \Omega}, \quad 0 \leq s \leq 2,$$

$$(3.5) \quad \left\| \frac{\partial \varphi_h}{\partial \nu} \right\|_{-1, \Gamma} \leq Ch^{\frac{5}{2}} \|\varphi_h\|_{2, \Omega}.$$

Furthermore, the following inverse inequality holds for all $\varphi_h \in X_h, K \in \mathcal{T}_h$:

$$(3.6) \quad \|\varphi_h\|_{2,\infty,K} \leq \hat{C}h^{-1}\|\varphi_h\|_{2,K}.$$

The constants in (3.4)–(3.6) only depend on Ω and ρ . Next, let us show that the bilinear form $B(v; \cdot, \cdot)$ is coercive on X_{h_0} .

Lemma 3.1. *Suppose that $v \in C^2(\bar{\Omega})$ satisfies $\|v\|_{2,\infty,\Omega} \leq M$. Then there exists $0 < h_1 \leq h_0$ depending on ρ , M and Ω such that*

$$(3.7) \quad B(v; \varphi_h, \varphi_h) \geq \delta \|\varphi_h\|_{2,\Omega}^2 \quad \text{for all } \varphi_h \in X_{h_0}, 0 < h \leq h_1.$$

Proof. We infer from (2.8) and (3.4) with $s = \frac{3}{2}$ that, for $\varphi_h \in X_{h_0}$,

$$B(v; \varphi_h, \varphi_h) \geq 2\delta \|\varphi_h\|_{2,\Omega}^2 - \mu \|\varphi_h\|_{\frac{2}{3},\Gamma}^2 \geq (2\delta - C\mu h^2) \|\varphi_h\|_{2,\Omega}^2 \geq \delta \|\varphi_h\|_{2,\Omega}^2,$$

provided that $0 < h \leq h_1$ and $C\mu h_1^2 \leq \delta$. $\square$

Next, for given $r_1 \in L^2(\Gamma), r_2 \in H^1(\Gamma)$ let $G : H^2(\Omega) \rightarrow \mathbb{R}$ be defined as

$$\langle G, \varphi \rangle := \int_{\Gamma} (r_1 \varphi + r_2 \frac{\partial \varphi}{\partial \nu}) dS, \quad \varphi \in H^2(\Omega).$$

We introduce an operator $P_h = P_h[v, G] : H^4(\Omega) \rightarrow X_h$ as follows: given $\psi \in H^4(\Omega)$ we define $P_h\psi$ by requiring that $P_h\psi - \Pi_h\psi \in X_{h_0}$ as well as

$$(3.8) \quad B(v; \varphi_h, P_h\psi) = B(v; \varphi_h, \psi) - \langle G, \varphi_h \rangle \quad \text{for all } \varphi_h \in X_{h_0}.$$

In view of the definition of B the operator $P_h[v, G]$ can be viewed as a Ritz projection associated with the linearization of W' at v , while G is included in order to handle the problem that $X_{h_0} \not\subset H_0^2(\Omega)$ later on. Lemma 3.1 implies the existence of $P_h\psi$; inserting $\varphi_h = P_h\psi - \Pi_h\psi$ into (3.8) we deduce with the help of (3.7), (3.8), (2.7), (3.1), (3.4) and (3.5) that

$$\begin{aligned} \delta \|P_h\psi - \Pi_h\psi\|_{2,\Omega}^2 &\leq B(v; P_h\psi - \Pi_h\psi, P_h\psi - \Pi_h\psi) \\ &= -\langle G, P_h\psi - \Pi_h\psi \rangle + B(v; P_h\psi - \Pi_h\psi, \psi - \Pi_h\psi) \\ &\leq \|r_1\|_{\Gamma} \|P_h\psi - \Pi_h\psi\|_{\Gamma} + \|r_2\|_{1,\Gamma} \|\frac{\partial}{\partial \nu}(P_h\psi - \Pi_h\psi)\|_{-1,\Gamma} \\ &\quad + Ch^2 \|\psi\|_{4,\Omega} \|P_h\psi - \Pi_h\psi\|_{2,\Omega} \\ &\leq Ch^{\frac{5}{2}} (\|r_1\|_{\Gamma} + \|r_2\|_{1,\Gamma}) \|P_h\psi - \Pi_h\psi\|_{2,\Omega} + Ch^2 \|\psi\|_{4,\Omega} \|P_h\psi - \Pi_h\psi\|_{2,\Omega}, \end{aligned}$$

from which we infer together with (3.1) that

$$(3.9) \quad \|\psi - P_h\psi\|_{2,\Omega} \leq Ch^2 (\|\psi\|_{4,\Omega} + \|r_1\|_{\Gamma} + \|r_2\|_{1,\Gamma}).$$

Lemma 3.2. *Suppose that $v \in C^4(\bar{\Omega})$ with $\|v\|_{4,\infty,\Omega} \leq M$, $r_1 \in L^2(\Gamma), r_2 \in H^1(\Gamma)$ and let $P_h = P_h[v, G]$ be defined by (3.8). Then we have for $0 < h \leq h_1$,*

$$\sum_{|\alpha|=0}^2 h^{|\alpha|} \|D^\alpha(\psi - P_h\psi)\|_{\Omega} \leq Ch^4 (\|\psi\|_{4,\Omega} + \|\psi\|_{4,\Gamma} + \|\frac{\partial \psi}{\partial \nu}\|_{4,\Gamma} + \|r_1\|_{\Gamma} + \|r_2\|_{1,\Gamma})$$

for all $\psi \in H^4(\Omega)$ such that $\psi|_{\Gamma} \in H^4(\Gamma), \frac{\partial \psi}{\partial \nu}|_{\Gamma} \in H^4(\Gamma)$. Here, the constant C depends on M, Ω and ρ .

Proof. We employ the usual duality argument in order to control $\|\psi - P_h\psi\|_\Omega$. Denoting by Φ the solution of (2.9), (2.10) for $f = \psi - P_h\psi$ we obtain, with the help of (2.12), (3.8), the fact that $\Phi = \frac{\partial\Phi}{\partial\nu} = 0$ on Γ , (2.7) and (2.14) that

$$\begin{aligned}
 \|\psi - P_h\psi\|_\Omega^2 &= B(v; \Phi, \psi - P_h\psi) + B_\Gamma(v; \Phi, \psi - P_h\psi) \\
 &= B(v; \Phi - \Pi_h\Phi, \psi - P_h\psi) + B_\Gamma(v; \Phi, \psi - P_h\psi) - \langle G, \Phi - \Pi_h\Phi \rangle \\
 &\leq C\|\Phi - \Pi_h\Phi\|_{2,\Omega}\|\psi - P_h\psi\|_{2,\Omega} \\
 (3.10) \quad &+ C\|\Phi\|_{4,\Omega}(\|\psi - P_h\psi\|_\Gamma + \|\frac{\partial}{\partial\nu}(\psi - P_h\psi)\|_{-1,\Gamma}) \\
 &+ \|r_1\|_\Gamma\|\Phi - \Pi_h\Phi\|_\Gamma + \|r_2\|_{1,\Gamma}\|\frac{\partial}{\partial\nu}(\Phi - \Pi_h\Phi)\|_{-1,\Gamma}.
 \end{aligned}$$

We deduce from (3.2) and (3.4) that

$$\begin{aligned}
 \|\psi - P_h\psi\|_\Gamma &\leq \|\psi - \Pi_h\psi\|_\Gamma + \|\Pi_h\psi - P_h\psi\|_\Gamma \leq Ch^4(\|\psi\|_{4,\Omega} + \|\psi\|_{4,\Gamma}) \\
 &+ Ch^{\frac{5}{2}}\|\Pi_h\psi - P_h\psi\|_{2,\Omega}.
 \end{aligned}$$

Arguing similarly for $\|\frac{\partial}{\partial\nu}(\psi - P_h\psi)\|_{-1,\Gamma}$ we may return to (3.10) and deduce with the help of (3.1) and (3.9) that

$$\|\psi - P_h\psi\|_\Omega^2 \leq Ch^4\|\Phi\|_{4,\Omega}(\|\psi\|_{4,\Omega} + \|\psi\|_{4,\Gamma} + \|\frac{\partial\psi}{\partial\nu}\|_{4,\Gamma} + \|r_1\|_\Gamma + \|r_2\|_{1,\Gamma}),$$

which together with (2.11) yields our result in the case $\alpha = 0$. The estimate for $|\alpha| = 1$ then follows together with (3.9) and an interpolation inequality. $\square$

We can now formulate the semidiscrete problem: find $u_h : \bar{\Omega} \times [0, T] \rightarrow \mathbb{R}$ such that $u_h(\cdot, t) \in X_h$, $0 \leq t \leq T$ and

$$(3.11) \quad \int_\Omega \frac{u_{h,t}\varphi_h}{Q[u_h]} dx + \langle W'(u_h), \varphi_h \rangle = 0,$$

$$\begin{aligned}
 (3.12) \quad u_h(a_i^e, t) &= g(a_i^e), \quad \nabla u_h(a_i^e, t) = \nabla_\Gamma g(a_i^e) + \tilde{g}(a_i^e)\nu(a_i^e), \\
 \frac{\partial u_h}{\partial\nu}(b^e, t) &= \tilde{g}(b^e),
 \end{aligned}$$

$$(3.13) \quad u_h(\cdot, 0) = u_0^h$$

for all $\varphi_h \in X_{h0}$, $e \subset \Gamma$, $i = 1, 2$ and $0 < t \leq T$. Here, $u_0^h = \hat{u}_h(\cdot, 0)$, where $\hat{u}_h$ will be defined in (4.2) below. Our main result are the following optimal (up to a logarithmic factor) error estimates for the semidiscrete scheme:

Theorem 3.3. *Suppose that (1.3), (1.4), (1.5) has a solution $u : \bar{\Omega} \times [0, T] \rightarrow \mathbb{R}$ satisfying (2.20). Then there exist $0 < \hat{h} \leq h_0$, $C > 0$ which depend on Ω, T, M, ρ and $\|g, \tilde{g}\|_{4,\Gamma}$ such that for all $0 < h \leq \hat{h}$ the discrete solution u_h exists on $[0, T]$ and*

$$\left(\int_0^T \|u_t - u_{ht}\|_\Omega^2 dt \right)^{\frac{1}{2}} + \max_{0 \leq t \leq T} \sum_{|\alpha|=0}^2 h^{|\alpha|} \|D^\alpha(u - u_h)(\cdot, t)\|_\Omega \leq Ch^4 \sqrt{|\log h|}.$$

4. PROOF OF THEOREM 3.3

Let us decompose the error $e_h := u - u_h$ into

$$(4.1) \quad e_h = (u - \hat{u}_h) + (\hat{u}_h - u_h) =: \eta_h + \theta_h,$$

where

$$(4.2) \quad \hat{u}_h(\cdot, t) := P_h[u(\cdot, t), G(u(\cdot, t))]u(\cdot, t)$$

and $G(u)$ is defined by (2.17). Note that $\hat{u}_h(\cdot, t) \in X_h, 0 \leq t \leq T$ and that $\|r_1(u)\|_\Gamma$ and $\|r_2(u)\|_{1,\Gamma}$ in (2.18) are bounded uniformly in time by a constant depending on Ω and M . In view of (3.8) η_h satisfies

$$(4.3) \quad B(u(\cdot, t); \varphi_h, \eta_h) = \langle G(u(\cdot, t)), \varphi_h \rangle, \quad \forall \varphi_h \in X_{h0}, t \in [0, T].$$

In what follows, we denote by C, h_2, h_3 constants which may depend on Ω, T, M, ρ and $\|g, \tilde{g}\|_{4,\Gamma}$. Let us begin by deriving bounds on η_h and $\eta_{h,t}$.

Lemma 4.1. *There exists a constant $C > 0$ so that*

$$\sum_{|\alpha|=0}^2 h^{|\alpha|} (\|D^\alpha \eta_h(\cdot, t)\|_\Omega + \|D^\alpha \eta_{h,t}(\cdot, t)\|_\Omega) \leq Ch^4, \quad 0 \leq t \leq T, 0 < h \leq h_1.$$

Proof. In view of Lemma 3.2 it is sufficient to derive the bounds for $\eta_{h,t}$. To do so, we abbreviate $a_{\alpha\beta}(x, t) = a_{\alpha\beta}(\nabla u(x, t), D^2 u(x, t))$. Differentiating (4.3) with respect to time yields for $\varphi_h \in X_{h0}$,

$$(4.4) \quad B(u(\cdot, t); \varphi_h, \eta_{h,t}) = - \sum_{1 \leq |\alpha|, |\beta| \leq 2} \int_\Omega a_{\alpha\beta,t} D^\alpha \varphi_h D^\beta \eta_h dx + \langle \partial_t [G(u(\cdot, t))], \varphi_h \rangle.$$

The choice $\varphi_h = \Pi_h u_t(\cdot, t) - \hat{u}_{ht}(\cdot, t)$ together with standard arguments implies

$$(4.5) \quad \|\eta_{h,t}(\cdot, t)\|_{2,\Omega} \leq Ch^2, \quad t \in [0, T].$$

As before we employ a duality argument in order to derive an optimal L^2 -bound. Denote by Φ the solution of (2.9), (2.10) with $f = \eta_{h,t}$. Then, using (2.12), (4.4),

integration by parts, and the fact that $\Phi = \frac{\partial \Phi}{\partial \nu} = 0$ on Γ , we obtain

$$\begin{aligned} \int_\Omega |\eta_{h,t}|^2 dx &= B(u(\cdot, t); \Phi, \eta_{h,t}) + B_\Gamma(u(\cdot, t); \Phi, \eta_{h,t}) \\ &= B(u(\cdot, t); \Phi - \Pi_h \Phi, \eta_{h,t}) - \sum_{1 \leq |\alpha|, |\beta| \leq 2} \int_\Omega a_{\alpha\beta,t} D^\alpha \Pi_h \Phi D^\beta \eta_h dx \\ &\quad + \langle \partial_t [G(u(\cdot, t))], \Pi_h \Phi \rangle + B_\Gamma(u(\cdot, t); \Phi, \eta_{h,t}) \\ &= B(u(\cdot, t); \Phi - \Pi_h \Phi, \eta_{h,t}) + \sum_{1 \leq |\alpha|, |\beta| \leq 2} \int_\Omega a_{\alpha\beta,t} D^\alpha (\Phi - \Pi_h \Phi) D^\beta \eta_h dx \\ &\quad - \sum_{1 \leq |\alpha|, |\beta| \leq 2} (-1)^{|\beta|} \int_\Omega D^\beta (a_{\alpha\beta,t} D^\alpha \Phi) \eta_h dx + \tilde{B}_\Gamma(u(\cdot, t); \Phi, \eta_h) \\ &\quad - \int_\Gamma (r_{1,t}(\Phi - \Pi_h \Phi) + r_{2,t} \frac{\partial}{\partial \nu} (\Phi - \Pi_h \Phi)) dS + B_\Gamma(u(\cdot, t); \Phi, \eta_{h,t}). \end{aligned}$$

Here, the term $\tilde{B}_\Gamma(u(\cdot, t); \Phi, \eta_h)$ contains boundary integrals resulting from the corresponding integration by parts and has the same structure as B_Γ in (2.13). It can be estimated by

$$|\tilde{B}_\Gamma(u(\cdot, t); \Phi, \varphi)| \leq C \|\Phi\|_{4,\Omega} (\|\varphi\|_\Gamma + \|\frac{\partial \varphi}{\partial \nu}\|_{-1,\Gamma}), \quad \varphi \in H^2(\Omega)$$

for some $C = C(\Omega, M)$. As a result we deduce from (3.1)–(3.5), (2.14) and (4.5) that

$$\begin{aligned} \|\eta_{h,t}\|_\Omega^2 &\leq Ch^2 (\|\eta_{h,t}\|_{2,\Omega} + \|\eta_h\|_{2,\Omega}) \|\Phi\|_{4,\Omega} + C \|\Phi\|_{4,\Omega} \|\eta_h\|_\Omega \\ &\quad + C \|\Phi\|_{4,\Omega} (\|\eta_h\|_\Gamma + \|\frac{\partial \eta_h}{\partial \nu}\|_{-1,\Gamma}) + C \|\Phi\|_{4,\Omega} (\|\eta_{h,t}\|_\Gamma + \|\frac{\partial \eta_{h,t}}{\partial \nu}\|_{-1,\Gamma}) \\ &\leq Ch^2 \|\Phi\|_{4,\Omega} (\|\eta_{h,t}\|_{2,\Omega} + \|\eta_h\|_{2,\Omega}) + Ch^4 \|\Phi\|_{4,\Omega} \leq Ch^4 \|\Phi\|_{4,\Omega}, \end{aligned}$$

which yields $\|\eta_{h,t}\|_\Omega \leq Ch^4$ in view of (2.11). Combining this estimate with (4.5) and an interpolation inequality we finish the proof. $\square$

Corollary 4.2. *There exists $0 < h_2 \leq h_1$ such that for all $0 < h \leq h_2$,*

$$(4.6) \quad \|\hat{u}_h(\cdot, t)\|_{2,\infty,\Omega}, \|\hat{u}_{ht}(\cdot, t)\|_{2,\infty,\Omega} \leq \hat{C} \left(\frac{1}{2} + M\right), \quad t \in [0, T],$$

where $\hat{C}$ is the constant in (3.6).

Proof. In view of (3.6), (2.20), and Lemma 4.1 we have for every $K \in \mathcal{T}_h$ and $t \in [0, T]$,

$$\begin{aligned} \|\hat{u}_h(\cdot, t)\|_{2,\infty,K} &\leq \hat{C} h^{-1} \|\hat{u}_h(\cdot, t)\|_{2,K} \leq \hat{C} h^{-1} (\|\eta_h(\cdot, t)\|_{2,K} + \|u(\cdot, t)\|_{2,K}) \\ &\leq \hat{C} h^{-1} Ch^2 + \hat{C} h^{-1} M |K|^{\frac{1}{2}} \leq \hat{C} Ch + \hat{C} M \leq \hat{C} \left(\frac{1}{2} + M\right), \end{aligned}$$

provided that $0 < h \leq h_2 \leq h_1$ and $Ch_2 \leq \frac{1}{2}$. The second bound is obtained in the same way. $\square$

Lemma 4.3. *The function $\hat{u}_h$ satisfies*

$$\langle W'(\hat{u}_h), \varphi_h \rangle = \langle W'(u), \varphi_h \rangle - \langle G(u), \varphi_h \rangle + \gamma \int_\Omega \eta_h \varphi_h \, dx + \langle R_h, \varphi_h \rangle, \quad \varphi_h \in X_{h0},$$

where

$$(4.7) \quad |\langle R_h, \varphi_h \rangle| + |\langle \partial_t R_h, \varphi_h \rangle| \leq Ch^4 \sqrt{|\log h|} \|\varphi_h\|_{2,\Omega}, \quad \varphi_h \in X_{h0}.$$

Proof. Recalling (2.4), the definitions of $B(v; \cdot, \cdot)$ and $\hat{u}_h$ as well as (4.3) we have

$$\begin{aligned} \langle W'(u), \varphi_h \rangle - \langle W'(\hat{u}_h), \varphi_h \rangle &= \sum_{1 \leq |\alpha| \leq 2} \int_\Omega (b_\alpha(\nabla u, D^2 u) - b_\alpha(\nabla \hat{u}_h, D^2 \hat{u}_h)) D^\alpha \varphi_h \, dx \\ &= \int_0^1 B(u - (1-s)\eta_h; \varphi_h, \eta_h) \, ds - \gamma \int_\Omega \eta_h \varphi_h \, dx \\ &= \int_0^1 (B(u - (1-s)\eta_h; \varphi_h, \eta_h) - B(u; \varphi_h, \eta_h)) \, ds + \langle G(u), \varphi_h \rangle - \gamma \int_\Omega \eta_h \varphi_h \, dx \\ &=: -\langle R_h, \varphi_h \rangle + \langle G(u), \varphi_h \rangle - \gamma \int_\Omega \eta_h \varphi_h \, dx. \end{aligned}$$

Since $a_{\alpha\beta}$ is independent of S for $|\alpha| = |\beta| = 2$ we obtain with the help of Hölder's inequality that

$$\begin{aligned} |\langle R_h, \varphi_h \rangle| &\leq C \int_{\Omega} (|\nabla \eta_h| + |D^2 \eta_h|) (|\nabla \eta_h| |\nabla \varphi_h| + |D^2 \eta_h| |\nabla \varphi_h| + |\nabla \eta_h| |D^2 \varphi_h|) dx \\ &\quad + C \int_{\Omega} |\nabla \eta_h| |D^2 \eta_h| |D^2 \varphi_h| dx \\ &\leq C \int_{\Omega} (|\nabla \eta_h|^2 + |D^2 \eta_h|^2) |\nabla \varphi_h| + C \int_{\Omega} |\nabla \eta_h| (|\nabla \eta_h| + |D^2 \eta_h|) |D^2 \varphi_h| dx \\ &\leq C \|\eta_h\|_{2,\Omega} \|\eta_h\|_{2,\frac{2p}{p-2},\Omega} \|\nabla \varphi_h\|_{0,p,\Omega} + C \|\eta_h\|_{2,\Omega} \|\nabla \eta_h\|_{0,p,\Omega} \|D^2 \varphi_h\|_{0,\frac{2p}{p-2},\Omega}. \end{aligned}$$

We infer from (3.6) that for every $K \in \mathcal{T}_h$ and every $\varphi_h \in X_h$,

$$\|\varphi_h\|_{2,\frac{2p}{p-2},K} \leq Ch^{\frac{p-2}{p}} \|\varphi_h\|_{2,\infty,K} \leq Ch^{-\frac{2}{p}} \|\varphi_h\|_{2,K}$$

and hence

$$\begin{aligned} \|\varphi_h\|_{2,\frac{2p}{p-2},\Omega} &\leq Ch^{-\frac{2}{p}} \left(\sum_{K \in \mathcal{T}_h} \|\varphi_h\|_{2,K}^{\frac{2p}{p-2}} \right)^{\frac{p-2}{2p}} \\ (4.8) \quad &\leq Ch^{-\frac{2}{p}} \left(\max_{K \in \mathcal{T}_h} \|\varphi_h\|_{2,K}^2 \right)^{\frac{1}{p}} \left(\sum_{K \in \mathcal{T}_h} \|\varphi_h\|_{2,K}^2 \right)^{\frac{p-2}{2p}} \leq Ch^{-\frac{2}{p}} \|\varphi_h\|_{2,\Omega}, \end{aligned}$$

since $\max_{K \in \mathcal{T}_h} \|\varphi_h\|_{2,K}^2 \leq \sum_{K \in \mathcal{T}_h} \|\varphi_h\|_{2,K}^2$. Note that the constant C only depends on Ω and ρ . Extending (5.15) in [26] to functions $\varphi \in H^1(\Omega)$ we can show that $\|\varphi\|_{0,p,\Omega} \leq C\sqrt{p}\|\varphi\|_{1,\Omega}$. Combining this estimate with (4.8) we obtain

$$|\langle R_h, \varphi_h \rangle| \leq Ch^{-\frac{2}{p}} \sqrt{p} \|\eta_h\|_{2,\Omega}^2 \|\varphi_h\|_{2,\Omega} \leq Ch^{4-\frac{2}{p}} \sqrt{p} \|\varphi_h\|_{2,\Omega}$$

in view of Lemma 4.1. The choice $p = |\log h|$ yields the estimate for $\langle R_h, \varphi_h \rangle$. The bound for $\langle \partial_t R_h, \varphi_h \rangle$ follows in the same way starting from

$$\begin{aligned} \langle \partial_t R_h, \varphi_h \rangle &= - \int_0^1 (B(u_t - (1-s)\eta_{h,t}; \varphi_h, \eta_h) - B(u_t; \varphi_h, \eta_h)) ds \\ &\quad - \int_0^1 (B(u - (1-s)\eta_h; \varphi_h, \eta_{h,t}) - B(u; \varphi_h, \eta_{h,t})) ds, \quad \varphi_h \in X_{h0}. \quad \square \end{aligned}$$

Next, we focus on bounds for $\theta_h = \hat{u}_h - u_h$. As θ_h essentially represents the difference between a suitable Ritz projection and the discrete solution our procedure can be seen as a nonlinear, fourth order version of a superconvergence estimate due to Wheeler for the heat equation (cf. [28]). We define

$$\begin{aligned} \hat{T}_h &:= \sup \{t \in [0, T] \mid u_h \text{ solves (3.11)–(3.13),} \\ &\quad \|u_h(\cdot, s)\|_{2,\infty,\Omega} \leq \hat{C}(1+M), 0 \leq s \leq t\}. \end{aligned}$$

Since $u_h(\cdot, 0) = \hat{u}_h(\cdot, 0)$, (4.6) implies that $\|u_h(\cdot, 0)\|_{2,\infty,\Omega} \leq \hat{C}(\frac{1}{2} + M)$, $0 < h \leq h_2$, so that we have $\hat{T}_h > 0$ by continuity. Note that

$$(4.9) \quad \|u_h(\cdot, t)\|_{2,\infty,\Omega} \leq \hat{C}(1+M) \quad t \in [0, \hat{T}_h].$$

We will first carry out the error analysis on $[0, \hat{T}_h)$ and then use the resulting error bounds in order to show that $\hat{T}_h = T$ provided that h is sufficiently small. If we

combine (2.16) with Lemma 4.3 we obtain

$$\int_{\Omega} \frac{u_t \varphi_h}{Q[u]} dx + \langle W'(\hat{u}_h), \varphi_h \rangle = \gamma \int_{\Omega} \eta_h \varphi_h dx + \langle R_h, \varphi_h \rangle, \quad \varphi_h \in X_{h0},$$

so that in view of (3.11), θ_h satisfies the relation

$$(4.10) \quad \begin{aligned} & \int_{\Omega} \frac{\theta_{h,t} \varphi_h}{Q[u_h]} dx + \langle W'(\hat{u}_h), \varphi_h \rangle - \langle W'(u_h), \varphi_h \rangle \\ &= \int_{\Omega} u_t \left(\frac{1}{Q[u_h]} - \frac{1}{Q[u]} \right) \varphi_h dx - \int_{\Omega} \frac{\eta_{h,t} \varphi_h}{Q[u_h]} dx + \gamma \int_{\Omega} \eta_h \varphi_h dx + \langle R_h, \varphi_h \rangle \end{aligned}$$

for all $\varphi_h \in X_{h0}$. Inserting $\varphi_h = \theta_{h,t}(\cdot, t) \in X_{h0}$ into (4.10) and observing that

$$Q[u_h] \leq \sqrt{1 + \|\nabla u_h\|_{0,\infty,\Omega}^2} \leq \sqrt{1 + \hat{C}^2(1 + M)^2} =: \frac{1}{c_0} \quad \text{in } \Omega \times [0, \hat{T}_h),$$

by (4.9) we obtain

$$(4.11) \quad \begin{aligned} & c_0 \|\theta_{h,t}\|_{\Omega}^2 + \langle W'(\hat{u}_h), \theta_{h,t} \rangle - \langle W'(u_h), \theta_{h,t} \rangle \\ & \leq \int_{\Omega} u_t \left(\frac{1}{Q[u_h]} - \frac{1}{Q[u]} \right) \theta_{h,t} dx - \int_{\Omega} \frac{\eta_{h,t} \theta_{h,t}}{Q[u_h]} dx + \gamma \int_{\Omega} \eta_h \theta_{h,t} dx + \langle R_h, \theta_{h,t} \rangle \\ & \equiv I + II + III + IV. \end{aligned}$$

In order to deal with the first term we write

$$\frac{1}{Q[u_h]} - \frac{1}{Q[u]} = -\frac{\nabla u}{Q[u]^3} \cdot \nabla(u_h - u) + S_h = \frac{\nabla u \cdot \nabla e_h}{Q[u]^3} + S_h$$

where $|S_h| \leq C|\nabla e_h|^2$. Hence,

$$I = \int_{\Omega} u_t \theta_{h,t} \frac{\nabla u \cdot \nabla e_h}{Q[u]^3} dx + \int_{\Omega} u_t \theta_{h,t} S_h dx = I_1 + I_2.$$

Integrating by parts and observing that $u_t(\cdot, t)|_{\Gamma} = 0$ we infer

$$\begin{aligned} I_1 &= - \int_{\Omega} \nabla \cdot \left(\frac{u_t \nabla u}{Q[u]^3} \right) e_h \theta_{h,t} dx - \int_{\Omega} e_h \frac{u_t}{Q[u]^3} \nabla u \cdot \nabla \theta_{h,t} dx \\ &= - \int_{\Omega} \nabla \cdot \left(\frac{u_t \nabla u}{Q[u]^3} \right) e_h \theta_{h,t} dx - \frac{d}{dt} \int_{\Omega} e_h \frac{u_t}{Q[u]^3} \nabla u \cdot \nabla \theta_h dx \\ &\quad + \int_{\Omega} e_h \partial_t \left(\frac{u_t \nabla u}{Q[u]^3} \right) \cdot \nabla \theta_h dx + \int_{\Omega} e_{h,t} \frac{u_t}{Q[u]^3} \nabla u \cdot \nabla \theta_h dx. \end{aligned}$$

To begin, we have

$$\begin{aligned} I_1 &\leq -\frac{d}{dt} \int_{\Omega} e_h \frac{u_t}{Q[u]^3} \nabla u \cdot \nabla \theta_h dx \\ &\quad + C(\|e_h\|_{\Omega} \|\theta_{h,t}\|_{\Omega} + \|e_h\|_{\Omega} \|\nabla \theta_h\|_{\Omega} + \|e_{h,t}\|_{\Omega} \|\nabla \theta_h\|_{\Omega}) \\ &\leq -\frac{d}{dt} \int_{\Omega} e_h \frac{u_t}{Q[u]^3} \nabla u \cdot \nabla \theta_h dx + \epsilon \|\theta_{h,t}\|_{\Omega}^2 + C_{\epsilon}(h^8 + \|\theta_h\|_{1,\Omega}^2), \end{aligned}$$

since $\|e_h\|_\Omega \leq Ch^4 + \|\theta_h\|_\Omega$ and $\|e_{h,t}\|_\Omega \leq Ch^4 + \|\theta_{h,t}\|_\Omega$ in view of (4.1) and Lemma 4.1. Next, using the continuous embedding $H^1(\Omega) \hookrightarrow L^4(\Omega)$ we infer

$$\begin{aligned} I_2 &\leq C \int_\Omega (|\nabla \eta_h|^2 + |\nabla \theta_h|^2) |\theta_{h,t}| \, dx \leq C (\|\nabla \eta_h\|_{0,4,\Omega}^2 + \|\nabla \theta_h\|_{0,4,\Omega}^2) \|\theta_{h,t}\|_\Omega \\ &\leq C (\|\eta_h\|_{2,\Omega}^2 + \|\theta_h\|_{2,\Omega}^2) \|\theta_{h,t}\|_\Omega \leq \epsilon \|\theta_{h,t}\|_\Omega^2 + C_\epsilon (h^8 + \|\theta_h\|_{2,\Omega}^2) \end{aligned}$$

in view of Lemma 4.1, (4.6) and (4.9). Clearly,

$$II + III \leq C (\|\eta_{h,t}\|_\Omega + \|\eta_h\|_\Omega) \|\theta_{h,t}\|_\Omega \leq \epsilon \|\theta_{h,t}\|_\Omega^2 + C_\epsilon h^8,$$

while we deduce from (4.7) that

$$\begin{aligned} IV &= \frac{d}{dt} \langle R_h, \theta_h \rangle - \langle \partial_t R_h, \theta_h \rangle \\ &\leq \frac{d}{dt} \langle R_h, \theta_h \rangle + Ch^4 \sqrt{|\log h|} \|\theta_h\|_{2,\Omega} \leq \frac{d}{dt} \langle R_h, \theta_h \rangle + \|\theta_h\|_{2,\Omega}^2 + Ch^8 |\log h|. \end{aligned}$$

Inserting the above estimates into (4.11) and choosing $\epsilon > 0$ sufficiently small we deduce that

$$\begin{aligned} (4.12) \quad &\frac{c_0}{2} \|\theta_{h,t}\|_\Omega^2 + \langle W'(\hat{u}_h), \theta_{h,t} \rangle - \langle W'(u_h), \theta_{h,t} \rangle \\ &\leq Ch^8 |\log h| + C \|\theta_h\|_{2,\Omega}^2 + \frac{d}{dt} \langle R_h, \theta_h \rangle - \frac{d}{dt} \int_\Omega e_h \frac{u_t}{Q[u]^3} \nabla u \cdot \nabla \theta_h \, dx. \end{aligned}$$

Let us next examine the terms on the left-hand side in more detail. We write

$$\begin{aligned} (4.13) \quad &\langle W'(\hat{u}_h), \theta_{h,t} \rangle - \langle W'(u_h), \theta_{h,t} \rangle \\ &= \langle W'(\hat{u}_h), \hat{u}_{ht} \rangle - \langle W'(\hat{u}_h), u_{h,t} \rangle - \langle W'(u_h), \hat{u}_{ht} \rangle + \langle W'(u_h), u_{h,t} \rangle. \end{aligned}$$

Clearly,

$$\begin{aligned} \langle W'(\hat{u}_h), \hat{u}_{ht} \rangle &= \frac{d}{dt} W(\hat{u}_h) = \frac{d}{dt} \int_\Omega \frac{1}{2} [E(\nabla \hat{u}_h) : D^2 \hat{u}_h]^2 \, dx \\ &= \frac{d}{dt} \int_\Omega \left(\frac{1}{2} [E(\nabla \hat{u}_h) : D^2 \hat{u}_h]^2 + [E(\nabla \hat{u}_h) : D^2 \hat{u}_h] \sum_{k=1}^2 \hat{u}_{h,x_k} [E_{p_k}(\nabla \hat{u}_h) : D^2 \hat{u}_h] \right) \, dx \\ &\quad - \int_\Omega [\partial_t (E(\nabla \hat{u}_h)) : D^2 \hat{u}_h] \left(\sum_{k=1}^2 \hat{u}_{h,x_k} [E_{p_k}(\nabla \hat{u}_h) : D^2 \hat{u}_h] + [E(\nabla \hat{u}_h) : D^2 \hat{u}_h] \right) \, dx \\ &\quad - \int_\Omega [E(\nabla \hat{u}_h) : D^2 \hat{u}_{ht}] \sum_{k=1}^2 \hat{u}_{h,x_k} [E_{p_k}(\nabla \hat{u}_h) : D^2 \hat{u}_h] \, dx \\ (4.14) \quad &\quad - \int_\Omega [E(\nabla \hat{u}_h) : D^2 \hat{u}_h] \sum_{k=1}^2 \hat{u}_{h,x_k} \partial_t ([E_{p_k}(\nabla \hat{u}_h) : D^2 \hat{u}_h]) \, dx. \end{aligned}$$

Next, in view of (2.3),

$$\begin{aligned}
 & \langle W'(\hat{u}_h), u_{ht} \rangle \\
 &= \int_{\Omega} [E(\nabla \hat{u}_h) : D^2 \hat{u}_h] \left([E(\nabla \hat{u}_h) : D^2 u_{h,t}] + \sum_{k=1}^2 u_{h,tx_k} [E_{p_k}(\nabla \hat{u}_h) : D^2 \hat{u}_h] \right) dx \\
 &= \frac{d}{dt} \int_{\Omega} [E(\nabla \hat{u}_h) : D^2 \hat{u}_h] \left([E(\nabla \hat{u}_h) : D^2 u_h] + \sum_{k=1}^2 u_{h,x_k} [E_{p_k}(\nabla \hat{u}_h) : D^2 \hat{u}_h] \right) dx \\
 &\quad - \int_{\Omega} \left\{ \partial_t \left([E(\nabla \hat{u}_h) : D^2 \hat{u}_h] E(\nabla \hat{u}_h) \right) : D^2 u_h \right\} dx \\
 (4.15) \quad &\quad - \int_{\Omega} \partial_t \left([E(\nabla \hat{u}_h) : D^2 \hat{u}_h] \sum_{k=1}^2 [E_{p_k}(\nabla \hat{u}_h) : D^2 \hat{u}_h] \right) u_{h,x_k} dx.
 \end{aligned}$$

Furthermore,

$$\begin{aligned}
 & \langle W'(u_h), \hat{u}_{ht} \rangle \\
 &= \int_{\Omega} [E(\nabla u_h) : D^2 u_h] \left([E(\nabla u_h) : D^2 \hat{u}_{ht}] + \sum_{k=1}^2 \hat{u}_{h,tx_k} [E_{p_k}(\nabla u_h) : D^2 u_h] \right) dx \\
 &= \int_{\Omega} [E(\nabla u_h) : D^2 u_h] \left([E(\nabla u_h) : D^2 \hat{u}_{ht}] + [\partial_t(E(\nabla \hat{u}_h)) : D^2 u_h] \right) dx \\
 &\quad + \int_{\Omega} [E(\nabla u_h) : D^2 u_h] \sum_{k=1}^2 \hat{u}_{h,tx_k} [(E_{p_k}(\nabla u_h) - E_{p_k}(\nabla \hat{u}_h)) : D^2 u_h] dx
 \end{aligned}$$

and finally,

$$(4.17) \quad \langle W'(u_h), u_{ht} \rangle = \frac{d}{dt} W(u_h) = \frac{d}{dt} \int_{\Omega} \frac{1}{2} [E(\nabla u_h) : D^2 u_h]^2 dx.$$

Inserting (4.14)–(4.17) into (4.13) we obtain, after some long but straightforward calculations,

$$\begin{aligned}
 & \langle W'(\hat{u}_h), \theta_{h,t} \rangle - \langle W'(u_h), \theta_{h,t} \rangle = \phi'_h(t) + \int_{\Omega} A_h : \partial_t(E(\nabla \hat{u}_h)) dx \\
 (4.18) \quad & \quad + \int_{\Omega} B_h : D^2 \hat{u}_{ht} dx + \sum_{k=1}^2 \int_{\Omega} C_h^k \hat{u}_{h,tx_k} dx,
 \end{aligned}$$

where

$$\begin{aligned}
 \phi_h(t) &= \int_{\Omega} \left(\frac{1}{2} [E(\nabla \hat{u}_h) : D^2 \hat{u}_h]^2 \right. \\
 &\quad \left. + [E(\nabla \hat{u}_h) : D^2 \hat{u}_h] \sum_{k=1}^2 \hat{u}_{h,x_k} [E_{p_k}(\nabla \hat{u}_h) : D^2 \hat{u}_h] \right) dx \\
 &\quad - \int_{\Omega} [E(\nabla \hat{u}_h) : D^2 \hat{u}_h] \left([E(\nabla \hat{u}_h) : D^2 u_h] + \sum_{k=1}^2 u_{h,x_k} [E_{p_k}(\nabla \hat{u}_h) : D^2 \hat{u}_h] \right) dx \\
 &\quad + \int_{\Omega} \frac{1}{2} [E(\nabla u_h) : D^2 u_h]^2 dx.
 \end{aligned}$$

Furthermore,

$$\begin{aligned} A_h = & -[E(\nabla \hat{u}_h) : D^2(u_h - \hat{u}_h)]D^2(u_h - \hat{u}_h) - [(E(\nabla u_h) - E(\nabla \hat{u}_h)) : D^2 u_h]D^2 u_h \\ & + \left(\sum_{k=1}^2 (u_{h,x_k} - \hat{u}_{h,x_k}) [E_{p_k}(\nabla \hat{u}_h) : D^2 \hat{u}_h] \right) D^2 \hat{u}_h, \end{aligned}$$

$$\begin{aligned} B_h = & \left(\sum_{k=1}^2 (u_{h,x_k} - \hat{u}_{h,x_k}) [E_{p_k}(\nabla \hat{u}_h) : D^2 \hat{u}_h] \right. \\ & \left. - [(E(\nabla u_h) - E(\nabla \hat{u}_h)) : D^2 u_h] \right) E(\nabla \hat{u}_h) \\ & + [E(\nabla \hat{u}_h) : D^2 \hat{u}_h] \sum_{k=1}^2 (u_{h,x_k} - \hat{u}_{h,x_k}) E_{p_k}(\nabla \hat{u}_h) \\ & - [E(\nabla u_h) : D^2 u_h] (E(\nabla u_h) - E(\nabla \hat{u}_h)), \end{aligned}$$

$$\begin{aligned} C_h^k = & [E(\nabla \hat{u}_h) : D^2 \hat{u}_h] \sum_{l=1}^2 (u_{h,x_l} - \hat{u}_{h,x_l}) [E_{p_k p_l}(\nabla \hat{u}_h) : D^2 \hat{u}_h] \\ & - [E(\nabla u_h) : D^2 u_h] [(E_{p_k}(\nabla u_h) - E_{p_k}(\nabla \hat{u}_h)) : D^2 u_h], \quad k = 1, 2. \end{aligned}$$

Using Taylor's expansion together with (4.6) and (4.9) it is not difficult to verify that

$$(4.19) \quad |A_h|, |B_h|, |C_h^k| \leq C(|\nabla \theta_h|^2 + |D^2 \theta_h|^2) \quad \text{in } \Omega \times [0, \hat{T}_h).$$

Inserting (4.18) and (4.19) into (4.12) and recalling (4.6) we derive

$$\frac{c_0}{2} \|\theta_{h,t}\|_{\Omega}^2 + \phi'_h(t) \leq Ch^8 |\log h| + C \|\theta_h\|_{2,\Omega}^2 + \frac{d}{dt} \langle R_h, \theta_h \rangle - \frac{d}{dt} \int_{\Omega} e_h \frac{u_t}{Q[u]^3} \nabla u \cdot \nabla \theta_h \, dx.$$

Integrating this inequality with respect to time taking into account that $\theta_h(\cdot, 0) = 0$, $\phi_h(0) = 0$ we obtain

$$\begin{aligned} (4.20) \quad & \frac{c_0}{2} \int_0^t \|\theta_{h,t}\|_{\Omega}^2 ds + \phi_h(t) \\ & \leq Ch^8 |\log h| + C \int_0^t \|\theta_h\|_{2,\Omega}^2 ds + \langle R_h, \theta_h \rangle(t) \\ & \quad - \int_{\Omega} (e_h \frac{u_t}{Q[u]^3} \nabla u \cdot \nabla \theta_h)(\cdot, t) \, dx \\ & \leq Ch^8 |\log h| + C \int_0^t \|\theta_h\|_{2,\Omega}^2 ds + Ch^4 \sqrt{|\log h|} \|\theta_h(\cdot, t)\|_{2,\Omega} \\ & \quad + C \|e_h(\cdot, t)\|_{\Omega} \|\nabla \theta_h(\cdot, t)\|_{\Omega} \\ & \leq \epsilon \|\theta_h(\cdot, t)\|_{2,\Omega}^2 + C \|\theta_h(\cdot, t)\|_{1,\Omega}^2 + C_{\epsilon} h^8 |\log h| + C \int_0^t \|\theta_h\|_{2,\Omega}^2 ds, \end{aligned}$$

where we used (4.7) and the fact that

$$\|e_h(\cdot, t)\|_{\Omega} \leq \|\eta_h(\cdot, t)\|_{\Omega} + \|\theta_h(\cdot, t)\|_{\Omega} \leq Ch^4 + \|\theta_h(\cdot, t)\|_{\Omega}.$$

Next, we show that the function ϕ_h essentially controls the H^2 -norm of θ_h .

Lemma 4.4. *There exists $0 < h_3 \leq h_2$ and $C \geq 0$ such that for all $0 < h \leq h_3$,*

$$\phi_h(t) \geq \frac{\delta}{8} \|\theta_h(\cdot, t)\|_{2,\Omega}^2 - C \|\theta_h(\cdot, t)\|_{1,\Omega}^2.$$

Proof. A long but straightforward calculation shows that

$$\begin{aligned} \phi_h(t) = & \int_{\Omega} \left\{ \frac{1}{2} [E(\nabla u_h) : D^2 \theta_h]^2 - \frac{1}{2} [(E(\nabla \hat{u}_h) - E(\nabla u_h)) : D^2 \hat{u}_h]^2 \right. \\ & + [E(\nabla \hat{u}_h) : D^2 \hat{u}_h] [(E(\nabla \hat{u}_h) - E(\nabla u_h)) : D^2 \theta_h] \\ & + ([E(\nabla \hat{u}_h) : D^2 \hat{u}_h] - [E(\nabla u_h) : D^2 u_h]) [(E(\nabla \hat{u}_h) - E(\nabla u_h)) : D^2 \hat{u}_h] \\ & + [E(\nabla \hat{u}_h) : D^2 \hat{u}_h] [(E(\nabla u_h) - E(\nabla \hat{u}_h)) : D^2 \hat{u}_h] \\ & \left. - \sum_{k=1}^2 (u_{h,x_k} - \hat{u}_{h,x_k}) E_{p_k}(\nabla \hat{u}_h) : D^2 \hat{u}_h \right\} dx \end{aligned}$$

from which we deduce with the help of (4.6) and (4.9) that

$$\begin{aligned} \phi_h(t) & \geq \frac{1}{2} \int_{\Omega} [E(\nabla u_h) : D^2 \theta_h]^2 dx - C \|\nabla \theta_h\|_{\Omega}^2 - C \|\nabla \theta_h\|_{\Omega} \|D^2 \theta_h\|_{\Omega} \\ (4.21) \quad & \geq \frac{1}{2} \int_{\Omega} [E(\nabla u_h) : D^2 \theta_h]^2 dx - \epsilon \|\theta_h\|_{2,\Omega}^2 - C_{\epsilon} \|\theta_h\|_{1,\Omega}^2. \end{aligned}$$

Clearly,

$$\begin{aligned} (4.22) \quad & \int_{\Omega} [E(\nabla u_h) : D^2 \theta_h]^2 dx \\ & \geq \frac{1}{2} \int_{\Omega} [E(\nabla u) : D^2 \theta_h]^2 dx - C \int_{\Omega} |\nabla e_h|^2 |D^2 \theta_h|^2 dx. \end{aligned}$$

Let us examine the first term on the right-hand side of (4.22). In view of (2.5), the definition of B and Lemma 3.1 we have

$$\begin{aligned} \int_{\Omega} [E(\nabla u) : D^2 \theta_h]^2 dx & = \sum_{|\alpha|=|\beta|=2} \int_{\Omega} a_{\alpha\beta}(\nabla u, D^2 u) D^{\alpha} \theta_h D^{\beta} \theta_h dx \\ & = B(u; \theta_h, \theta_h) - \sum_{2 \leq |\alpha|+|\beta| < 4} \int_{\Omega} a_{\alpha\beta}(\nabla u, D^2 u) D^{\alpha} \theta_h D^{\beta} \theta_h dx - \gamma \|\theta_h\|_{\Omega}^2 \\ (4.23) \quad & \geq \delta \|\theta_h\|_{2,\Omega}^2 - C \|\theta_h\|_{1,\Omega} \|\theta_h\|_{2,\Omega}. \end{aligned}$$

On the other hand, (4.8) with $p = 4$ and the elementary inequality $\|f\|_{0,4,\Omega}^2 \leq \epsilon \|f\|_{1,\Omega}^2 + C_{\epsilon} \|f\|_{\Omega}^2$, $f \in H^1(\Omega)$ imply

$$\begin{aligned} \int_{\Omega} |\nabla e_h|^2 |D^2 \theta_h|^2 dx & \leq \|\nabla e_h\|_{0,4,\Omega}^2 \|D^2 \theta_h\|_{0,4,\Omega}^2 \\ & \leq 2 \|\nabla \eta_h\|_{0,4,\Omega}^2 \|D^2 \theta_h\|_{0,4,\Omega}^2 + 2 \|\nabla \theta_h\|_{0,4,\Omega}^2 \|D^2 \theta_h\|_{0,4,\Omega}^2 \\ & \leq C h^{-1} \|\eta_h\|_{2,\Omega}^2 \|\theta_h\|_{2,\Omega}^2 + (\epsilon \|\theta_h\|_{2,\Omega}^2 + C_{\epsilon} \|\theta_h\|_{1,\Omega}^2) \|D^2 \theta_h\|_{0,\infty,\Omega}^2 \\ (4.24) \quad & \leq C(h^3 + \epsilon) \|\theta_h\|_{2,\Omega}^2 + C_{\epsilon} \|\theta_h\|_{1,\Omega}^2, \end{aligned}$$

taking into account that $\|D^2\theta_h\|_{0,\infty,\Omega} \leq \|D^2\hat{u}_h\|_{0,\infty,\Omega} + \|D^2u_h\|_{0,\infty,\Omega} \leq C$ by (4.6) and (4.9). Inserting (4.22), (4.23) and (4.24) into (4.21) we deduce

$$\phi_h(t) \geq \left(\frac{\delta}{4} - Ch^3 - C\epsilon\right)\|\theta_h\|_{2,\Omega}^2 - C_\epsilon\|\theta_h\|_{1,\Omega}^2$$

and the result follows by choosing ϵ and $h_3 \leq h_2$ sufficiently small. $\square$

Lemma 4.4 together with (4.20) imply

$$\begin{aligned} (4.25) \quad & \frac{c_0}{2} \int_0^t \|\theta_{h,t}\|_{\Omega}^2 ds + \frac{\delta}{8} \|\theta_h(\cdot, t)\|_{2,\Omega}^2 \\ & \leq \epsilon \|\theta_h(\cdot, t)\|_{2,\Omega}^2 + C \|\theta_h(\cdot, t)\|_{1,\Omega}^2 + C_\epsilon h^8 |\log h| + C \int_0^t \|\theta_h\|_{2,\Omega}^2 ds \\ & \leq 2\epsilon \|\theta_h(\cdot, t)\|_{2,\Omega}^2 + C_\epsilon \|\theta_h(\cdot, t)\|_{\Omega}^2 + C_\epsilon h^8 |\log h| + C \int_0^t \|\theta_h\|_{2,\Omega}^2 ds, \end{aligned}$$

where we used an interpolation inequality for the last estimate. Furthermore,

$$\|\theta_h(\cdot, t)\|_{\Omega}^2 \leq 2 \int_0^t \|\theta_h\|_{\Omega} \|\theta_{h,t}\|_{\Omega} ds \leq \epsilon \int_0^t \|\theta_{h,t}\|_{\Omega}^2 ds + C_\epsilon \int_0^t \|\theta_h\|_{\Omega}^2 ds, \quad t \in [0, \hat{T}_h].$$

Inserting this bound into (4.25) we deduce after choosing $\epsilon > 0$ sufficiently small that

$$\frac{c_0}{4} \int_0^t \|\theta_{h,t}\|_{\Omega}^2 ds + \frac{\delta}{16} \|\theta_h(\cdot, t)\|_{2,\Omega}^2 \leq Ch^8 |\log h| + C \int_0^t \|\theta_h\|_{2,\Omega}^2 ds, \quad t \in [0, \hat{T}_h].$$

Gronwall's inequality yields

$$(4.26) \quad \int_0^{\hat{T}_h} \|\theta_{h,t}\|_{\Omega}^2 ds + \sup_{0 \leq t < \hat{T}_h} \|\theta_h(\cdot, t)\|_{2,\Omega}^2 \leq Ch^8 |\log h|.$$

We can use this bound to show that $\hat{T}_h = T$. Suppose that $\hat{T}_h < T$. We infer from (3.6), (4.26) and Lemma 4.1 that for every $K \in \mathcal{T}_h$:

$$\begin{aligned} \|u_h\|_{2,\infty,K} & \leq \hat{C}h^{-1}\|u_h\|_{2,K} \leq \hat{C}h^{-1}(\|u\|_{2,K} + \|\theta_h\|_{2,K} + \|\eta_h\|_{2,K}) \\ & \leq \hat{C}h^{-1}(|K|^{\frac{1}{2}}M + Ch^4\sqrt{|\log h|} + Ch^2) \\ & \leq \hat{C}(M + Ch). \end{aligned}$$

As a consequence,

$$\|u_h(\cdot, \hat{T}_h)\|_{2,\infty,\Omega} \leq \hat{C}\left(\frac{1}{2} + M\right),$$

provided that $0 < h \leq \hat{h}$, where $0 < \hat{h} \leq h_3$ is chosen so small that $C\hat{h} \leq \frac{1}{2}$. Therefore we can extend the solution u_h to an interval $[0, \hat{T}_h + \epsilon)$ with

$$\|u_h(\cdot, s)\|_{2,\infty,\Omega} \leq \hat{C}(1 + M), \quad 0 \leq s \leq \hat{T}_h + \epsilon,$$

contradicting the definition of $\hat{T}_h$. This shows that $\hat{T}_h = T$. The error bound in Theorem 3.3 now follows from (4.26) and Lemma 4.1.

5. TIME DISCRETIZATION

In this section, we describe how the semidiscrete solution $u_h(\cdot, t) \in X_h$, $t \in [0, T]$, of (3.11)–(3.13) is approximated in time by a fully discrete solution $u_{h,\tau}$. To this end, we decompose the time interval $I := (0, T]$ into N disjoint subintervals $I_n := (t_{n-1}, t_n]$, where $n = 1, \dots, N$ and $0 = t_0 < t_1 < \dots < t_N = T$. We denote by $\tau_n := t_n - t_{n-1}$ the local time step size of interval I_n . For some order $k \geq 0$ and a Banach space B , let

$$\mathbb{P}_k(I_n, B) := \{w : I_n \rightarrow B \mid w(t) = \sum_{j=0}^k w_j t^j \quad \forall t \in I_n, w_j \in B \forall j\}$$

denote the space of B -valued polynomials in time of order k on the time interval I_n . Furthermore, let $C^0(\bar{I}, B)$ denote the space of all functions $w : \bar{I} \rightarrow B$ that are continuous on $\bar{I}$. Then, by setting $u_h(t) := u_h(\cdot, t) \in X_h$, we can consider our semidiscrete solution u_h as a function in time with values in X_h and we will approximate $u_h \in C^0(\bar{I}, X_h)$ by means of a fully discrete solution $u_{h,\tau} \in X_{h,\tau}$, where

$$X_{h,\tau} := \{w \in C^0(\bar{I}, X_h) \mid w|_{I_n} \in \mathbb{P}_2(I_n, X_h) \quad \forall n = 1, \dots, N\}$$

denotes the space of continuous X_h -valued functions on $\bar{I}$ which are locally a polynomial of second order in time on each subinterval I_n . We determine the quadratic polynomial $u_{h,\tau}|_{I_n}$ by means of the following three conditions: the continuity at t_{n-1} and two collocation conditions at the integration points $t_{n,1} := t_{n-1} + \tau_n/3$ and $t_{n,2} := t_n$ of the 2-point Gauß-Radau quadrature formula. Note, that by definition, $u_{h,\tau}(t_{n-1})$ denotes, for $n > 1$, the left-sided value that comes from subinterval I_{n-1} at $t = t_{n-1}$ or, for $n = 1$, the initial value $u_0^h \in X_h$ from (3.13). We will denote by $u_{h,\tau}(t_{n-1}^+)$ the right-sided limit $\lim_{t \downarrow t_{n-1}} u_{h,\tau}(t)$ that comes from interval I_n .

Then, the fully discrete solution $u_{h,\tau} \in X_{h,\tau}$ with the initial condition $u_{h,\tau}(0) = u_0^h$ is determined by successively solving for $n = 1, \dots, N$ the following local problem on I_n :

Find $u_{h,\tau}|_{I_n} \in \mathbb{P}_2(I_n, X_h)$ such that $u_{h,\tau}(t_{n-1}^+) = u_{h,\tau}(t_{n-1})$ and there holds for $t \in \{t_{n,1}, t_{n,2}\}$:

$$(5.1) \quad \int_{\Omega} \frac{\partial_t u_{h,\tau}(t) \varphi_h}{Q[u_{h,\tau}(t)]} dx + \langle W'(u_{h,\tau}(t)), \varphi_h \rangle = 0, \quad \forall \varphi_h \in X_{h,0},$$

$$(5.2) \quad \begin{aligned} u_{h,\tau}(a_i^e, t) &= g(a_i^e), \quad \nabla u_{h,\tau}(a_i^e, t) = \nabla_{\Gamma} g(a_i^e) + \tilde{g}(a_i^e) \nu(a_i^e), \\ \frac{\partial u_{h,\tau}}{\partial \nu}(b^e, t) &= \tilde{g}(b^e), \quad \forall e \in \Gamma, i = 1, 2. \end{aligned}$$

Remark 5.1. The fully discrete scheme (5.1)–(5.2) can be derived in the following way from a known time discretization scheme in the literature. At first, we write the semidiscrete problem (3.11)–(3.13) in an equivalent way as a system of ordinary differential equations for the time-dependent coefficients $u_j(t)$ in the basis representation $u_h(\cdot, t) = \sum_j u_j(t) \varphi_j(\cdot)$ of the semidiscrete solution $u_h(\cdot, t) \in X_h$. Then, we apply to this system the collocation method based on the integration points of the 2-point Gauß-Radau quadrature formula (see [15, chap. II.7, def. 7.6]) and reformulate the scheme back into the form (5.1)–(5.2). The consistency error of the method is of the order $\mathcal{O}(\tau^3)$ where $\tau := \max_{1 \leq n \leq N} \tau_n$. The scheme is A-stable and even L-stable, i.e., high frequency errors are damped strongly. It can be shown [18, Lemma 7, Thm. 10] that the scheme (5.1)–(5.2) generates the

same values $u_{h,\tau}(t_{n,k}) \in X_h$, $k = 1, 2$, at the Gauß-Radau points $t_{n,1}$ and $t_{n,2} = t_n$ as the well-known discontinuous Galerkin method with piecewise linear ansatz in time (dG(1)-method, [26]).

Remark 5.2. The scheme (5.1)-(5.2) has the nice property that the Willmore energy $t \mapsto W(u_{h,\tau}(t))$ as a function in time is strictly monotonic decreasing in the neighborhood of each Gauß-Radau point $t_{n,k}$, $k = 1, 2$, unless $u_{h,\tau}(t_{n,k})$ is a stationary point of the Willmore functional in a discrete sense. This can be seen as follows. In (5.1) we choose $t = t_{n,k}$, $k = 1, 2$, and $\varphi_h = \partial_t u_{h,\tau}(t_{n,k}) \in X_{h0}$ (since the boundary data is constant in time) and obtain

$$\frac{d}{dt} \left(W(u_{h,\tau}(t)) \right) \Big|_{t=t_{n,k}} = \langle W'(u_{h,\tau}(t_{n,k})), \partial_t u_{h,\tau}(t_{n,k}) \rangle = - \int_{\Omega} \frac{(\partial_t u_{h,\tau}(t_{n,k}))^2}{Q[u_{h,\tau}(t_{n,k})]} dx.$$

As long as $\partial_t u_{h,\tau}(t_{n,k}) \neq 0$ the right-hand side is negative, saying that the Willmore energy is strongly monotonic decreasing in the neighborhood of $t_{n,k}$. In the case when $\partial_t u_{h,\tau}(t_{n,k}) = 0$, it follows from (5.1) that $W'(u_{h,\tau}(t_{n,k})) = 0$ in the dual space X'_{h0} , i.e., that $u_{h,\tau}(t_{n,k})$ is a stationary point of the Willmore functional in a discrete sense.

The local problem (5.1)-(5.2) on interval I_n is equivalent to a coupled system for the two finite element functions $u_{hn}^j := u_{h,\tau}(t_{n,j}) \in X_h$, $j = 1, 2$. To see this, we introduce the notation $t_{n,0} := t_{n-1}$, $u_{hn}^0 := u_{h,\tau}(t_{n,0}) \in X_h$ and write $u_{h,\tau}|_{I_n} \in \mathbb{P}_2(I_n, X_h)$ in the basis representation for all $t \in I_n$, i.e.,

$$(5.3) \quad u_{h,\tau}(t) = \sum_{j=0}^2 u_{hn}^j \psi_{n,j}(t) \quad \text{with} \quad \psi_{n,j}(t_{n,i}) = \delta_{i,j}, \quad \forall i, j \in \{0, 1, 2\},$$

where $\delta_{i,j}$ denotes the Kronecker symbol and the $\psi_{n,j}$, $j = 0, 1, 2$, form a Lagrange basis of the polynomial space $\mathbb{P}_2(I_n, \mathbb{R})$. Then, the continuity condition $u_{h,\tau}(t_{n-1}^+) = u_{h,\tau}(t_{n-1})$ is already satisfied due to this representation with the known value $u_{hn}^0 \in X_h$. If we differentiate the representation (5.3) with respect to t and introduce the constants $\hat{\alpha}_{k,j} := \tau_n \psi'_{n,j}(t_{n,k})$ for $k = 1, 2$, $j = 0, 1, 2$, we get the following problem for the remaining coefficients $u_{hn}^1, u_{hn}^2 \in X_h$ in the representation (5.3) of the fully discrete solution $u_{h,\tau}$ on interval I_n :

Find $u_{hn}^1, u_{hn}^2 \in X_h$ such that with $u_{hn}^0 = u_{h,\tau}(t_{n-1}) \in X_h$ there holds for $k = 1, 2$,

$$(5.4) \quad \sum_{j=0}^2 \frac{\hat{\alpha}_{k,j}}{\tau_n} \int_{\Omega} \frac{u_{hn}^j \varphi_h}{Q[u_{hn}^k]} dx + \langle W'(u_{hn}^k), \varphi_h \rangle = 0, \quad \forall \varphi_h \in X_{h0},$$

$$(5.5) \quad \begin{aligned} u_{hn}^k(a_i^e) &= g(a_i^e), \quad \nabla u_{hn}^k(a_i^e) = \nabla_{\Gamma} g(a_i^e) + \tilde{g}(a_i^e) \nu(a_i^e), \\ \frac{\partial u_{hn}^k}{\partial \nu}(b^e) &= \tilde{g}(b^e), \quad \forall e \in \Gamma, \quad i = 1, 2. \end{aligned}$$

For the constants $\hat{\alpha}_{k,j}$, we obtain

$$(\hat{\alpha}_{k,j}) = \begin{pmatrix} -2 & 3/2 & 1/2 \\ 2 & -9/2 & 5/2 \end{pmatrix}, \quad k = 1, 2, \quad j = 0, 1, 2.$$

FIGURE 2. Meshes for grid levels $\ell = 1, 2, 3$.

6. NUMERICAL RESULTS

In this section, we will apply our fully discrete scheme (5.1)-(5.2) at first to a test example where the exact solution is known such that we can compute the discretization error $u - u_{h,\tau}$ in several norms. In order to verify the order of the error $u - u_h$ of the semidiscretization, we choose the time step size sufficiently small such that the error $u_{h,\tau} - u_h$ of the discretization error in time can be neglected.

For the construction of our test problem, we prescribe an exact height function $u : \bar{\Omega} \times [0, T] \rightarrow \mathbb{R}$ with $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ in the form

$$u(x, y, t) = \sqrt{R^2(t) - x^2 - y^2} - \sqrt{R^2(t) - 1}, \quad \text{where} \quad R(t) = 2 - 0.7 \sin(2\pi t),$$

which represents part of an evolving sphere with midpoint $(0, 0, -\sqrt{R^2(t) - 1})$ and radius $R(t) > 1$. The prescribed solution u satisfies the generalized variational formulation

$$(6.1) \quad \int_{\Omega} \frac{u_t \varphi}{Q[u]} dx + \langle W'(u), \varphi \rangle = \int_{\Omega} r \varphi dx, \quad \text{for all } \varphi \in H_0^2(\Omega), 0 < t \leq T,$$

which differs from (2.19) by the nonhomogeneous right hand side with the data

$$r(x, y, t) := R'(t) \left(1 - \sqrt{\frac{R^2(t) - x^2 - y^2}{R^2(t) - 1}} \right).$$

The Dirichlet boundary data g and $\tilde{g}$ in (1.4) as well as the initial data u_0 in (1.5) are defined in such a way that they fit to the prescribed solution u . The semi-discrete problem (3.11)-(3.13) for $u_h(\cdot, t) \in X_h$ as well as the fully discrete problem (5.1)-(5.2) for $u_{h,\tau} \in X_{h,\tau}$ are modified according to (6.1) by changing the right-hand side of (3.11) and (5.1), respectively, into $\int_{\Omega} r(\cdot, t) \varphi_h dx$.

The mesh $\mathcal{T}_{h_\ell}$ for a grid level $\ell \geq 1$ is generated by means of $\ell - 1$ uniform refinement steps applied recursively to a coarsest mesh $\mathcal{T}_{h_1}$. The cells of the next finer mesh level $k + 1$ are created by subdividing each cell of grid level k into four new cells using the edge midpoints and an interior point as new vertices. Figure 2 shows the used grids for the levels $\ell = 1, 2, 3$.

In order to solve the fully discrete problem (5.1)-(5.2) for $u_{h,\tau}$ on a subinterval I_n , we use the representation (5.3) and solve the coupled problem (5.4)-(5.5) for the unknown finite element functions $u_{hn}^1, u_{hn}^2 \in X_h$. To this end, we represent $u_{hn}^k \in X_h$ by means of the basis functions φ_j of X_h and a nodal vector $U^{n,k} \in \mathbb{R}^d$, $d = \dim(X_h)$, as

$$u_{hn}^k(x) = \sum_{j=1}^d U_j^{n,k} \varphi_j(x), \quad \forall x \in \Omega,$$

where we assume that the node numbers $j = \tilde{d} + 1, \dots, d$ are associated with the boundary nodes. To each nodal vector $U^{n,k} \in \mathbb{R}^d$ we assign the subvector

$\tilde{U}^{n,k} \in \mathbb{R}^{\tilde{d}}$ of the “free” (interior) nodal values and the subvector $U^{\Gamma,n,k} \in \mathbb{R}^{d-\tilde{d}}$ of the boundary nodal values, i.e.,

$$U^{n,k} = \begin{bmatrix} \tilde{U}^{n,k} \\ U^{\Gamma,n,k} \end{bmatrix} \quad \text{where} \quad \begin{cases} \tilde{U}_j^{n,k} := U_j^{n,k}, & j = 1, \dots, \tilde{d}, \\ U_j^{\Gamma,n,k} := U_{\tilde{d}+j}^{n,k}, & j = 1, \dots, d - \tilde{d}. \end{cases}$$

Since $U^{n,k}$, $k = 1, 2$, is the nodal vector of the finite element approximation $u_{hn}^k = u_{h,\tau}(\cdot, t_{n,k}) \in X_h$ at time $t = t_{n,k}$, the boundary nodal values $U_j^{\Gamma,n,k}$ are known from the boundary conditions (5.5) where the boundary data g and $\tilde{g}$ depend on the time $t = t_{n,k}$ for our academic test problem with the prescribed time-dependent solution.

Now, the coupled problem (5.4)–(5.5) for the unknown finite element functions $u_{hn}^1, u_{hn}^2 \in X_h$ on interval I_n is equivalent to a nonlinear coupled system of total size $2\tilde{d}$ for the two block-vectors $\tilde{U}^{n,1}$ and $\tilde{U}^{n,2}$ related to the interior nodal values at time $t_{n,1}$ and $t_{n,2} = t_n$, respectively. In our computations, we have solved these nonlinear systems by means of a fixed-point iteration with Anderson acceleration [27]. As the starting vector for $\tilde{U}^{n,1}$ and $\tilde{U}^{n,2}$ we used, in the case $n > 1$, the computed solution vector $\tilde{U}^{n-1,2}$ from the previous time interval I_{n-1} and, in the case $n = 1$, the vector $\tilde{U}^0 \in \mathbb{R}^{\tilde{d}}$ of the interior nodal values of the discrete initial value $u_0^h \in X_h$. For a sufficiently small time step size τ , the convergence of this iteration has been relatively fast, i.e., in most cases, 3 iterations have been sufficient to fulfill the stopping criterion that the maximum norm of the residual is less than 10^{-10} and the reduction of the residual of the starting vector in the maximum norm is less than $10^{-3}\tau$.

In the following, we give some heuristic arguments for the choice of the time step size τ in dependence of the mesh size h which should guarantee that the error part of the fully discrete solution caused by the time discretization does not dominate the error due to the spatial discretization. It is known that the accuracy of our time discretization scheme based on collocation at the integration points of the 2-point Gauß-Radau quadrature formula is of order τ^3 at the discrete time points t_n , i.e.,

$$(6.2) \quad \max_{0 \leq n \leq N} \|u_h(\cdot, t_n) - u_{h,\tau}(\cdot, t_n)\|_{L^2(\Omega)} \leq C_1 \tau^3,$$

where we assume here that the constant C_1 is independent of h . Applying an inverse inequality, Theorem 3.3 suggests the following estimate for the full discretization error $e_{h,\tau} := u - u_{h,\tau}$ in the norms $\|\cdot\|_i$, $i = 0, 1, 2$:

$$(6.3) \quad \|e_{h,\tau}\|_i := \max_{0 \leq n \leq N} \|u(\cdot, t_n) - u_{h,\tau}(\cdot, t_n)\|_{H^i(\Omega)} \leq \left(C_1 \tau^3 + C_2 h^4 |\log h|^{1/2} \right) h^{-i}.$$

In order to investigate numerically the order of the space discretization error $e_h := u - u_h$ with respect to the mesh-size h , we choose in our numerical experiment the time step size τ in such a way that the time discretization error in (6.2) is bounded by Ch^4 . Then, the norms of the full discretization error $\|e_{h,\tau}\|_i$ for a sequence of refined meshes show the order of the space discretization error $\|e_h\|_i$. Therefore, we choose τ asymptotically like $h^{4/3}$, i.e., for the mesh on grid level ℓ with a mesh-size $h_\ell \approx h_1 2^{-(\ell-1)}$, we choose the corresponding time step size τ_ℓ as

$$\tau_\ell := \frac{1}{100} 2^{-\frac{4(\ell-1)}{3}}.$$

Table 1 shows, for the above described test example with $T = 0.5$, the error norms $\|e_{h,\tau}\|_i$ of the full discretization error at different grid levels ℓ and the corresponding

TABLE 1. Error norms $\|e_{h,\tau}\|_i$ for the test problem with known analytical solution and experimental orders of convergence (EOC).

ℓ	h	$\ e_{h,\tau}\ _0$	EOC	$\ e_{h,\tau}\ _1$	EOC	$\ e_{h,\tau}\ _2$	EOC
1	$7.709 e - 1$	$8.842 e - 3$		$2.301 e - 2$		$2.736 e - 1$	
2	$4.463 e - 1$	$9.246 e - 4$	4.130	$4.244 e - 3$	3.092	$9.911 e - 2$	1.858
3	$2.401 e - 1$	$8.082 e - 5$	3.931	$6.949 e - 4$	2.918	$3.158 e - 2$	1.845
4	$1.233 e - 1$	$5.766 e - 6$	3.962	$9.753 e - 5$	2.947	$8.802 e - 3$	1.917
5	$6.269 e - 2$	$3.708 e - 7$	4.058	$1.270 e - 5$	3.014	$2.287 e - 3$	1.993

experimental orders of convergence (EOC). The results confirm the error estimates of Theorem 3.3.

Finally, we use the discrete flow (3.11)-(3.13) in order to investigate numerically whether multiple graph solutions of the (stationary) Willmore Dirichlet boundary value problem

$$(6.4) \quad \Delta_{\Gamma} H + \frac{1}{2} H^3 - 2HK = 0 \text{ in } \Omega, \quad u = g, \quad \frac{\partial u}{\partial \nu} = \tilde{g} \text{ on } \Gamma$$

exist. Consider $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ together with the Dirichlet boundary data

$$g(\cos(\varphi), \sin(\varphi)) = \lambda \cos(4\varphi), \quad \tilde{g}(\cos(\varphi), \sin(\varphi)) = 0, \quad \varphi \in [0, 2\pi).$$

Here, $\lambda > 0$ plays the role of a bifurcation parameter. By linearizing around $u \equiv 0$ it is not difficult to see that (6.4) has a unique solution provided that λ is sufficiently small. Starting from the initial values $u_0^{(1)}, u_0^{(2)}$ with

$$\begin{aligned} u_0^{(1)}(x, y) &= \lambda \left(1 - 8x^2y^2(2 - x^2 - y^2)^2 + \frac{1}{2} (1 - \cos(2\pi(x^2 + y^2))) \right), \\ u_0^{(2)}(x, y) &= -u_0^{(1)} \left(\frac{x-y}{\sqrt{2}}, \frac{x+y}{\sqrt{2}} \right) \end{aligned}$$

we examined whether the corresponding discrete solutions of the Willmore flow approached a stationary state for large times. For small λ up to $\lambda = \frac{1}{4}$ we observed convergence towards the same stationary solution, while for $\lambda = \frac{1}{2}$ the flow evolved to two different stationary states. Figure 3 shows the graphs of these solutions on grid level 5 where the nodal vectors $U^{n,k}$ consist of $\bar{d} = 15107$ free nodes and 512 given boundary nodes. In Figure 4, we present the behaviour of the Willmore energy over time for the discrete solution $u_h^{(1)}(t)$ which is nearly identical to that of $u_h^{(2)}(t)$.

The numerical results suggest the possibility of a bifurcation as λ increases from 0 to $\frac{1}{2}$.

FIGURE 3. The different stationary solutions corresponding to the initial values $u_0^{(1)}$ (left) and $u_0^{(2)}$ (right). The color indicates the size of the mean curvature.

FIGURE 4. Willmore energy over the time for the discrete solution $u_h^{(1)}(t)$.

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