# ZEROS OF THE DILOGARITHM 

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#### Abstract

We show that the dilogarithm has at most one zero on each branch, that each zero is close to a root of unity, and that they may be found to any precision with Newton's method. This work is motivated by applications to the asymptotics of coefficients in partial fraction decompositions considered by Rademacher. We also survey what is known about zeros of polylogarithms in general.


## 1. Introduction

In the recent resolution of an old conjecture of Rademacher, described below in Section 1.2, the location of a particular zero, $w_{0}$, of the dilogarithm played an important role. It has been known since LR00 that the only zero of the dilogarithm on its principal branch is at 0 . The zero $w_{0}$ is on the next branch. Zeros on further branches were also needed in O'Sa, and in this paper we locate all zeros on every branch.

The dilogarithm is initially defined as

$$
\begin{equation*}
\operatorname{Li}_{2}(z):=\sum_{n=1}^{\infty} \frac{z^{n}}{n^{2}} \quad \text { for }|z| \leqslant 1 \tag{1.1}
\end{equation*}
$$

(see for example Max03, Zag07), with an analytic continuation given by

$$
\begin{equation*}
-\int_{0}^{z} \log (1-u) \frac{d u}{u} \tag{1.2}
\end{equation*}
$$

The principal branch of the logarithm has $-\pi<\arg z \leqslant \pi$ with a branch cut $(-\infty, 0]$. From (1.2), the corresponding principal branch of the dilogarithm has branch points at $1, \infty$ and branch cut $[1, \infty)$. Crossing this branch cut from below, it is easy to show that the dilogarithm increases by $2 \pi i \log (z)$ over its principal value. On this new sheet there is now an additional branch point at 0 coming from the logarithm. In general, the dilogarithm is a multi-valued holomorphic function with branch points at $1, \infty$ and off the principal branch another branch point at 0 . For clarity, the notation $\operatorname{Li}_{2}(z)$ and $\log (z)$ will mean the principal branches of these functions from this point. We also note that $\mathrm{Li}_{2}(z)$ may be expressed as a ${ }_{3} F_{2}$ hypergeometric function (see AAR99, Sect. 2.6]), with $\operatorname{Li}_{2}(z) / z={ }_{3} F_{2}(1,1,1 ; 2,2 ; z)$.

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1.1. Main results. On any branch it follows (see for example LR00 Max03), that the dilogarithm must take the form

$$
\phi_{A, B}(z):=\operatorname{Li}_{2}(z)+4 \pi^{2} A+2 \pi i B \log (z) \quad(A, B \in \mathbb{Z}, z \in \mathbb{C})
$$

and so we want to know when $\phi_{A, B}(z)=0$.
Theorem 1.1. For $A, B \in \mathbb{Z}$, the function $\phi_{A, B}(z)$ has no zeros in $\mathbb{C}$ unless
(i) $B=0$ and $A \geqslant 0$ or
(ii) $-|B| / 2<A \leqslant|B| / 2$.

If (i) or (ii) holds, then $\phi_{A, B}(z)$ has exactly one zero and it is simple.
On the principal branch we will see, as noted earlier, that $\operatorname{Li}_{2}(z)=\phi_{0,0}(z)$ has just the zero at $z=0$.

Theorem 1.2. For $A \in \mathbb{Z}_{\geqslant 1}$, if $\phi_{A, 0}(\rho)=0$, then

$$
\rho=-\exp (\pi \sqrt{8 A-1 / 3})+O(1 / \sqrt{A})
$$

With the initial value $-\exp (\pi \sqrt{8 A-1 / 3})$, Newton's method applied to $\phi_{A, 0}$ produces a sequence converging quadratically to $\rho$.

Note that by conjugation, $\phi_{A, B}(\rho)=0$ if and only if $\phi_{A,-B}(\bar{\rho})=0$. So for $B \neq 0$ we may assume $B \geqslant 1$ without loss of generality. The second Bernoulli polynomial is defined as $B_{2}(x):=x^{2}-x+1 / 6$ and Clausen's integral is

$$
\begin{equation*}
\mathrm{Cl}_{2}(\theta):=-\int_{0}^{\theta} \log |2 \sin (x / 2)| d x \tag{1.3}
\end{equation*}
$$

both shown in Figure 1
Theorem 1.3. Let $A$ and $B$ be integers satisfying $-B / 2<A \leqslant B / 2$ and suppose $\phi_{A, B}(\rho)=0$. The sequence $c_{0}, c_{1}, \cdots$ defined as

$$
c_{0}:=\left\{\begin{array}{ll}
\exp (2 \pi i A / B) & \text { if } A \neq 0  \tag{1.4}\\
\exp (\pi i /(12 B)) & \text { if } A=0,
\end{array} \quad c_{n+1}:=c_{n}-\frac{\phi_{A, B}\left(c_{n}\right)}{\phi_{A, B}^{\prime}\left(c_{n}\right)} \quad \text { for } \quad n \geqslant 0,\right.
$$

converges quadratically to $\rho$. Using $c_{1}$, we have

$$
\begin{equation*}
\rho=e^{2 \pi i A / B}\left(1+\frac{-\mathrm{Cl}_{2}(2 \pi A / B)+i \pi^{2} B_{2}(|A| / B)}{2 \pi B}\right)+O\left(\frac{1+\log B}{B^{2}}\right) \tag{1.5}
\end{equation*}
$$

for an absolute implied constant.
The above three theorems are proved in Sections 3-7 after some preliminary results are reviewed in Section 2. The dilogarithm is the case $s=2$ of the polylogarithm $\operatorname{Li}_{s}(z)$. We put our results in context in Section 8 by describing what is known about the zeros of $\mathrm{Li}_{s}(z)$ for a general fixed $s \in \mathbb{C}$. It turns out that for $\operatorname{Re}(s) \leqslant 0$ very much is known, at least for zeros on the principal branch, due to work of Le Roy, Frobenius, Reisz, Peyerimhoff, Sobolev, Gawronski and Stadtmüller among others.
1.2. Rademacher's conjecture. We briefly describe here the work that led to our study of dilogarithm zeros. Rademacher conjectured in Rad73, p. 302] that the coefficients $C_{h k \ell}(N)$ in the partial fraction decomposition

$$
\begin{equation*}
\prod_{j=1}^{N} \frac{1}{1-q^{j}}=\frac{C_{011}(N)}{q-1}+\cdots+\frac{C_{h k \ell}(N)}{\left(q-e^{2 \pi i h / k}\right)^{\ell}}+\cdots+\frac{C_{01 N}(N)}{(q-1)^{N}} \tag{1.6}
\end{equation*}
$$

converge as $N \rightarrow \infty$ to the corresponding partial fraction coefficients of the infinite product $\prod_{j=1}^{\infty} 1 /\left(1-q^{j}\right)$. Rademacher had previously given a partial fraction decomposition of this infinite product using his famous exact formula for the partition function $p(n)$, of which it is the generating function; see [Rad73, pp. 292-302]. Of course, $\prod_{j=1}^{N} 1 /\left(1-q^{j}\right)$ is the generating function for $p_{N}(n)$, the number of partitions of $n$ into at most $N$ parts.

Sills and Zeilberger in SZ13] obtained numerical evidence that the conjecture was not correct, and in O'S15 the true asymptotic behavior of $C_{011}(N)$ was conjectured to be

$$
\begin{equation*}
C_{011}(N)=\operatorname{Re}\left[\left(-2 z_{0} e^{-\pi i z_{0}}\right) \frac{w_{0}^{-N}}{N^{2}}\right]+O\left(\frac{\left|w_{0}\right|^{-N}}{N^{3}}\right) \tag{1.7}
\end{equation*}
$$

for $w_{0}$ the dilogarithm zero satisfying $\phi_{0,-1}\left(w_{0}\right)=0$ and $z_{0}$ given by $w_{0}=1-e^{2 \pi i z_{0}}$ for $1 / 2<\operatorname{Re}\left(z_{0}\right)<3 / 2$. With Theorem 1.3 we find

$$
\begin{aligned}
w_{0} & \approx 0.91619781620686260140-0.18245889720714117505 i \\
z_{0} & \approx 1.18147496973270876764+0.25552764641754743773 i
\end{aligned}
$$

Then $\left|w_{0}\right|<1$ (see the conjugate of $w_{0}$ in Figure 3), and (1.7) implies that $C_{011}(N)$ oscillates with exponentially growing amplitude and diverges. Slightly weaker forms of (1.7), enough to disprove Rademacher's conjecture, were independently shown in DG14 and O'Sa with different proofs, but both employing the saddle-point method. The formulation of the main term in (1.7) is a little different in [DG14.

The proof in O'Sa is based on breaking up $C_{011}(N)$ into manageable components that are similar to Sylvester waves. Let $w(A, B)$ denote the zero of $\phi_{A, B}$ when it exists. The appearance of the dilogarithm zero $w_{0}=w(0,-1)$ in $\mathrm{O}^{\prime} \mathrm{Sa}$, as well as $w(0,-2)$ and $w(1,-3)$, comes from estimating the following sums, (1.8), (1.9) and (1.10), which correspond to some of the largest components of $C_{011}(N)$. Set

$$
Q_{h k 1}(N):=2 \pi i \operatorname{Res}_{z=h / k} \frac{e^{2 \pi i z}}{\left(1-e^{2 \pi i 1 z}\right)\left(1-e^{2 \pi i 2 z}\right) \cdots\left(1-e^{2 \pi i N z}\right)}
$$

and

$$
z_{1}:=2+\log (1-w(0,-2)) /(2 \pi i), \quad z_{3}:=3+\log (1-w(1,-3)) /(2 \pi i)
$$

Then it is proved in $\mathrm{O}^{\prime} \mathrm{Sa}, \mathrm{O}^{\prime} \mathrm{Sb}$ that
(1.8) $2 \operatorname{Re} \sum_{N / 2<k \leqslant N} Q_{1 k 1}(N)=\operatorname{Re}\left[2 z_{0} e^{-\pi i z_{0}} \frac{w(0,-1)^{-N}}{N^{2}}\right]+O\left(\frac{|w(0,-1)|^{-N}}{N^{3}}\right)$,

$$
\begin{equation*}
2 \operatorname{Re} \sum_{N / 3<k \leqslant N / 2} Q_{1 k 1}(N)=\operatorname{Re}\left[-\frac{3 z_{1}}{2} e^{-\pi i z_{1}} \frac{w(0,-2)^{-N}}{N^{2}}\right]+O\left(\frac{|w(0,-2)|^{-N}}{N^{3}}\right) \tag{1.9}
\end{equation*}
$$

$2 \operatorname{Re} \sum_{N / 2<k \leqslant N, k \text { odd }} Q_{2 k 1}(N)=\operatorname{Re}\left[-\frac{z_{3}}{4} e^{-\pi i z_{3}} \frac{w(1,-3)^{-N}}{N^{2}}\right]+O\left(\frac{|w(1,-3)|^{-N}}{N^{3}}\right)$.
In fact these are special cases of the results in $\mathrm{O}^{\prime} \mathrm{Sa}, \mathrm{O}^{\prime} \mathrm{Sb}$. See Sections 1 and 6 of ['Sa for a more detailed account. In forthcoming work we also show that $w_{0}=w(0,-1)$ similarly controls the asymptotics of the first Sylvester waves.

## 2. Some properties of the dilogarithm

We have seen that $\phi_{A, B}(z)$ is defined as a single-valued function on $\mathbb{C}$. Away from the cuts $(-\infty, 0]$ and $[1, \infty)$ it is holomorphic with

$$
\begin{equation*}
z \phi_{A, B}^{\prime}(z)=-\log (1-z)+2 \pi i B \quad \text { for } \quad z \notin(-\infty, 0] \cup[1, \infty) \tag{2.1}
\end{equation*}
$$

We will also use (2.1) for $z \in \mathbb{C}-[1, \infty)$ when $B=0$. For $x \in(-\infty, 0)$, as usual,

$$
\begin{equation*}
\lim _{y \rightarrow 0^{+}} \log (x+i y)=\log (x), \quad \lim _{y \rightarrow 0^{-}} \log (x+i y)=\log (x)-2 \pi i \tag{2.2}
\end{equation*}
$$

Similarly, for $x \in[1, \infty)$,

$$
\begin{equation*}
\lim _{y \rightarrow 0^{+}} \operatorname{Li}_{2}(x+i y)=\operatorname{Li}_{2}(x)+2 \pi i \log (x), \quad \lim _{y \rightarrow 0^{-}} \operatorname{Li}_{2}(x+i y)=\operatorname{Li}_{2}(x) \tag{2.3}
\end{equation*}
$$

Thus we see that if $z$ makes a full rotation in the positive direction about 0 , then $\phi_{A, B} \rightarrow \phi_{A+B, B}$. A full rotation in the negative direction about 1 means $\phi_{A, B} \rightarrow$ $\phi_{A, B+1}$. We have the natural matrix representations

$$
\begin{align*}
& \left(\begin{array}{l}
A \\
B \\
1
\end{array}\right) \longrightarrow\left(\begin{array}{lll}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right)\left(\begin{array}{l}
A \\
B \\
1
\end{array}\right)=\left(\begin{array}{c}
A+B \\
B \\
1
\end{array}\right)  \tag{2.4}\\
& \left(\begin{array}{l}
A \\
B \\
1
\end{array}\right) \longrightarrow\left(\begin{array}{lll}
1 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{array}\right)\left(\begin{array}{l}
A \\
B \\
1
\end{array}\right)=\left(\begin{array}{c}
A \\
B+1 \\
1
\end{array}\right) \tag{2.5}
\end{align*}
$$

and the $3 \times 3$ matrices in (2.4), (2.5) generate the well-known monodromy group of the dilogarithm. This is the non-abelian Heisenberg group

$$
H_{3}(\mathbb{Z}):=\left\{\left.\left(\begin{array}{ccc}
1 & x & z \\
0 & 1 & y \\
0 & 0 & 1
\end{array}\right) \right\rvert\, x, y, z \in \mathbb{Z}\right\}
$$

as described in Vep08, p. 244], for example.

For all $z \in \mathbb{C}$, except for the given restrictions, the dilogarithm satisfies the functional equations

$$
\begin{align*}
\mathrm{Li}_{2}(1 / z) & =-\mathrm{Li}_{2}(z)-\operatorname{Li}_{2}(1)-\frac{1}{2} \log ^{2}(-z), & & z \notin[0,1),  \tag{2.6}\\
\mathrm{Li}_{2}(1-z) & =-\operatorname{Li}_{2}(z)+\mathrm{Li}_{2}(1)-\log (z) \log (1-z), & & z \neq 0,1 . \tag{2.7}
\end{align*}
$$

Note that $\mathrm{Li}_{2}(1)=\zeta(2)=\pi^{2} / 6$. Equations (2.6) and (2.7) are shown in Max03, Sect. 3], for example, for $z \notin[0, \infty)$ and $z \notin(-\infty, 0] \cup[1, \infty)$ respectively. Then use (2.2), (2.3) to obtain (2.6) and (2.7). For the excluded $z$ values, (2.6) becomes

$$
\operatorname{Li}_{2}(1 / x)-2 \pi i \log (x)=-\operatorname{Li}_{2}(x)-\operatorname{Li}_{2}(1)-\frac{1}{2} \log ^{2}(-x), \quad x \in(0,1)
$$

Recall $B_{2}(x)=x^{2}-x+1 / 6$ and $\mathrm{Cl}_{2}(\theta)$ from (1.3). As in [Max03, Sect. 8] we have

$$
\begin{array}{ll}
\operatorname{Re}\left(\operatorname{Li}_{2}\left(e^{2 \pi i x}\right)\right)=\sum_{n=1}^{\infty} \frac{\cos (2 \pi n x)}{n^{2}}=\pi^{2} B_{2}(x-\lfloor x\rfloor) & (x \in \mathbb{R}) \\
\operatorname{Im}\left(\operatorname{Li}_{2}\left(e^{2 \pi i x}\right)\right)=\sum_{n=1}^{\infty} \frac{\sin (2 \pi n x)}{n^{2}}=\mathrm{Cl}_{2}(2 \pi x) & (x \in \mathbb{R}) \tag{2.9}
\end{array}
$$



Figure 1. Real and imaginary parts of $\operatorname{Li}_{2}\left(e^{2 \pi i x}\right)$
The function $\mathrm{Cl}_{2}(\theta)$ is odd, has period $2 \pi$ and satisfies $2 \mathrm{Cl}_{2}(\theta)-2 \mathrm{Cl}_{2}(\pi-\theta)=$ $\mathrm{Cl}_{2}(2 \theta)$. Then

$$
\begin{equation*}
\mathrm{Cl}_{2}^{\prime}(\theta)=-\log |2 \sin (\theta / 2)|, \quad \mathrm{Cl}_{2}^{\prime \prime}(\theta)=-\frac{1}{2} \cot (\theta / 2) \tag{2.10}
\end{equation*}
$$

and $\left|\mathrm{Cl}_{2}(\theta)\right|$ has its maximum at $\theta=\pi / 3$, for example, with maximum value

$$
\begin{equation*}
\kappa:=\sum_{n=1}^{\infty} \frac{\sin (\pi n / 3)}{n^{2}} \approx 1.0149416 \tag{2.11}
\end{equation*}
$$

From [Rad73, (11.1)],

$$
\begin{equation*}
\cot (z)=\frac{1}{z}-\sum_{n=1}^{\infty} \frac{2^{2 n}\left|B_{2 n}\right|}{(2 n)!} z^{2 n-1} \quad(0<|z|<\pi) \tag{2.12}
\end{equation*}
$$

for $B_{2 n}$ the Bernoulli number. Integrating (2.12) twice we find

$$
\begin{equation*}
\mathrm{Cl}_{2}(\theta)=\theta-\theta \log |\theta|+\sum_{n=1}^{\infty} \frac{\left|B_{2 n}\right|}{2 n(2 n+1)!} \theta^{2 n+1} \quad(-2 \pi<\theta<2 \pi) \tag{2.13}
\end{equation*}
$$

and as in [O'Sa, Lemma 3.7] it follows from (2.13) that

$$
\begin{equation*}
\left|\mathrm{Cl}_{2}(\theta)\right| \leqslant|\theta|-|\theta| \log |\theta|+|\theta|^{3} / 54 \quad(-\pi<\theta<\pi) \tag{2.14}
\end{equation*}
$$

## 3. Zeros of $\phi_{A, 0}$

In this section we locate the zeros of $\phi_{A, 0}(z)=\operatorname{Li}_{2}(z)+4 \pi^{2} A$.

Proof. We first note that $\overline{\operatorname{Li}_{2}(\bar{z})}=\operatorname{Li}_{2}(z)$ is true for $|z| \leqslant 1$ by (1.1), so it must be true for all $z$ in $\mathbb{C}-[1, \infty)$ by analytic continuation. It follows that

$$
2 i \operatorname{Im} \operatorname{Li}_{2}(z)=\operatorname{Li}_{2}(z)-\overline{\operatorname{Li}_{2}(z)}=\operatorname{Li}_{2}(z)-\operatorname{Li}_{2}(\bar{z}) \quad(z \in \mathbb{C}-[1, \infty))
$$

and hence

$$
\begin{equation*}
\operatorname{Im~Li}_{2}(z)=0 \quad \text { for } z \in(-\infty, 1] \tag{3.1}
\end{equation*}
$$

With (3.1) and (2.6) we can see that

$$
\begin{equation*}
\operatorname{Im}_{\operatorname{Li}_{2}}(z)=-\pi \log z \neq 0 \quad \text { for } z \in(1, \infty) \tag{3.2}
\end{equation*}
$$

Write $z \in \mathbb{C}$ as $z=r e^{i \theta}$ for $r \geqslant 0$ and $-\pi<\theta \leqslant \pi$. From (1.2) we have that

$$
\frac{d}{d r} \operatorname{Li}_{2}\left(r e^{i \theta}\right)=-\frac{1}{r} \log \left(1-r e^{i \theta}\right)
$$

and so

$$
\begin{equation*}
\frac{d}{d r} \operatorname{Im} \operatorname{Li}_{2}\left(r e^{i \theta}\right)=-\operatorname{Im}\left(\frac{1}{r} \log \left(1-r e^{i \theta}\right)\right)=-\frac{1}{r} \arg \left(1-r e^{i \theta}\right) \tag{3.3}
\end{equation*}
$$

Hence $\operatorname{Im} \operatorname{Li}_{2}\left(r e^{i \theta}\right)$ is a strictly increasing function of $r$ for $0<\theta<\pi$ and strictly decreasing for $-\pi<\theta<0$. Since $\operatorname{Im} \operatorname{Li}_{2}(0)=0$ it follows that $\operatorname{Im~}_{2}(z) \neq 0$ for $z \in \mathbb{C}-\mathbb{R}$. Combining this with (3.1), (3.2) completes the proof.

Proposition 3.2. Let $A \in \mathbb{Z}$. Then $\phi_{A, 0}(z)$ has one zero if $A \geqslant 0$ and no zeros otherwise.

Proof. With Lemma 3.1 we see the only possible solutions to $\phi_{A, 0}(z)=0$ have $z=x \in(-\infty, 1]$. We know $\operatorname{Li}_{2}(x)$ is continuous on $(-\infty, 1]$ and it is real-valued for these $x$ by (3.1). We have

$$
\begin{equation*}
\frac{d}{d x} \operatorname{Li}_{2}(x)=-\frac{1}{x} \log (1-x)>0 \quad \text { for } \quad x \in(-\infty, 1) \tag{3.4}
\end{equation*}
$$

so that $\operatorname{Li}_{2}(x)$ is strictly increasing. If $\phi_{A, 0}(x)=0$, then

$$
\pi^{2} / 6=\operatorname{Li}_{2}(1) \geqslant \operatorname{Li}_{2}(x)=-4 \pi^{2} A
$$

implying that $\phi_{A, 0}$ has no zeros for $A<0$. For $A=0$ there is the necessarily unique zero at $x=0$.

Now we assume $A \geqslant 1$. Then

$$
\begin{equation*}
\phi_{A, 0}(-x)=0 \Longrightarrow-x \in(-\infty,-1) \quad \text { since } \quad \mathrm{Li}_{2}(-1)=-\pi^{2} / 12 \tag{3.5}
\end{equation*}
$$

From the functional equation (2.6) we see that

$$
\begin{equation*}
\mathrm{Li}_{2}(-x)=-\frac{\pi^{2}}{6}-\frac{1}{2} \log ^{2}(x)-\mathrm{Li}_{2}\left(-\frac{1}{x}\right) \quad(x>0) \tag{3.6}
\end{equation*}
$$

Note that

$$
\begin{equation*}
\left|\operatorname{Li}_{2}(z)\right| \leqslant \sum_{n=1}^{\infty} \frac{|z|^{n}}{n^{2}} \leqslant|z| \sum_{n=1}^{\infty} \frac{1}{n^{2}}=|z| \frac{\pi^{2}}{6} \quad(|z| \leqslant 1) \tag{3.7}
\end{equation*}
$$

and so (3.6) and (3.7) imply that, for $x \geqslant 1$,

$$
\begin{equation*}
\operatorname{Li}_{2}(-x)=-\frac{\pi^{2}}{6}-\frac{1}{2} \log ^{2}(x)+\varepsilon \tag{3.8}
\end{equation*}
$$

with $\varepsilon \in \mathbb{R}$ satisfying $|\varepsilon| \leqslant \pi^{2} /(6 x)$. Therefore $\operatorname{Li}_{2}(-x) \rightarrow-\infty$ as $-x \rightarrow-\infty$ and $\mathrm{Li}_{2}(-x)=-4 \pi^{2} A$ has a single solution, as required.

Proposition 3.3. For $A \in \mathbb{Z}_{\geqslant 1}$, if $\phi_{A, 0}(\rho)=0$, then

$$
|\rho+\exp (\pi \sqrt{8 A-1 / 3})|<1 / \sqrt{A}
$$

Proof. It follows from (3.8) that

$$
\begin{equation*}
\rho=-\exp \left(\pi \sqrt{8 A-1 / 3+\varepsilon^{\prime}}\right) \quad \text { for } \quad\left|\varepsilon^{\prime}\right|=\frac{2|\varepsilon|}{\pi^{2}} \leqslant \frac{1}{3|\rho|} . \tag{3.9}
\end{equation*}
$$

We have $-1 / 3 \leqslant \varepsilon^{\prime} \leqslant 1 / 3$ by (3.5) and therefore

$$
\rho \leqslant-\exp (\pi \sqrt{8 A-2 / 3})
$$

which in turn implies

$$
\left|\varepsilon^{\prime}\right| \leqslant \exp (-\pi \sqrt{8 A-2 / 3}) / 3
$$

Next write

$$
\exp \left(\pi \sqrt{8 A-1 / 3+\varepsilon^{\prime}}\right)=\exp \left(\pi \sqrt{8 A-1 / 3}(1+u)^{1 / 2}\right)
$$

with $u=\varepsilon^{\prime} /(8 A-1 / 3)$. If $u \in \mathbb{C}$ satisfies $|u| \leqslant 1 / 2$, say, we have the simple bounds

$$
\begin{align*}
(1+u)^{1 / 2}=1+w & \Longrightarrow \quad|w| \leqslant|u|  \tag{3.10}\\
\exp (u)=1+w \quad & \Longrightarrow \quad|w| \leqslant 2|u| . \tag{3.11}
\end{align*}
$$

Hence

$$
\begin{aligned}
\exp \left(\pi \sqrt{8 A-1 / 3}(1+u)^{1 / 2}\right) & =\exp (\pi \sqrt{8 A-1 / 3}(1+w)) \\
& =\exp (\pi \sqrt{8 A-1 / 3}) \exp (w \cdot \pi \sqrt{8 A-1 / 3}) \\
& =\exp (\pi \sqrt{8 A-1 / 3})\left(1+w^{\prime}\right)
\end{aligned}
$$

for $\left|w^{\prime}\right| \leqslant 2|w \cdot \pi \sqrt{8 A-1 / 3}| \leqslant 2|u \cdot \pi \sqrt{8 A-1 / 3}|$ using (3.10) and (3.11). Then the error has the bounds

$$
\begin{aligned}
\left|w^{\prime} \exp (\pi \sqrt{8 A-1 / 3})\right| & \leqslant \frac{2 \pi}{3 \sqrt{8 A-1 / 3}} \exp (\pi \sqrt{8 A-1 / 3}-\pi \sqrt{8 A-2 / 3}) \\
& \leqslant \frac{2 \pi}{3 \sqrt{8 A-1 / 3}} \exp (\pi \sqrt{8-1 / 3}-\pi \sqrt{8-2 / 3})<\frac{1}{\sqrt{A}}
\end{aligned}
$$

as required, where we used that $\sqrt{x+c}-\sqrt{x}$ is a decreasing function of $x$ for $x+c, x \geqslant 0$.

## 4. Newton's method for $\phi_{A, 0}$

Proposition 4.1. Let $z$ and $c$ be two points in $\mathbb{C}$ with $\operatorname{Re}(z), \operatorname{Re}(c) \leqslant 1-s$ for $s>0$. Then

$$
\left|\operatorname{Li}_{2}(z)-\operatorname{Li}_{2}(c)-(z-c) \operatorname{Li}_{2}^{\prime}(c)\right| \leqslant|z-c|^{2} \frac{1}{2 s}
$$

Proof. We need to bound the remainder term in the Taylor expansion of $\operatorname{Li}_{2}(z)$ at c,

$$
\operatorname{Li}_{2}(z)=\operatorname{Li}_{2}(c)+(z-c) \operatorname{Li}_{2}^{\prime}(c)+\frac{(z-c)^{2}}{2 \pi i} \int_{C} \frac{\operatorname{Li}_{2}(w)}{(w-c)^{2}(w-z)} d w
$$

where $C$ is the circular path of radius $T$ centered at 1 that avoids the branch cut $[1, \infty)$ as shown in Figure 2.


Figure 2. The path of integration $C$

The path $C$ makes a small circle of radius $\varepsilon$ about 1 that is connected to a path from $1+\varepsilon$ to $1+T$ just above the cut and a path from $1+T$ to $1+\varepsilon$ just below the cut. Letting the horizontal paths meet the branch cut, we find the values of $\operatorname{Li}_{2}(w)$ with (2.3). Use (2.6) and (2.7) to see that the growth of $\mathrm{Li}_{2}(z)$ is logarithmic on the circles of radius $T$ and $\varepsilon$ so that as $T \rightarrow \infty$ and $\varepsilon \rightarrow 0$ their contributions to $\int_{C}$ go to zero. Therefore

$$
\frac{1}{2 \pi i} \int_{C} \frac{\operatorname{Li}_{2}(w)}{(w-c)^{2}(w-z)} d w=\int_{0}^{\infty} \frac{\log (t+1)}{(t+1-c)^{2}(t+1-z)} d t
$$

Since $|t+1-c|,|t+1-z| \geqslant t+s$ and $|\log (t+1)| \leqslant t$, the proposition follows from

$$
\left|\int_{0}^{\infty} \frac{\log (t+1)}{(t+1-c)^{2}(t+1-z)} d t\right| \leqslant \int_{0}^{\infty} \frac{t}{(t+s)^{3}} d t=\frac{1}{2 s}
$$

Write $D_{A}:=\exp (\pi \sqrt{8 A-1 / 3})$ and $c^{*}:=c-\phi_{A, 0}(c) / \phi_{A, 0}^{\prime}(c)$.
Lemma 4.2. Let $A \in \mathbb{Z}_{\geqslant 1}$. Suppose $\rho$ and $c$ are in the interval $\left[-D_{A}-1 / \sqrt{A}\right.$, $\left.-D_{A}+1 / \sqrt{A}\right]$ and $\phi_{A, 0}(\rho)=0$. Then

$$
\begin{equation*}
\left|\rho-c^{*}\right|<\frac{|\rho-c|^{2}}{2 \pi \sqrt{A}} \tag{4.1}
\end{equation*}
$$

Proof. Let $z=\rho$ and $s=D_{A}$ in Proposition 4.1 to get

$$
\begin{equation*}
\left|\phi_{A, 0}(\rho)-\phi_{A, 0}(c)-(\rho-c) \phi_{A, 0}^{\prime}(c)\right| \leqslant|\rho-c|^{2} \frac{1}{2 D_{A}} . \tag{4.2}
\end{equation*}
$$

Now $c \cdot \phi_{A, 0}^{\prime}(c)=-\log (1-c)$ as in (2.1), and dividing (4.2) by $\phi_{A, 0}^{\prime}(c)$ finds

$$
\left|\rho-c^{*}\right| \leqslant|\rho-c|^{2} \frac{|c|}{2 D_{A}|\log (1-c)|} \leqslant|\rho-c|^{2} \frac{D_{A}+1 / \sqrt{A}}{2 D_{A} \cdot \pi \sqrt{8 A-1 / 3}}<\frac{|\rho-c|^{2}}{2 \pi \sqrt{A}} .
$$

Let $c_{0}:=-D_{A}=-\exp (\pi \sqrt{8 A-1 / 3})$ and $c_{n+1}:=c_{n}-\phi_{A, 0}(c) / \phi_{A, 0}^{\prime}(c)$ for $n \geqslant 0$.

Proposition 4.3. Suppose $A \in \mathbb{Z}_{\geqslant 1}$ and $\rho$ satisfies $\phi_{A, 0}(\rho)=0$. Then the sequence $c_{0}, c_{1}, \cdots$ above converges to $\rho$ with

$$
\left|\rho-c_{n}\right| \leqslant 2 \pi \sqrt{A} \cdot(2 \pi A)^{-2^{n}}
$$

Proof. With Proposition 3.3 we know that $\rho$ is in the interval $\left[-D_{A}-1 / \sqrt{A}\right.$, $\left.-D_{A}+1 / \sqrt{A}\right]$. Lemma 4.2 implies that for any $c \in\left[-D_{A}-1 / \sqrt{A},-D_{A}+1 / \sqrt{A}\right]$ we have $\left|\rho-c^{*}\right|<|\rho-c|$ so that $c^{*}$ remains in the interval. In this way, starting with $c_{0}$, we may keep applying Lemma 4.2 to each term in the sequence. We find

$$
\frac{\left|\rho-c_{n}\right|}{2 \pi \sqrt{A}}<\left(\frac{\left|\rho-c_{n-1}\right|}{2 \pi \sqrt{A}}\right)^{2}<\left(\frac{\left|\rho-c_{n-2}\right|}{2 \pi \sqrt{A}}\right)^{4}<\cdots<\left(\frac{\left|\rho-c_{0}\right|}{2 \pi \sqrt{A}}\right)^{2^{n}}
$$

and $\left|\rho-c_{0}\right| \leqslant 1 / \sqrt{A}$ completes the proof.
For example, with $A=1$ the sequence produced by Proposition 4.3 has initial value $c_{0} \approx-5994.97063$ and the next term $c_{1} \approx-5995.08558$. Taking more terms, we find $\phi_{1,0}(\rho)=0$ for $\rho \approx-5995.08558$, which is correct to the precision shown and already given by $c_{1}$.

With the results in Sections 3 and 4 we have proved Theorem 1.2 and the $B=0$ case of Theorem 1.1. We prove the simplicity of all zeros in Proposition 7.4.

## 5. Zeros of $\phi_{A, B}$

Lemma 5.1. For $A \in \mathbb{Z}, B \in \mathbb{Z}_{\geqslant 1}$ and $x \in \mathbb{R}$ we have $\phi_{A, B}(x) \neq 0$.
Proof. For $x>1$, the imaginary part of $\phi_{A, B}(x)$ equals $\pi(-1+2 B) \log x \neq 0$ by (3.2). For $0 \leqslant x \leqslant 1$ the imaginary part of $\phi_{A, B}(x)$ equals $2 \pi B \log x$ by (3.1). This is zero only if $x=1$, but then

$$
\phi_{A, B}(1)=\mathrm{Li}_{2}(1)+4 \pi^{2} A+2 \pi i B \log (1)=\pi^{2}(1 / 6+4 A) \neq 0 .
$$

For the remaining case of $x<0$ we have $\log x=\log |x|+\pi i$ so that the imaginary part of $\phi_{A, B}(x)$ equals $2 \pi B \log |x|$. This is zero only if $x=-1$, but then

$$
\phi_{A, B}(-1)=\operatorname{Li}_{2}(-1)+4 \pi^{2} A-2 \pi^{2} B=\pi^{2}(-1 / 12+4 A-2 B) \neq 0 .
$$

So we look for solutions to $\phi_{A, B}(z)=0$ with $z \in \mathbb{C}-\mathbb{R}$. The main idea to locate these zeros is to consider the vanishing of the real and imaginary parts of $\phi_{A, B}(z)$ separately. We will see that $\operatorname{Im} \phi_{A, B}(z)=0$ makes a curve near the unit circle and $\operatorname{Re} \phi_{A, B}(z)=0$ makes a curve close to the ray from the origin through $e^{2 \pi i A / B}$ when $A$ is small enough.

For $B \in \mathbb{Z} \geqslant 1$ fixed, write the imaginary part of

$$
\begin{equation*}
\phi_{A, B}\left(r e^{i \theta}\right)=\mathrm{Li}_{2}\left(r e^{i \theta}\right)+4 \pi^{2} A+2 \pi i B \log \left(r e^{i \theta}\right) \tag{5.1}
\end{equation*}
$$

as the function

$$
\begin{equation*}
I_{\theta}(r):=\operatorname{Im}_{L_{2}}\left(r e^{i \theta}\right)+2 \pi B \log r . \tag{5.2}
\end{equation*}
$$

As in (3.3),

$$
\begin{align*}
& \frac{\partial I}{\partial r}=\frac{1}{r}\left(2 \pi B-\arg \left(1-r e^{i \theta}\right)\right)  \tag{5.3}\\
& \frac{\partial I}{\partial \theta}=-\log \left|1-r e^{i \theta}\right|
\end{align*}
$$

Lemma 5.2. Fix $B \in \mathbb{Z}_{\geqslant 1}$. For each $\theta$ with $0<|\theta|<\pi$ there exists a unique $r>0$ so that $I_{\theta}(r)=0$.

Proof. With (5.3) we have $\frac{\partial I}{\partial r}>0$ so that $I_{\theta}$ is a strictly increasing function of $r$. It is easy to see that $\lim _{r \rightarrow 0} I_{\theta}(r)=-\infty$. With (2.6),

$$
\begin{aligned}
\operatorname{Li}_{2}\left(r e^{i \theta}\right) & =-\operatorname{Li}_{2}\left(\frac{1}{r} e^{-i \theta}\right)-\frac{\pi^{2}}{6}-\frac{1}{2} \log ^{2}\left(\frac{1}{r} e^{i(\pi-\theta)}\right) \\
& =-\operatorname{Li}_{2}\left(\frac{1}{r} e^{-i \theta}\right)-\frac{\pi^{2}}{6}-\frac{1}{2}(-\log r+i(\operatorname{sgn} \theta \cdot \pi-\theta))^{2}
\end{aligned}
$$

and hence

$$
\begin{equation*}
I_{\theta}(r)=-\operatorname{Im} \operatorname{Li}_{2}\left(\frac{1}{r} e^{-i \theta}\right)+(2 \pi B+\operatorname{sgn} \theta \cdot \pi-\theta) \log r \tag{5.5}
\end{equation*}
$$

It follows that $\lim _{r \rightarrow \infty} I_{\theta}(r)=\infty$ and so exactly one $r$ makes $I_{\theta}(r)=0$.


Figure 3. The polar rectangle $\mathcal{R}_{0,1}$ containing the zero $w(0,1) \approx$ $0.916+0.182 i$

We may therefore define a function $g(\theta)=g_{B}(\theta)$, with domain $(-\pi, 0) \cup(0, \pi)$, equalling the unique $r$ such that $I_{\theta}(r)=0$. This lets us parameterize the curve where $\operatorname{Im} \phi_{A, B}(z)=0$ for $z \in \mathbb{C}-\mathbb{R}$ as $(g(\theta), \theta)$ in polar coordinates. Recall $\kappa \approx 1.015$ defined in (2.11).

Proposition 5.3. The function $g(\theta)$ is smooth and satisfies

$$
\begin{align*}
1<g(\theta)<\exp \left(\frac{\kappa}{\pi(2 B-1)}\right) & \text { for } \quad \theta \in(-\pi, 0)  \tag{5.6}\\
\exp \left(\frac{-\kappa}{\pi(2 B)}\right)<g(\theta)<1 & \text { for } \quad \theta \in(0, \pi) \tag{5.7}
\end{align*}
$$

Proof. For any $\theta_{0}$ in the domain of $g$, set $r_{0}=g\left(\theta_{0}\right)$. Then $\left(r_{0}, \theta_{0}\right)$ is a solution to $I_{\theta}(r)=0$. By the implicit function theorem, $r$ satisfying $I_{\theta}(r)=0$ is a function of $\theta$ in a neighborhood of $\theta_{0}$ provided

$$
\begin{equation*}
\left.\frac{\partial I}{\partial r}\right|_{(r, \theta)=\left(r_{0}, \theta_{0}\right)} \neq 0 \tag{5.8}
\end{equation*}
$$

As we have already seen with (5.3), the left side of (5.8) is always $>0$. Thus, the implicit function theorem confirms that $r$ is a function of $\theta$ and that $g$ is differentiable as many times as $I$ is. Hence $g$ is a smooth function of $\theta$. By implicit differentiation we have, for example, using (5.3), (5.4),

$$
\begin{equation*}
g^{\prime}(\theta)=-\frac{\partial I}{\partial \theta} / \frac{\partial I}{\partial r}=r \frac{\log \left|1-r e^{i \theta}\right|}{2 \pi B-\arg \left(1-r e^{i \theta}\right)} \quad \text { for } \quad r=g(\theta) \tag{5.9}
\end{equation*}
$$

Next we prove the bounds (5.6), (5.7). For $r=1$ we have $I_{\theta}(1)=\mathrm{Cl}_{2}(\theta)$. Since we have seen that $I_{\theta}$ is a strictly increasing function of $r$ it follows that (see Figures 1 and 3)

$$
\begin{array}{lll}
\theta \in(-\pi, 0) & \Longrightarrow \mathrm{Cl}_{2}(\theta)<0 & \Longrightarrow g(\theta)>1, \\
\theta \in(0, \pi) & \Longrightarrow \mathrm{Cl}_{2}(\theta)>0 \quad \Longrightarrow g(\theta)<1, \tag{5.11}
\end{array}
$$

giving the lower bound in (5.6) and the upper bound in (5.7).
For $\theta \in(-\pi, 0), I_{\theta}(r)=0$ implies that $r>1$ by (5.10) and, employing (5.5),

$$
(2 \pi B-\pi-\theta) \log r=\operatorname{Im}_{\operatorname{Li}}\left(\frac{1}{r} e^{-i \theta}\right)<\operatorname{Im} \operatorname{Li}_{2}\left(1 e^{-i \theta}\right) \leqslant \kappa
$$

since $\operatorname{Im} \operatorname{Li}_{2}\left(t e^{-i \theta}\right)$ is increasing in $t$ as we saw after (3.3). The upper bound in (5.6) follows. For $\theta \in(0, \pi), I_{\theta}(r)=0$ implies that $r<1$ by (5.11) and so

$$
2 \pi B \log r=-\operatorname{Im}_{L_{2}}\left(r e^{i \theta}\right)>-\operatorname{Im}_{\operatorname{Li}_{2}}\left(1 e^{i \theta}\right) \geqslant-\kappa .
$$

This gives the lower bound in (5.7) and completes the proof.
With Proposition 5.3. we may restrict our attention to $r$ in the interval $(0.8,2)$ since

$$
0.85 \approx \exp \left(\frac{-\kappa}{2 \pi}\right)<r<\exp \left(\frac{\kappa}{\pi}\right) \approx 1.38
$$

corresponding to (5.6), (5.7) with $B=1$.
For $A, B$ fixed, write the real part of (5.1) as the function

$$
\begin{equation*}
R_{r}(\theta):=\operatorname{Re}^{\operatorname{Li}}\left(r e^{i \theta}\right)+4 \pi^{2} A-2 \pi B \theta \tag{5.12}
\end{equation*}
$$

We see with (2.3) that $R_{r}(\theta)$ is a continuous function of $\theta \in[-\pi, \pi]$. It is always smooth for $\theta \in(-\pi, 0) \cup(0, \pi)$, but it is not differentiable at $\theta=0$ when $r \geqslant 1$.

Similarly to (5.3), (5.4) we have

$$
\begin{align*}
& \frac{\partial R}{\partial \theta}=-2 \pi B+\arg \left(1-r e^{i \theta}\right),  \tag{5.13}\\
& \frac{\partial R}{\partial r}=-\frac{1}{r} \log \left|1-r e^{i \theta}\right| \tag{5.14}
\end{align*}
$$

Lemma 5.4. Fix $B \in \mathbb{Z}_{\geqslant 1}$ and $r$ in the interval $(0.8,2)$.

- If $-B / 2<A \leqslant B / 2$, then there exists a unique $\theta \in(-\pi, 0) \cup(0, \pi)$ so that $R_{r}(\theta)=0$.
- Otherwise, if $A \leqslant-B / 2$ or $B / 2<A$, then there is no solution to $R_{r}(\theta)=$ 0 with $\theta \in(-\pi, 0) \cup(0, \pi)$.
Proof. With (5.13) we have $\frac{\partial R}{\partial \theta}<0$ for $\theta \in(-\pi, 0) \cup(0, \pi)$ so that $R_{r}$ is a strictly decreasing function of $\theta$. Therefore

$$
R_{r}(\pi)<R_{r}(\theta)<R_{r}(-\pi) \quad \text { for } \quad-\pi<\theta<\pi
$$

where

$$
\begin{align*}
R_{r}(\pi) & =\operatorname{ReLi}_{2}(-r)+4 \pi^{2} A-2 \pi^{2} B,  \tag{5.15}\\
R_{r}(-\pi) & =\operatorname{ReLi}_{2}(-r)+4 \pi^{2} A+2 \pi^{2} B . \tag{5.16}
\end{align*}
$$

Hence, for $r>0$ fixed, $R_{r}(\theta)=0$ has a solution $\theta$ if and only if $R_{r}(\pi)<0$ and $R_{r}(-\pi)>0$. With (5.15), (5.16) these conditions are equivalent to

$$
\begin{equation*}
-\frac{B}{2}<A+\frac{\operatorname{Re}^{2} \mathrm{Li}_{2}(-r)}{4 \pi^{2}}<\frac{B}{2} \tag{5.17}
\end{equation*}
$$

Note that $\operatorname{Re}_{\operatorname{Li}}^{2}(x)$ is increasing by (3.4) so that

$$
-1.44 \approx \operatorname{ReLi}_{2}(-2)<\operatorname{ReLi}_{2}(-r)<\operatorname{ReLi}_{2}(-0.8) \approx-0.68
$$

and the result follows.
For $A, B \in \mathbb{Z}$ with $-B / 2<A \leqslant B / 2$, we may therefore define a function $h(r)=h_{A, B}(r)$, with domain $(0.8,2)$, equalling the unique $\theta \in(-\pi, 0) \cup(0, \pi)$ such that $R_{r}(\theta)=0$. Then $(r, h(r))$ parameterizes a part of the curve defined by $\operatorname{Re} \phi_{A, B}(z)=0$.
Proposition 5.5. For $r \in(0.8,2)$ the function $h(r)=h_{A, B}(r)$ is smooth and satisfies

$$
\begin{equation*}
\frac{\pi}{B}\left(2 A-\frac{1}{8}\right)<h_{A, B}(r)<\frac{\pi}{B}\left(2 A+\frac{1}{8}\right) \quad(-B / 2<A<B / 2) \tag{5.18}
\end{equation*}
$$

For $A=0$ we have the improvement on (5.18):

$$
\begin{equation*}
\frac{\pi}{24 B}<h_{0, B}(r)<\frac{\pi}{8 B} \tag{5.19}
\end{equation*}
$$

and for $A=B / 2$ with $B$ even:

$$
\begin{equation*}
\pi-\frac{\pi}{8 B}<h_{B / 2, B}(r)<\pi . \tag{5.20}
\end{equation*}
$$

Proof. For any $r_{0}$ in the domain of $h$, set $\theta_{0}=h\left(r_{0}\right)$. Then $\left(r_{0}, \theta_{0}\right)$ is a solution to $R_{r}(\theta)=0$. By the implicit function theorem, $\theta$ satisfying $R_{r}(\theta)=0$ is a function of $r$ in a neighborhood of $r_{0}$ since

$$
\begin{equation*}
\left.\frac{\partial R}{\partial \theta}\right|_{(r, \theta)=\left(r_{0}, \theta_{0}\right)}<0 \tag{5.21}
\end{equation*}
$$

by (5.13). The theorem also says that $h$ is differentiable as many times as $R$ is. Hence $h$ is a smooth function of $r$. By implicit differentiation we have, for example, using (5.13), (5.14),

$$
\begin{equation*}
h^{\prime}(r)=-\frac{\partial R}{\partial r} / \frac{\partial R}{\partial \theta}=-\frac{1}{r} \frac{\log \left|1-r e^{i \theta}\right|}{2 \pi B-\arg \left(1-r e^{i \theta}\right)} \quad \text { for } \quad \theta=h(r) \tag{5.22}
\end{equation*}
$$

The bounds (5.18), (5.19) and (5.20) follow from (5.12) and estimates for $\operatorname{Re}^{\operatorname{Li}} 2_{2}\left(r e^{i \theta}\right)$, which we work out next. First we note that $\operatorname{Re}^{2} \mathrm{Li}_{2}\left(r e^{i \theta}\right)$ is continuous for $r \geqslant 0$ and $-\pi \leqslant \theta \leqslant \pi$. As in Lemma 3.1 we can show $\operatorname{ReLi}_{2}\left(r e^{-i \theta}\right)=$ $\operatorname{Re}^{\operatorname{Li}} 2\left(r e^{i \theta}\right)$. Also $\operatorname{Re}^{2} \operatorname{Li}_{2}\left(r e^{i \theta}\right)$ is strictly decreasing as a function of $\theta$ when $0 \leqslant$ $\theta \leqslant \pi$, and as a function of $r$ it is increasing for $r<2 \cos \theta$ and decreasing for $r>2 \cos \theta$.

The maximum value of $\operatorname{Re}^{\operatorname{Li}}\left(r e^{i \theta}\right)$ occurs at $(r, \theta)=(2,0)$ and equals $\pi^{2} / 4$. Since $\operatorname{Re}^{\operatorname{Li}} 2(-2)>-\pi^{2} / 4$ it follows that

$$
\begin{equation*}
\left|\operatorname{Re}^{\operatorname{Li}}\left(r e^{i \theta}\right)\right|<\pi^{2} / 4 \quad(r<2) \tag{5.23}
\end{equation*}
$$

Then (5.18) is implied by (5.12) and (5.23). For $A=0$ this means $-\frac{\pi}{8 B}<h_{0, B}(r)<$ $\frac{\pi}{8 B}$. However, for $r \in(0.8,2)$ and $-\pi / 8<\theta<\pi / 8$ we have $\operatorname{ReLi}_{2}\left(r e^{i \theta}\right)>$ $\operatorname{ReLi}_{2}\left(0.8 e^{-i \pi / 8}\right)>\pi^{2} / 12$. This gives the improved lower bound in (5.19). Finally, (5.20) is a consequence of

$$
\begin{equation*}
\operatorname{Re}_{\operatorname{Li}_{2}}\left(r e^{i \theta}\right)<0 \quad \text { for } \quad r>0,|\theta| \geqslant \pi / 2 \tag{5.24}
\end{equation*}
$$

The bound (5.24) is true since $\operatorname{Re}^{\operatorname{Li}}{ }_{2}\left(r e^{i \pi / 2}\right)$ is decreasing from the value 0 at $r=0$.

Let $B \in \mathbb{Z}_{\geqslant 1}$. For each $A \in \mathbb{Z}$ satisfying $-B / 2<A \leqslant B / 2$, based on the bounds in Propositions 5.3 and 5.5 define the following polar rectangles:

$$
\mathcal{R}_{A, B}:=\left\{(r, \theta) \mid r_{1} \leqslant r \leqslant r_{2}, \theta_{1} \leqslant \theta \leqslant \theta_{2}\right\}
$$

where

$$
\begin{aligned}
& r_{1}=r_{1}(A, B):=\left\{\begin{array}{lll}
1 & \text { for } & -B / 2<A<0 \\
\exp (-\kappa /(2 \pi B)) & \text { for } & 0 \leqslant A \leqslant B / 2
\end{array}\right. \\
& r_{2}=r_{2}(A, B):=\left\{\begin{array}{lll}
\exp (\kappa /(\pi(2 B-1))) & \text { for } \quad-B / 2<A<0 \\
1 & \text { for } & 0 \leqslant A \leqslant B / 2
\end{array}\right. \\
& \theta_{1}=\theta_{1}(A, B):=\left\{\begin{array}{lll}
\pi(2 A-1 / 8) / B & \text { for } & A \neq 0 \\
\pi /(24 B) & \text { for } & A=0
\end{array}\right. \\
& \theta_{2}=\theta_{2}(A, B):=\left\{\begin{array}{lll}
\pi(2 A+1 / 8) / B & \text { for } & A \neq B / 2 \\
\pi & \text { for } & A=B / 2 .
\end{array}\right.
\end{aligned}
$$

For example, see Figure 3 for $\mathcal{R}_{0,1}$.
Theorem 5.6. For $A \in \mathbb{Z}$ and $B \in \mathbb{Z} \geqslant 1$, we have solutions to $\phi_{A, B}(z)=0$ if and only if $-B / 2<A \leqslant B / 2$. For such a pair $A, B$ the solution $z$ is unique and contained in the interior of the polar rectangle $\mathcal{R}_{A, B}$.

Proof. Let $r_{1}, r_{2}, \theta_{1}, \theta_{2}$ be defined as above, giving the boundaries of $\mathcal{R}_{A, B}$. With Propositions 5.3 and 5.5 we have seen that there exist continuously differentiable functions $g$, $h$ with

$$
\begin{equation*}
g:\left[\theta_{1}, \theta_{2}\right] \rightarrow\left(r_{1}, r_{2}\right), \quad h:\left[r_{1}, r_{2}\right] \rightarrow\left(\theta_{1}, \theta_{2}\right) \tag{5.25}
\end{equation*}
$$

and

$$
\begin{equation*}
g^{\prime}(\theta)=r \cdot \frac{\log \left|1-r e^{i \theta}\right|}{2 \pi B-\arg \left(1-r e^{i \theta}\right)}, \quad h^{\prime}(r)=-\frac{1}{r} \cdot \frac{\log \left|1-r e^{i \theta}\right|}{2 \pi B-\arg \left(1-r e^{i \theta}\right)} . \tag{5.26}
\end{equation*}
$$

Solutions to $\phi_{A, B}(z)=0$ must be points in the intersection

$$
\begin{equation*}
(g(\theta), \theta)_{\theta_{1} \leqslant \theta \leqslant \theta_{2}} \cap(r, h(r))_{r_{1} \leqslant r \leqslant r_{2}} . \tag{5.27}
\end{equation*}
$$

The denominators $2 \pi B-\arg \left(1-r e^{i \theta}\right)$ in (5.26) are positive and bounded below by $\pi(2 B-1)$. We have $\log \left|1-r e^{i \theta}\right|=0$ if and only if $r=2 \cos \theta$, and it is straightforward to show that the curve $r=2 \cos \theta$ intersects $\mathcal{R}_{A, B}$ if and only if $A= \pm B / 6$.

Suppose first that $-B / 6<A<B / 6$. Then we have $\log \left|1-r e^{i \theta}\right|<0$ for all $(r, \theta) \in \mathcal{R}_{A, B}$, and therefore $h^{\prime}(r)>0$ and $g^{\prime}(\theta)<0$ for all $r$ and $\theta$ in the domains (5.25). By the inverse function theorem, $g$ has an inverse, $g^{-1}(r)$, a strictly decreasing continuously differentiable function on $\left[g\left(\theta_{1}\right), g\left(\theta_{2}\right)\right]$. Set $f(\theta)$ to be the difference $h(r)-g^{-1}(r)$. Then $f$ is continuous and strictly increasing on $\left[g\left(\theta_{1}\right), g\left(\theta_{2}\right)\right]$ with $f\left(g\left(\theta_{1}\right)\right)<0$ and $f\left(g\left(\theta_{2}\right)\right)>0$. By the intermediate value theorem, there is a unique $r^{*}$ so that $f\left(r^{*}\right)=0$. Set $\theta^{*}=h\left(r^{*}\right)$. Then $\left(r^{*}, \theta^{*}\right)$ is the unique element of (5.27) and it lies in the interior of $\mathcal{R}_{A, B}$.

The cases with $A<-B / 6$ or $A>B / 6$ are handled very similarly to the above, the only difference being that now $g^{\prime}(\theta)>0$ and $h^{\prime}(r)<0$.

For the last case we have $A= \pm B / 6$. With (5.26) we may easily show that $\left|g^{\prime}(\theta)\right|,\left|h^{\prime}(r)\right|<1$. Since $h^{\prime}$ may be zero, it does not necessarily have an inverse. Consider $\mathcal{R}_{A, B}$ as a rectangle in the $r \theta$ plane and rotate it about the origin, say, by an angle of $\pi / 4$. The curves corresponding to the rotated curves $(r, h(r))$ and $(g(\theta), \theta)$ may now be expressed as graphs of functions (of positive and negative slope, respectively) and our previous argument applies.

Theorem 5.6 establishes the $B \neq 0$ case of Theorem 1.1 (except for simplicity) and shows that when $\phi_{A, B}$ has a zero it is close to $e^{2 \pi i A / B}$. In the next section, we find each zero more precisely.

## 6. Newton's method for $\phi_{A, B}$

Proposition 6.1. Let $L$ be a line in $\mathbb{C}$ that is a distance $r>0$ from the origin. Suppose $z$ and $c$ are two points in the half plane that is bounded by $L$ and does not contain the origin. If the ray $(-\infty, 0]$ intersects this half plane, we also assume $z$ and $c$ are on the same side of $(-\infty, 0]$. With these assumptions

$$
\left|\log (z)-\log (c)-\frac{z-c}{c}\right| \leqslant|z-c|^{2} \frac{1}{2 r^{2}} .
$$

Proof. As in Proposition 4.1, express $\log (z)$ with the first two terms of its Taylor expansion at $c$ to get

$$
\log (z)=\log (c)+\log ^{\prime}(c)(z-c)+\frac{(z-c)^{2}}{2 \pi i} \int_{C} \frac{\log (w)}{(w-c)^{2}(w-z)} d w
$$

with $C$ a positively oriented circular curve of radius $T$ containing $z$ and $c$, similar to Figure 2 but this time avoiding the branch cut $(-\infty, 0]$. Let the horizontal paths above and below the cut coincide. The values of $\log$ on these paths differ by $2 \pi i$, as in (2.2). We now rotate these horizontal paths away from $z$ and $c$ (using values of the continued $\log$ that keep the difference on the two paths as $2 \pi i$ ) until they are perpendicular to $L$, not intersecting it. To keep track of the angle, suppose these paths now pass through $\alpha$ with $|\alpha|=1$. Letting $T \rightarrow \infty$ and $\varepsilon \rightarrow 0$ we find

$$
\begin{equation*}
\frac{1}{2 \pi i} \int_{C} \frac{\log (w)}{(w-c)^{2}(w-z)} d w=-\int_{0}^{\infty} \frac{\alpha}{(\alpha t-c)^{2}(\alpha t-z)} d t \tag{6.1}
\end{equation*}
$$

Since $|\alpha t-c|,|\alpha t-z| \geqslant t+r$, the right side of (6.1) is bounded in absolute value by $\int_{0}^{\infty}(t+r)^{-3} d t=1 /\left(2 r^{2}\right)$ as required.

Define

$$
M(s):=\left(\frac{2 \log (4 / 3)}{s}+8 \pi\right) \frac{1}{(4 s+1)^{2}}+3
$$

Proposition 6.2. Let $L$ be a line in $\mathbb{C}$ that is a distance $s>0$ from the point 1. Suppose $z$ and $c$ are two points in the half plane that is bounded by $L$ and does not contain 1. If the ray $[1, \infty)$ intersects this half plane, we assume $z$ and $c$ are on the same side of $[1, \infty)$. With these assumptions

$$
\left|\operatorname{Li}_{2}(z)-\operatorname{Li}_{2}(c)-(z-c) \operatorname{Li}_{2}^{\prime}(c)\right| \leqslant|z-c|^{2} M(s)
$$

Proof. As in Propositions 4.1 and 6.1 we must bound the remainder term in

$$
\operatorname{Li}_{2}(z)=\operatorname{Li}_{2}(c)+(z-c) \operatorname{Li}_{2}^{\prime}(c)+\frac{(z-c)^{2}}{2 \pi i} \int_{C} \frac{\operatorname{Li}_{2}(w)}{(w-c)^{2}(w-z)} d w
$$

where $C$ is the circular path of radius $T$ containing $z$ and $c$ in Figure 2, Let the paths above and below the branch cut $[1, \infty)$ coincide; the difference between values of $\operatorname{Li}_{2}(w)$ with $w$ coming from above and below the branch cut is $2 \pi i \log (w)$ by (2.3). Similarly to Proposition 6.1, we rotate these horizontal paths away from $z$ and $c$ until they are perpendicular to $L$, using values of the continued dilogarithm that keep the difference at a point $w$ on the two paths as $2 \pi i \log (w)$. Suppose these paths now pass through $1+\alpha$ with $|\alpha|=1$. Letting $T \rightarrow \infty$ and $\varepsilon \rightarrow 0$ we find

$$
\frac{1}{2 \pi i} \int_{C} \frac{\operatorname{Li}_{2}(w)}{(w-c)^{2}(w-z)} d w=\int_{0}^{\infty} \frac{\alpha \log (\alpha t+1)}{(\alpha t+1-c)^{2}(\alpha t+1-z)} d t
$$

Then $|\alpha t+1-c|,|\alpha t+1-z| \geqslant t+s$ imply

$$
\begin{equation*}
\left|\int_{0}^{\infty} \frac{\alpha \log (\alpha t+1)}{(\alpha t+1-c)^{2}(\alpha t+1-z)} d t\right| \leqslant \int_{0}^{\infty} \frac{|\log (\alpha t+1)|}{(t+s)^{3}} d t \tag{6.2}
\end{equation*}
$$

With the straightforward inequality

$$
\begin{equation*}
|\log (z+1)| \leqslant \frac{1}{Y} \log \left(\frac{1}{1-Y}\right) \cdot|z| \quad(|z| \leqslant Y<1) \tag{6.3}
\end{equation*}
$$

we have $|\log (\alpha t+1)| \leqslant 4 \log (4 / 3) \cdot t$ for $|t| \leqslant 1 / 4$. Also

$$
\begin{equation*}
|\log (\alpha t+1)| \leqslant|\operatorname{Im} \log (\alpha t+1)|+|\operatorname{Re} \log (\alpha t+1)| \leqslant \pi+\log (t+1) \tag{6.4}
\end{equation*}
$$

so the right side of (6.2) is bounded by

$$
\begin{align*}
\int_{0}^{1 / 4} \frac{4 \log (4 / 3) t}{(t+s)^{3}} d t+\int_{1 / 4}^{\infty} & \frac{\pi}{(t+s)^{3}} d t+\int_{1 / 4}^{\infty} \frac{\log (t+1)}{(t+s)^{3}} d t  \tag{6.5}\\
& \leqslant \frac{2 \log (4 / 3)}{s(4 s+1)^{2}}+\frac{8 \pi}{(4 s+1)^{2}}+\int_{1 / 4}^{\infty} \frac{\log (t+1)}{t^{3}} d t
\end{align*}
$$

The last integral on the right of (6.5) equals $2+\frac{15}{2} \log (5)-16 \log (2)<3$.
Let $A, B$ be integers with $-B / 2<A \leqslant B / 2$. We know that $\phi_{A, B}(\rho)=0$ for some unique $\rho \in \mathcal{R}_{A, B}$. Let $\mathcal{R}_{A, B}^{\prime}$ be a convex version of $\mathcal{R}_{A, B}$ with the boundary arc of radius $r_{1}(A, B)$ replaced by a straight line between the corners. Let $c$ be any point in $\mathcal{R}_{A, B}^{\prime}$. Then

$$
\begin{equation*}
\left|c \cdot \phi_{A, B}^{\prime}(c)\right|=|-\log (1-c)+2 \pi i B| \geqslant \pi(2 B-1) \tag{6.6}
\end{equation*}
$$

and in particular $\phi_{A, B}^{\prime}(c) \neq 0$, so that $c^{*}:=c-\phi_{A, B}(c) / \phi_{A, B}^{\prime}(c)$ makes sense.
Theorem 6.3. Let $A, B$ be integers with $-B / 2<A \leqslant B / 2$. If $\rho, c \in \mathcal{R}_{A, B}^{\prime}$ with $\phi_{A, B}(\rho)=0$, then

$$
\left|\rho-c^{*}\right|<B|\rho-c|^{2} \times\left\{\begin{array}{ll}
0.76 & \text { if }  \tag{6.7}\\
2.51 & \text { if }
\end{array} \quad A=0\right.
$$

Proof. Suppose all points in $\mathcal{R}_{A, B}^{\prime}$ are at least a distance $r$ from 0 and $s$ from 1 . Then Propositions 6.1 and 6.2 imply

$$
\begin{equation*}
\left|\phi_{A, B}(\rho)-\phi_{A, B}(c)-(z-c) \phi_{A, B}^{\prime}(c)\right| \leqslant|\rho-c|^{2}\left(\pi B / r^{2}+M(s)\right) . \tag{6.8}
\end{equation*}
$$

Combining (6.8) with (6.6) shows that

$$
\begin{equation*}
\left|\rho-c^{*}\right| \leqslant|\rho-c|^{2} \cdot B \cdot N(A, B) \tag{6.9}
\end{equation*}
$$

for

$$
\begin{equation*}
N(A, B):=r_{2}(A, B) \frac{\pi B / r^{2}+M(s)}{\pi B(2 B-1)} \tag{6.10}
\end{equation*}
$$

since $|c| \leqslant r_{2}(A, B)$. Looking at the easier cases with $A \neq 0$ and $B \geqslant 2$ first, it is routine to show that

$$
\begin{equation*}
r_{2}(A, B)<6 / 5, \quad r>9 / 10, \quad s>2 / B \tag{6.11}
\end{equation*}
$$

With these numbers, for $A \neq 0$,

$$
\begin{equation*}
N(A, B)<\frac{6}{5 \pi(2 B-1)}\left(\frac{100 \pi}{81}+\left[\log \left(\frac{4}{3}\right)+\frac{8 \pi}{B}\right]\left(\frac{B}{B+8}\right)^{2}+\frac{3}{B}\right) \tag{6.12}
\end{equation*}
$$

Then $N(A, 2)<0.76, N(A, 3)<0.43, N(A, 4)<0.30$, and if we replace $\frac{B}{B+8}$ by 1 in (6.12) we see that $N(A, B)<0.42$ for all $B \geqslant 5$. This proves the theorem for $A \neq 0$.

To treat the cases with $A=0$ we need to rework Proposition 6.2 a little. The set $\mathcal{R}_{0, B}^{\prime}$ has all its points at least $\sin (\pi /(24 B)>1 /(8 B)$ vertically above $\mathbb{R}$. Taking $\alpha=-i$ in Proposition 6.2 means we need to bound $|\log (-i t+1)|=|\log (i t+1)|$ in (6.2).

Lemma 6.4. We have $|\log (i t+1)| \leqslant t$ for all $t \geqslant 0$.

Proof. Let $\theta=\arg (i t+1)$, and our desired inequality is equivalent to

$$
g(\theta):=\tan ^{2} \theta-\theta^{2}-\log ^{2}(\cos \theta) \geqslant 0 \quad(0 \leqslant \theta<\pi / 2)
$$

With [Rad73, (11.3)] write

$$
\tan \theta=\frac{1}{\theta} \sum_{n=2}^{\infty} a_{n} \theta^{n} \quad \text { for } \quad a_{n}= \begin{cases}2^{n}\left(2^{n}-1\right)\left|B_{n}\right| / n!>0 & \text { if } n \text { is even } \\ 0 & \text { if } n \text { is odd }\end{cases}
$$

This expansion is valid for $|\theta|<\pi / 2$. Therefore

$$
\begin{equation*}
\tan ^{2} \theta=\frac{1}{\theta^{2}} \sum_{n=4}^{\infty} \theta^{n} \sum_{j=0}^{n} a_{j} a_{n-j} . \tag{6.13}
\end{equation*}
$$

Since $\log (\cos \theta)=\sum_{n=2}^{\infty} \frac{a_{n}}{n+1} \theta^{n}$ we find

$$
\begin{equation*}
\log ^{2}(\cos \theta)=\sum_{n=4}^{\infty} \theta^{n} \sum_{j=0}^{n} \frac{a_{j} a_{n-j}}{(j+1)(n-j+1)} . \tag{6.14}
\end{equation*}
$$

Comparing (6.13) and (6.14) shows that

$$
\tan ^{2} \theta \geqslant \frac{5 \log ^{2}(\cos \theta)}{\theta^{2}}
$$

and so

$$
g(\theta) \geqslant-\theta^{2}+\left(\frac{5}{\theta^{2}}-1\right) \log ^{2}(\cos \theta)
$$

Since $\log ^{2}(\cos \theta)=\theta^{4} / 4+\cdots$ it follows that $g(\theta) \geqslant 0$ for $0 \leqslant \theta \leqslant 1$. When $1 \leqslant \theta<\pi / 2$ it is straightforward to verify that $g(\theta) \geqslant 0$ since $g^{\prime}(\theta)>0$ in this range.

Now the remainder term in (6.2) is

$$
\begin{equation*}
\int_{0}^{\infty} \frac{|\log (-i t+1)|}{(t+s)^{3}} d t \leqslant \int_{0}^{\infty} \frac{t}{(t+s)^{3}} d t=\frac{1}{2 s} \tag{6.15}
\end{equation*}
$$

The quantities we need for $\mathcal{R}_{0, B}^{\prime}$ satisfy

$$
\begin{equation*}
r_{2}(0, B)=1, \quad r>9 / 10, \quad s>1 /(8 B) \tag{6.16}
\end{equation*}
$$

Then (6.9) is true with $M(s)$ in (6.10) replaced by $1 /(2 s)$ from (6.15). We obtain

$$
\begin{equation*}
N(0, B)<\left(\frac{100 \pi}{81}+4\right) \frac{1}{\pi(2 B-1)} \tag{6.17}
\end{equation*}
$$

so that $N(0, B) \leqslant N(0,1)<2.51$ for $B \geqslant 1$. This completes the proof of Theorem 6.3 .

Theorem 6.5. Let $A$ and $B$ be integers satisfying $-B / 2<A \leqslant B / 2$. Let $\rho$ be the unique zero of $\phi_{A, B}$. Then the sequence $c_{0}, c_{1}, \cdots$ in (1.4) converges to $\rho$ with

$$
\begin{equation*}
\left|\rho-c_{n}\right|<1.25(0.95)^{2^{n}} \tag{6.18}
\end{equation*}
$$

Proof. Suppose first that $A \neq 0$. The distance between any two points in $\mathcal{R}_{A, B}^{\prime}$ may be shown to be less than $6 /(5 B)$. It then follows from Theorem 6.3 that for any $c \in \mathcal{R}_{A, B}^{\prime}$ we have $\left|\rho-c^{*}\right|<|\rho-c|$ so that $c^{*}$ remains in $\mathcal{R}_{A, B}^{\prime}$. We have $c_{0} \in \mathcal{R}_{A, B}^{\prime}$, and repeated applications of Theorem 6.3 show that

$$
0.8 B\left|\rho-c_{n}\right|<\left(0.8 B\left|\rho-c_{n-1}\right|\right)^{2}<\left(0.8 B\left|\rho-c_{n-2}\right|\right)^{4}<\cdots<\left(0.8 B\left|\rho-c_{0}\right|\right)^{2^{n}}
$$

The inequality (6.18) now follows using $\left|\rho-c_{0}\right|<6 /(5 B)$.
The case with $A=0$ is proved similarly. Use that the distance between any two points in $\mathcal{R}_{0, B}^{\prime}$ is less than $1 /(3 B)$.

## 7. Further results

Lemma 7.1. We have

$$
\log \left(1-e^{i \theta}\right)=-\mathrm{Cl}_{2}^{\prime}(\theta)+\frac{i}{2}\left(\theta-\pi \frac{\theta}{|\theta|}\right) \quad(0<|\theta|<\pi)
$$

Proof. We may write $1-e^{i \theta}=-e^{i \theta / 2} 2 i \sin (\theta / 2)=e^{i(\theta-\pi) / 2} 2 \sin (\theta / 2)$. Taking logs with appropriate branches and using $\mathrm{Cl}_{2}^{\prime}(\theta)=-\log |2 \sin (\theta / 2)|$ from (2.10) completes the proof.

Lemma 7.2. For $0<|\theta| \leqslant \pi$,

$$
\left|\mathrm{Cl}_{2}^{\prime}(\theta)\right| \leqslant \log \left(\max \left\{2, \frac{\pi}{3|\theta|}\right\}\right)
$$

Proof. When $\pi / 3 \leqslant|\theta| \leqslant \pi$ we have $1 \leqslant|2 \sin (\theta / 2)| \leqslant 2$ so that $|\log | 2 \sin (\theta / 2)|\mid \leqslant$ $\log 2$. When $0<|\theta|<\pi / 3$ we have $|2 \sin (\theta / 2)|<1$, and using the inequality $|\sin x| \geqslant 3|x| / \pi$ for $|x| \leqslant \pi / 6$ implies

$$
|\log | 2 \sin (\theta / 2)\left|\left\lvert\,=\log \left(\frac{1}{|2 \sin (\theta / 2)|}\right) \leqslant \log \frac{\pi}{3|\theta|}\right.\right.
$$

Theorem 7.3. Let $A$ and $B$ be integers satisfying $-B / 2<A \leqslant B / 2$ and suppose $\phi_{A, B}(\rho)=0$. Then

$$
\begin{equation*}
\rho=e^{2 \pi i A / B}\left(1+\frac{-\mathrm{Cl}_{2}(2 \pi A / B)+i \pi^{2} B_{2}(|A| / B)}{2 \pi B}\right)+O\left(\frac{1+\log B}{B^{2}}\right) . \tag{7.1}
\end{equation*}
$$

Proof. Let $c_{0}$ and $c_{1}$ be as in Theorem 6.5. With (6.12) and (6.17) it is easy to see that $B \cdot N(A, B)$ is bounded by an absolute constant. Also $\left|\rho-c_{0}\right|<6 /(5 B)$, as we have seen in the proof of Theorem 6.5) so that (6.9) implies

$$
\begin{equation*}
\left|\rho-c_{1}\right|=O\left(B^{-2}\right) . \tag{7.2}
\end{equation*}
$$

Now for $A \neq 0$,

$$
\phi_{A, B}\left(c_{0}\right)=-\operatorname{Li}_{2}\left(e^{2 \pi i A / B}\right)=\pi^{2} B_{2}(|A| / B)+i \mathrm{Cl}_{2}(2 \pi A / B)
$$

with (2.8) and (2.9). Also
$c_{0} \phi_{A, B}^{\prime}\left(c_{0}\right)=2 \pi i B-\log \left(1-e^{2 \pi i A / B}\right)=2 \pi i B+\mathrm{Cl}_{2}^{\prime}(2 \pi A / B)+2 \pi i\left(-\frac{A}{2 B}+\frac{A}{4|A|}\right)$
using Lemma 7.1. Therefore

$$
\begin{equation*}
-\frac{\phi_{A, B}\left(c_{0}\right)}{c_{0} \phi_{A, B}^{\prime}\left(c_{0}\right)}=\frac{-\mathrm{Cl}_{2}(2 \pi A / B)+i \pi^{2} B_{2}(|A| / B)}{2 \pi B} \frac{1}{1+X} \tag{7.3}
\end{equation*}
$$

for

$$
X:=\frac{1}{B}\left(-\frac{A}{2 B}+\frac{A}{4|A|}\right)+\frac{\mathrm{Cl}_{2}^{\prime}(2 \pi A / B)}{2 \pi i B} .
$$

We have

$$
\begin{equation*}
\left|\mathrm{Cl}_{2}^{\prime}(2 \pi A / B)\right| \leqslant \log (2 B) \tag{7.4}
\end{equation*}
$$

in an easy corollary to Lemma 7.2, Use (7.4) to see that

$$
\begin{equation*}
|X| \leqslant \frac{\pi / 2+\log (2 B)}{2 \pi B} \leqslant \frac{1}{2} \quad \Longrightarrow \quad\left|\frac{1}{1+X}-1\right| \leqslant 2|X| \leqslant \frac{\pi+2 \log (2 B)}{2 \pi B} \tag{7.5}
\end{equation*}
$$

and hence (7.3) and (7.5) together imply

$$
\begin{equation*}
1-\frac{\phi_{A, B}\left(c_{0}\right)}{c_{0} \phi_{A, B}^{\prime}\left(c_{0}\right)}=1+\frac{-\mathrm{Cl}_{2}(2 \pi A / B)+i \pi^{2} B_{2}(|A| / B)}{2 \pi B}+O\left(\frac{1+\log B}{B^{2}}\right) \tag{7.6}
\end{equation*}
$$

Multiplying both sides of (7.6) by $c_{0}$ and using (7.2) completes the proof of (7.1).
In the case $A=0$, our goal (7.1) becomes $\rho=1+\pi i /(12 B)+O\left((1+\log B) / B^{2}\right)$. Similarly to (7.6) we find

$$
1-\frac{\phi_{0, B}\left(c_{0}\right)}{c_{0} \phi_{0, B}^{\prime}\left(c_{0}\right)}=1+O\left(\frac{1+\log B}{B^{2}}\right)
$$

by using (2.14). Multiplying by $c_{0}$ then shows that

$$
\begin{equation*}
c_{1}=e^{\pi i /(12 B)}+O\left(\frac{1+\log B}{B^{2}}\right) \tag{7.7}
\end{equation*}
$$

Finally, (7.7) and (7.2) prove (7.1).
Theorems 6.5 and 7.3 above establish Theorem 1.3 . The following result completes the last part of the proof of Theorem 1.1.

Proposition 7.4. For $A$ and $B \in \mathbb{Z}$, all zeros of $\phi_{A, B}(z)$ are simple.
Proof. The zero of $\operatorname{Li}_{2}(z)=\phi_{0,0}(z)$ at $z=0$ is clearly simple by (1.1). For $B=0$ and $A \geqslant 1$, we have by (2.1) and Proposition 3.3 that

$$
\rho \cdot \phi_{A, 0}^{\prime}(\rho)=-\log (1-\rho)<-\pi \sqrt{8 A-1 / 3}<0
$$

which shows that $\rho$ is simple. For $-B / 2<A \leqslant B / 2$, if $\phi_{A, B}^{\prime}(\rho)=0$, then $2 \pi i B=$ $\log (1-\rho)$ by (2.1). But this is impossible since $-\pi<\operatorname{Im} \log (1-\rho) \leqslant \pi$.

Another way to verify the simplicity of the zeros of $\phi_{A, B}(z)$, as well as their existence and uniqueness, is with the argument principle. Let $C$ be the contour shown in Figure 2, but also avoiding the branch cut $(-\infty, 0]$ if $B \neq 0$. Then for $A, B \in \mathbb{Z}$,

$$
\Psi_{C}(A, B):=\frac{1}{2 \pi i} \int_{C} \frac{\phi_{A, B}^{\prime}(z)}{\phi_{A, B}(z)} d z
$$

counts the number of zeros of $\phi_{A, B}(z)$ in the interior of $C$ with multiplicity. Letting the horizontal paths coincide, the large radii of $C$ go to infinity and the small radii go to zero, we define $\Psi(A, B)$ which counts all the zeros of $\phi_{A, B}(z)$. With (2.2) and (2.3) we obtain
$2 \pi i \Psi(A, B)=\int_{1}^{\infty}\left(\frac{\phi_{A, B+1}^{\prime}(z)}{\phi_{A, B+1}(z)}-\frac{\phi_{A, B}^{\prime}(z)}{\phi_{A, B}(z)}\right) d z+\int_{-\infty}^{0}\left(\frac{\phi_{A, B}^{\prime}(z)}{\phi_{A, B}(z)}-\frac{\phi_{A+B, B}^{\prime}(z)}{\phi_{A+B, B}(z)}\right) d z$.
Note that $\Psi(A, B)$ correctly counts the zeros of $\phi_{A, B}(z)$ when $B=0$; in that case there is no branch cut at $(-\infty, 0]$ to avoid and the second integral in (7.8) is zero.

Theorem 1.1 implies

$$
\Psi(A, B)= \begin{cases}1 & \text { if } B=0, A \geqslant 0  \tag{7.9}\\ 1 & \text { if }-|B| / 2<A \leqslant|B| / 2 \\ 0 & \text { otherwise }\end{cases}
$$

with the right side of (7.9) indicating a single simple zero with 1 and no zero with 0 . Checking formula (7.9) numerically, we confirm that it is true for $|A|,|B| \leqslant 50$. Perhaps (7.9) can be proved directly from (7.8).

## 8. Zeros of polylogarithms

The dilogarithm $\mathrm{Li}_{2}(z)$ is a special case of the polylogarithm, also known as Jonquière's function Jon89,

$$
\operatorname{Li}_{s}(z):=\sum_{n=1}^{\infty} \frac{z^{n}}{n^{s}} \quad \text { for } \quad|z|<1, s \in \mathbb{C} .
$$

As a function of $z$, it has an analytic continuation to all of $\mathbb{C}$ and in general will be multi-valued with branch points at 0,1 and $\infty$; see for example Sections 4 and 11 of Vep08. It satisfies

$$
\begin{equation*}
z \frac{d}{d z} \operatorname{Li}_{s}(z)=\operatorname{Li}_{s-1}(z) \tag{8.1}
\end{equation*}
$$

8.1. Zeros of $\operatorname{Li}_{s}(z)$ for $\operatorname{Re}(s)>0$. Here we discuss what is known about the zeros of $\operatorname{Li}_{s}(z)$ for $s \in \mathbb{C}$ with $\operatorname{Re}(s)>0$.
Theorem 8.1 (Le Roy [R00]). For $r>0, \operatorname{Li}_{r}(z)$ has exactly one zero for $z \in$ $\mathbb{C}-[1, \infty)$. It is at $z=0$ and is simple.

Proof. We use Jonquière's representation Jon89,

$$
\begin{equation*}
\operatorname{Li}_{s}(z)=\frac{z}{\Gamma(s)} \int_{0}^{\infty} \frac{t^{s-1}}{e^{t}-z} d t \quad(\operatorname{Re}(s)>0, z \in \mathbb{C}-[1, \infty)) \tag{8.2}
\end{equation*}
$$

Then for $s=r>0$ and $z=x+i y$, the imaginary part of $\operatorname{Li}_{r}(z) / z$ is

$$
\frac{y}{\Gamma(r)} \int_{0}^{\infty} \frac{t^{r-1}}{\left(e^{t}-x\right)^{2}+y^{2}} d t
$$

and therefore non-zero for $y \neq 0$. For $y=0$ we have $z=x<0$ and so, clearly, $\operatorname{Li}_{r}(x) / x>0$. We have shown that $\operatorname{Li}_{r}(z) / z$ is finite and non-zero for $z \in \mathbb{C}-[1, \infty)$ as required.

For $s=2$ we have a clear picture of the zeros of $\operatorname{Li}_{2}(z)$ with Theorems 1.11.3. When $s=1$ we have the simple case $\operatorname{Li}_{1}(z)=-\log (1-z)$. Clearly $\operatorname{Li}_{1}(0)=0$ gives the only zero on the principal branch and $\mathrm{Li}_{1}(z)$ is non-zero on every other branch. Going in the other direction, if we let $\operatorname{Li}_{3}(z)$ denote the trilogarithm on its principal branch, it may be shown (see Vep08, p. 246]) that on any branch it has the form

$$
\begin{equation*}
\mathrm{Li}_{3}(z)+4 \pi^{3} i A+2 \pi^{2} B \log (z)-\pi i C \log ^{2}(z) \quad(A, B, C \in \mathbb{Z}) \tag{8.3}
\end{equation*}
$$

By studying when the real and imaginary parts of (8.3) vanish, as in Section [5 it should be possible to approximately determine the zeros. Looking at some cases, it seems there may be up to two zeros on each branch and these zeros are close to the unit circle.

For general $s$ with $\operatorname{Re}(s)>0$, the only result on the zeros of $\operatorname{Li}_{s}(z)$ seems to be that they are finite in number Gaw79] for $z$ on the principal branch. Vepštas gives an efficient method to compute polylogarithms in Vep08 and displays the zeros of $\mathrm{Li}_{1 / 2+80 i}(z), \mathrm{Li}_{1 / 2+15 i}(z)$ and $\mathrm{Li}_{6 / 5+14 i}(z)$ in the phase plots Vep08, Figs. 8-10]. In these cases the zeros lie near the unit circle. If $s_{n}$ is a zero of the Riemann zeta function $\zeta(s)$, then $\operatorname{Li}_{s_{n}}(z)$ has zeros at $z= \pm 1$. As $s$ moves continuously near $s_{n}$ the corresponding $z$ zeros of $\operatorname{Li}_{s}(z)$ show interesting behavior moving near $\pm 1$. This connection is explored in FK75.

For another example, Figure 4 shows the zeros of $\operatorname{Li}_{s}(z)$ for $s=10+44 i$. The zeros were found numerically by combining a phase plot with Newton's method. The spiraling curve that these zeros are making seems to be of a similar form to (8.20). It would be interesting to identify it exactly.


Figure 4. The zeros of $\operatorname{Li}_{s}(z)$ for $s=10+44 i$
8.2. $\operatorname{Zeros}$ of $\operatorname{Li}_{s}(z)$ for $\operatorname{Re}(s) \leqslant 0$. Much more is known about the zeros of $\operatorname{Li}_{s}(z)$ for $\operatorname{Re}(s) \leqslant 0$. We look at the case when $s$ is an integer first. Applying (8.1) to $\mathrm{Li}_{1}(z)$ shows $\mathrm{Li}_{0}(z)=z /(1-z)$, a rational function. For $-m \in \mathbb{Z}_{\leqslant 0}$ write

$$
\begin{equation*}
\operatorname{Li}_{-m}(z)=\frac{z \cdot A_{m}(z)}{(1-z)^{m+1}} \tag{8.4}
\end{equation*}
$$

Then (8.1) implies

$$
\begin{equation*}
A_{m+1}(z)=(1-z)^{m+2} \frac{d}{d z}\left(\frac{z \cdot A_{m}(z)}{(1-z)^{m+1}}\right) . \tag{8.5}
\end{equation*}
$$

We may therefore recursively define the functions $A_{m}$ by

$$
\begin{equation*}
A_{0}(z):=1, \quad A_{m+1}(z):=(1+m z) A_{m}(z)+z(1-z) A_{m}^{\prime}(z) \quad\left(m \in \mathbb{Z}_{\geqslant 0}\right) \tag{8.6}
\end{equation*}
$$

Hence $A_{m}(z)$ is a polynomial and for $m \in \mathbb{Z}_{\geqslant 1}$ it has degree $m-1$. These are the Eulerian polynomials, introduced by Euler in connection with evaluating the Riemann zeta function at negative integers. For example
$A_{1}(z)=1, \quad A_{2}(z)=1+z, \quad A_{3}(z)=1+4 z+z^{2}, \quad A_{4}(z)=1+11 z+11 z^{2}+z^{3}$.
The coefficient of $z^{k}$ in $A_{m}(z)$ has a combinatorial interpretation as the number of permutations of $\{1,2, \ldots, m\}$ with $k$ ascents; see [GKP94, Sect. 6.2]. Frobenius
also showed in Fro10 that

$$
A_{m}(z)=\sum_{k=0}^{m} k!\left\{\begin{array}{c}
m \\
k
\end{array}\right\}(z-1)^{m-k} \quad\left(m \in \mathbb{Z}_{\geqslant 0}\right)
$$

with the Stirling number $\left\{\begin{array}{c}m \\ k\end{array}\right\}$ indicating the number of partitions of $m$ elements into $k$ non-empty subsets.

We have

$$
\begin{equation*}
z^{m-1} A_{m}(1 / z)=A_{m}(z) \quad\left(m \in \mathbb{Z}_{\geqslant 1}\right) \tag{8.7}
\end{equation*}
$$

which is equivalent to

$$
\begin{equation*}
(-1)^{m+1} \operatorname{Li}_{-m}(1 / z)=\operatorname{Li}_{-m}(z) \quad\left(m \in \mathbb{Z}_{\geqslant 1}\right) \tag{8.8}
\end{equation*}
$$

and (8.8) may be easily established using (8.1) and induction.
Thus by (8.4), $\mathrm{Li}_{-m}(z)$ is a rational function and its zeros are at $z=0$ along with the zeros of the Eulerian polynomial $A_{m}(z)$. Frobenius showed in Fro10 that the zeros of $A_{m}(z)$ are always distinct and negative. For completeness, we give a short proof that also shows their interlacing property. This proof is based on [Pey66, Thm. 4] in a special case; see also [Bón12, Thm. 1.34] for a slightly weaker result.

Theorem 8.2. For $m \in \mathbb{Z}_{\geqslant 1}$ the Eulerian polynomial $A_{m}(z)$ has $m-1$ distinct zeros. If $A_{m}(\rho)=0$, then $\rho<0$ and $A_{m}(1 / \rho)=0$. Also, exactly one zero of $A_{m}(z)$ lies between each pair of consecutive zeros of $A_{m+1}(z)$.

Proof. Suppose for our induction hypothesis that $A_{m}(z)$ has $m-1$ distinct negative zeros. Note that by (8.6), $z=1$ is not a zero of $A_{m+1}(z)$. Therefore, by (8.5), the zeros of $A_{m+1}(z)$ must be the zeros of the derivative of

$$
\begin{equation*}
z \cdot A_{m}(z) \cdot(1-z)^{-m-1} \tag{8.9}
\end{equation*}
$$

By Rolle's Theorem, this derivative has zeros between the zeros of $A_{m}(z)$ as well as one between the greatest zero of $A_{m}(z)$ and 0 . Since (8.9) goes to 0 as $z \rightarrow-\infty$, the derivative is also zero at a value less than the least zero of $A_{m}(z)$. This accounts for all $m$ zeros of $A_{m+1}(z)$ and completes the induction. That the zeros come in reciprocal pairs follows from (8.7).

In Pey66, Thm. 4], the above theorem is extended to all negative real numbers:
Theorem 8.3 ([Pey66, Thm. 4]). For $r<0, \operatorname{Li}_{r}(z)$ has $-\lfloor r\rfloor$ simple zeros for $z \in \mathbb{C}-[1, \infty)$ and they are all $\leqslant 0$.

Sobolev, in Sob77, seems to have been the first to locate the zeros of $A_{m}(z)$ for $m$ large, with an explicit form for the error given in Sir78. We give this result next along with the proof, since Sob77, Sir78 give only brief summaries. Further results on these zeros appear in Sob78, Sob79a, Sob79b.

Theorem 8.4. Fix $M>1$. Suppose $m$ is large enough that

$$
K:=\left(1+\frac{1}{4} \sqrt{9 \pi^{2}+\log ^{2} M}\right)\left(\frac{\pi^{2}+\log ^{2} M}{9 \pi^{2}+\log ^{2} M}\right)^{(m+1) / 2} \leqslant 1 / 3
$$

Label the zeros of $A_{m}(z)$ as $\lambda_{m-1}<\lambda_{m-2}<\cdots<\lambda_{1}<0$. Then for each $j$ with $-M \leqslant \lambda_{j} \leqslant-1 / M$ we have

$$
\lambda_{j}=-\exp \left(-\pi \cot \left(\frac{\pi\left(2 j+1+\varepsilon_{j}\right)}{2(m+1)}\right)\right) \quad \text { for } \quad\left|\varepsilon_{j}\right| \leqslant 2 K / 3
$$

Proof. Lipschitz's formula (an application of Poisson summation) gives

$$
\begin{equation*}
\frac{\pi^{1-s}}{\Gamma(1-s)} \sum_{n=1}^{\infty} \frac{e^{\pi n z}}{n^{s}}=\sum_{k \in \mathbb{Z}}(2 k i-z)^{s-1} \quad(\operatorname{Re}(z), \operatorname{Re}(s)<0) \tag{8.10}
\end{equation*}
$$

in a special case; see Rad73, Sect. 37]. With $z=i+t$ and $s=-m$ in (8.10) we obtain

$$
\begin{equation*}
\frac{\pi^{1+m}}{m!} \operatorname{Li}_{-m}\left(-e^{\pi t}\right)=2 \operatorname{Re} \sum_{k=1}^{\infty} \frac{1}{((2 k-1) i-t)^{m+1}} \quad\left(m \in \mathbb{Z}_{\geqslant 1}\right) \tag{8.11}
\end{equation*}
$$

Equation (8.11) is now valid for all $t \in \mathbb{R}$, because of the convergence of the right side of (8.11), and clearly both sides of (8.11) are smooth functions of $t$. For $m$ large, the term with $k=1$ on the right side of (8.11) is largest. Let $R_{m}(t)$ be the rest of the series. Then

$$
\begin{equation*}
\operatorname{Li}_{-m}\left(-e^{\pi t}\right)=0 \Longleftrightarrow 2 \operatorname{Re} \frac{1}{(i-t)^{m+1}}+R_{m}(t)=0 \tag{8.12}
\end{equation*}
$$

where

$$
\begin{aligned}
\left|R_{m}(t)\right| & \leqslant \sum_{k=2}^{\infty} \frac{2}{\left((2 k-1)^{2}+t^{2}\right)^{(m+1) / 2}} \\
& \leqslant \frac{2}{\left(9+t^{2}\right)^{(m+1) / 2}}+\int_{2}^{\infty} \frac{2 d x}{\left((2 x-1)^{2}+t^{2}\right)^{(m+1) / 2}}
\end{aligned}
$$

This last integral is

$$
\begin{aligned}
\int_{3}^{\infty} \frac{d u}{\left(u^{2}+t^{2}\right)^{(m+1) / 2}} & \leqslant \frac{1}{\left(9+t^{2}\right)^{(m-1) / 2}} \int_{3}^{\infty} \frac{d u}{u^{2}+t^{2}} \\
& \leqslant \frac{1}{\left(9+t^{2}\right)^{(m-1) / 2}} \int_{0}^{\infty} \frac{d u}{u^{2}+9+t^{2}}=\frac{\pi}{2\left(9+t^{2}\right)^{m / 2}}
\end{aligned}
$$

and we have shown that

$$
\begin{equation*}
\left|R_{m}(t)\right| \leqslant \frac{1}{\left(9+t^{2}\right)^{(m+1) / 2}}\left(2+\frac{\pi}{2} \sqrt{9+t^{2}}\right) \tag{8.13}
\end{equation*}
$$

Next write

$$
\text { it }+1=\sqrt{1+t^{2}} \cdot e^{i \pi \theta} \quad \text { with } \quad t=\tan (\pi \theta) \quad \text { for } \quad-\frac{\pi}{2}<\theta<\frac{\pi}{2}
$$

to get

$$
2 \operatorname{Re} \frac{1}{(i-t)^{m+1}}=\frac{2 \cos (\pi(m+1)(\theta+1 / 2))}{\left(1+t^{2}\right)^{(m+1) / 2}}
$$

Let

$$
R_{m}^{*}(t):=\frac{1}{2}\left(1+t^{2}\right)^{(m+1) / 2} R_{m}(t)
$$

and (8.12) implies that $\mathrm{Li}_{-m}\left(-e^{\pi \tan (\pi \theta)}\right)=0$ if and only if

$$
\begin{equation*}
\cos (\pi(m+1)(\theta+1 / 2))+R_{m}^{*}(\tan (\pi \theta))=0 \tag{8.14}
\end{equation*}
$$

The values of $t$ we are interested in have $-M \leqslant-e^{\pi t} \leqslant-1 / M$, which is equivalent to $\pi|t| \leqslant \log M$, and we see from (8.13) that $\left|R_{m}^{*}(t)\right| \leqslant K \leqslant 1 / 3$ for these values of $t$. The corresponding range of $\theta$ is $-X \leqslant \theta \leqslant X$ for

$$
X:=\frac{1}{\pi} \arctan \left(\frac{\log M}{\pi}\right), \quad 0<X<\frac{\pi}{2} .
$$

So now we study the left side of (8.14) for $-X \leqslant \theta \leqslant X$. Write $\theta$ uniquely as

$$
\begin{equation*}
\theta=\frac{2 j-m+\varepsilon}{2(m+1)} \quad \text { for } \quad j \in \mathbb{Z},-1<\varepsilon \leqslant 1, \tag{8.15}
\end{equation*}
$$

and all the zeros of $\cos (\pi(m+1)(\theta+1 / 2))$ occur for $j \in \mathbb{Z}$ and $\varepsilon=0$ since

$$
\begin{equation*}
\cos (\pi(m+1)(\theta+1 / 2))=(-1)^{j+1} \sin (\pi \varepsilon / 2) \tag{8.16}
\end{equation*}
$$

Lemma 8.5. If $\operatorname{Li}_{-m}\left(-e^{\pi \tan (\pi \theta)}\right)=0$ for $-X \leqslant \theta \leqslant X$, then $\theta$ is of the form (8.15) for some $j \in \mathbb{Z}$ and $\varepsilon$ satisfying $|\varepsilon| \leqslant 2 K / 3$.

Proof. With (8.16) and (8.14)

$$
|\sin (\pi \varepsilon / 2)|=|\cos (\pi(m+1)(\theta+1 / 2))|=\left|R_{m}^{*}(\tan (\pi \theta))\right| \leqslant K
$$

Recalling that $|\sin (\pi x)| \geqslant 3|x|$ for $|x| \leqslant 1 / 6$ gives the desired inequality for $\varepsilon$.
Set $\theta_{j}:=\frac{2 j-m}{2(m+1)}$ for $j \in \mathbb{Z}$. We see now that, for $-X \leqslant \theta \leqslant X$, the left side of (8.14) is possibly zero only for $\theta$ in intervals of the form $\left[\theta_{j}-\frac{K}{3(m+1)}, \theta_{j}+\frac{K}{3(m+1)}\right]$. It is also clear from Lemma 8.5 and (8.16) that outside these intervals the left side of (8.14) alternates $>0$ and $<0$. Therefore there is at least one zero in each such interval.

The next lemma shows there is at most one zero for $\theta$ in each of these intervals; this point was not addressed in Sob77, Sir78.
Lemma 8.6. The left side of (8.14) is strictly increasing or decreasing for $-X \leqslant$ $\theta \leqslant X$ when $\theta$ is in the interval $\left[\theta_{j}-\frac{K}{3(m+1)}, \theta_{j}+\frac{K}{3(m+1)}\right]$.
Proof. A short computation, using $\cos ^{2}(\pi \theta)=1+t^{2}$, shows that $\frac{d}{d \theta}$ of the left of (8.14) may be expressed as

$$
\begin{equation*}
\pi(m+1)\left[-\sin (\pi(m+1)(\theta+1 / 2))+\frac{t}{\left(1+t^{2}\right)^{2}} R_{m}^{*}(t)+\frac{1}{\left(1+t^{2}\right)^{2}} R_{m+1}^{*}(t)\right] . \tag{8.17}
\end{equation*}
$$

Write $\theta$ in the form (8.15) with $|\varepsilon| \leqslant 2 K / 3$. Since

$$
\begin{gathered}
|\sin (\pi(m+1)(\theta+1 / 2))|=|\cos (\pi \varepsilon / 2)| \\
\left|\frac{t}{\left(1+t^{2}\right)^{2}} R_{m}^{*}(t)+\frac{1}{\left(1+t^{2}\right)^{2}} R_{m+1}^{*}(t)\right|<2 K
\end{gathered}
$$

we see that (8.17) is non-zero if $|\cos (\pi \varepsilon / 2)|>2 K$ and this is equivalent to $1-$ $\sin ^{2}(\pi \varepsilon / 2)>4 K^{2}$. With $|\sin (x)| \leqslant|x|$ we see that $|\sin (\pi \varepsilon / 2)| \leqslant \pi K / 3$ for $|\varepsilon| \leqslant$ $2 K / 3$. Hence

$$
\begin{equation*}
K^{2}<1 /\left(4+\pi^{2} / 3^{2}\right) \tag{8.18}
\end{equation*}
$$

implies (8.17) is non-zero. Inequality (8.18) is true by our assumption $K \leqslant 1 / 3$.

We have therefore shown that every zero $\lambda$ of $\operatorname{Li}_{-m}(z)$, for $-M \leqslant \lambda \leqslant-1 / M$, is of the form

$$
\begin{align*}
\lambda & =-\exp \left(\pi \tan \left(\pi \frac{2 j-m+\varepsilon}{2(m+1)}\right)\right) \\
& =-\exp \left(-\pi \cot \left(\pi \frac{2 j+1+\varepsilon}{2(m+1)}\right)\right) \quad \text { with } \quad|\varepsilon| \leqslant 2 K / 3 \tag{8.19}
\end{align*}
$$

and also that there is exactly one zero of the form (8.19) when the interval $\left[\theta_{j}-\frac{K}{3(m+1)}, \theta_{j}+\frac{K}{3(m+1)}\right]$ is contained in $(-X, X)$. It only remains to match (8.19) with the ordering of the zeros $\lambda_{m-1}<\cdots<\lambda_{1}$. (Recall from Theorem 8.2 that $\lambda_{m-j}=1 / \lambda_{j}$.) If $m$ is even, then $j=m / 2$ in (8.19) gives the zero of $\operatorname{Li}_{-m}(z)$ closest to $z=-1$. In fact $z=\lambda_{m / 2}=-1$ is the middle zero in this case. Also $j=m / 2-1$ in (8.19) gives the next zero to the right and $j=m / 2+1$ the next to the left. Hence $\lambda_{j}$ is given by (8.19). Similarly for $m$ odd. This completes the proof of Theorem 8.4.

The above theorem is also essentially equivalent to [GS84, Thm. 4], and we have used their ordering of the zeros since it generalizes to other $s$ values more readily. As an example, take $M=1000$ and $m=10$, giving $K \approx 0.034$. The leftmost two zeros of $A_{10}(z)$ are $\lambda_{9} \approx-963.85, \lambda_{8} \approx-37.54$ and all zeros are between $-M$ and $-1 / M$. By comparison, the values of $-\exp \left(-\pi \cot \left(\pi \frac{2 j+1}{2(m+1)}\right)\right)$ for $j=9,8$ are $-971.78,-37.55$. We have $\left|\varepsilon_{j}\right|<0.0032$ for $1 \leqslant j \leqslant 9$ and this is less than $2 K / 3 \approx 0.023$.


Figure 5. The zeros of $\operatorname{Li}_{s}(z)$ for $s=-10-44 i$

Gawronski and Stadtmüller in [GS84 generalize Theorems 8.4 and 8.3 to the zeros of $\operatorname{Li}_{s}(z)$ for $s \in \mathbb{C}$ with $\operatorname{Re}(s)<0$. The required computations become much more elaborate in this general case. Though the work in GS84 is independent of Sob77, it is based on the same essential idea. Starting with (8.10), replace $z$ with $z+i$ and ignore the terms with $k \neq 0,1$ to show that

$$
\frac{\pi^{1-s}}{\Gamma(1-s)} \operatorname{Li}_{s}\left(-e^{\pi z}\right) \approx \frac{1}{(i-z)^{1-s}}+\frac{1}{(-i-z)^{1-s}}
$$

where the error may be explicitly bounded. Therefore

$$
\begin{aligned}
\operatorname{Li}_{s}\left(-e^{\pi z}\right)=0 & \Longleftrightarrow 1+\left(\frac{i-z}{-i-z}\right)^{1-s} \approx 0 \\
& \Longleftrightarrow \frac{i-z}{-i-z} \approx \exp \left(\frac{\pi i(2 j+1)}{1-s}\right) \quad(j \in \mathbb{Z}) \\
& \Longleftrightarrow z \approx-\cot \left(\frac{\pi(2 j+1)}{2(1-s)}\right) \quad(j \in \mathbb{Z}),
\end{aligned}
$$

and we expect the zeros of $\operatorname{Li}_{s}(z)$ to approximately take the form

$$
\begin{equation*}
-\exp \left(-\pi \cot \left(\frac{\pi(2 j+1)}{2(1-s)}\right)\right) . \tag{8.20}
\end{equation*}
$$

Due to branch considerations, (8.20) will correspond to zeros of $\mathrm{Li}_{s}(z)$ only for certain integers $j \geqslant 0$; see GS84, Lemma 2, Thm. 2].

Figure 5 shows the zeros of $\operatorname{Li}_{s}(z)$ for $s=-10-44 i$, found numerically. They are indistinguishable, at the scale of the figure, from the points 0 and (8.20) for $j=0,1, \cdots, 139$. This $s$ satisfies $s=(1+v)(1+4 i)-1$ for $v=10$. If we change $s$ by increasing $v$, then the number of zeros of $\operatorname{Li}_{s}(z)$ increases, all getting closer to the spiral shown in Figure 5 and filling it more densely. See [GS84, Thm. 2] for precise statements.

It should be possible to extend the results in GS84 to all branches. We have seen that $\mathrm{Li}_{2}(z)$ has infinitely many zeros if we include those of every branch, while $\mathrm{Li}_{1}(z)$ has only one zero. A natural question arises: for any fixed $s \in \mathbb{C}$, how many zeros does $\operatorname{Li}_{s}(z)$ have on all branches?

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