# AN $h p$-VERSION LEGENDRE-JACOBI SPECTRAL COLLOCATION METHOD FOR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS WITH SMOOTH AND WEAKLY SINGULAR KERNELS 

ZHONG-QING WANG, YU-LING GUO, AND LI-JUN YI


#### Abstract

In this paper, we present an $h p$-version Legendre-Jacobi spectral collocation method for Volterra integro-differential equations with smooth and weakly singular kernels. We establish several new approximation results of the Legendre/Jacobi polynomial interpolations for both smooth and singular functions. As applications of these approximation results, we derive $h p$-version error bounds of the Legendre-Jacobi collocation method under the $H^{1}$-norm for the Volterra integro-differential equations with smooth solutions on arbitrary meshes and singular solutions on quasi-uniform meshes. We also show the exponential rates of convergence for singular solutions by using geometric time partitions and linearly increasing polynomial degrees. Numerical experiments are included to illustrate the theoretical results.


## 1. Introduction

In this paper, we consider the numerical solution of the linear Volterra integrodifferential equation (VIDE) of the form

$$
\left\{\begin{array}{l}
u^{\prime}(t)+a(t) u(t)+\int_{0}^{t}(t-s)^{-\mu} b(s) u(s) d s=f(t), \quad t \in(0, T]  \tag{1.1}\\
u(0)=u_{0}
\end{array}\right.
$$

where $\mu<1$ (i.e., the kernel is weakly singular if $0<\mu<1$, and in particular, the kernel is smooth if $\left.\mu \in \mathbb{N}_{0}^{-}:=\{0,-1,-2, \cdots\}\right)$. Moreover, the real functions $a(t)$, $b(t)$ and $f(t)$ are continuous on $I:=[0, T]$, and $u_{0}$ is the initial data.

Over the last few decades, various numerical methods have been proposed and analyzed for linear VIDEs of the form (1.1) with smooth and weakly singular kernels; see, for example, Runge-Kutta methods [1,25], collocation methods [2, 19], continuous and discontinuous Galerkin methods [11,12]. We also refer to the monographs [3, 5 and the references therein.

The works mentioned above are mainly concerned with the so-called $h$-version method, which means that the convergence is obtained by decreasing the size of the time steps at a fixed order of approximation, and usually the resulting error bounds

[^0]do not deal with the explicit dependence of the constant $c$ on the approximation order. To address this issue, some high order methods, for instance, the $p$-version, the $h p$-version and the spectral (element) methods, need to be studied. Such methods usually approximate the problem under consideration by increasing the approximation order on a fixed time partition ( $p$-approach) or, alternatively, on variable time steps ( $h p$-approach). Since the $h p$-version method allows for locally varying time steps and approximation orders, it can approximate smooth solutions with possible local singularities at high algebraic or even exponential rates of convergence. Due to their high accuracy, the high order methods for integral or integro-differential equations of Volterra type has received considerable attention in recent years. For example, the $h p$-version discontinuous Galerkin time-stepping method has been developed for VIDEs in [4] and for parabolic VIDEs in [13], and the $h p$-version continuous Petrov-Galerkin time-stepping method was recently analyzed for linear and nonlinear VIDEs in [23, 24]. Meanwhile, the $p$-version spectral Galerkin and collocation methods were also studied for integral or integro-differential equations of Volterra type; see, e.g., 6, 8 -10, 20, 21] and the references therein. Furthermore, the $h p$-version Legendre spectral collocation methods were proposed for nonlinear Volterra integral equations with smooth kernels in [18] and for nonlinear Volterra integral equations with smooth kernels and variable delays in [22]. However, to the best of our knowledge, there are no theoretical results available for the weakly singular VIDEs by $h p$-version collocation methods, which are much more difficult to analyze than spectral Galerkin methods. The main difficulties encountered in the $h p$-version of collocation methods for VIDEs include: (i) how to design an efficient algorithm ensuring the optimal convergence of the $h p$-version; (ii) how to analyze the convergence of the $h p$-version for smooth solutions, on account of the influence of the weakly singular kernels; (iii) how to analyze the convergence of the $h p$-version for singular solutions, and especially, the exponential convergence for singular solutions under geometric time partitions.

In the present work, we introduce and analyze an $h p$-version Legendre-Jacobi spectral collocation method for the VIDE (1.1) with smooth and weakly singular kernels. The $h p$-version Legendre-Jacobi spectral collocation scheme is constructed based on three kinds of polynomial interpolations (see (2.21)), i.e., the LegendreGauss, Legendre-Gauss-Lobatto, and Jacobi-Gauss interpolations. Particularly, in the numerical scheme (2.21), we choose a special weight $(\alpha, \beta)=(-\mu, 0)$ for the Jacobi-Gauss interpolation $\mathcal{I}_{t, M_{n}}^{\alpha, \beta} v$ (see the definition (2.10)), which corresponds to the kernel $(t-s)^{-\mu}$ in the VIDE (1.1). We carry out a rigorous error analysis of the proposed method for both smooth and singular solutions and present some numerical experiments to verify the theoretical results. The main features and contributions of this paper are highlighted as follows.

- For analyzing the numerical errors, we derive several new approximation results of the Legendre-Jacobi polynomial interpolations for both smooth and singular functions. These approximation results can also be applied to pseudo-spectral and spectral collocation methods for other problems, especially for those with solutions of $t^{\nu}$-type singularity.
- The key feature of the $h p$-version Legendre-Jacobi spectral collocation is its great flexibility with respect to the size of the time steps and the local approximation orders, which enable us to cope with problems with nonsmooth solutions. Indeed, our numerical results show that for analytic
solutions with start-up singularities, exponential rates of convergence can be achieved for the $h p$-version collocation method with geometric time steps and linearly increasing approximation orders.
- We establish a priori error estimate under the $H^{1}$-norm that is explicit in the time steps and the approximation orders. In particular, for the VIDE (1.1) with weakly singular kernel and singular solutions of $t^{\nu}$-type, we derive optimal error bounds under the $H^{1}$-norm for the $h p$-version collocation method with quasi-uniform meshes. It is shown that the $p$-version gives twice the rate of convergence as the $h$-version for singular solutions on quasi-uniform meshes, which coincides with the well-known phenomenon in the $p$-version of the finite element method for elliptic problems with corner singularities (see, e.g., 16]). We also show the exponential rates of convergence for singular solutions by using geometric time partitions and linearly increasing polynomial degrees.

The remainder of the paper is organized as follows. In Section 2, we introduce some basic properties of the shifted Legendre-Jacobi polynomial interpolations and propose the $h p$-version Legendre-Jacobi spectral collocation method for the VIDE (1.1). In Section 3, we establish some new approximation results of the LegendreJacobi polynomial interpolations for both smooth and singular functions, which are very useful for the convergence analysis. In Section 4, we derive the $h p$-version error bounds of the Legendre-Jacobi collocation methods, for smooth solutions on an arbitrary mesh and for singular solutions on a quasi-uniform mesh. We also show the exponential rates of convergence for singular solutions under geometric time partitions. Our theoretical results are verified by the numerical experiments in Section 5. Some concluding remarks are given in the last section.

## 2. The $h p$-Version Legendre-Jacobi spectral collocation method

In this section, we shall introduce some basic properties of the shifted LegendreJacobi polynomial interpolations, and propose an $h p$-version Legendre-Jacobi spectral collocation method for the VIDE (1.1).
2.1. Preliminaries. Let $I_{h}$ be a mesh on the interval $I$,

$$
I_{h}:=\left\{t_{n}: 0=t_{0}<t_{1}<\cdots<t_{N}=T\right\} .
$$

We set $h_{n}=t_{n}-t_{n-1}, I_{n}=\left(t_{n-1}, t_{n}\right]$, and denote by $u^{n}(t)$ the solution of (1.1) on the $n$-th element, namely,

$$
u^{n}(t):=u(t), \quad t \in I_{n}, \quad 1 \leq n \leq N .
$$

From (1.1) we have that for any $t \in I_{n}$,
$\frac{d}{d t} u^{n}(t)+a(t) u^{n}(t)+\sum_{k=1}^{n-1} \int_{I_{k}}(t-s)^{-\mu} b(s) u^{k}(s) d s+\int_{t_{n-1}}^{t}(t-\xi)^{-\mu} b(\xi) u^{n}(\xi) d \xi=f(t)$.
In order to transfer the integral interval $\left(t_{n-1}, t\right]$ to $I_{n}$, we make the following transformation:

$$
\begin{equation*}
\xi=\sigma(\lambda, t):=t_{n-1}+\frac{\left(\lambda-t_{n-1}\right)\left(t-t_{n-1}\right)}{h_{n}}, \quad \lambda \in I_{n} . \tag{2.2}
\end{equation*}
$$

Then equation (2.1) becomes

$$
\begin{align*}
\frac{d}{d t} u^{n}(t) & +a(t) u^{n}(t)+\sum_{k=1}^{n-1} \int_{I_{k}}(t-s)^{-\mu} b(s) u^{k}(s) d s  \tag{2.3}\\
& +\left(\frac{t-t_{n-1}}{h_{n}}\right)^{1-\mu} \int_{I_{n}}\left(t_{n}-\lambda\right)^{-\mu} b(\sigma(\lambda, t)) u^{n}(\sigma(\lambda, t)) d \lambda=f(t)
\end{align*}
$$

Hereafter, for a given interval $\Lambda$ and a certain weight function $\chi(x)$, we define

$$
L_{\chi}^{2}(\Lambda)=\left\{v \mid v \text { is measurable and }\|v\|_{L_{\chi}^{2}(\Lambda)}<\infty\right\}
$$

with the norm $\|v\|_{L_{\chi}^{2}(\Lambda)}=\left(\int_{\Lambda}|v(x)|^{2} \chi(x) d x\right)^{\frac{1}{2}}$. We also denote by $H_{\chi}^{r}(\Lambda)$ the usual weighted Sobolev space.

### 2.2. The shifted polynomial interpolation on $I_{n}$.

2.2.1. The shifted Jacobi-Gauss interpolation on $I_{n}$. For $\alpha, \beta>-1$, let $J_{k}^{\alpha, \beta}(x)$, $x \in(-1,1)$, be the standard Jacobi polynomial of degree $k$, and denote the weight function $\chi^{\alpha, \beta}(x)=(1-x)^{\alpha}(1+x)^{\beta}$. The set of Jacobi polynomials is a complete $L_{\chi^{\alpha, \beta}}^{2}(-1,1)$-orthogonal system, i.e.,

$$
\begin{equation*}
\int_{-1}^{1} J_{k}^{\alpha, \beta}(x) J_{j}^{\alpha, \beta}(x) \chi^{\alpha, \beta}(x) d x=\gamma_{k}^{\alpha, \beta} \delta_{k, j} \tag{2.4}
\end{equation*}
$$

where $\delta_{k, j}$ is the Kronecker function, and

$$
\gamma_{k}^{\alpha, \beta}= \begin{cases}\frac{2^{\alpha+\beta+1} \Gamma(\alpha+1) \Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}, & k=0 \\ \frac{2^{\alpha+\beta+1}}{(2 k+\alpha+\beta+1)} \frac{\Gamma(k+\alpha+1) \Gamma(k+\beta+1)}{k!\Gamma(k+\alpha+\beta+1)}, & k \geq 1\end{cases}
$$

In particular, $J_{0}^{\alpha, \beta}(x)=1$ and

$$
\begin{equation*}
\int_{-1}^{1} \frac{d}{d x} J_{k}^{\alpha, \beta}(x) \frac{d}{d x} J_{j}^{\alpha, \beta}(x) \chi^{\alpha+1, \beta+1}(x) d x=k(k+\alpha+\beta+1) \gamma_{k}^{\alpha, \beta} \delta_{k, j} \tag{2.5}
\end{equation*}
$$

The shifted Jacobi polynomial of degree $k$ is defined by

$$
\begin{equation*}
J_{n, k}^{\alpha, \beta}(t)=J_{k}^{\alpha, \beta}\left(\frac{2 t-t_{n-1}-t_{n}}{h_{n}}\right), \quad t \in I_{n}, \quad k \geq 0 . \tag{2.6}
\end{equation*}
$$

The set of $J_{n, k}^{\alpha, \beta}(t), k \geq 0$, is a complete $L_{\chi_{n}^{\alpha, \beta}}^{2}\left(I_{n}\right)$-orthogonal system with the weight function $\chi_{n}^{\alpha, \beta}(t)=\left(t_{n}-t\right)^{\alpha}\left(t-t_{n-1}\right)^{\beta}$, namely,

$$
\begin{equation*}
\int_{I_{n}} J_{n, k}^{\alpha, \beta}(t) J_{n, j}^{\alpha, \beta}(t) \chi_{n}^{\alpha, \beta}(t) d t=\left(\frac{h_{n}}{2}\right)^{\alpha+\beta+1} \gamma_{k}^{\alpha, \beta} \delta_{k, j} \tag{2.7}
\end{equation*}
$$

We now turn to the Jacobi-Gauss interpolation. For any given integer $M_{n} \geq 0$, we denote by $\left\{x_{n, j}^{\alpha, \beta}, \omega_{n, j}^{\alpha, \beta}\right\}_{j=0}^{M_{n}}$ the nodes and the corresponding Christoffel numbers of the standard Jacobi-Gauss interpolation on the interval $(-1,1)$. Let $\mathcal{P}_{M_{n}}\left(I_{n}\right)$ be the set of polynomials of degree at most $M_{n}$ on the interval $I_{n}$, and let $t_{n, j}^{\alpha, \beta}$ be the shifted Jacobi-Gauss quadrature nodes on the interval $I_{n}$,

$$
t_{n, j}^{\alpha, \beta}=\frac{1}{2}\left(h_{n} x_{n, j}^{\alpha, \beta}+t_{n-1}+t_{n}\right), \quad 0 \leq j \leq M_{n}
$$

Due to the property of the standard Jacobi-Gauss quadrature, it follows that for any $\phi(t) \in \mathcal{P}_{2 M_{n}+1}\left(I_{n}\right)$,

$$
\begin{align*}
\int_{I_{n}} \phi(t) \chi_{n}^{\alpha, \beta}(t) d t & =\left(\frac{h_{n}}{2}\right)^{\alpha+\beta+1} \int_{-1}^{1} \phi\left(\frac{h_{n} x+t_{n-1}+t_{n}}{2}\right) \chi^{\alpha, \beta}(x) d x \\
& =\left(\frac{h_{n}}{2}\right)^{\alpha+\beta+1} \sum_{j=0}^{M_{n}} \phi\left(\frac{h_{n} x_{n, j}^{\alpha, \beta}+t_{n-1}+t_{n}}{2}\right) \omega_{n, j}^{\alpha, \beta}  \tag{2.8}\\
& =\left(\frac{h_{n}}{2}\right)^{\alpha+\beta+1} \sum_{j=0}^{M_{n}} \phi\left(t_{n, j}^{\alpha, \beta}\right) \omega_{n, j}^{\alpha, \beta} .
\end{align*}
$$

By (2.7) and (2.8), we further obtain that for any $0 \leq p+q \leq 2 M_{n}+1$,

$$
\begin{equation*}
\sum_{j=0}^{M_{n}} J_{n, p}^{\alpha, \beta}\left(t_{n, j}^{\alpha, \beta}\right) J_{n, q}^{\alpha, \beta}\left(t_{n, j}^{\alpha, \beta}\right) \omega_{n, j}^{\alpha, \beta}=\gamma_{p}^{\alpha, \beta} \delta_{p, q} \tag{2.9}
\end{equation*}
$$

We denote by $\mathcal{I}_{t, M_{n}}^{\alpha, \beta}: C\left(I_{n}\right) \rightarrow \mathcal{P}_{M_{n}}\left(I_{n}\right)$ the shifted Jacobi-Gauss interpolation operator in the $t$-direction, such that

$$
\begin{equation*}
\mathcal{I}_{t, M_{n}}^{\alpha, \beta} v\left(t_{n, j}^{\alpha, \beta}\right)=v\left(t_{n, j}^{\alpha, \beta}\right), \quad 0 \leq j \leq M_{n} \tag{2.10}
\end{equation*}
$$

2.2.2. The shifted Legendre-Gauss interpolation on $I_{n}$. In the special case where $\alpha=\beta=0$, the shifted Jacobi polynomial $J_{n, k}^{0,0}(t)$ is reduced to the shifted Legendre polynomial $L_{n, k}(t)$. Accordingly, we write $t_{n, j}:=t_{n, j}^{0,0}$ and $\omega_{n, j}:=\omega_{n, j}^{0,0}$. Moreover, we denote by $\mathcal{I}_{t, M_{n}}:=\mathcal{I}_{t, M_{n}}^{0,0}$ the shifted Legendre-Gauss interpolation operator in the $t$-direction. According to the properties of the standard Legendre polynomials, we have (cf. [22])

$$
\begin{equation*}
\int_{I_{n}} L_{n, k}(t) L_{n, j}(t) d t=\frac{h_{n}}{2 k+1} \delta_{k, j}, \tag{2.11}
\end{equation*}
$$

$(k+1) L_{n, k+1}(t)-h_{n}^{-1}(2 k+1)\left(2 t-t_{n-1}-t_{n}\right) L_{n, k}(t)+k L_{n, k-1}(t)=0, \quad k \geq 1$,

$$
\begin{equation*}
L_{n, k+1}^{\prime}(t)-L_{n, k-1}^{\prime}(t)=\frac{4 k+2}{h_{n}} L_{n, k}(t), \quad k \geq 1 . \tag{2.13}
\end{equation*}
$$

In particular,

$$
\begin{align*}
& L_{n, 0}(t)=1, \quad L_{n, 1}(t)=\frac{2 t-t_{n-1}-t_{n}}{h_{n}} \\
& L_{n, 2}(t)=\frac{6 t^{2}-6\left(t_{n-1}+t_{n}\right) t+4 t_{n-1} t_{n}+t_{n-1}^{2}+t_{n}^{2}}{h_{n}^{2}} \tag{2.14}
\end{align*}
$$

Moreover, by taking $\alpha=\beta=0$ in (2.8) and (2.9), we get that

$$
\begin{equation*}
\int_{I_{n}} \phi(t) d t=\frac{h_{n}}{2} \sum_{j=0}^{M_{n}} \phi\left(t_{n, j}\right) \omega_{n, j}, \quad \forall \phi \in \mathcal{P}_{2 M_{n}+1}\left(I_{n}\right), \tag{2.15}
\end{equation*}
$$

and

$$
\begin{equation*}
\sum_{j=0}^{M_{n}} L_{n, p}\left(t_{n, j}\right) L_{n, q}\left(t_{n, j}\right) \omega_{n, j}=\frac{2}{2 p+1} \delta_{p, q}, \quad \forall 0 \leq p+q \leq 2 M_{n}+1 \tag{2.16}
\end{equation*}
$$

2.2.3. The shifted Legendre-Gauss-Lobatto interpolation on $I_{n}$. For $M_{n} \geq 0$, let $\left\{x_{n, j}^{L}, \omega_{n, j}^{L}\right\}_{j=0}^{M_{n}+1}$ be the nodes and the corresponding Christoffel numbers of the standard Legendre-Gauss-Lobatto interpolation on the interval $[-1,1]$, and

$$
t_{n, j}^{L}:=\frac{1}{2}\left(h_{n} x_{n, j}^{L}+t_{n-1}+t_{n}\right), \quad 0 \leq j \leq M_{n}+1 .
$$

We also denote by $\mathcal{I}_{t, M_{n}+1}^{L}$ the shifted Legendre-Gauss-Lobatto interpolation operator in the $t$-direction with $\mathcal{I}_{t, M_{n}+1}^{L} v \in \mathcal{P}_{M_{n}+1}\left(I_{n}\right)$ and

$$
\begin{equation*}
\mathcal{I}_{t, M_{n}+1}^{L} v\left(t_{n, j}^{L}\right)=v\left(t_{n, j}^{L}\right), \quad 0 \leq j \leq M_{n}+1 \tag{2.17}
\end{equation*}
$$

According to the properties of the standard Legendre-Gauss-Lobatto quadrature formulas, we get

$$
\begin{equation*}
\int_{I_{n}} \phi(t) d t=\frac{h_{n}}{2} \sum_{j=0}^{M_{n}+1} \phi\left(t_{n, j}^{L}\right) \omega_{n, j}^{L}, \quad \forall \phi \in \mathcal{P}_{2 M_{n}+1}\left(I_{n}\right), \tag{2.18}
\end{equation*}
$$

and

$$
\begin{equation*}
\sum_{j=0}^{M_{n}+1} L_{n, p}\left(t_{n, j}^{L}\right) L_{n, q}\left(t_{n, j}^{L}\right) \omega_{n, j}^{L}=\frac{2}{2 p+1} \delta_{p, q}, \quad \forall 0 \leq p+q \leq 2 M_{n}+1 \tag{2.19}
\end{equation*}
$$

As mentioned in [3], due to the presence of the weak singularity (e.g., $\mu \in(0,1)$ ), it is natural to consider the weighted interpolatory quadrature formulas whose weights depend on the weakly singular factor $(t-s)^{-\mu}$ in the kernel. For any $\phi(s) \in \mathcal{P}_{M_{k}+1}\left(I_{k}\right)$, we introduce the weighted interpolatory quadrature formulas, defined by

$$
\begin{equation*}
\int_{I_{k}}(t-s)^{-\mu} \phi(s) d s=\sum_{j=0}^{M_{k}+1} \phi\left(t_{k, j}^{L}\right) \widetilde{\omega}_{k, j}^{L}(t), \quad t \in I_{n}, \quad k<n, \tag{2.20}
\end{equation*}
$$

where $\widetilde{\omega}_{k, j}^{L}(t)=\int_{I_{k}}(t-s)^{-\mu} l_{k, j}(s) d s$ and $\left\{l_{k, j}(s)\right\}_{j=0}^{M_{k}+1}$ are the Lagrange fundamental polynomials corresponding to the collocation points $\left\{t_{k, j}^{L}\right\}_{j=0}^{M_{k}+1}$. The function $\widetilde{\omega}_{k, j}^{L}(t)$ can be calculated precisely according to the properties of Legendre polynomials.
2.3. The $h p$-version Legendre-Jacobi spectral collocation scheme. The $h p$ version Legendre-Jacobi spectral collocation scheme for solving (2.3) is to seek $U^{n}(t) \in \mathcal{P}_{M_{n}+1}\left(I_{n}\right)$, such that

$$
\left\{\begin{array}{l}
\mathcal{I}_{t, M_{n}}\left(\frac{d}{d t} U^{n}(t)+a(t) U^{n}(t)+\sum_{k=1}^{n-1} \int_{I_{k}}(t-s)^{-\mu} \mathcal{I}_{s, M_{k}+1}^{L}\left(b(s) U^{k}(s)\right) d s\right.  \tag{2.21}\\
\left.+\left(\frac{t-t_{n-1}}{h_{n}}\right)^{1-\mu} \int_{I_{n}}\left(t_{n}-\lambda\right)^{-\mu} \mathcal{I}_{\lambda, M_{n}+1}^{-\mu, 0}\left(b(\sigma(\lambda, t)) U^{n}(\sigma(\lambda, t))\right) d \lambda\right) \\
\quad=\mathcal{I}_{t, M_{n}} f(t), \quad t \in I_{n}, \\
U^{n}\left(t_{n-1}\right)=U^{n-1}\left(t_{n-1}\right), \quad U^{1}\left(t_{0}\right)=u_{0}
\end{array}\right.
$$

where $U^{k}(t) \in \mathcal{P}_{M_{k}+1}\left(I_{k}\right)$ is the numerical solution of $u^{k}(t)$ on the interval $I_{k}$. We now describe the numerical implementations and present an algorithm for scheme (2.21). To this end, we set

$$
\begin{aligned}
& U^{n}(t)=\sum_{p=0}^{M_{n}+1} u_{p}^{n} L_{n, p}(t), \quad \Longrightarrow \quad \frac{d}{d t} U^{n}(t)=\sum_{p=1}^{M_{n}+1} u_{p}^{n} L_{n, p}^{\prime}(t), \\
& \mathcal{I}_{t, M_{n}}\left(a(t) U^{n}(t)\right)=\sum_{p=0}^{M_{n}} a_{p}^{n} L_{n, p}(t), \quad \mathcal{I}_{t, M_{n}} f(t)=\sum_{p=0}^{M_{n}} f_{p}^{n} L_{n, p}(t), \\
& \mathcal{I}_{t, M_{n}} \mathcal{I}_{\lambda, M_{n}+1}^{-\mu, 0}\left(\left(t-t_{n-1}\right)^{1-\mu} b(\sigma(\lambda, t)) U^{n}(\sigma(\lambda, t))\right) \\
& \quad=\sum_{p=0}^{M_{n}} \sum_{p^{\prime}=0}^{M_{n}+1} d_{p, p^{\prime}}^{n} L_{n, p}(t) J_{n, p^{\prime}}^{-\mu, 0}(\lambda), \\
& \mathcal{I}_{t, M_{n}} \widetilde{\omega}_{k, p^{\prime}}^{L}(t)=\sum_{p=0}^{M_{n}} \widehat{w}_{p^{\prime}, p}^{(k)} L_{n, p}(t) .
\end{aligned}
$$

Then by (2.22) and (2.13), a direct computation leads to

$$
\begin{aligned}
\mathcal{I}_{t, M_{n}}\left(a(t) U^{n}(t)\right)= & a_{0}^{n}+h_{n} \sum_{p=1}^{M_{n}} \frac{a_{p}^{n}}{4 p+2}\left(L_{n, p+1}^{\prime}(t)-L_{n, p-1}^{\prime}(t)\right) \\
= & a_{0}^{n}-\frac{h_{n} a_{2}^{n}}{10} L_{n, 1}^{\prime}(t)+h_{n} \sum_{p=2}^{M_{n}-1}\left(\frac{a_{p-1}^{n}}{4 p-2}-\frac{a_{p+1}^{n}}{4 p+6}\right) L_{n, p}^{\prime}(t) \\
& +\frac{h_{n} a_{M_{n}-1}^{n}}{4 M_{n}-2} L_{n, M_{n}}^{\prime}(t)+\frac{h_{n} a_{M_{n}}^{n}}{4 M_{n}+2} L_{n, M_{n}+1}^{\prime}(t) \\
= & h_{n} \sum_{p=1}^{M_{n}-1}\left(\frac{a_{p-1}^{n}}{4 p-2}-\frac{a_{p+1}^{n}}{4 p+6}\right) L_{n, p}^{\prime}(t)+\frac{h_{n} a_{M_{n}-1}^{n}}{4 M_{n}-2} L_{n, M_{n}}^{\prime}(t) \\
& +\frac{h_{n} a_{M_{n}}^{n}}{4 M_{n}+2} L_{n, M_{n}+1}^{\prime}(t)=: h_{n} \sum_{p=1}^{M_{n}+1} \widehat{a}_{p}^{n} L_{n, p}^{\prime}(t) .
\end{aligned}
$$

Similarly,

$$
\begin{align*}
\mathcal{I}_{t, M_{n}} f(t)= & h_{n} \sum_{p=1}^{M_{n}-1}\left(\frac{f_{p-1}^{n}}{4 p-2}-\frac{f_{p+1}^{n}}{4 p+6}\right) L_{n, p}^{\prime}(t) \\
& +\frac{h_{n} f_{M_{n}-1}^{n}}{4 M_{n}-2} L_{n, M_{n}}^{\prime}(t)+\frac{h_{n} f_{M_{n}}^{n}}{4 M_{n}+2} L_{n, M_{n}+1}^{\prime}(t)  \tag{2.24}\\
= & h_{n} \sum_{p=1}^{M_{n}+1} \widehat{f}_{p}^{n} L_{n, p}^{\prime}(t) .
\end{align*}
$$

Moreover, with the aid of $(\sqrt{2.20)},(\sqrt{2.22)}$ and $(2.13)$, for $1 \leq k \leq n-1$, we get (2.25)

$$
\begin{aligned}
& \mathcal{I}_{t, M_{n}} \int_{I_{k}}(t-s)^{-\mu} \mathcal{I}_{s, M_{k}+1}^{L}\left(b(s) U^{k}(s)\right) d s=\mathcal{I}_{t, M_{n}} \sum_{p^{\prime}=0}^{M_{k}+1} b\left(t_{k, p^{\prime}}^{L}\right) U^{k}\left(t_{k, p^{\prime}}^{L}\right) \widetilde{\omega}_{k, p^{\prime}}^{L}(t) \\
& =\sum_{p^{\prime}=0}^{M_{k}+1} b\left(t_{k, p^{\prime}}^{L}\right) U^{k}\left(t_{k, p^{\prime}}^{L}\right) \mathcal{I}_{t, M_{n}} \widetilde{\omega}_{k, p^{\prime}}^{L}(t)=\sum_{p^{\prime}=0}^{M_{k}+1} b\left(t_{k, p^{\prime}}^{L}\right) U^{k}\left(t_{k, p^{\prime}}^{L}\right) \sum_{p=0}^{M_{n}} \widehat{w}_{p^{\prime}, p}^{(k)} L_{n, p}(t) \\
& =h_{n} \sum_{p^{\prime}=0}^{M_{k}+1} b\left(t_{k, p^{\prime}}^{L}\right) U^{k}\left(t_{k, p^{\prime}}^{L}\right)\left(\sum_{p=1}^{M_{n}-1}\left(\frac{\widehat{w}_{p^{\prime}, p-1}^{(k)}}{4 p-2}-\frac{\widehat{w}_{p^{\prime}, p+1}^{(k)}}{4 p+6}\right) L_{n, p}^{\prime}(t)\right. \\
& \left.\quad+\frac{\widehat{w}_{p^{\prime}, M_{n}-1}^{(k)}}{4 M_{n}-2} L_{n, M_{n}}^{\prime}(t)+\frac{\widehat{w}_{p^{\prime}, M_{n}}^{(k)}}{4 M_{n}+2} L_{n, M_{n}+1}^{\prime}(t)\right)=: h_{n} \sum_{p=1}^{M_{n}+1} \widehat{b}_{p}^{k} L_{n, p}^{\prime}(t),
\end{aligned}
$$

where

$$
\begin{aligned}
& \widehat{b}_{p}^{k}=\sum_{p^{\prime}=0}^{M_{k}+1} b\left(t_{k, p^{\prime}}^{L}\right) U^{k}\left(t_{k, p^{\prime}}^{L}\right)\left(\frac{\widehat{w}_{p^{\prime}, p-1}^{(k)}}{4 p-2}-\frac{\widehat{w}_{p^{\prime}, p+1}^{(k)}}{4 p+6}\right), \quad 1 \leq p \leq M_{n}-1 \\
& \widehat{b}_{M_{n}}^{k}=\frac{1}{4 M_{n}-2} \sum_{p^{\prime}=0}^{M_{k}+1} b\left(t_{k, p^{\prime}}^{L}\right) U^{k}\left(t_{k, p^{\prime}}^{L}\right) \widehat{w}_{p^{\prime}, M_{n}-1}^{(k)} \\
& \widehat{b}_{M_{n}+1}^{k}=\frac{1}{4 M_{n}+2} \sum_{p^{\prime}=0}^{M_{k}+1} b\left(t_{k, p^{\prime}}^{L}\right) U^{k}\left(t_{k, p^{\prime}}^{L}\right) \widehat{w}_{p^{\prime}, M_{n}}^{(k)}
\end{aligned}
$$

Further, by using (2.22), (2.7) and a similar argument as before, we obtain that

$$
\begin{aligned}
& \int_{I_{n}}\left(t_{n}-\lambda\right)^{-\mu} \mathcal{I}_{t, M_{n}} \mathcal{I}_{\lambda, M_{n}+1}^{-\mu, 0}\left(\left(t-t_{n-1}\right)^{1-\mu} b(\sigma(\lambda, t)) U^{n}(\sigma(\lambda, t))\right) d \lambda \\
& =\sum_{p=0}^{M_{n}} \sum_{p^{\prime}=0}^{M_{n}+1} d_{p, p^{\prime}}^{n} L_{n, p}(t) \int_{I_{n}}\left(t_{n}-\lambda\right)^{-\mu} J_{n, p^{\prime}}^{-\mu, 0}(\lambda) d \lambda \\
& =\frac{h_{n}^{1-\mu}}{1-\mu} \sum_{p=0}^{M_{n}} d_{p, 0}^{n} L_{n, p}(t) \\
& =\frac{h_{n}^{2-\mu}}{1-\mu}\left(\sum_{p=1}^{M_{n}-1}\left(\frac{d_{p-1,0}^{n}}{4 p-2}-\frac{d_{p+1,0}^{n}}{4 p+6}\right) L_{n, p}^{\prime}(t)+\frac{d_{M_{n}-1,0}^{n}}{4 M_{n}-2} L_{n, M_{n}}^{\prime}(t)\right. \\
& \left.=: \frac{d_{M_{n}, 0}^{n}}{4 M_{n}+2} L_{n, M_{n}+1}^{\prime}(t)\right) \\
& =: \frac{h_{n}^{2-\mu}}{1-\mu} \sum_{p=1}^{M_{n}+1} \widehat{d}_{p, 0}^{n} L_{n, p}^{\prime}(t)
\end{aligned}
$$

Applying (2.11), (2.15), (2.7) and (2.8) (with $M_{n}+1$ instead of $M_{n}$ ) to (2.22), one can verify readily that

$$
\begin{aligned}
& a_{p}^{n}=\frac{2 p+1}{2} \sum_{j=0}^{M_{n}} a\left(t_{n, j}\right) U^{n}\left(t_{n, j}\right) L_{n, p}\left(t_{n, j}\right) \omega_{n, j}, \\
& \widehat{w}_{p^{\prime}, p}^{(k)}=\frac{2 p+1}{2} \sum_{j=0}^{M_{n}} \widetilde{\omega}_{k, p^{\prime}}^{L}\left(t_{n, j}\right) L_{n, p}\left(t_{n, j}\right) \omega_{n, j}, \\
& d_{p, 0}^{n}=\frac{(1-\mu)(2 p+1)}{2^{2-\mu}} \sum_{i=0}^{M_{n}} \sum_{j=0}^{M_{n}+1}\left(t_{n, i}-t_{n-1}\right)^{1-\mu} b\left(\sigma\left(t_{n, j}^{-\mu, 0}, t_{n, i}\right)\right) \\
& \\
& \quad \times U^{n}\left(\sigma\left(t_{n, j}^{-\mu, 0}, t_{n, i}\right)\right) L_{n, p}\left(t_{n, i}\right) \omega_{n, i} \omega_{n, j}^{-\mu, 0}, \\
& f_{p}^{n}=\frac{2 p+1}{2} \sum_{j=0}^{M_{n}} f\left(t_{n, j}\right) L_{n, p}\left(t_{n, j}\right) \omega_{n, j} .
\end{aligned}
$$

Next, by using (2.21) $-(2.26)$, we deduce that

$$
\begin{align*}
& \sum_{p=1}^{M_{n}+1} u_{p}^{n} L_{n, p}^{\prime}(t)+h_{n} \sum_{p=1}^{M_{n}+1} \widehat{a}_{p}^{n} L_{n, p}^{\prime}(t)+h_{n} \sum_{p=1}^{M_{n}+1}\left(\sum_{k=1}^{n-1} \widehat{b}_{p}^{k}\right) L_{n, p}^{\prime}(t) \\
& +\frac{h_{n}}{1-\mu} \sum_{p=1}^{M_{n}+1} \widehat{d}_{p, 0}^{n} L_{n, p}^{\prime}(t)=h_{n} \sum_{p=1}^{M_{n}+1} \widehat{f}_{p}^{n} L_{n, p}^{\prime}(t) \tag{2.27}
\end{align*}
$$

According to the property of the standard Legendre polynomials, we know that $\left\{L_{n, p}^{\prime}(t)\right\}_{p \geq 1}$ are mutually orthogonal with respect to the weight $\left(t_{n}-t\right)\left(t-t_{n-1}\right)$. Hence, we compare the expansion coefficients of (2.27) to obtain that

$$
\begin{equation*}
u_{p}^{n}=h_{n} \widehat{f}_{p}^{n}-h_{n} \widehat{a}_{p}^{n}-h_{n} \sum_{k=1}^{n-1} \widehat{b}_{p}^{k}-\frac{h_{n}}{1-\mu} \widehat{d}_{p, 0}^{n}, \quad 1 \leq p \leq M_{n}+1 . \tag{2.28}
\end{equation*}
$$

Furthermore, due to $L_{n, p}\left(t_{n-1}\right)=(-1)^{p}, U^{1}(0)=u_{0}$ and $U^{n}\left(t_{n-1}\right)=U^{n-1}\left(t_{n-1}\right)$ for $2 \leq n \leq N$, we get from the first formula of (2.22) that

$$
\begin{equation*}
u_{0}^{1}=u_{0}-\sum_{p=1}^{M_{n}+1}(-1)^{p} u_{p}^{1}, \quad u_{0}^{n}=U^{n-1}\left(t_{n-1}\right)-\sum_{p=1}^{M_{n}+1}(-1)^{p} u_{p}^{n}, \quad n \geq 2 . \tag{2.29}
\end{equation*}
$$

The system (2.28) can be solved directly, based on matrix factorizations such as LU decomposition.

## 3. Some useful approximation results

In this section, we present some approximation results (mainly shown in Theorems 3.1 3.5), which will be useful for convergence analysis. Denote by $c$ a generic positive constant independent of $h_{k}$ and $M_{k}$.
3.1. The Jacobi and Legendre interpolation approximations for smooth functions. In this subsection, we focus on the interpolation approximations for smooth functions on the interval $I_{n}$.

Theorem 3.1. For any $v \in H_{\chi_{n}^{\alpha, \beta}}^{m}\left(I_{n}\right)$ with $\alpha, \beta>-1$ and integer $1 \leq m \leq M_{n}+1$,

$$
\begin{equation*}
\left\|v-\mathcal{I}_{t, M_{n}}^{\alpha, \beta} v\right\|_{L_{\chi_{n}^{\alpha}, \beta}^{\alpha}\left(I_{n}\right)} \leq c \sqrt{\frac{\Gamma\left(M_{n}+2-m\right)}{\Gamma\left(M_{n}+2+m\right)}}\left\|\partial_{t}^{m} v\right\|_{L_{\chi_{n}^{\alpha}}^{\alpha+m, \beta+m}\left(I_{n}\right)} \tag{3.1}
\end{equation*}
$$

where $H_{\chi_{n}^{\alpha, \beta}}^{m}\left(I_{n}\right)$ is the weighted Sobolev space with weight

$$
\chi_{n}^{\alpha, \beta}(t)=\left(t_{n}-t\right)^{\alpha}\left(t-t_{n-1}\right)^{\beta} .
$$

In particular, for any fixed $m$, we further get

$$
\begin{align*}
\left\|v-\mathcal{I}_{t, M_{n}}^{\alpha, \beta} v\right\|_{L_{\chi_{n}^{\alpha}, \beta}^{\alpha}\left(I_{n}\right)} & \leq c\left(M_{n}+1\right)^{-m}\left\|\partial_{t}^{m} v\right\|_{L_{\chi_{n}^{\alpha}}^{\alpha+m, \beta+m}}\left(I_{n}\right)  \tag{3.2}\\
& \leq c h_{n}^{m}\left(M_{n}+1\right)^{-m}\left\|\partial_{t}^{m} v\right\|_{L_{\chi_{n}^{\alpha}, \beta}^{2}\left(I_{n}\right)} .
\end{align*}
$$

Proof. Let $\pi_{M_{n}}^{\alpha, \beta}, \alpha, \beta>-1$, be the standard Jacobi-Gauss interpolation operator with respect to the nodes $\left\{x_{n, j}^{\alpha, \beta}\right\}_{j=0}^{M_{n}}$ on the interval $(-1,1)$. According to Theorem 3.41 and (3.261) of [17, for any function $u(x)$ satisfying $\partial_{x}^{k} u(x) \in L_{\chi^{\alpha+k, \beta+k}}^{2}(-1,1)$ with integers $0 \leq k \leq m$ and $1 \leq m \leq M_{n}+1$, we get

$$
\begin{align*}
& \left\|\pi_{M_{n}}^{\alpha, \beta} u-u\right\|_{L_{\chi^{\alpha, \beta}}^{2}(-1,1)} \\
& \quad \leq c \sqrt{\frac{\Gamma\left(M_{n}-m+2\right)}{\Gamma\left(M_{n}+1\right)}}\left(M_{n}+m\right)^{-(m+1) / 2}\left\|\partial_{x}^{m} u\right\|_{L_{\chi^{\alpha+m, \beta+m}}^{2}(-1,1)}  \tag{3.3}\\
& \quad \leq c \sqrt{\frac{\Gamma\left(M_{n}+2-m\right)}{\Gamma\left(M_{n}+2+m\right)}}\left\|\partial_{x}^{m} u\right\|_{L_{\chi^{\alpha+m, \beta+m}}^{2}(-1,1)}
\end{align*}
$$

Next let $u(x):=\left.v(t)\right|_{t=\frac{h_{n} x+t_{n-1}+t_{n}}{2}}$. Then we have

$$
\mathcal{I}_{t, M_{n}}^{\alpha, \beta} v\left(t_{n, j}^{\alpha, \beta}\right)=v\left(t_{n, j}^{\alpha, \beta}\right)=u\left(x_{n, j}^{\alpha, \beta}\right)=\pi_{M_{n}}^{\alpha, \beta} u\left(x_{n, j}^{\alpha, \beta}\right), \quad 0 \leq j \leq M_{n} .
$$

Since $\left.\mathcal{I}_{t, M_{n}}^{\alpha, \beta} v(t)\right|_{t=\frac{h_{n} x+t_{n-1}+t_{n}}{2}}$ and $\pi_{M_{n}}^{\alpha, \beta} u(x)$ belong to $\mathcal{P}_{M_{n}}(-1,1)$ in the variable $x$, hence

$$
\begin{equation*}
\left.\mathcal{I}_{t, M_{n}}^{\alpha, \beta} v(t)\right|_{t=\frac{h_{n} x+t_{n-1}+t_{n}}{2}}=\pi_{M_{n}}^{\alpha, \beta} u(x) \tag{3.4}
\end{equation*}
$$

The above with (3.3) gives

$$
\begin{align*}
\| v- & \mathcal{I}_{t, M_{n}}^{\alpha, \beta} v \|_{L_{x_{n}^{\alpha}, \beta}^{\alpha}\left(I_{n}\right)}^{2} \\
& =\left(\frac{h_{n}}{2}\right)^{\alpha+\beta+1} \int_{-1}^{1}\left(u(x)-\pi_{M_{n}}^{\alpha, \beta} u(x)\right)^{2}(1-x)^{\alpha}(1+x)^{\beta} d x \\
& \leq c h_{n}^{\alpha+\beta+1} \frac{\Gamma\left(M_{n}+2-m\right)}{\Gamma\left(M_{n}+2+m\right)} \int_{-1}^{1}\left(\partial_{x}^{m} u(x)\right)^{2}(1-x)^{\alpha+m}(1+x)^{\beta+m} d x  \tag{3.5}\\
& \leq c \frac{\Gamma\left(M_{n}+2-m\right)}{\Gamma\left(M_{n}+2+m\right)} \int_{I_{n}}\left(\partial_{t}^{m} v(t)\right)^{2}\left(t_{n}-t\right)^{\alpha+m}\left(t-t_{n-1}\right)^{\beta+m} d t .
\end{align*}
$$

This leads to (3.1). The result (3.2) can be derived easily from (3.1).
By (3.2) and the triangle inequality, we further obtain the following result.

Corollary 3.1. For any $v \in H_{\chi_{n}^{\alpha, \beta}}^{1}\left(I_{n}\right)$ with $\alpha, \beta>-1$,

$$
\begin{align*}
\left\|\mathcal{I}_{t, M_{n}}^{\alpha, \beta} v\right\|_{L_{n}^{\alpha, \beta}}^{L_{n}^{2}\left(I_{n}\right)} & \leq\|v\|_{L_{\chi_{n}^{\alpha, \beta}}^{2}\left(I_{n}\right)}+c\left(M_{n}+1\right)^{-1}\left\|\partial_{t} v\right\|_{L_{\chi_{n}^{\alpha+1, \beta+1}}^{2}\left(I_{n}\right)}  \tag{3.6}\\
& \leq\|v\|_{L_{\chi_{n}^{\alpha, \beta}}^{2}\left(I_{n}\right)}+c h_{n}\left(M_{n}+1\right)^{-1}\left\|\partial_{t} v\right\|_{L_{\chi_{n}^{2}, \beta}^{2}\left(I_{n}\right)} .
\end{align*}
$$

As results of Theorem 3.1 and Corollary 3.1 with $\alpha=\beta=0$, we have the following two corollaries.

Corollary 3.2 (see also [22]). For any $v \in H^{m}\left(I_{n}\right)$ with any fixed integer $1 \leq m \leq$ $M_{n}+1$,

$$
\begin{align*}
\left\|v-\mathcal{I}_{t, M_{n}} v\right\|_{L^{2}\left(I_{n}\right)} & \leq c\left(M_{n}+1\right)^{-m}\left\|\partial_{t}^{m} v\right\|_{L_{x_{n}^{m, m}}^{2}\left(I_{n}\right)}  \tag{3.7}\\
& \leq c h_{n}^{m}\left(M_{n}+1\right)^{-m}\left\|\partial_{t}^{m} v\right\|_{L^{2}\left(I_{n}\right)},
\end{align*}
$$

where $H^{m}\left(I_{n}\right)$ is the usual Sobolev space.
Corollary 3.3. For any $v \in H^{1}\left(I_{n}\right)$,

$$
\begin{align*}
\left\|\mathcal{I}_{t, M_{n}} v\right\|_{L^{2}\left(I_{n}\right)} & \leq\|v\|_{L^{2}\left(I_{n}\right)}+c\left(M_{n}+1\right)^{-1}\left\|\partial_{t} v\right\|_{L_{x_{n}^{2}}^{2}, 1}\left(I_{n}\right)  \tag{3.8}\\
& \leq\|v\|_{L^{2}\left(I_{n}\right)}+c h_{n}\left(M_{n}+1\right)^{-1}\left\|\partial_{t} v\right\|_{L^{2}\left(I_{n}\right)} .
\end{align*}
$$

Moreover, one can verify readily that
Theorem 3.2. For any $v \in H^{m}\left(I_{n}\right)$ with integer $1 \leq m \leq M_{n}+2$ and $M_{n} \geq 0$,

$$
\begin{equation*}
\left\|v-\mathcal{I}_{t, M_{n}+1}^{L} v\right\|_{L^{2}\left(I_{n}\right)} \leq c h_{n} \sqrt{\frac{\Gamma\left(M_{n}+3-m\right)}{\Gamma\left(M_{n}+3+m\right)}}\left\|\partial_{t}^{m} v\right\|_{L_{\chi_{n}^{m-1, m-1}}\left(I_{n}\right)} \tag{3.9}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|\left(v-\mathcal{I}_{t, M_{n}+1}^{L} v\right)^{\prime}\right\|_{L^{2}\left(I_{n}\right)} \leq c\left(M_{n}+1\right) \sqrt{\frac{\Gamma\left(M_{n}+3-m\right)}{\Gamma\left(M_{n}+3+m\right)}}\left\|\partial_{t}^{m} v\right\|_{x_{x_{n}^{2}}^{m-1, m-1}\left(I_{n}\right)} \tag{3.10}
\end{equation*}
$$

In particular, for any fixed $m$, we further get

$$
\begin{align*}
\left\|v-\mathcal{I}_{t, M_{n}+1}^{L} v\right\|_{L^{2}\left(I_{n}\right)} & \leq c h_{n}\left(M_{n}+1\right)^{-m}\left\|\partial_{t}^{m} v\right\|_{L_{x_{n}^{m-1, m-1}}^{2}\left(I_{n}\right)}  \tag{3.11}\\
& \leq c h_{n}^{m}\left(M_{n}+1\right)^{-m}\left\|\partial_{t}^{m} v\right\|_{L^{2}\left(I_{n}\right)}
\end{align*}
$$

and

$$
\begin{align*}
\left\|\left(v-\mathcal{I}_{t, M_{n}+1}^{L} v\right)^{\prime}\right\|_{L^{2}\left(I_{n}\right)} & \leq c\left(M_{n}+1\right)^{1-m}\left\|\partial_{t}^{m} v\right\|_{L_{\chi_{m}^{2}}^{2}-1, m-1}\left(I_{n}\right)  \tag{3.12}\\
& \leq c h_{n}^{m-1}\left(M_{n}+1\right)^{1-m}\left\|\partial_{t}^{m} v\right\|_{L^{2}\left(I_{n}\right)} .
\end{align*}
$$

Proof. Let $M_{n} \geq 0$ and let $\pi_{M_{n}+1}^{L}$ be the standard Legendre-Gauss-Lobatto interpolation operator with respect to the nodes $\left\{x_{n, j}^{L}\right\}_{j=0}^{M_{n}+1}$ on the interval $[-1,1]$.

According to Theorem 3.44 and (3.261) of [17], we obtain that for any $\partial_{x}^{m} u \in$ $L_{\chi^{m-1, m-1}}^{2}(-1,1)$ with integer $1 \leq m \leq M_{n}+2$,

$$
\begin{align*}
\| u- & \pi_{M_{n}+1}^{L} u \|_{L^{2}(-1,1)} \\
& \leq c \sqrt{\frac{\Gamma\left(M_{n}-m+3\right)}{\Gamma\left(M_{n}+2\right)}} \frac{\left(M_{n}+m+1\right)^{(1-m) / 2}}{\left(M_{n}+1\right)}\left\|\partial_{x}^{m} u\right\|_{L_{\chi^{m-1, m-1}}^{2}(-1,1)}  \tag{3.13}\\
& \leq c \sqrt{\frac{\Gamma\left(M_{n}+3-m\right)}{\Gamma\left(M_{n}+3+m\right)}}\left\|\partial_{x}^{m} u\right\|_{L_{\chi^{m-1, m-1}}^{2}(-1,1)}
\end{align*}
$$

and

$$
\begin{equation*}
\left\|\partial_{x}\left(u-\pi_{M_{n}+1}^{L} u\right)\right\|_{L^{2}(-1,1)} \leq c\left(M_{n}+1\right) \sqrt{\frac{\Gamma\left(M_{n}+3-m\right)}{\Gamma\left(M_{n}+3+m\right)}}\left\|\partial_{x}^{m} u\right\|_{x^{2}+1, m-1}(-1,1) . \tag{3.14}
\end{equation*}
$$

Next let $u(x)=\left.v(t)\right|_{t=\frac{h_{n} x+t_{n-1}+t_{n}}{2}}$. Then we have

$$
\mathcal{I}_{t, M_{n}+1}^{L} v\left(t_{n, j}^{L}\right)=v\left(t_{n, j}^{L}\right)=u\left(x_{n, j}^{L}\right)=\pi_{M_{n}+1}^{L} u\left(x_{n, j}^{L}\right), \quad 0 \leq j \leq M_{n}+1 .
$$

Since $\left.\mathcal{I}_{t, M_{n}+1}^{L} v(t)\right|_{t=\frac{h_{n} x+t_{n-1}+t_{n}}{2}}$ and $\pi_{M_{n}+1}^{L} u(x)$ belong to $\mathcal{P}_{M_{n}+1}(-1,1)$ in the variable $x$, hence

$$
\begin{equation*}
\left.\mathcal{I}_{t, M_{n}+1}^{L} v(t)\right|_{t=\frac{h_{n} x+t_{n-1}+t_{n}}{2}}=\pi_{M_{n}+1}^{L} u(x) . \tag{3.15}
\end{equation*}
$$

The above with (3.13) gives

$$
\begin{align*}
\| v- & \mathcal{I}_{t, M_{n}+1}^{L} v \|_{L^{2}\left(I_{n}\right)}^{2}=\frac{h_{n}}{2} \int_{-1}^{1}\left(u(x)-\pi_{M_{n}+1}^{L} u(x)\right)^{2} d x \\
& \leq c h_{n} \frac{\Gamma\left(M_{n}+3-m\right)}{\Gamma\left(M_{n}+3+m\right)} \int_{-1}^{1}\left(\partial_{x}^{m} u(x)\right)^{2}\left(1-x^{2}\right)^{m-1} d x  \tag{3.16}\\
& \leq \operatorname{ch}_{n}^{2} \frac{\Gamma\left(M_{n}+3-m\right)}{\Gamma\left(M_{n}+3+m\right)} \int_{I_{n}}\left(\partial_{t}^{m} v(t)\right)^{2}\left(t_{n}-t\right)^{m-1}\left(t-t_{n-1}\right)^{m-1} d t
\end{align*}
$$

Similarly, by (3.14) we obtain

$$
\begin{align*}
& \left\|\partial_{t}\left(v-\mathcal{I}_{t, M_{n}+1}^{L} v\right)\right\|_{L^{2}\left(I_{n}\right)}^{2}=\frac{2}{h_{n}} \int_{-1}^{1}\left(\partial_{x} u(x)-\partial_{x} \pi_{M_{n}+1}^{L} u(x)\right)^{2} d x  \tag{3.17}\\
\leq & c h_{n}^{-1}\left(M_{n}+1\right)^{2} \frac{\Gamma\left(M_{n}+3-m\right)}{\Gamma\left(M_{n}+3+m\right)} \int_{-1}^{1}\left(\partial_{x}^{m} u(x)\right)^{2}\left(1-x^{2}\right)^{m-1} d x \\
\leq & c\left(M_{n}+1\right)^{2} \frac{\Gamma\left(M_{n}+3-m\right)}{\Gamma\left(M_{n}+3+m\right)} \int_{I_{n}}\left(\partial_{t}^{m} v(t)\right)^{2}\left(t_{n}-t\right)^{m-1}\left(t-t_{n-1}\right)^{m-1} d t
\end{align*}
$$

This leads to the desired results (3.9) and (3.10). Finally, using (3.9) and (3.10) yields (3.11) and (3.12).

Corollary 3.4. For any $v \in H^{1}\left(I_{n}\right)$ and $M_{n} \geq 0$,

$$
\begin{equation*}
\left\|\partial_{t} \mathcal{I}_{t, M_{n}+1}^{L} v\right\|_{L^{2}\left(I_{n}\right)} \leq c\left\|\partial_{t} v\right\|_{L^{2}\left(I_{n}\right)} . \tag{3.18}
\end{equation*}
$$

3.2. The Jacobi and Legendre interpolation approximations for singular functions. In this subsection, we focus on the interpolation approximations for the $t^{\nu}$-type singular functions on the interval $I$. To this end, we first consider the interpolation approximations for the $(1+x)^{\nu}$-type singular functions on the interval $(-1,1)$.
3.2.1. The $(1+x)^{\nu}$-type singularity on the interval $(-1,1)$. Let $\nu$ be a noninteger and denote $u_{\nu}(x)=(1+x)^{\nu}$.
(i) The Jacobi-Gauss interpolation approximation. Assume that the Jacobi expansion of $u_{\nu}(x)$ is

$$
u_{\nu}(x)=\sum_{n=0}^{\infty} a_{n} J_{n}^{\alpha, 0}(x), \quad \alpha>-1 .
$$

The following lemma gives the expression for the coefficients of the Jacobi expansion.
Lemma 3.1. If $\nu>-1$ is a noninteger and $\alpha>-1$, then

$$
\begin{align*}
a_{0}= & \frac{2^{\nu} \Gamma(\alpha+2) \Gamma(\nu+1)}{\Gamma(\nu+\alpha+2)}, \\
a_{n}= & \frac{2^{\nu}(2 n+\alpha+1) \Gamma(\nu+1) \Gamma(n+\alpha+1)}{n!\Gamma(n+\nu+\alpha+2)}  \tag{3.19}\\
& \times \nu(\nu-1)(\nu-2) \cdots(\nu-n+1), \quad n \geq 1 .
\end{align*}
$$

Proof. According to (2.4), we have

$$
\begin{equation*}
a_{n}=\frac{2 n+\alpha+1}{2^{\alpha+1}} \int_{-1}^{1} u_{\nu}(x) J_{n}^{\alpha, 0}(x)(1-x)^{\alpha} d x \tag{3.20}
\end{equation*}
$$

By the definition of Jacobi polynomial, we know

$$
(1-x)^{\alpha} J_{n}^{\alpha, 0}(x)=\frac{(-1)^{n}}{2^{n} n!} \frac{d^{n}}{d x^{n}}\left((1-x)^{n+\alpha}(1+x)^{n}\right)
$$

and hence

$$
\begin{equation*}
a_{n}=\frac{(-1)^{n}(2 n+\alpha+1)}{2^{n+\alpha+1} n!} \int_{-1}^{1}(1+x)^{\nu} \frac{d^{n}}{d x^{n}}\left((1-x)^{n+\alpha}(1+x)^{n}\right) d x \tag{3.21}
\end{equation*}
$$

This, along with (2.4), leads to the expression of $a_{0}$. Next, owing to

$$
\lim _{x \rightarrow \pm 1}(1+x)^{\nu-k+1} \frac{d^{n-k}}{d x^{n-k}}\left((1-x)^{n+\alpha}(1+x)^{n}\right)=0, \quad \forall \nu>-1,1 \leq k \leq n
$$

we use (3.21), (2.4) and integration by parts to obtain

$$
\begin{align*}
a_{n}= & \frac{(2 n+\alpha+1)}{2^{n+\alpha+1} n!} \nu(\nu-1) \cdots(\nu-n+1) \int_{-1}^{1}(1-x)^{n+\alpha}(1+x)^{\nu} d x \\
= & \frac{2^{\nu}(2 n+\alpha+1) \Gamma(\nu+1) \Gamma(n+\alpha+1)}{n!\Gamma(n+\nu+\alpha+2)}  \tag{3.22}\\
& \times \nu(\nu-1)(\nu-2) \cdots(\nu-n+1), \quad \forall n \geq 1 .
\end{align*}
$$

This ends the proof.
By using the Stirling's formula we can easily prove the following lemma (cf. [7]).

Lemma 3.2. For $n \rightarrow \infty$,

$$
\begin{equation*}
\frac{\Gamma(n+\gamma)}{\Gamma(n+\delta)}=\frac{1}{n^{\delta-\gamma}}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right) \tag{3.23}
\end{equation*}
$$

where $\mathcal{O}\left(\frac{1}{n}\right)$ depends on $\gamma$ and $\delta$.
The following lemma is concerned with the asymptotic behaviour of the coefficients obtained in (3.19).

Lemma 3.3. Let $a_{n}$ be the coefficients of the Jacobi expansion of $(1+x)^{\nu}$ and let $\nu>-1$ be a noninteger. Then for $n \rightarrow \infty$,

$$
\begin{equation*}
a_{n}=(-1)^{n-1} \frac{C_{0}(\nu)}{n^{2 \nu+1}}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right) \tag{3.24}
\end{equation*}
$$

with

$$
C_{0}(\nu)=\frac{2^{\nu+1} \sin (\pi \nu)}{\pi} \Gamma^{2}(\nu+1)
$$

Proof. Clearly, by Euler's reflection formula $\Gamma(1-z) \Gamma(z)=\frac{\pi}{\sin (\pi z)}$, we deduce $\nu(\nu-1)(\nu-2) \cdots(\nu-n+1)=(-1)^{n} \frac{\Gamma(n-\nu)}{\Gamma(-\nu)}=(-1)^{n-1} \frac{\sin (\pi \nu)}{\pi} \Gamma(n-\nu) \Gamma(\nu+1)$.
Hence, we can rewrite (3.19) as

$$
\begin{equation*}
a_{n}=(-1)^{n-1} C_{0}(\nu)\left(n+\frac{\alpha+1}{2}\right) \frac{\Gamma(n-\nu) \Gamma(n+\alpha+1)}{\Gamma(n+1) \Gamma(n+\nu+\alpha+2)}, \quad n \geq 1 \tag{3.25}
\end{equation*}
$$

This, along with Lemma 3.2 leads to the desired result.
Remark 3.1. Gui and Babuška [7] considered the Legendre expansion of $u_{\nu}(x)$, and derived some results similar to Lemmas 3.1 and 3.3.

To establish the result of the Jacobi-Gauss interpolation approximation for the singular function $u_{\nu}(x)$, we first need to consider the Jacobi orthogonal projection. For any $\alpha>-1$ and integer $M \geq 0$, the Jacobi orthogonal projection $P_{M}^{\alpha, 0}$ : $L_{\chi^{\alpha, 0}}^{2}(-1,1) \rightarrow \mathcal{P}_{M}(-1,1)$ is defined by

$$
\int_{-1}^{1}\left(P_{M}^{\alpha, 0} v(x)-v(x)\right) \phi(x)(1-x)^{\alpha} d x=0, \quad \forall \phi \in \mathcal{P}_{M}(-1,1)
$$

Clearly,

$$
\begin{equation*}
P_{M}^{\alpha, 0} u_{\nu}(x)=\sum_{n=0}^{M} a_{n} J_{n}^{\alpha, 0}(x) \tag{3.26}
\end{equation*}
$$

Lemma 3.4. If $\nu>-\frac{1}{2}$ is a noninteger, $\alpha>-1$ and $M \geq 0$, then

$$
\begin{equation*}
\left\|u_{\nu}-P_{M}^{\alpha, 0} u_{\nu}\right\|_{L_{\chi^{\alpha, 0}}^{2}(-1,1)} \leq c(M+1)^{-2 \nu-1} \tag{3.27}
\end{equation*}
$$

If, in addition, $\nu>0$, then

$$
\begin{equation*}
\left\|\left(u_{\nu}-P_{M}^{\alpha, 0} u_{\nu}\right)^{\prime}\right\|_{L_{\chi^{\alpha+1,1}}^{2}(-1,1)} \leq c(M+1)^{-2 \nu} \tag{3.28}
\end{equation*}
$$

Proof. According to (3.26) and (2.4), we have (3.29)

$$
\left\|u_{\nu}-P_{M}^{\alpha, 0} u_{\nu}\right\|_{L_{\chi^{\alpha, 0}}^{2}(-1,1)}^{2}=\left\|\sum_{n=M+1}^{\infty} a_{n} J_{n}^{\alpha, 0}\right\|_{L_{\chi^{\alpha, 0}}^{2}(-1,1)}^{2}=\sum_{n=M+1}^{\infty} \frac{2^{\alpha+1} a_{n}^{2}}{2 n+\alpha+1}
$$

This, together with (3.24), gives that for $\nu>-\frac{1}{2}$,

$$
\begin{align*}
\left\|u_{\nu}-P_{M}^{\alpha, 0} u_{\nu}\right\|_{L^{\alpha}, 0}^{2}(-1,1) & =\sum_{n=M+1}^{\infty} \frac{2^{\alpha+1} C_{0}^{2}(\nu)}{(2 n+\alpha+1) n^{4 \nu+2}}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right)  \tag{3.30}\\
& \leq c \sum_{n=M+1}^{\infty} \frac{1}{n^{4 \nu+3}}=\frac{c}{(M+1)^{4 \nu+3}}+c \sum_{n=M+2}^{\infty} \frac{1}{n^{4 \nu+3}} \\
& \leq \frac{c}{(M+1)^{4 \nu+3}}+c \int_{M+1}^{\infty} x^{-4 \nu-3} d x \\
& \leq c(M+1)^{-4 \nu-2} .
\end{align*}
$$

Similarly, by (3.26), (2.5) and (3.24) we obtain that for $\nu>0$,

$$
\begin{align*}
& \left\|\left(u_{\nu}-P_{M}^{\alpha, 0} u_{\nu}\right)^{\prime}\right\|_{L_{\chi^{\alpha+1,1}}^{2}(-1,1)}^{2}=\sum_{n=M+1}^{\infty} a_{n}^{2}\left\|\frac{d}{d x} J_{n}^{\alpha, 0}\right\|_{L_{\chi^{\alpha+1,1}}^{2}(-1,1)}^{2}  \tag{3.31}\\
& =\sum_{n=M+1}^{\infty} n(n+\alpha+1) a_{n}^{2} \gamma_{n}^{\alpha, 0}=\sum_{n=M+1}^{\infty} \frac{n(n+\alpha+1) 2^{\alpha+1} C_{0}^{2}(\nu)}{(2 n+\alpha+1) n^{4 \nu+2}}\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right) \\
& \leq c \sum_{n=M+1}^{\infty} \frac{1}{n^{4 \nu+1}} \leq c(M+1)^{-4 \nu} .
\end{align*}
$$

This leads to the desired result.

We now consider the Jacobi-Gauss interpolation for the singular function $u_{\nu}(x)$. As in the proof of Theorem 3.1] we denote by $\pi_{M}^{\alpha, 0}: C(-1,1) \rightarrow \mathcal{P}_{M}(-1,1)$ the standard Jacobi-Gauss interpolation operator on the interval $(-1,1)$.

Lemma 3.5. If $\nu>0$ is a noninteger and $M \geq 0$, then

$$
\begin{equation*}
\left\|u_{\nu}-\pi_{M}^{\alpha, 0} u_{\nu}\right\|_{L_{\chi^{\alpha, 0}}^{2}(-1,1)} \leq c(M+1)^{-2 \nu-1}, \quad \forall \alpha>-1 . \tag{3.32}
\end{equation*}
$$

Proof. Clearly,

$$
\left\|u_{\nu}-\pi_{M}^{\alpha, 0} u_{\nu}\right\|_{L_{\chi^{\alpha, 0}}^{2}(-1,1)} \leq\left\|u_{\nu}-P_{M}^{\alpha, 0} u_{\nu}\right\|_{L_{\chi^{\alpha, 0}}^{2}(-1,1)}+\left\|\pi_{M}^{\alpha, 0}\left(u_{\nu}-P_{M}^{\alpha, 0} u_{\nu}\right)\right\|_{L_{\chi^{\alpha, 0}}^{2}(-1,1)}
$$

Moreover, according to Lemma 3.8 of [17],

$$
\left\|\pi_{M}^{\alpha, 0} v\right\|_{L_{\chi^{\alpha, 0}}^{2}(-1,1)} \leq c\|v\|_{L_{\chi^{\alpha, 0}}^{2}(-1,1)}+c(M+1)^{-1}\left\|v^{\prime}\right\|_{L_{\chi^{\alpha+1,1}}^{2}(-1,1)} .
$$

The previous two inequalities, along with (3.27) and (3.28), yield

$$
\begin{aligned}
\left\|u_{\nu}-\pi_{M}^{\alpha, 0} u_{\nu}\right\|_{L^{\alpha, 0}}^{2}(-1,1) \leq & c\left\|u_{\nu}-P_{M}^{\alpha, 0} u_{\nu}\right\|_{L_{\chi^{\alpha, 0}}^{2}(-1,1)} \\
& +c(M+1)^{-1}\left\|\left(u_{\nu}-P_{M}^{\alpha, 0} u_{\nu}\right)^{\prime}\right\|_{L^{\alpha}}^{2}+1,1 \\
\leq & c(M+1)^{-2 \nu-1}
\end{aligned}
$$

This leads to (3.32).
(ii). The Legendre-Gauss-Lobatto interpolation approximation. We now consider the Legendre-Gauss-Lobatto interpolation for the singular function $u_{\nu}(x)$. As in the proof of Theorem 3.2, for any integer $M \geq 0$, we denote by $\pi_{M+1}^{L}$ : $C[-1,1] \rightarrow \mathcal{P}_{M+1}[-1,1]$ the standard Legendre-Gauss-Lobatto interpolation operator on the interval $[-1,1]$.
Lemma 3.6. If $\nu>0$ is a noninteger and $M \geq 0$, then

$$
\begin{equation*}
\left\|u_{\nu}-\pi_{M+1}^{L} u_{\nu}\right\|_{L^{2}(-1,1)} \leq c(M+1)^{-2 \nu-1} . \tag{3.33}
\end{equation*}
$$

If, in addition, $\nu>\frac{1}{2}$, then

$$
\begin{equation*}
\left\|\left(u_{\nu}-\pi_{M+1}^{L} u_{\nu}\right)^{\prime}\right\|_{L^{2}(-1,1)} \leq c(M+1)^{-2 \nu+1} \tag{3.34}
\end{equation*}
$$

Proof. We first verify the result (3.33). Obviously,
$\left\|u_{\nu}-\pi_{M+1}^{L} u_{\nu}\right\|_{L^{2}(-1,1)} \leq\left\|u_{\nu}-P_{M+1}^{0,0} u_{\nu}\right\|_{L^{2}(-1,1)}+\left\|\pi_{M+1}^{L}\left(u_{\nu}-P_{M+1}^{0,0} u_{\nu}\right)\right\|_{L^{2}(-1,1)}$.
Owing to (3.317) of [17], we know that

$$
\begin{align*}
& \left\|\pi_{M+1}^{L}\left(u_{\nu}-P_{M+1}^{0,0} u_{\nu}\right)\right\|_{L^{2}(-1,1)}  \tag{3.36}\\
& \leq c(M+1)^{-1}\left(\left|u_{\nu}(1)-P_{M+1}^{0,0} u_{\nu}(1)\right|+\left|u_{\nu}(-1)-P_{M+1}^{0,0} u_{\nu}(-1)\right|\right) \\
& \quad+c\left\|u_{\nu}-P_{M+1}^{0,0} u_{\nu}\right\|_{L^{2}(-1,1)}+c(M+1)^{-1}\left\|\left(u_{\nu}-P_{M+1}^{0,0} u_{\nu}\right)^{\prime}\right\|_{L_{\chi^{1}, 1}^{2}(-1,1)}
\end{align*}
$$

Further, by (3.26) with $\alpha=0$, we get

$$
\left|u_{\nu}(1)-P_{M+1}^{0,0} u_{\nu}(1)\right|=\left|\sum_{n=M+2}^{\infty} a_{n} J_{n}^{0,0}(1)\right|=\left|\sum_{n=M+2}^{\infty} a_{n}\right| .
$$

This, along with (3.24), leads to

$$
\begin{equation*}
\left|u_{\nu}(1)-P_{M+1}^{0,0} u_{\nu}(1)\right| \leq c \sum_{n=M+2}^{\infty} \frac{1}{n^{2 \nu+1}} \leq c(M+1)^{-2 \nu}, \quad \forall \nu>0 \tag{3.37}
\end{equation*}
$$

Similarly,

$$
\begin{equation*}
\left|u_{\nu}(-1)-P_{M+1}^{0,0} u_{\nu}(-1)\right| \leq c(M+1)^{-2 \nu}, \quad \forall \nu>0 \tag{3.38}
\end{equation*}
$$

Hence, by (3.35), (3.36), (3.27), (3.28), (3.37) and (3.38), we obtain the result (3.33).

It remains to estimate (3.34). To this end, we define the following operator:

$$
\Pi_{M+1} u(x):=\int_{-1}^{x} P_{M}^{0,0} u^{\prime}(\xi) d \xi \in \mathcal{P}_{M+1}(-1,1)
$$

Then we have

$$
\begin{aligned}
\left\|\left(u_{\nu}-\pi_{M+1}^{L} u_{\nu}\right)^{\prime}\right\|_{L^{2}(-1,1)} \leq & \left\|\left(u_{\nu}-\Pi_{M+1} u_{\nu}\right)^{\prime}\right\|_{L^{2}(-1,1)} \\
& +\left\|\partial_{x} \pi_{M+1}^{L}\left(u_{\nu}-\Pi_{M+1} u_{\nu}\right)\right\|_{L^{2}(-1,1)}
\end{aligned}
$$

Moreover, by (3.14) with $m=1$ we know that for any $v \in H^{1}(-1,1)$,

$$
\left\|\partial_{x} \pi_{M+1}^{L} v\right\|_{L^{2}(-1,1)} \leq c\left\|\partial_{x} v\right\|_{L^{2}(-1,1)}
$$

The previous two inequalities imply that
$\left\|\left(u_{\nu}-\pi_{M+1}^{L} u_{\nu}\right)^{\prime}\right\|_{L^{2}(-1,1)} \leq c\left\|\left(u_{\nu}-\Pi_{M+1} u_{\nu}\right)^{\prime}\right\|_{L^{2}(-1,1)}=c\left\|u_{\nu}^{\prime}-P_{M}^{0,0} u_{\nu}^{\prime}\right\|_{L^{2}(-1,1)}$.
The above, together with (3.27), gives that for $\nu>\frac{1}{2}$,

$$
\begin{equation*}
\left\|\left(u_{\nu}-\pi_{M+1}^{L} u_{\nu}\right)^{\prime}\right\|_{L^{2}(-1,1)} \leq c(M+1)^{-2 \nu+1} . \tag{3.40}
\end{equation*}
$$

This leads to (3.34).
3.2.2. The $t^{\nu}$-type singularity on the interval $I$. We next consider the $t^{\nu}$-type singularity on the interval $I$.

Theorem 3.3. Let $M_{n}=M \geq 0$ and let $I_{h}$ be a quasi-uniform mesh ( $h_{n} \approx h$ ). Assume that $v(t)=t^{\nu}$ with $\nu>0$ being a noninteger. Then for $M \geq \nu-1$,

$$
\begin{equation*}
\left\|v-\mathcal{I}_{t, M}^{\alpha, 0} v\right\|_{L_{\chi_{n}^{\alpha, 0}}^{2}\left(I_{n}\right)} \leq c h^{\nu+\frac{\alpha+1}{2}}(M+1)^{-2 \nu-1}, \quad \forall \alpha>-1, \tag{3.41}
\end{equation*}
$$

where the weight $\chi_{n}^{\alpha, 0}(t)=\left(t_{n}-t\right)^{\alpha}$ and $\mathcal{I}_{t, M}^{\alpha, 0}$ is the intervalwise Jacobi-Gauss interpolation operator defined in (2.10) for each subinterval.

Proof. Since $v(t)$ has singularity at the endpoint $t=0$ of the first element $I_{1}$, we shall first focus on the approximation error of $\mathcal{I}_{t, M}^{\alpha, 0} v$ in $I_{1}$. For this purpose, let

$$
\begin{equation*}
u(x):=\left.v(t)\right|_{t=\frac{h_{1}(x+1)}{2}}=\left(\frac{h_{1}}{2}\right)^{\nu} u_{\nu}(x) \quad \text { with } \quad u_{\nu}(x)=(x+1)^{\nu} \tag{3.42}
\end{equation*}
$$

Then by (3.4) we know

$$
\begin{equation*}
\left.\mathcal{I}_{t, M}^{\alpha, 0} v(t)\right|_{t=\frac{h_{1}(x+1)}{2}}=\pi_{M}^{\alpha, 0} u(x)=\left(\frac{h_{1}}{2}\right)^{\nu} \pi_{M}^{\alpha, 0} u_{\nu}(x) \tag{3.43}
\end{equation*}
$$

This, together with (3.32), gives that for $\nu>0$,

$$
\begin{align*}
\left\|v-\mathcal{I}_{t, M}^{\alpha, 0} v\right\|_{L_{\chi_{1}^{\alpha}}^{2}\left(I_{1}\right)} & =\left(\frac{h_{1}}{2}\right)^{\nu+\frac{\alpha+1}{2}}\left\|u_{\nu}-\pi_{M}^{\alpha, 0} u_{\nu}\right\|_{L_{\chi^{\alpha, 0}}^{2}(-1,1)}  \tag{3.44}\\
& \leq c h^{\nu+\frac{\alpha+1}{2}}(M+1)^{-2 \nu-1}
\end{align*}
$$

where $\chi_{1}^{\alpha, 0}(t)=\left(t_{1}-t\right)^{\alpha}$ and $\chi^{\alpha, 0}(x)=(1-x)^{\alpha}$.
We next deal with the approximation error of $\mathcal{I}_{t, M}^{\alpha, 0} v$ in $I_{n}$ with $n>1$. Note that $v(t)$ is analytic in the interval $\left[t_{1}, T\right]$, the regularity exponents $m$ can be chosen
arbitrarily large. Thereby, we use (3.2) to deduce that for any fixed integer $1 \leq$ $m \leq M+1$ and $m \geq \nu$,

$$
\begin{align*}
\| v- & \mathcal{I}_{t, M}^{\alpha, 0} v\left\|_{L_{\chi_{n}^{2}, 0}^{2}\left(I_{n}\right)} \leq c h^{m}(M+1)^{-m}\right\| \partial_{t}^{m} v \|_{L_{\chi_{n}^{\alpha, 0}}^{2}\left(I_{n}\right)} \\
& =c h^{m}(M+1)^{-m}|\nu(\nu-1) \cdots(\nu-m+1)|\left(\int_{I_{n}} t^{2 \nu-2 m}\left(t_{n}-t\right)^{\alpha} d t\right)^{\frac{1}{2}} \\
& \leq c h^{\nu}(M+1)^{-m}|\nu(\nu-1) \cdots(\nu-m+1)|\left(\int_{I_{n}}\left(t_{n}-t\right)^{\alpha} d t\right)^{\frac{1}{2}}  \tag{3.45}\\
& \leq c h^{\nu+\frac{\alpha+1}{2}}(M+1)^{-m}|\nu(\nu-1) \cdots(\nu-m+1)| .
\end{align*}
$$

If $2 \nu \leq M$, we select an integer $m$ satisfying $2 \nu+1 \leq m \leq M+1$. Then, by (3.45) we have

$$
\begin{equation*}
\left\|v-\mathcal{I}_{t, M}^{\alpha, 0} v\right\|_{L_{n}^{2}, 0}\left(I_{n}\right) \leq c h^{\nu+\frac{\alpha+1}{2}}(M+1)^{-m} \leq c h^{\nu+\frac{\alpha+1}{2}}(M+1)^{-2 \nu-1} \tag{3.46}
\end{equation*}
$$

If $\nu-1 \leq M<2 \nu$, we take $m=M+1 \geq \nu$. Since $M$ is bounded, and hence the result (3.46) is still satisfied. This ends the proof.

Theorem 3.4. Let $M_{n}=M \geq 0$ and let $I_{h}$ be a quasi-uniform mesh ( $h_{n} \approx h$ ). Assume that $v(t)=t^{\nu}$ with $\nu>0$ being a noninteger. Then for $M>\nu-\frac{3}{2}$,

$$
\begin{equation*}
\left\|v-\mathcal{I}_{t, M+1}^{L} v\right\|_{L^{2}(I)} \leq c h^{\nu+\frac{1}{2}}(M+1)^{-2 \nu-1} \tag{3.47}
\end{equation*}
$$

If, in addition, $\nu>\frac{1}{2}$, then

$$
\begin{equation*}
\left\|\left(v-\mathcal{I}_{t, M+1}^{L} v\right)^{\prime}\right\|_{L^{2}(I)} \leq c h^{\nu-\frac{1}{2}}(M+1)^{-2 \nu+1} \tag{3.48}
\end{equation*}
$$

Here, $\mathcal{I}_{t, M+1}^{L}$ is the intervalwise Legendre-Gauss-Lobatto interpolation operator defined in (2.17) for each subinterval.

Proof. We first focus on the approximation error of $\mathcal{I}_{t, M+1}^{L} v$ in $I_{1}$. Let $u(x)$ be the same as that in (3.42). Then by (3.15) we know

$$
\begin{equation*}
\left.\mathcal{I}_{t, M+1}^{L} v(t)\right|_{t=\frac{h_{1}(x+1)}{2}}=\pi_{M+1}^{L} u(x)=\left(\frac{h_{1}}{2}\right)^{\nu} \pi_{M+1}^{L} u_{\nu}(x) \tag{3.49}
\end{equation*}
$$

This, along with (3.33), gives that for $\nu>0$,

$$
\begin{equation*}
\left\|v-\mathcal{I}_{t, M+1}^{L} v\right\|_{L^{2}\left(I_{1}\right)}=\left(\frac{h_{1}}{2}\right)^{\nu+\frac{1}{2}}\left\|u_{\nu}-\pi_{M+1}^{L} u_{\nu}\right\|_{L^{2}(-1,1)} \leq c h^{\nu+\frac{1}{2}}(M+1)^{-2 \nu-1} . \tag{3.50}
\end{equation*}
$$

Similarly, by (3.34) we get that for $\nu>\frac{1}{2}$,

$$
\begin{align*}
\left\|\left(v-\mathcal{I}_{t, M+1}^{L} v\right)^{\prime}\right\|_{L^{2}\left(I_{1}\right)} & =\left(\frac{h_{1}}{2}\right)^{\nu-\frac{1}{2}}\left\|\partial_{x}\left(u_{\nu}-\pi_{M+1}^{L} u_{\nu}\right)\right\|_{L^{2}(-1,1)}  \tag{3.51}\\
& \leq c h^{\nu-\frac{1}{2}}(M+1)^{-2 \nu+1}
\end{align*}
$$

We next deal with the approximation error of $\mathcal{I}_{t, M+1}^{L} v$ in $\left[t_{1}, T\right]$. Note that $v(t)$ is analytic in the interval $\left[t_{1}, T\right]$, we use (3.11) to deduce that for any fixed integer
$1 \leq m \leq M+2$ and $m>\nu+\frac{1}{2}$,

$$
\begin{align*}
\| v- & \mathcal{I}_{t, M+1}^{L} v\left\|_{L^{2}\left(t_{1}, T\right)}^{2} \leq c h^{2 m}(M+1)^{-2 m}\right\| \partial_{t}^{m} v \|_{L^{2}\left(t_{1}, T\right)}^{2} \\
& =c h^{2 m}(M+1)^{-2 m}|\nu(\nu-1) \cdots(\nu-m+1)|^{2} \int_{t_{1}}^{T} t^{2 \nu-2 m} d t  \tag{3.52}\\
& \leq c h^{2 \nu+1}(M+1)^{-2 m}\left(m-\nu-\frac{1}{2}\right)^{-1}|\nu(\nu-1) \cdots(\nu-m+1)|^{2} .
\end{align*}
$$

If $2 \nu-1 \leq M$, we select an integer $m$ satisfying $2 \nu+1 \leq m \leq M+2$. Then, by (3.52) we have

$$
\begin{equation*}
\left\|v-\mathcal{I}_{t, M+1}^{L} v\right\|_{L^{2}\left(t_{1}, T\right)} \leq c h^{\nu+\frac{1}{2}}(M+1)^{-m} \leq c h^{\nu+\frac{1}{2}}(M+1)^{-2 \nu-1} . \tag{3.53}
\end{equation*}
$$

If $\nu-\frac{3}{2}<M<2 \nu-1$, we take $m=M+2>\nu+\frac{1}{2}$. Since $M$ is bounded, the result (3.53) is still satisfied. A combination of (3.50) and (3.53) leads to (3.47).

Analogously, we use (3.12) to deduce that for any fixed integer $1 \leq m \leq M+2$ and $m>\nu+\frac{1}{2}$,

$$
\begin{align*}
\|(v- & \left.\mathcal{I}_{t, M+1}^{L} v\right)^{\prime}\left\|_{L^{2}\left(t_{1}, T\right)}^{2} \leq c h^{2 m-2}(M+1)^{2-2 m}\right\| \partial_{t}^{m} v \|_{L^{2}\left(t_{1}, T\right)}^{2} \\
& \leq c h^{2 \nu-1}(M+1)^{2-2 m}\left(m-\nu-\frac{1}{2}\right)^{-1}|\nu(\nu-1) \cdots(\nu-m+1)|^{2} \tag{3.54}
\end{align*}
$$

If $2 \nu-2 \leq M$, we select an integer $m$ satisfying $2 \nu \leq m \leq M+2$. Then, by (3.54) we get

$$
\begin{equation*}
\left\|\left(v-\mathcal{I}_{t, M+1}^{L} v\right)^{\prime}\right\|_{L^{2}\left(t_{1}, T\right)} \leq c h^{\nu-\frac{1}{2}}(M+1)^{-2 \nu+1} . \tag{3.55}
\end{equation*}
$$

If $\nu-\frac{3}{2}<M<2 \nu-2$, we take $m=M+2>\nu+\frac{1}{2}$. Since $M$ is bounded, hence the result (3.55) is still satisfied. Finally, by (3.51) and (3.55) we obtain the result (3.48).

By using a similar argument as in Theorem [3.4, we obtain
Theorem 3.5. Let $M_{n}=M \geq 0$ and let $I_{h}$ be a quasi-uniform mesh $\left(h_{n} \approx h\right)$. Assume that $v(t)=t^{\nu}$ with $\nu>0$ being a noninteger. Then for $M>\nu-\frac{1}{2}$,

$$
\begin{equation*}
\left\|v-\mathcal{I}_{t, M} v\right\|_{L^{2}(I)} \leq c h^{\nu+\frac{1}{2}}(M+1)^{-2 \nu-1} \tag{3.56}
\end{equation*}
$$

where $\mathcal{I}_{t, M}$ is the intervalwise Legendre-Gauss interpolation operator.

## 4. Error analysis

In this section, we shall analyze and characterize the $h p$-convergence of scheme (2.21). We first study the error bounds for smooth solutions on an arbitrary mesh. Then we consider the error bounds for singular solutions on a quasi-uniform mesh. Finally, we show the exponential convergence for singular solutions on a geometric mesh.

To this end, we introduce two lemmas. The first one is about the Poincaré inequality stated below.

Lemma 4.1. Let $\alpha<1$ and $\kappa \geq t_{n-1}$ be any given constants. For any $u \in$ $H_{\omega}^{1}\left(0, t_{n-1}\right)$ and $u(0)=0$ with $\omega(t)=(\kappa-t)^{-\alpha}$, we have

$$
\begin{equation*}
\int_{0}^{t_{n-1}} u^{2}(t)(\kappa-t)^{-\alpha} d t \leq \frac{4}{(1-\alpha)^{2}} \int_{0}^{t_{n-1}}\left(u^{\prime}(t)\right)^{2}(\kappa-t)^{2-\alpha} d t \tag{4.1}
\end{equation*}
$$

Proof. Clearly,

$$
u^{2}(t)(\kappa-t)^{1-\alpha}=\int_{0}^{t} \partial_{y}\left(u^{2}(y)(\kappa-y)^{1-\alpha}\right) d y
$$

Hence, for $t \in\left[0, t_{n-1}\right]$,

$$
\begin{align*}
u^{2}(t)(\kappa-t)^{1-\alpha} & +(1-\alpha) \int_{0}^{t} u^{2}(y)(\kappa-y)^{-\alpha} d y=2 \int_{0}^{t} u(y) u^{\prime}(y)(\kappa-y)^{1-\alpha} d y  \tag{4.2}\\
& \leq 2\left(\int_{0}^{t} u^{2}(y)(\kappa-y)^{-\alpha} d y\right)^{\frac{1}{2}}\left(\int_{0}^{t}\left(u^{\prime}(y)\right)^{2}(\kappa-y)^{2-\alpha} d y\right)^{\frac{1}{2}}
\end{align*}
$$

which implies that for any $\alpha<1$ and $t \in\left[0, t_{n-1}\right]$,

$$
\begin{equation*}
\int_{0}^{t} u^{2}(y)(\kappa-y)^{-\alpha} d y \leq \frac{4}{(1-\alpha)^{2}} \int_{0}^{t}\left(u^{\prime}(y)\right)^{2}(\kappa-y)^{2-\alpha} d y \tag{4.3}
\end{equation*}
$$

Letting $t \rightarrow t_{n-1}$ in the above inequality leads to the desired result.
The second one is about Gronwall's inequality given in [18].
Lemma 4.2. Assume that $\left\{k_{j}\right\}$ and $\left\{\rho_{j}\right\}(j \geq 0)$ are given nonnegative sequences, and the sequence $\left\{\varepsilon_{n}\right\}$ satisfies $\varepsilon_{0} \leq \rho_{0}$ and

$$
\varepsilon_{n} \leq \rho_{n}+\sum_{j=0}^{n-1} q_{j}+\sum_{j=0}^{n-1} k_{j} \varepsilon_{j}, \quad n \geq 1
$$

with $q_{j} \geq 0(j \geq 0)$. Then

$$
\varepsilon_{n} \leq \rho_{n}+\sum_{j=0}^{n-1}\left(q_{j}+k_{j} \rho_{j}\right) \exp \left(\sum_{j=0}^{n-1} k_{j}\right), \quad n \geq 1
$$

We now begin with the error analysis. For convenience, we denote

$$
e_{k}(t):=u^{k}(t)-U^{k}(t), \quad 1 \leq k \leq n
$$

Clearly,

$$
\begin{equation*}
\left\|e_{n}^{\prime}(t)\right\|_{L^{2}\left(I_{n}\right)}^{2} \leq 2\left\|\mathcal{I}_{t, M_{n}} \partial_{t} u^{n}-\partial_{t} U^{n}\right\|_{L^{2}\left(I_{n}\right)}^{2}+2\left\|\partial_{t} u^{n}-\mathcal{I}_{t, M_{n}} \partial_{t} u^{n}\right\|_{L^{2}\left(I_{n}\right)}^{2} \tag{4.4}
\end{equation*}
$$

We next estimate the term $\left\|\mathcal{I}_{t, M_{n}} \partial_{t} u^{n}-\partial_{t} U^{n}\right\|_{L^{2}\left(I_{n}\right)}^{2}$.
Lemma 4.3. The following inequality holds:

$$
\begin{equation*}
\left\|\mathcal{I}_{t, M_{n}} \partial_{t} u^{n}-\partial_{t} U^{n}\right\|_{L^{2}\left(I_{n}\right)}^{2} \leq 3 \sum_{j=1}^{3}\left\|B_{j}\right\|_{L^{2}\left(I_{n}\right)}^{2}, \tag{4.5}
\end{equation*}
$$

where

$$
\begin{aligned}
B_{1}(t)= & \mathcal{I}_{t, M_{n}}\left(a(t)\left(U^{n}(t)-u^{n}(t)\right)\right) \\
B_{2}(t)= & \sum_{k=1}^{n-1} \mathcal{I}_{t, M_{n}} \int_{I_{k}}(t-s)^{-\mu}\left(\mathcal{I}_{s, M_{k}+1}^{L}\left(b(s) U^{k}(s)\right)-b(s) u^{k}(s)\right) d s \\
B_{3}(t)= & \mathcal{I}_{t, M_{n}}\left[( \frac { t - t _ { n - 1 } } { h _ { n } } ) ^ { 1 - \mu } \int _ { I _ { n } } ( t _ { n } - \lambda ) ^ { - \mu } \left(\mathcal{I}_{\lambda, M_{n}+1}^{-\mu, 0}\left(b(\sigma(\lambda, t)) U^{n}(\sigma(\lambda, t))\right)\right.\right. \\
& \left.\left.-b(\sigma(\lambda, t)) u^{n}(\sigma(\lambda, t))\right) d \lambda\right]
\end{aligned}
$$

Proof. By (2.3) we have

$$
\begin{align*}
& \mathcal{I}_{t, M_{n}}\left(\frac{d}{d t} u^{n}(t)+a(t) u^{n}(t)+\sum_{k=1}^{n-1} \int_{I_{k}}(t-s)^{-\mu} b(s) u^{k}(s) d s\right.  \tag{4.6}\\
& \left.+\left(\frac{t-t_{n-1}}{h_{n}}\right)^{1-\mu} \int_{I_{n}}\left(t_{n}-\lambda\right)^{-\mu} b(\sigma(\lambda, t)) u^{n}(\sigma(\lambda, t)) d \lambda\right)=\mathcal{I}_{t, M_{n}} f(t) .
\end{align*}
$$

By subtracting (2.21) from (4.6), we derive the desired result.

Lemma 4.4. For $a(t) \in C\left[t_{n-1}, t_{n}\right]$ and $u^{n} \in H^{1}\left(I_{n}\right)$, we have

$$
\begin{equation*}
\left\|B_{1}\right\|_{L^{2}\left(I_{n}\right)}^{2} \leq c\left\|e_{n}\right\|_{L^{2}\left(I_{n}\right)}^{2}+c h_{n}^{2}\left(M_{n}+1\right)^{-2}\left\|e_{n}^{\prime}\right\|_{L^{2}\left(I_{n}\right)}^{2} \tag{4.7}
\end{equation*}
$$

Proof. By (2.15) and (3.8), we deduce that

$$
\begin{align*}
\left\|B_{1}\right\|_{L^{2}\left(I_{n}\right)}^{2}= & \int_{I_{n}}\left(\mathcal{I}_{t, M_{n}}\left(a(t)\left(U^{n}(t)-u^{n}(t)\right)\right)\right)^{2} d t \\
= & \frac{h_{n}}{2} \sum_{j=0}^{M_{n}} a^{2}\left(t_{n, j}\right)\left(U^{n}\left(t_{n, j}\right)-u^{n}\left(t_{n, j}\right)\right)^{2} \omega_{n, j} \\
\leq & c h_{n} \sum_{j=0}^{M_{n}}\left(U^{n}\left(t_{n, j}\right)-u^{n}\left(t_{n, j}\right)\right)^{2} \omega_{n, j}  \tag{4.8}\\
\leq & c\left\|\mathcal{I}_{t, M_{n}}\left(U^{n}-u^{n}\right)\right\|_{L^{2}\left(I_{n}\right)}^{2} \leq c\left\|U^{n}-u^{n}\right\|_{L^{2}\left(I_{n}\right)}^{2} \\
& +c h_{n}^{2}\left(M_{n}+1\right)^{-2}\left\|\partial_{t}\left(U^{n}-u^{n}\right)\right\|_{L^{2}\left(I_{n}\right)}^{2} .
\end{align*}
$$

This ends the proof.

Lemma 4.5. Assume that $b(t) \in C^{1}\left[t_{k-1}, t_{k}\right], 1 \leq k \leq n-1, b(t) \in H^{1}\left(0, t_{n-1}\right)$ and $u \in H^{1}\left(0, t_{n-1}\right)$. Then we have

$$
\begin{equation*}
\left\|B_{2}\right\|_{L^{2}\left(I_{n}\right)}^{2} \leq c h_{n} T^{3-2 \mu} \sum_{k=1}^{n-1}\left(\left\|e_{k}\right\|_{H^{1}\left(I_{k}\right)}^{2}+\left\|\left(\mathcal{I}_{s, M_{k}+1}^{L}\left(b u^{k}\right)-b u^{k}\right)^{\prime}\right\|_{L^{2}\left(I_{k}\right)}^{2}\right) \tag{4.9}
\end{equation*}
$$

Proof. Let $V(s)$ and $W(s)$ be the global functions defined on $\left[0, t_{n-1}\right]$, such that

$$
\begin{aligned}
\left.V(s)\right|_{s \in I_{k}} & :=\mathcal{I}_{s, M_{k}+1}^{L}\left(b(s) U^{k}(s)\right)-b(s) u^{k}(s),\left.\quad W(s)\right|_{s \in I_{k}} \\
& :=\mathcal{I}_{s, M_{k}+1}^{L}\left(b(s) U^{k}(s)\right), \quad 1 \leq k \leq n-1 .
\end{aligned}
$$

Clearly, $W(s)$ is a piecewise polynomial. Since $\mathcal{I}_{s, M_{k}+1}^{L}$ is the Legendre-GaussLobatto interpolation operator, we can verify readily that $W(s) \in C\left(0, t_{n-1}\right)$. Hence, $W(s) \in H^{1}\left(0, t_{n-1}\right)$. Accordingly, we have $V(s) \in H^{1}\left(0, t_{n-1}\right)$ and $V(0)=$ 0 . Next, by (2.15), the Cauchy-Schwarz inequality and the definition of $B_{2}$, we
obtain that

$$
\begin{align*}
\left\|B_{2}\right\|_{L^{2}\left(I_{n}\right)}^{2} & =\left\|\mathcal{I}_{t, M_{n}} \sum_{k=1}^{n-1} \int_{I_{k}}(t-s)^{-\mu}\left(\mathcal{I}_{s, M_{k}+1}^{L}\left(b(s) U^{k}(s)\right)-b(s) u^{k}(s)\right) d s\right\|_{L^{2}\left(I_{n}\right)}^{2}  \tag{4.10}\\
& =\left\|\mathcal{I}_{t, M_{n}} \int_{0}^{t_{n-1}}(t-s)^{-\mu} V(s) d s\right\|_{L^{2}\left(I_{n}\right)}^{2} \\
& =\frac{h_{n}}{2} \sum_{j=0}^{M_{n}} \omega_{n, j}\left(\int_{0}^{t_{n-1}}\left(t_{n, j}-s\right)^{-\mu} V(s) d s\right)^{2} \\
& \leq \frac{h_{n}}{2} \sum_{j=0}^{M_{n}} \omega_{n, j} \int_{0}^{t_{n-1}}\left(t_{n, j}-s\right)^{-\mu} d s \int_{0}^{t_{n-1}}\left(t_{n, j}-s\right)^{-\mu} V^{2}(s) d s \\
& \leq c h_{n} T^{1-\mu} \sum_{j=0}^{M_{n}} \omega_{n, j} \int_{0}^{t_{n-1}}\left(t_{n, j}-s\right)^{-\mu} V^{2}(s) d s
\end{align*}
$$

Due to (4.1) and the fact $\sum_{j=0}^{M_{n}} \omega_{n, j}=2$, we further get

$$
\begin{align*}
\left\|B_{2}\right\|_{L^{2}\left(I_{n}\right)}^{2} & \leq c h_{n} T^{1-\mu} \sum_{j=0}^{M_{n}} \omega_{n, j} \int_{0}^{t_{n-1}}\left(t_{n, j}-s\right)^{2-\mu}\left(V^{\prime}(s)\right)^{2} d s  \tag{4.11}\\
& \leq c h_{n} T^{3-2 \mu} \int_{0}^{t_{n-1}}\left(V^{\prime}(s)\right)^{2} d s
\end{align*}
$$

Hence,
(4.12)

$$
\begin{aligned}
\left\|B_{2}\right\|_{L^{2}\left(I_{n}\right)}^{2} \leq & c h_{n} T^{3-2 \mu} \sum_{k=1}^{n-1} \int_{I_{k}}\left|\partial_{s} \mathcal{I}_{s, M_{k}+1}^{L}\left(b(s) U^{k}(s)\right)-\partial_{s}\left(b(s) u^{k}(s)\right)\right|^{2} d s \\
\leq & c h_{n} T^{3-2 \mu} \sum_{k=1}^{n-1} \int_{I_{k}}\left|\partial_{s} \mathcal{I}_{s, M_{k}+1}^{L}\left(b(s) U^{k}(s)-b(s) u^{k}(s)\right)\right|^{2} d s \\
& +c h_{n} T^{3-2 \mu} \sum_{k=1}^{n-1} \int_{I_{k}}\left|\partial_{s} \mathcal{I}_{s, M_{k}+1}^{L}\left(b(s) u^{k}(s)\right)-\partial_{s}\left(b(s) u^{k}(s)\right)\right|^{2} d s
\end{aligned}
$$

Applying (3.18) to (4.12), we derive that
(4.13)

$$
\begin{aligned}
\left\|B_{2}\right\|_{L^{2}\left(I_{n}\right)}^{2} \leq & c h_{n} T^{3-2 \mu} \sum_{k=1}^{n-1} \int_{I_{k}}\left|\partial_{s}\left(b(s) U^{k}(s)-b(s) u^{k}(s)\right)\right|^{2} d s \\
& +c h_{n} T^{3-2 \mu} \sum_{k=1}^{n-1} \int_{I_{k}}\left|\partial_{s} \mathcal{I}_{s, M_{k}+1}^{L}\left(b(s) u^{k}(s)\right)-\partial_{s}\left(b(s) u^{k}(s)\right)\right|^{2} d s \\
\leq & c h_{n} T^{3-2 \mu} \sum_{k=1}^{n-1}\left(\left\|e_{k}\right\|_{H^{1}\left(I_{k}\right)}^{2}+\left\|\left(\mathcal{I}_{s, M_{k}+1}^{L}\left(b u^{k}\right)-b u^{k}\right)^{\prime}\right\|_{L^{2}\left(I_{k}\right)}^{2}\right)
\end{aligned}
$$

This leads to the desired result.

Lemma 4.6. Assume that $b(t) \in C\left[t_{n-1}, t_{n}\right]$ and $u^{n} \in H^{1}\left(I_{n}\right)$. Then the following inequality holds:

$$
\begin{align*}
& \left\|B_{3}\right\|_{L^{2}\left(I_{n}\right)}^{2} \leq c h_{n}^{2-2 \mu}\left\|e_{n}\right\|_{L^{2}\left(I_{n}\right)}^{2}+c h_{n}^{4-2 \mu}\left\|e_{n}^{\prime}\right\|_{L^{2}\left(I_{n}\right)}^{2}  \tag{4.14}\\
& \quad+c h_{n}^{2-\mu} \sum_{j=0}^{M_{n}} \omega_{n, j} \int_{I_{n}}\left(t_{n}-\lambda\right)^{-\mu}\left|\left(\mathcal{I}_{\lambda, M_{n}+1}^{-\mu, 0}-\mathcal{I}\right)\left(b\left(\sigma\left(\lambda, t_{n, j}\right)\right) u^{n}\left(\sigma\left(\lambda, t_{n, j}\right)\right)\right)\right|^{2} d \lambda,
\end{align*}
$$

where $\mathcal{I}$ is the identity operator.
Proof. By (2.15) we have

$$
\begin{aligned}
\left\|B_{3}\right\|_{L^{2}\left(I_{n}\right)}^{2}= & \left.\frac{h_{n}}{2} \sum_{j=0}^{M_{n}} \omega_{n, j} \right\rvert\,\left(\frac{t_{n, j}-t_{n-1}}{h_{n}}\right)^{1-\mu} \\
& \times \int_{I_{n}}\left(t_{n}-\lambda\right)^{-\mu}\left(\mathcal{I}_{\lambda, M_{n}+1}^{-\mu, 0}\left(b\left(\sigma\left(\lambda, t_{n, j}\right)\right) U^{n}\left(\sigma\left(\lambda, t_{n, j}\right)\right)\right)\right. \\
& \left.-b\left(\sigma\left(\lambda, t_{n, j}\right)\right) u^{n}\left(\sigma\left(\lambda, t_{n, j}\right)\right)\right)\left.d \lambda\right|^{2} r \\
\leq & c h_{n} \sum_{j=0}^{M_{n}} \omega_{n, j} \mid \int_{I_{n}}\left(t_{n}-\lambda\right)^{-\mu}\left(\mathcal{I}_{\lambda, M_{n}+1}^{-\mu, 0}\left(b\left(\sigma\left(\lambda, t_{n, j}\right)\right) U^{n}\left(\sigma\left(\lambda, t_{n, j}\right)\right)\right)\right. \\
\leq & \quad c h_{n} \sum_{j=0}^{M_{n}} \omega_{n, j}\left(D_{1 j}+D_{2 j}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
& D_{1 j}=\left|\int_{I_{n}}\left(t_{n}-\lambda\right)^{-\mu} \mathcal{I}_{\lambda, M_{n}+1}^{-\mu, 0}\left(b\left(\sigma\left(\lambda, t_{n, j}\right)\right)\left(U^{n}\left(\sigma\left(\lambda, t_{n, j}\right)\right)-u^{n}\left(\sigma\left(\lambda, t_{n, j}\right)\right)\right)\right) d \lambda\right|^{2} \\
& D_{2 j}=\left|\int_{I_{n}}\left(t_{n}-\lambda\right)^{-\mu}\left(\mathcal{I}_{\lambda, M_{n}+1}^{-\mu, 0}-\mathcal{I}\right)\left(b\left(\sigma\left(\lambda, t_{n, j}\right)\right) u^{n}\left(\sigma\left(\lambda, t_{n, j}\right)\right)\right) d \lambda\right|^{2}
\end{aligned}
$$

We next estimate the terms $D_{1 j}$ and $D_{2 j}$. Clearly, by (2.8) (using $M_{n}+1$ instead of $M_{n}$ ) and the Cauchy-Schwarz inequality, we have

$$
\begin{align*}
D_{1 j}= & \left\lvert\,\left(\frac{h_{n}}{2}\right)^{1-\mu} \sum_{i=0}^{M_{n}+1} b\left(\sigma\left(t_{n, i}^{-\mu, 0}, t_{n, j}\right)\right)\left(U^{n}\left(\sigma\left(t_{n, i}^{-\mu, 0}, t_{n, j}\right)\right)\right.\right.  \tag{4.15}\\
& \left.\quad-u^{n}\left(\sigma\left(t_{n, i}^{-\mu, 0}, t_{n, j}\right)\right)\right)\left.\omega_{n, i}^{-\mu, 0}\right|^{2} \\
\leq & c h_{n}^{2-2 \mu} \sum_{i=0}^{M_{n}+1}\left(U^{n}\left(\sigma\left(t_{n, i}^{-\mu, 0}, t_{n, j}\right)\right)-u^{n}\left(\sigma\left(t_{n, i}^{-\mu, 0}, t_{n, j}\right)\right)\right)^{2} \omega_{n, i}^{-\mu, 0} \\
\leq & c h_{n}^{1-\mu} \int_{I_{n}}\left|\mathcal{I}_{\lambda, M_{n}+1}^{-\mu, 0}\left(U^{n}\left(\sigma\left(\lambda, t_{n, j}\right)\right)-u^{n}\left(\sigma\left(\lambda, t_{n, j}\right)\right)\right)\right|^{2}\left(t_{n}-\lambda\right)^{-\mu} d \lambda .
\end{align*}
$$

The above with (3.6) gives that

$$
\begin{aligned}
& D_{1 j} \leq c h_{n}^{1-\mu} \int_{I_{n}}\left|U^{n}\left(\sigma\left(\lambda, t_{n, j}\right)\right)-u^{n}\left(\sigma\left(\lambda, t_{n, j}\right)\right)\right|^{2}\left(t_{n}-\lambda\right)^{-\mu} d \lambda \\
& +c h_{n}^{1-\mu}\left(M_{n}+1\right)^{-2} \int_{I_{n}} \partial_{\lambda}\left(U^{n}\left(\sigma\left(\lambda, t_{n, j}\right)\right)\right. \\
& \left.-u^{n}\left(\sigma\left(\lambda, t_{n, j}\right)\right)\right)\left.\right|^{2}\left(t_{n}-\lambda\right)^{1-\mu}\left(\lambda-t_{n-1}\right) d \lambda \\
& \leq c h_{n}^{1-\mu} \int_{I_{n}} \mid U^{n}\left(\sigma\left(\lambda, t_{n, j}\right)\right)-u^{n}\left(\sigma\left(\lambda, t_{n, j}\right)\right) \\
& -U^{n}\left(t_{n, j}\right)+\left.u^{n}\left(t_{n, j}\right)\right|^{2}\left(t_{n}-\lambda\right)^{-\mu} d \lambda \\
& \text { (4.16) }+c h_{n}^{2-2 \mu}\left|U^{n}\left(t_{n, j}\right)-u^{n}\left(t_{n, j}\right)\right|^{2} \\
& +c h_{n}^{3-2 \mu}\left(M_{n}+1\right)^{-2} \int_{I_{n}}\left|\partial_{\lambda}\left(U^{n}\left(\sigma\left(\lambda, t_{n, j}\right)\right)-u^{n}\left(\sigma\left(\lambda, t_{n, j}\right)\right)\right)\right|^{2} d \lambda \\
& \leq c h_{n}^{2-\mu} \int_{I_{n}} \mid U^{n}\left(\sigma\left(\lambda, t_{n, j}\right)\right)-u^{n}\left(\sigma\left(\lambda, t_{n, j}\right)\right) \\
& -U^{n}\left(t_{n, j}\right)+\left.u^{n}\left(t_{n, j}\right)\right|^{2}\left(t_{n}-\lambda\right)^{-\mu-1} d \lambda \\
& +c h_{n}^{2-2 \mu}\left|U^{n}\left(t_{n, j}\right)-u^{n}\left(t_{n, j}\right)\right|^{2} \\
& +\operatorname{ch}_{n}^{3-2 \mu}\left(M_{n}+1\right)^{-2} \int_{I_{n}}\left|\partial_{\lambda}\left(U^{n}\left(\sigma\left(\lambda, t_{n, j}\right)\right)-u^{n}\left(\sigma\left(\lambda, t_{n, j}\right)\right)\right)\right|^{2} d \lambda .
\end{aligned}
$$

Owing to (2.2), we assert that

$$
\left.\left(U^{n}\left(\sigma\left(\lambda, t_{n, j}\right)\right)-u^{n}\left(\sigma\left(\lambda, t_{n, j}\right)\right)\right)\right|_{\lambda=t_{n}}=U^{n}\left(t_{n, j}\right)-u^{n}\left(t_{n, j}\right)
$$

Hence, we use the Hardy inequality to obtain that

$$
\begin{align*}
D_{1 j} \leq & c h_{n}^{2-\mu} \int_{I_{n}}\left|\partial_{\lambda}\left(U^{n}\left(\sigma\left(\lambda, t_{n, j}\right)\right)-u^{n}\left(\sigma\left(\lambda, t_{n, j}\right)\right)\right)\right|^{2}\left(t_{n}-\lambda\right)^{1-\mu} d \lambda  \tag{4.17}\\
& +c h_{n}^{2-2 \mu}\left|U^{n}\left(t_{n, j}\right)-u^{n}\left(t_{n, j}\right)\right|^{2} \\
& +c h_{n}^{3-2 \mu}\left(M_{n}+1\right)^{-2} \int_{I_{n}}\left|\partial_{\lambda}\left(U^{n}\left(\sigma\left(\lambda, t_{n, j}\right)\right)-u^{n}\left(\sigma\left(\lambda, t_{n, j}\right)\right)\right)\right|^{2} d \lambda \\
\leq & c h_{n}^{2-2 \mu}\left|U^{n}\left(t_{n, j}\right)-u^{n}\left(t_{n, j}\right)\right|^{2} \\
& +c h_{n}^{3-2 \mu} \int_{I_{n}}\left|\partial_{\lambda}\left(U^{n}\left(\sigma\left(\lambda, t_{n, j}\right)\right)-u^{n}\left(\sigma\left(\lambda, t_{n, j}\right)\right)\right)\right|^{2} d \lambda
\end{align*}
$$

Finally, by (4.17), (2.15), (2.2) and (3.8) we get

$$
\begin{align*}
h_{n} \sum_{j=0}^{M_{n}} \omega_{n, j} D_{1 j} \leq & c h_{n}^{3-2 \mu} \sum_{j=0}^{M_{n}} \omega_{n, j}\left|U^{n}\left(t_{n, j}\right)-u^{n}\left(t_{n, j}\right)\right|^{2}  \tag{4.18}\\
& +c h_{n}^{4-2 \mu} \sum_{j=0}^{M_{n}} \omega_{n, j} \int_{I_{n}}\left|\partial_{\lambda}\left(U^{n}\left(\sigma\left(\lambda, t_{n, j}\right)\right)-u^{n}\left(\sigma\left(\lambda, t_{n, j}\right)\right)\right)\right|^{2} d \lambda \\
\leq & c h_{n}^{2-2 \mu} \int_{I_{n}}\left|\mathcal{I}_{t, M_{n}}\left(U^{n}(t)-u^{n}(t)\right)\right|^{2} d t \\
& +c h_{n}^{4-2 \mu} \sum_{j=0}^{M_{n}} \omega_{n, j} \frac{t_{n, j}-t_{n-1}}{h_{n}} \int_{I_{n}}\left|\partial_{\xi}\left(U^{n}(\xi)-u^{n}(\xi)\right)\right|^{2} d \xi \\
\leq & c h_{n}^{2-2 \mu} \int_{I_{n}}\left(U^{n}(t)-u^{n}(t)\right)^{2} d t \\
& +c h_{n}^{4-2 \mu}\left(M_{n}+1\right)^{-2} \int_{I_{n}}\left(\partial_{t}\left(U^{n}(t)-u^{n}(t)\right)\right)^{2} d t \\
& +c h_{n}^{4-2 \mu} \int_{I_{n}}\left(\partial_{t}\left(U^{n}(t)-u^{n}(t)\right)\right)^{2} d t \\
\leq & c h_{n}^{2-2 \mu}\left\|e_{n}\right\|_{L^{2}\left(I_{n}\right)}^{2}+c h_{n}^{4-2 \mu}\left\|e_{n}^{\prime}\right\|_{L^{2}\left(I_{n}\right)}^{2} .
\end{align*}
$$

It remains to estimate $D_{2 j}$. By the Cauchy-Schwarz inequality, we deduce that (4.19)

$$
D_{2 j} \leq \operatorname{ch}_{n}^{1-\mu} \int_{I_{n}}\left(t_{n}-\lambda\right)^{-\mu}\left|\left(\mathcal{I}_{\lambda, M_{n}+1}^{-\mu, 0}-\mathcal{I}\right)\left(b\left(\sigma\left(\lambda, t_{n, j}\right)\right) u^{n}\left(\sigma\left(\lambda, t_{n, j}\right)\right)\right)\right|^{2} d \lambda
$$

Thereby,

$$
\begin{align*}
& h_{n} \sum_{j=0}^{M_{n}} \omega_{n, j} D_{2 j}  \tag{4.20}\\
\leq & c h_{n}^{2-\mu} \sum_{j=0}^{M_{n}} \omega_{n, j} \int_{I_{n}}\left(t_{n}-\lambda\right)^{-\mu}\left|\left(\mathcal{I}_{\lambda, M_{n}+1}^{-\mu, 0}-\mathcal{I}\right)\left(b\left(\sigma\left(\lambda, t_{n, j}\right)\right) u^{n}\left(\sigma\left(\lambda, t_{n, j}\right)\right)\right)\right|^{2} d \lambda
\end{align*}
$$

This, along with (4.18), leads to the desired result.

### 4.1. Error bounds for smooth solutions on an arbitrary mesh.

Theorem 4.1. Assume $a(t) \in C(I), b(t) \in H^{1}(I),\left.b(t)\right|_{t \in\left[t_{n-1}, t_{n}\right]} \in C^{m_{n}}\left[t_{n-1}, t_{n}\right]$, $f(t) \in C(I), u \in H^{1}(I)$ and $\left.u\right|_{t \in I_{n}} \in H^{m_{n}}\left(I_{n}\right)$ with $1 \leq n \leq N$ and integers $2 \leq m_{n} \leq M_{n}+2$. Then for $h_{n}$ sufficiently small (cf. (4.27)), we get

$$
\begin{align*}
\left\|u^{n}-U^{n}\right\|_{H^{1}\left(I_{n}\right)}^{2} \leq & \operatorname{ch}_{n}^{2 m_{n}-2}\left(M_{n}+1\right)^{2-2 m_{n}}\|u\|_{H^{m_{n}\left(I_{n}\right)}}^{2}  \tag{4.21}\\
& +h_{n} \exp \left(c T^{4-2 \mu}\right) \sum_{k=1}^{n-1} h_{k}^{2 m_{k}-2}\left(M_{k}+1\right)^{2-2 m_{k}}\|u\|_{H^{m_{k}\left(I_{k}\right)}}^{2}
\end{align*}
$$

Proof. By (3.7) we get that for integer $2 \leq m_{n} \leq M_{n}+2$,

$$
\begin{equation*}
\left\|\partial_{t} u^{n}-\mathcal{I}_{t, M_{n}} \partial_{t} u^{n}\right\|_{L^{2}\left(I_{n}\right)}^{2} \leq c h_{n}^{2 m_{n}-2}\left(M_{n}+1\right)^{2-2 m_{n}}\left\|\partial_{t}^{m_{n}} u^{n}\right\|_{L^{2}\left(I_{n}\right)}^{2} \tag{4.22}
\end{equation*}
$$

Moreover, applying (3.12) to (4.9), we derive that for $\left.b(t)\right|_{t \in\left[t_{k-1}, t_{k}\right]} \in C^{m_{k}}\left[t_{k-1}, t_{k}\right]$ and $1 \leq m_{k} \leq M_{k}+2$,

$$
\begin{equation*}
\left\|B_{2}\right\|_{L^{2}\left(I_{n}\right)}^{2} \leq c h_{n} T^{3-2 \mu} \sum_{k=1}^{n-1}\left(\left\|e_{k}\right\|_{H^{1}\left(I_{k}\right)}^{2}+h_{k}^{2 m_{k}-2}\left(M_{k}+1\right)^{2-2 m_{k}}\left\|u^{k}\right\|_{H^{m_{k}\left(I_{k}\right)}}^{2}\right) \tag{4.23}
\end{equation*}
$$

Further, by (3.2) and (2.2), we obtain that for $1 \leq m_{n} \leq M_{n}+2$,
(4.24)

$$
\begin{aligned}
& \int_{I_{n}}\left(t_{n}-\lambda\right)^{-\mu}\left|\left(\mathcal{I}_{\lambda, M_{n}+1}^{-\mu, 0}-\mathcal{I}\right)\left(b\left(\sigma\left(\lambda, t_{n, j}\right)\right) u^{n}\left(\sigma\left(\lambda, t_{n, j}\right)\right)\right)\right|^{2} d \lambda \\
& \leq c\left(M_{n}+1\right)^{-2 m_{n}} \int_{I_{n}}\left(t_{n}-\lambda\right)^{m_{n}-\mu}\left(\lambda-t_{n-1}\right)^{m_{n}} \\
& \quad \times\left|\partial_{\lambda}^{m_{n}}\left(b\left(\sigma\left(\lambda, t_{n, j}\right)\right) u^{n}\left(\sigma\left(\lambda, t_{n, j}\right)\right)\right)\right|^{2} d \lambda \\
& \leq c h_{n}^{2 m_{n}-\mu}\left(M_{n}+1\right)^{-2 m_{n}} \int_{t_{n-1}}^{t_{n, j}}\left(\frac{t_{n, j}-t_{n-1}}{h_{n}}\right)^{2 m_{n}-1}\left|\partial_{\xi}^{m_{n}}\left(b(\xi) u^{n}(\xi)\right)\right|^{2} d \xi \\
& \leq c h_{n}^{2 m_{n}-\mu}\left(M_{n}+1\right)^{-2 m_{n}} \int_{I_{n}}\left|\partial_{\xi}^{m_{n}}\left(b(\xi) u^{n}(\xi)\right)\right|^{2} d \xi
\end{aligned}
$$

Therefore, by (4.24) and (4.14) we have

$$
\begin{align*}
\left\|B_{3}\right\|_{L^{2}\left(I_{n}\right)}^{2} \leq & c h_{n}^{2-2 \mu}\left\|e_{n}\right\|_{L^{2}\left(I_{n}\right)}^{2}+c h_{n}^{4-2 \mu}\left\|e_{n}^{\prime}\right\|_{L^{2}\left(I_{n}\right)}^{2} \\
& +c h_{n}^{2+2 m_{n}-2 \mu}\left(M_{n}+1\right)^{-2 m_{n}}\left\|u^{n}\right\|_{H^{m_{n}\left(I_{n}\right)}}^{2} . \tag{4.25}
\end{align*}
$$

Thus, by (4.4), (4.5), (4.7), (4.22), (4.23) and (4.25), we deduce that for $2 \leq m_{k} \leq$ $M_{k}+2$, we get

$$
\begin{align*}
& \left\|e_{n}^{\prime}\right\|_{L^{2}\left(I_{n}\right)}^{2} \\
& \leq c\left\|e_{n}\right\|_{L^{2}\left(I_{n}\right)}^{2}+c h_{n}^{2}\left(M_{n}+1\right)^{-2}\left\|e_{n}^{\prime}\right\|_{L^{2}\left(I_{n}\right)}^{2} \\
& \quad+c h_{n}^{2 m_{n}-2}\left(M_{n}+1\right)^{2-2 m_{n}}\left\|\partial_{t}^{m_{n}} u^{n}\right\|_{L^{2}\left(I_{n}\right)}^{2} \\
& \quad+c h_{n} T^{3-2 \mu} \sum_{k=1}^{n-1}\left(\left\|e_{k}\right\|_{H^{1}\left(I_{k}\right)}^{2}+h_{k}^{2 m_{k}-2}\left(M_{k}+1\right)^{2-2 m_{k}}\left\|u^{k}\right\|_{H^{m_{k}\left(I_{k}\right)}}^{2}\right)  \tag{4.26}\\
& \quad+c h_{n}^{2-2 \mu}\left\|e_{n}\right\|_{L^{2}\left(I_{n}\right)}^{2}+c h_{n}^{4-2 \mu}\left\|e_{n}^{\prime}\right\|_{L^{2}\left(I_{n}\right)}^{2} \\
& \quad+c h_{n}^{2+2 m_{n}-2 \mu}\left(M_{n}+1\right)^{-2 m_{n}}\left\|u^{n}\right\|_{H^{m_{n}}\left(I_{n}\right)}^{2} .
\end{align*}
$$

Assume that $h_{n}$ is sufficiently small such that

$$
\begin{equation*}
c h_{n}^{2}+c h_{n}^{4-2 \mu} \leq \eta<1 \tag{4.27}
\end{equation*}
$$

Then we may rewrite (4.26) as

$$
\begin{equation*}
\left\|e_{n}^{\prime}\right\|_{L^{2}\left(I_{n}\right)}^{2} \leq c\left\|e_{n}\right\|_{L^{2}\left(I_{n}\right)}^{2}+c h_{n} T^{3-2 \mu} \sum_{k=1}^{n-1}\left\|e_{k}\right\|_{H^{1}\left(I_{k}\right)}^{2}+Q_{n} \tag{4.28}
\end{equation*}
$$

where

$$
\begin{aligned}
Q_{n}= & c_{n}^{2 m_{n}-2}\left(M_{n}+1\right)^{2-2 m_{n}}\left\|u^{n}\right\|_{H^{m_{n}\left(I_{n}\right)}}^{2} \\
& +c h_{n} T^{3-2 \mu} \sum_{k=1}^{n-1} h_{k}^{2 m_{k}-2}\left(M_{k}+1\right)^{2-2 m_{k}}\left\|u^{k}\right\|_{H^{m_{k}\left(I_{k}\right)}}^{2}
\end{aligned}
$$

It is clear that

$$
\left\{\begin{array}{l}
e_{k}^{2}\left(t_{k}\right)-e_{k}^{2}\left(t_{k-1}\right)=2 \int_{I_{k}} e_{k}^{\prime}(t) e_{k}(t) d t \leq\left\|e_{k}\right\|_{H^{1}\left(I_{k}\right)}^{2} \\
e_{k}\left(t_{k-1}\right)=e_{k-1}\left(t_{k-1}\right), \quad e_{1}\left(t_{0}\right)=0
\end{array}\right.
$$

Summing up all these inequalities, we obtain

$$
\begin{equation*}
e_{n-1}^{2}\left(t_{n-1}\right) \leq \sum_{k=1}^{n-1}\left\|e_{k}\right\|_{H^{1}\left(I_{k}\right)}^{2} \tag{4.29}
\end{equation*}
$$

Therefore,

$$
\begin{align*}
\left\|e_{n}\right\|_{L^{2}\left(I_{n}\right)}^{2} & =\int_{I_{n}}\left(\int_{t_{n-1}}^{t} e_{n}^{\prime}(s) d s+e_{n}\left(t_{n-1}\right)\right)^{2} d t \leq 2 h_{n}^{2}\left\|e_{n}^{\prime}\right\|_{L^{2}\left(I_{n}\right)}^{2}+2 h_{n} e_{n}^{2}\left(t_{n-1}\right)  \tag{4.30}\\
& =2 h_{n}^{2}\left\|e_{n}^{\prime}\right\|_{L^{2}\left(I_{n}\right)}^{2}+2 h_{n} e_{n-1}^{2}\left(t_{n-1}\right) \leq 2 h_{n}^{2}\left\|e_{n}^{\prime}\right\|_{L^{2}\left(I_{n}\right)}^{2}+2 h_{n} \sum_{k=1}^{n-1}\left\|e_{k}\right\|_{H^{1}\left(I_{k}\right)}^{2}
\end{align*}
$$

This, along with (4.28) and (4.27), yields

$$
\begin{equation*}
\left\|e_{n}\right\|_{H^{1}\left(I_{n}\right)}^{2} \quad \leq c h_{n} T^{3-2 \mu} \sum_{k=1}^{n-1}\left\|e_{k}\right\|_{H^{1}\left(I_{k}\right)}^{2}+Q_{n} \tag{4.31}
\end{equation*}
$$

By taking $\varepsilon_{k}=h_{k}^{-1}\left\|e_{k}\right\|_{H^{1}\left(I_{k}\right)}^{2}$ in Lemma 4.2 we get

$$
\begin{aligned}
\left\|e_{n}\right\|_{H^{1}\left(I_{n}\right)}^{2} \leq & c h_{n}^{2 m_{n}-2}\left(M_{n}+1\right)^{2-2 m_{n}}\|u\|_{H^{m_{n}}\left(I_{n}\right)}^{2} \\
& +c h_{n} \exp \left(c T^{4-2 \mu}\right) \sum_{k=1}^{n-1} h_{k}^{2 m_{k}-2}\left(M_{k}+1\right)^{2-2 m_{k}}\|u\|_{H^{m_{k}\left(I_{k}\right)}}^{2}
\end{aligned}
$$

This leads to the desired result.
Let $U(t)$ be the global numerical solution of (1.1), which is given by

$$
U(t):=U^{n}(t), \quad t \in I_{n}, \quad 1 \leq n \leq N
$$

Then, by Theorem 4.1 we further obtain
Theorem 4.2. Assume that $\mu<1, a(t) \in C(I), b(t) \in H^{1}(I),\left.b(t)\right|_{t \in\left[t_{n-1}, t_{n}\right]} \in$ $C^{m_{n}}\left[t_{n-1}, t_{n}\right], f(t) \in C(I), u \in H^{1}(I)$ and $\left.u\right|_{t \in I_{n}} \in H^{m_{n}}\left(I_{n}\right)$ with $1 \leq n \leq N$ and integers $2 \leq m_{n} \leq M_{n}+2$. Then for $h_{k}$ sufficiently small (cf. (4.27)), the following inequality holds:

$$
\begin{equation*}
\|u-U\|_{H^{1}(I)}^{2} \leq \exp \left(c T^{4-2 \mu}\right) \sum_{k=1}^{N} h_{k}^{2 m_{k}-2}\left(M_{k}+1\right)^{2-2 m_{k}}\|u\|_{H^{m_{k}}\left(I_{k}\right)}^{2} \tag{4.32}
\end{equation*}
$$

Remark 4.1. The estimate in Theorem 4.2 shows that the spectral collocation method converges either as the time-steps $h_{k}$ are decreased or as the degrees of the polynomials $M_{k}$ are increased. Moreover, we also observe that the $H^{1}$-estimate in Theorem 4.2 is optimal in both $h_{k}$ and $M_{k}$.

Remark 4.2. If the exact solution $u$ of (1.1) is analytic on $[0, T]$, then by (4.21) and the standard approximation theory for analytic functions, we have the error bound $\|u-U\|_{H^{1}(I)} \leq c \exp (-d(M+1))$, with the degree $M=\min _{1 \leq k \leq N} M_{k} \geq 0$ and the constants $c, d>0$.

Remark 4.3. We may also consider the nonlinear VIDE with the weakly singular kernel,

$$
\left\{\begin{array}{l}
u^{\prime}(t)=f(t, u(t))+\int_{0}^{t}(t-s)^{-\mu} K(t, s, u(s)) d s, \quad t \in(0, T]  \tag{4.33}\\
u(0)=u_{0}
\end{array}\right.
$$

where $0<\mu<1$, and the functions $f(t, y)$ and $K(t, s, u)$ are continuous.
The Legendre-Jacobi spectral collocation scheme is to seek $U^{n}(t) \in \mathcal{P}_{M_{n}+1}\left(I_{n}\right)$, such that

$$
\left\{\begin{array}{l}
\frac{d}{d t} U^{n}(t)=\mathcal{I}_{t, M_{n}}\left(f\left(t, U^{n}(t)\right)+\sum_{k=1}^{n-1} \int_{I_{k}}(t-s)^{-\mu} \mathcal{I}_{s, M_{k}+1}^{-\mu, 0} K\left(t, s, U^{k}(s)\right) d s\right.  \tag{4.34}\\
\left.+\left(\frac{t-t_{n-1}}{h_{n}}\right)^{1-\mu} \int_{I_{n}}\left(t_{n}-\lambda\right)^{-\mu} \mathcal{I}_{\lambda, M_{n}+1}^{-\mu, 0} K\left(t, \sigma(\lambda, t), U^{n}(\sigma(\lambda, t))\right) d \lambda\right) \\
U^{n}\left(t_{n-1}\right)=U^{n-1}\left(t_{n-1}\right), \quad U^{1}\left(t_{0}\right)=u_{0}
\end{array}\right.
$$

By using the analysis techniques developed here, we can derive almost the same convergence property as in Theorem 4.2, stated below:

Let $0<\mu<1$ and integers $2 \leq m_{n} \leq M_{n}+2$. Assume that $f(t, y), K(t, s, y(s))$ are continuous and satisfy the Lipschitz conditions:

$$
\begin{gathered}
\left|f\left(t, u_{1}\right)-f\left(t, u_{2}\right)\right| \leq \gamma_{f}\left|u_{1}-u_{2}\right|, \quad \gamma_{f} \geq 0 \\
\left|K\left(t, s, u_{1}\right)-K\left(t, s, u_{2}\right)\right| \leq \gamma_{K}\left|u_{1}-u_{2}\right|, \quad \gamma_{K} \geq 0
\end{gathered}
$$

Then for $h_{n}$ sufficiently small, the following inequality holds:

$$
\begin{aligned}
\|u-U\|_{H^{1}(I)}^{2} \lesssim & \sum_{k=1}^{N} h_{k}^{2 m_{k}-2}\left(M_{k}+1\right)^{2-2 m_{k}} \\
& \times\left(\left\|\partial_{t}^{m_{k}} u\right\|_{L^{2}\left(I_{k}\right)}^{2}+\left\|\partial_{s}^{m_{k}} K(t, \cdot, u(\cdot))\right\|_{L^{\infty}\left(I ; L^{2}\left(I_{k}\right)\right)}^{2}\right) .
\end{aligned}
$$

4.2. Error bounds for singular solutions on a quasi-uniform mesh. In general, the solutions of (1.1) with $\mu \in(0,1)$ will not be smooth at $t=0^{+}$, even if one has smooth data. Hence we need to analyze the $h p$-version convergence of the Legendre-Jacobi spectral collocation method for nonsmooth solutions. We recall the following regularity results.
Lemma 4.7 (see [3). Assume that $a, b, f \in C^{m}(I)(m \geq 1)$, and let $\mu \in(0,1)$. Then:
(1) The regularity of the solution $u$ of the equation (1.1) is described by

$$
u \in C^{1}(I) \cap C^{m+1}(0, T], \quad \text { with }\left|u^{\prime \prime}(t)\right| \leq c t^{-\mu} \text { for } t \in(0, T] .
$$

(2) The solution $u$ can be written in the form

$$
\begin{equation*}
u(t)=\sum_{(j, k)_{\mu}} \gamma_{j, k}(\mu) t^{j+k(2-\mu)}+Y_{m+1}(t, \mu), \quad t \in I, \tag{4.35}
\end{equation*}
$$

where $(j, k)_{\mu}:=\left\{(j, k): j, k \in \mathbb{N}_{0}, j+k(2-\mu)<m+1\right\}$. Moreover, $Y_{m+1}(\cdot, \mu) \in C^{m+1}(I)$, and the coefficients $\gamma_{j, k}(\mu)$ are dependent on $j, k$ and $\mu$.

The following theorem establishes the $h p$-version convergence of the LegendreJacobi spectral collocation method for singular solutions on a quasi-uniform mesh.

Theorem 4.3. Let $M_{n}=M$ and let $I_{h}$ be a quasi-uniform mesh $\left(h_{n} \approx h\right)$. Assume that $\mu \in(0,1)$, the functions $a, f \in C^{m}(I)$ and $b \in C^{m+1}(I)$. Then for $1 \leq m \leq$ $M+1, M>\frac{1}{2}-\mu$ and $h$ sufficiently small (cf. (4.46)),

$$
\begin{equation*}
\|u-U\|_{H^{1}(I)}^{2} \leq \exp \left(c T^{4-2 \mu}\right)\left(h^{3-2 \mu}(M+1)^{4 \mu-6}+h^{2 m}(M+1)^{-2 m}\right) \tag{4.36}
\end{equation*}
$$

Proof. It can be seen from (4.35) that the solution $u$ has $t^{\nu}$-type singularity near $t=0$ for noninteger $\nu$. Note that the most singular term in (4.35) is $\gamma_{0,1}(\mu) t^{2-\mu}$, which governs the convergence rate. Hence, without loss of generality, we may assume that $u$ can be written as

$$
u=C_{\mu} t^{2-\mu}+Y_{m+1}(t, \mu)=: u_{1}(t)+u_{2}(t),
$$

where $Y_{m+1} \in C^{m+1}(I)$ and $C_{\mu}$ is a constant depending on $\mu$. We will approximate the functions $u_{1}$ and $u_{2}$, respectively. By (3.56) we obtain

$$
\begin{equation*}
\left\|\partial_{t} u_{1}-\mathcal{I}_{t, M} \partial_{t} u_{1}\right\|_{L^{2}(I)}^{2} \leq c h^{3-2 \mu}(M+1)^{4 \mu-6} \tag{4.37}
\end{equation*}
$$

Moreover, by (3.7) we know that for $1 \leq m \leq M+1$,

$$
\begin{equation*}
\left\|\partial_{t} u_{2}-\mathcal{I}_{t, M} \partial_{t} u_{2}\right\|_{L^{2}\left(I_{n}\right)}^{2} \leq c h^{2 m}(M+1)^{-2 m}\left\|\partial_{t}^{m+1} u_{2}\right\|_{L^{2}\left(I_{n}\right)}^{2} . \tag{4.38}
\end{equation*}
$$

Combining (4.37) and (4.38) yields

$$
\begin{equation*}
\left\|\partial_{t} u-\mathcal{I}_{t, M} \partial_{t} u\right\|_{L^{2}(I)}^{2} \leq c h^{3-2 \mu}(M+1)^{4 \mu-6}+c h^{2 m}(M+1)^{-2 m}\left\|\partial_{t}^{m+1} u_{2}\right\|_{L^{2}(I)}^{2} . \tag{4.39}
\end{equation*}
$$

Since the most singular term in $b(t) u(t)$ is still $u_{1}(t)$, we use (3.48) and (3.12) to get that for $b \in C^{m+1}(I), 0 \leq m \leq M+1$ and $M>\frac{1}{2}-\mu$,

$$
\begin{align*}
& \sum_{k=1}^{n-1}\left\|\left(\mathcal{I}_{s, M+1}^{L}(b u)-b u\right)^{\prime}\right\|_{L^{2}\left(I_{k}\right)}^{2}  \tag{4.40}\\
\leq & 2 \sum_{k=1}^{n-1}\left(\left\|\left(\mathcal{I}_{s, M+1}^{L}\left(b u_{1}\right)-b u_{1}\right)^{\prime}\right\|_{L^{2}\left(I_{k}\right)}^{2}+\left\|\left(\mathcal{I}_{s, M+1}^{L}\left(b u_{2}\right)-b u_{2}\right)^{\prime}\right\|_{L^{2}\left(I_{k}\right)}^{2}\right) \\
\leq & c h^{3-2 \mu}(M+1)^{4 \mu-6}+c h^{2 m}(M+1)^{-2 m}\left\|\partial_{t}^{m+1}\left(b u_{2}\right)\right\|_{L^{2}(I)}^{2} \\
\leq & c h^{3-2 \mu}(M+1)^{4 \mu-6}+c h^{2 m}(M+1)^{-2 m}\left\|u_{2}\right\|_{H^{m+1}(I)}^{2} .
\end{align*}
$$

Applying (4.40) to (4.9), we derive that

$$
\begin{align*}
\left\|B_{2}\right\|_{L^{2}\left(I_{n}\right)}^{2} \leq & c h T^{3-2 \mu} \sum_{k=1}^{n-1}\left\|e_{k}\right\|_{H^{1}\left(I_{k}\right)}^{2}  \tag{4.41}\\
& +c h T^{3-2 \mu}\left(h^{3-2 \mu}(M+1)^{4 \mu-6}+h^{2 m}(M+1)^{-2 m}\left\|u_{2}\right\|_{H^{m+1}(I)}^{2}\right) .
\end{align*}
$$

Next, by (2.2) and an argument similar to Theorem 3.3) and (4.24), we deduce that (4.42)
$\int_{I_{n}}\left(t_{n}-\lambda\right)^{-\mu}\left|\left(\mathcal{I}_{\lambda, M+1}^{-\mu, 0}-I\right)\left(b\left(\sigma\left(\lambda, t_{n, j}\right)\right) u_{1}\left(\sigma\left(\lambda, t_{n, j}\right)\right)\right)\right|^{2} d \lambda \leq c h^{5-3 \mu}(M+1)^{4 \mu-10}$.
Further, by (4.24) we have that for $0 \leq m \leq M+1$,

$$
\begin{align*}
& \int_{I_{n}}\left(t_{n}-\lambda\right)^{-\mu}\left|\left(\mathcal{I}_{\lambda, M+1}^{-\mu, 0}-I\right)\left(b\left(\sigma\left(\lambda, t_{n, j}\right)\right) u_{2}\left(\sigma\left(\lambda, t_{n, j}\right)\right)\right)\right|^{2} d \lambda  \tag{4.43}\\
& \leq c h^{2 m-\mu+2}(M+1)^{-2 m-2}\left\|\partial_{t}^{m+1}\left(b u_{2}\right)\right\|_{L^{2}\left(I_{n}\right)}^{2}
\end{align*}
$$

Therefore, by (4.14), (4.42) and (4.43) we get that for $b \in C^{m+1}(I)$,

$$
\begin{align*}
& \left\|B_{3}\right\|_{L^{2}\left(I_{n}\right)}^{2} \leq c h^{2-2 \mu}\left\|e_{n}\right\|_{L^{2}\left(I_{n}\right)}^{2}+c h^{4-2 \mu}\left\|e_{n}^{\prime}\right\|_{L^{2}\left(I_{n}\right)}^{2}  \tag{4.44}\\
& \quad+c h^{2-\mu}\left(h^{5-3 \mu}(M+1)^{4 \mu-10}+h^{2 m-\mu+2}(M+1)^{-2 m-2}\left\|u_{2}\right\|_{H^{m+1}\left(I_{n}\right)}^{2}\right)
\end{align*}
$$

Thus, by (4.4), (4.5), (4.7), (4.41) and (4.44), we deduce that

$$
\begin{align*}
& \left\|e_{n}^{\prime}\right\|_{L^{2}\left(I_{n}\right)}^{2}  \tag{4.45}\\
& \quad \leq c\left\|e_{n}\right\|_{L^{2}\left(I_{n}\right)}^{2}+c h^{2}(M+1)^{-2}\left\|e_{n}^{\prime}\right\|_{L^{2}\left(I_{n}\right)}^{2}+2\left\|\partial_{t} u-\mathcal{I}_{t, M} \partial_{t} u\right\|_{L^{2}\left(I_{n}\right)}^{2} \\
& \quad+c h T^{3-2 \mu} \sum_{k=1}^{n-1}\left\|e_{k}\right\|_{H^{1}\left(I_{k}\right)}^{2} \\
& \quad+c h T^{3-2 \mu}\left(h^{3-2 \mu}(M+1)^{4 \mu-6}+h^{2 m}(M+1)^{-2 m}\left\|u_{2}\right\|_{H^{m+1}(I)}^{2}\right) \\
& \quad+c h^{2-2 \mu}\left\|e_{n}\right\|_{L^{2}\left(I_{n}\right)}^{2}+c h^{4-2 \mu}\left\|e_{n}^{\prime}\right\|_{L^{2}\left(I_{n}\right)}^{2} \\
& \quad+c h^{2-\mu}\left(h^{5-3 \mu}(M+1)^{4 \mu-10}+h^{2 m-\mu+2}(M+1)^{-2 m-2}\left\|u_{2}\right\|_{H^{m+1}\left(I_{n}\right)}^{2}\right) .
\end{align*}
$$

Assume that $h$ is sufficiently small such that

$$
\begin{equation*}
c h^{2}+c h^{4-2 \mu} \leq \eta<1 \tag{4.46}
\end{equation*}
$$

Then we may rewrite (4.45) as

$$
\begin{equation*}
\left\|e_{n}^{\prime}\right\|_{L^{2}\left(I_{n}\right)}^{2} \leq c\left\|e_{n}\right\|_{L^{2}\left(I_{n}\right)}^{2}+c h T^{3-2 \mu} \sum_{k=1}^{n-1}\left\|e_{k}\right\|_{H^{1}\left(I_{k}\right)}^{2}+\mathcal{Q}_{n} \tag{4.47}
\end{equation*}
$$

where

$$
\begin{aligned}
\mathcal{Q}_{n}= & 2\left\|\partial_{t} u-\mathcal{I}_{t, M} \partial_{t} u\right\|_{L^{2}\left(I_{n}\right)}^{2} \\
& +c h T^{3-2 \mu}\left(h^{3-2 \mu}(M+1)^{4 \mu-6}+h^{2 m}(M+1)^{-2 m}\left\|u_{2}\right\|_{H^{m+1}(I)}^{2}\right) \\
& +c h^{2-\mu}\left(h^{5-3 \mu}(M+1)^{4 \mu-10}+h^{2 m-\mu+2}(M+1)^{-2 m-2}\left\|u_{2}\right\|_{H^{m+1}\left(I_{n}\right)}^{2}\right) .
\end{aligned}
$$

This, along with (4.30) and (4.46), yields

$$
\begin{equation*}
\left\|e_{n}\right\|_{H^{1}\left(I_{n}\right)}^{2} \leq c h T^{3-2 \mu} \sum_{k=1}^{n-1}\left\|e_{k}\right\|_{H^{1}\left(I_{k}\right)}^{2}+\mathcal{Q}_{n} \tag{4.48}
\end{equation*}
$$

By taking $\varepsilon_{k}=h^{-1}\left\|e_{k}\right\|_{H^{1}\left(I_{k}\right)}^{2}$ in Lemma 4.2, we get

$$
\left\|e_{n}\right\|_{H^{1}\left(I_{n}\right)}^{2} \leq \mathcal{Q}_{n}+h \exp \left(c T^{4-2 \mu}\right) \sum_{j=1}^{n-1} \mathcal{Q}_{j}
$$

Thereby,

$$
\begin{equation*}
\|u-U\|_{H^{1}(I)}^{2} \quad \leq \exp \left(c T^{4-2 \mu}\right) \sum_{n=1}^{N} \mathcal{Q}_{n} \tag{4.49}
\end{equation*}
$$

Finally, by (4.39) and (4.49) we obtain

$$
\begin{equation*}
\|u-U\|_{H^{1}(I)}^{2} \leq \exp \left(c T^{4-2 \mu}\right)\left(h^{3-2 \mu}(M+1)^{4 \mu-6}+h^{2 m}(M+1)^{-2 m}\right) \tag{4.50}
\end{equation*}
$$

This ends the proof.
4.3. Exponential convergence for singular solutions on a geometric mesh. In this subsection, we show that the $h p$-version of spectral collocation method with geometrically time partitions and linearly increasing polynomial degrees yields exponential rates of convergence.

We assume that $\mu \in(0,1)$, the data $a(t)$ and $b(t)$ are analytic functions, and $f(t)$ has the form

$$
\begin{equation*}
f(t)=f_{1}(t)+t^{\beta} f_{2}(t), \quad \beta>1, \quad \beta \notin \mathbb{N} \tag{4.51}
\end{equation*}
$$

where $f_{1}$ and $f_{2}$ are analytic functions. Recall that an analytic function $g$ can be characterized by analyticity constants $C_{g}, d_{g}>0$ and the growth conditions (see [16, pp. 78-79] for details),

$$
\left|g^{(s)}(t)\right| \leq C_{g} d_{g}^{s} \Gamma(s+1), \quad t \in[0, T], \quad s \geq 0
$$

The following result describes the analyticity properties of the exact solution $u$.
Lemma 4.8 ([4, Theorem 4.1]). Assume that the data $a(t), b(t)$ are analytic functions and $f(t)$ has the form (4.51). Then there exist constants $c, d>0$ depending only on the analyticity constants of $a, b, f_{1}$ and $f_{2}$ such that the solution $u$ of (1.1) satisfies

$$
\left|u^{(s)}(t)\right| \leq c d^{s} \Gamma(s+1) t^{2-\mu-s}, \quad t \in(0, T], \quad s \in \mathbb{N}_{0}
$$

Definition 4.1. A geometric partition $\mathcal{T}_{N, \rho}=\left\{I_{n}\right\}_{n=1}^{N}$ of $(0, T)$ with grading factor $\rho \in(0,1)$ and $N$ levels of refinement is given by

$$
t_{0}=0, \quad t_{n}=T \rho^{N-n}, \quad 1 \leq n \leq N .
$$

For $2 \leq n \leq N$, the time steps $h_{n}=t_{n}-t_{n-1}$ satisfy $h_{n}=\lambda_{0} t_{n-1}$ with $\lambda_{0}=\frac{1-\rho}{\rho}$.
Definition 4.2. Let $\mathcal{T}_{N, \rho}$ be a geometric mesh of $(0, T)$ as defined in Definition 4.1. An approximation degree vector $\mathbf{M}$ on $\mathcal{T}_{N, \rho}$ is called linear with slope $\theta>0$ if $M_{1}=1, M_{n}=\max \{1,\lfloor\theta n\rfloor\}, 2 \leq n \leq N$, on the geometrically refined elements $\left\{I_{n}\right\}_{n=1}^{N}$.
Lemma 4.9. Assume that the data $a(t), b(t)$ are analytic functions and $f(t)$ has the form (4.51). Let $\mathcal{T}_{N, \rho}$ be a geometric mesh with $\left\{I_{n}\right\}_{n=1}^{N}$ denoting the geometric refinement of $(0, T)$. Then the solution $u$ of (1.1) satisfies

$$
\|u\|_{W^{1, \infty}\left(I_{1}\right)}^{2} \leq c
$$

and for $s \geq 0$,

$$
\|u\|_{W^{s+1, \infty}\left(I_{n}\right)}^{2} \leq c d^{2 s} \Gamma(2 s+1) \rho^{2(N-n+1)(1-\mu-s)}, \quad 2 \leq n \leq N
$$

where the constants $c, d>0$ are independent of $n, N$ and $s$.

Proof. The assertions follow from Lemma 4.8, Definition 4.1 and the property of the Gamma function.

The following result establishes the exponential convergence rate of the spectral collocation method for the VIDE (1.1).

Theorem 4.4. Assume that $\mu \in(0,1)$, the data $a(t), b(t)$ are analytic functions and $f(t)$ has the form (4.51). Let $\mathcal{T}_{N, \rho}$ be a geometric mesh of $(0, T)$. Then there exists a slope $\theta_{1}>0$ solely depending on $\rho, \mu, \beta$ and the constants $c$ and $d$ in Lemma 4.9 such that for all linear polynomial degree vectors $\mathbf{M}$ with slope $\theta \geq \theta_{1}$, the spectral collocation approximation $U$ of (1.1) obtained by (2.21) satisfies the error estimate

$$
\begin{equation*}
\|u-U\|_{H^{1}(I)} \leq c e^{-b_{0} \sqrt{D O F}} \tag{4.52}
\end{equation*}
$$

where the constants $c, b_{0}>0$ are independent of the degrees of freedom DOF.
Proof. The error bound (4.52) can be proved using similar techniques as in 4, 14, 15. For convenience, we select $\theta \geq \frac{1}{2}$ such that $M_{n}=\lfloor\theta n\rfloor \geq 1$ on the geometrically refined intervals $\left\{I_{n}\right\}_{n=2}^{N}$. Since $\beta>1$, we have $f \in C^{1}(I)$, and hence the results of Lemma 4.7 with $m=1$ hold. Without loss of generality, we may assume that $u$ can be written as

$$
u=C_{\mu} t^{2-\mu}+Y_{2}(t, \mu)=: u_{1}(t)+u_{2}(t)
$$

where $Y_{2} \in C^{2}(I)$ and $C_{\mu}$ is a constant depending on $\mu$.
We first estimate the term $\left\|\partial_{t} u-\mathcal{I}_{t, M_{n}} \partial_{t} u\right\|_{L^{2}\left(I_{n}\right)}$. By (3.41) with $\alpha=0$ we obtain

$$
\begin{equation*}
\left\|\partial_{t} u_{1}-\mathcal{I}_{t, M_{1}} \partial_{t} u_{1}\right\|_{L^{2}\left(I_{1}\right)}^{2} \leq \operatorname{ch}_{1}^{3-2 \mu}\left(M_{1}+1\right)^{4 \mu-6} \tag{4.53}
\end{equation*}
$$

Moreover, by (3.7) we know that

$$
\begin{align*}
\left\|\partial_{t} u_{2}-\mathcal{I}_{t, M_{1}} \partial_{t} u_{2}\right\|_{L^{2}\left(I_{1}\right)}^{2} & \leq c h_{1}^{2}\left(M_{1}+1\right)^{-2}\left\|\partial_{t}^{2} u_{2}\right\|_{L^{2}\left(I_{1}\right)}^{2}  \tag{4.54}\\
& \leq \operatorname{ch}_{1}^{3}\left(M_{1}+1\right)^{-2}\left\|\partial_{t}^{2} u_{2}\right\|_{L^{\infty}\left(I_{1}\right)}^{2} .
\end{align*}
$$

Combining (4.53) and (4.54) yields

$$
\begin{align*}
\left\|\partial_{t} u-\mathcal{I}_{t, M_{1}} \partial_{t} u\right\|_{L^{2}\left(I_{1}\right)}^{2} & \leq c h_{1}^{3-2 \mu}\left(M_{1}+1\right)^{4 \mu-6}+c h_{1}^{3}\left(M_{1}+1\right)^{-2}\left\|\partial_{t}^{2} u_{2}\right\|_{L^{\infty}\left(I_{1}\right)}^{2}  \tag{4.55}\\
& \leq c\left(\rho^{(3-2 \mu)(N-1)}+\rho^{3(N-1)}\right) \leq c \rho^{(3-2 \mu)(N-1)}
\end{align*}
$$

According to Lemma 4.8, the solution $u$ is analytic away from $t=0$. Thus, the regularity exponents $m_{n}$ can be chosen arbitrarily large for $2 \leq n \leq N$. Therefore, on the subintervals $I_{n}$ for $2 \leq n \leq N$, we use (3.1) with $\alpha=\beta=0$, Lemma 4.9 and
the equality $h_{n}=\lambda_{0} t_{n-1}$ with $t_{n-1}=T \rho^{N-n+1}$ to obtain that for $1 \leq m_{n} \leq M_{n}+1$,

$$
\begin{aligned}
\| \partial_{t} u & -\mathcal{I}_{t, M_{n}} \partial_{t} u\left\|_{L^{2}\left(I_{n}\right)}^{2} \leq c h_{n}^{2 m_{n}} \frac{\Gamma\left(M_{n}+2-m_{n}\right)}{\Gamma\left(M_{n}+2+m_{n}\right)}\right\| \partial_{t}^{m_{n}+1} u \|_{L^{2}\left(I_{n}\right)}^{2} \\
& \leq c h_{n}^{2 m_{n}+1} \frac{\Gamma\left(M_{n}+2-m_{n}\right)}{\Gamma\left(M_{n}+2+m_{n}\right)}\left\|\partial_{t}^{m_{n}+1} u\right\|_{L^{\infty}\left(I_{n}\right)}^{2} \\
& \leq c\left(\lambda_{0} T \rho^{N-n+1}\right)^{2 m_{n}+1} \frac{\Gamma\left(M_{n}+2-m_{n}\right)}{\Gamma\left(M_{n}+2+m_{n}\right)} d^{2 m_{n}} \Gamma\left(2 m_{n}+1\right) \rho^{2(N-n+1)\left(1-\mu-m_{n}\right)} \\
& \leq c \rho^{(3-2 \mu)(N-n+1)}\left(\lambda_{0} T\right)^{2 m_{n}+1} \frac{\Gamma\left(M_{n}+2-m_{n}\right)}{\Gamma\left(M_{n}+2+m_{n}\right)} d^{2 m_{n}} \Gamma\left(2 m_{n}+1\right) \\
& \leq c \rho^{(3-2 \mu)(N-n+1)}\left(\lambda_{0} T d\right)^{2 m_{n}} \frac{\Gamma\left(M_{n}+2-m_{n}\right)}{\Gamma\left(M_{n}+2+m_{n}\right)} \Gamma\left(2 m_{n}+1\right) .
\end{aligned}
$$

Now setting $m_{n}=\varepsilon_{n}\left(M_{n}+1\right)$ with $\varepsilon_{n} \in(0,1)$, with Stirling's formula we get

$$
\begin{aligned}
\| \partial_{t} u & -\mathcal{I}_{t, M_{n}} \partial_{t} u \|_{L^{2}\left(I_{n}\right)}^{2} \\
& \leq c \rho^{(3-2 \mu)(N-n+1)}\left(M_{n}+1\right)^{\frac{1}{2}}\left(\left(\lambda_{0} T d\right)^{2 \varepsilon_{n}} \frac{\left(1-\varepsilon_{n}\right)^{1-\varepsilon_{n}}}{\left(1+\varepsilon_{n}\right)^{1+\varepsilon_{n}}}\right)^{M_{n}+1} .
\end{aligned}
$$

Note that the function $g_{\lambda_{0}, T, d}(\varepsilon)=\left(\lambda_{0} T d\right)^{2 \varepsilon} \frac{(1-\varepsilon)^{1-\varepsilon}}{(1+\varepsilon)^{1+\varepsilon}}$ satisfies

$$
0<\inf _{0<\varepsilon<1} g_{\lambda_{0}, T, d}(\varepsilon)=: g_{\lambda_{0}, T, d}\left(\varepsilon_{\min }\right)<1 \quad \text { with } \varepsilon_{\min }=\frac{1}{\sqrt{1+\lambda_{0}^{2} T^{2} d^{2}}}
$$

and thus, setting $g_{\min }=: g_{\lambda_{0}, T, d}\left(\varepsilon_{\min }\right)$ and choosing $\varepsilon_{n}=\varepsilon_{\min }$ for $2 \leq n \leq N$, we conclude that

$$
\begin{aligned}
\left\|\partial_{t} u-\mathcal{I}_{t, M_{n}} \partial_{t} u\right\|_{L^{2}\left(I_{n}\right)}^{2} & \leq c \rho^{(3-2 \mu)(N-n+1)}\left(M_{n}+1\right)^{\frac{1}{2}} g_{\min }^{M_{n}+1} \\
& \leq c \rho^{(3-2 \mu) N}(\theta N)^{\frac{1}{2}}\left(\rho^{(3-2 \mu)(1-n)} g_{\min }^{\theta n}\right)
\end{aligned}
$$

Next, let

$$
\theta_{0}=\max \left\{\frac{(3-2 \mu) \log \rho}{\log \left(g_{\min }\right)}, \frac{1}{2}\right\} .
$$

Then, for $\theta \geq \theta_{0}$, we have $g_{\min }^{\theta n} \leq \rho^{(3-2 \mu) n}$, and hence,
$\left\|\partial_{t} u-\mathcal{I}_{t, M_{n}} \partial_{t} u\right\|_{L^{2}\left(I_{n}\right)}^{2} \leq c \rho^{(3-2 \mu) N}(\theta N)^{\frac{1}{2}} \rho^{3-2 \mu} \leq c \rho^{(3-2 \mu) N}(\theta N)^{\frac{1}{2}}, \quad 2 \leq n \leq N$.
Combining the estimates in (4.55) and (4.56) leads to

$$
\begin{equation*}
\left\|\partial_{t} u-\mathcal{I}_{t, M_{n}} \partial_{t} u\right\|_{L^{2}\left(I_{n}\right)}^{2} \leq c e^{-b_{0} N}, \quad 1 \leq n \leq N \tag{4.57}
\end{equation*}
$$

as $N \rightarrow \infty$, where we have absorbed the term $(\theta N)^{\frac{1}{2}}$ into the constants $c$ and $b_{0}$.

We next estimate the term $\left\|B_{2}\right\|_{L^{2}\left(I_{n}\right)}$. Since the most singular term in $b(t) u(t)$ is still $u_{1}(t)$, we use (3.51), (3.12) and a similar argument as in (4.55) to get that

$$
\begin{align*}
& \left\|\left(\mathcal{I}_{s, M_{1}+1}^{L}(b u)-b u\right)^{\prime}\right\|_{L^{2}\left(I_{1}\right)}^{2} \\
\leq & 2\left\|\left(\mathcal{I}_{s, M_{1}+1}^{L}\left(b u_{1}\right)-b u_{1}\right)^{\prime}\right\|_{L^{2}\left(I_{1}\right)}^{2}+2\left\|\left(\mathcal{I}_{s, M_{1}+1}^{L}\left(b u_{2}\right)-b u_{2}\right)^{\prime}\right\|_{L^{2}\left(I_{1}\right)}^{2} \\
\leq & c h_{1}^{3-2 \mu}\left(M_{1}+1\right)^{4 \mu-6}+c h_{1}^{2}\left(M_{1}+1\right)^{-2}\left\|\partial_{t}^{2}\left(b u_{2}\right)\right\|_{L^{2}\left(I_{1}\right)}^{2}  \tag{4.58}\\
\leq & c h_{1}^{3-2 \mu}\left(M_{1}+1\right)^{4 \mu-6}+c h_{1}^{2}\left(M_{1}+1\right)^{-2}\left\|u_{2}\right\|_{H^{2}(I)}^{2} \\
\leq & c \rho^{(3-2 \mu)(N-1)} .
\end{align*}
$$

Moreover, by (3.10) and an argument similar to (4.56), we obtain that for $2 \leq n \leq$ $N$,

$$
\begin{align*}
& \left\|\left(\mathcal{I}_{s, M_{n}+1}^{L}(b u)-b u\right)^{\prime}\right\|_{L^{2}\left(I_{n}\right)}^{2} \\
& \quad \leq c h_{n}^{2 m_{n}}\left(M_{n}+1\right)^{2} \frac{\Gamma\left(M_{n}+2-m_{n}\right)}{\Gamma\left(M_{n}+4+m_{n}\right)}\left\|\partial_{t}^{m_{n}+1}(b u)\right\|_{L^{2}\left(I_{n}\right)}^{2} \\
& \quad \leq c h_{n}^{2 m_{n}} \frac{\Gamma\left(M_{n}+2-m_{n}\right)}{\Gamma\left(M_{n}+2+m_{n}\right)}\|u\|_{H^{m_{n}+1}\left(I_{n}\right)}^{2}  \tag{4.59}\\
& \quad \leq c \rho^{(3-2 \mu) N}(\theta N)^{\frac{1}{2}} .
\end{align*}
$$

Applying (4.58), (4.59) and (4.57) to (4.9), we derive that

$$
\begin{equation*}
\left\|B_{2}\right\|_{L^{2}\left(I_{n}\right)}^{2} \leq c h_{n} T^{3-2 \mu} \sum_{k=1}^{n-1}\left\|e_{k}\right\|_{H^{1}\left(I_{k}\right)}^{2}+c h_{n} e^{-b_{0} N} \tag{4.60}
\end{equation*}
$$

We now estimate the term $\left\|B_{3}\right\|_{L^{2}\left(I_{n}\right)}$. By (4.42) we know

$$
\begin{align*}
& \int_{I_{1}}\left(t_{1}-\lambda\right)^{-\mu}\left|\left(\mathcal{I}_{\lambda, M_{1}+1}^{-\mu, 0}-I\right)\left(b\left(\sigma\left(\lambda, t_{1, j}\right)\right) u_{1}\left(\sigma\left(\lambda, t_{1, j}\right)\right)\right)\right|^{2} d \lambda  \tag{4.61}\\
& \quad \leq c h_{1}^{5-3 \mu}\left(M_{1}+1\right)^{4 \mu-10}
\end{align*}
$$

Further, by (4.24) we have

$$
\begin{align*}
& \int_{I_{1}}\left(t_{1}-\lambda\right)^{-\mu}\left|\left(\mathcal{I}_{\lambda, M_{1}+1}^{-\mu, 0}-I\right)\left(b\left(\sigma\left(\lambda, t_{1, j}\right)\right) u_{2}\left(\sigma\left(\lambda, t_{1, j}\right)\right)\right)\right|^{2} d \lambda  \tag{4.62}\\
\leq & c h_{1}^{4-\mu}\left(M_{1}+1\right)^{-4}\left\|\partial_{t}^{2}\left(b u_{2}\right)\right\|_{L^{2}\left(I_{1}\right)}^{2} \leq c h_{1}^{4-\mu}\left(M_{1}+1\right)^{-4}\left\|u_{2}\right\|_{H^{2}\left(I_{1}\right)}^{2}
\end{align*}
$$

Hence,

$$
\begin{align*}
& \int_{I_{1}}\left(t_{1}-\lambda\right)^{-\mu}\left|\left(\mathcal{I}_{\lambda, M_{1}+1}^{-\mu, 0}-I\right)\left(b\left(\sigma\left(\lambda, t_{1, j}\right)\right) u\left(\sigma\left(\lambda, t_{1, j}\right)\right)\right)\right|^{2} d \lambda  \tag{4.63}\\
\leq & c \rho^{(5-3 \mu)(N-1)}+c \rho^{(4-\mu)(N-1)} \leq c e^{-b_{0} N}
\end{align*}
$$

Moreover, by (3.1) and similar arguments as for (4.24) and (4.57), we obtain that for $2 \leq n \leq N$ and $\theta \geq \theta_{1}=\max \left\{\frac{(5-3 \mu) \log \rho}{\log \left(g_{\min }\right)}, \frac{1}{2}\right\}$,

$$
\begin{align*}
& \int_{I_{n}}\left(t_{n}-\lambda\right)^{-\mu}\left|\left(\mathcal{I}_{\lambda, M_{n}+1}^{-\mu, 0}-I\right)\left(b\left(\sigma\left(\lambda, t_{n, j}\right)\right) u\left(\sigma\left(\lambda, t_{n, j}\right)\right)\right)\right|^{2} d \lambda \\
\leq & c h_{n}^{2 m_{n}-\mu+2} \frac{\Gamma\left(M_{n}+2-m_{n}\right)}{\Gamma\left(M_{n}+4+m_{n}\right)}\left\|\partial_{t}^{m_{n}+1}(b u)\right\|_{L^{2}\left(I_{n}\right)}^{2}  \tag{4.64}\\
\leq & c h_{n}^{2 m_{n}-\mu+2}\left(M_{n}+1\right)^{-2} \frac{\Gamma\left(M_{n}+2-m_{n}\right)}{\Gamma\left(M_{n}+2+m_{n}\right)}\|u\|_{H^{m_{n}+1}\left(I_{n}\right)}^{2} \\
\leq & c \rho^{(5-3 \mu) N}(\theta N)^{-\frac{3}{2}} \leq c e^{-b_{0} N}
\end{align*}
$$

Therefore, by (4.14), (4.63) and (4.64) we get that for all $1 \leq n \leq N$,

$$
\begin{equation*}
\left\|B_{3}\right\|_{L^{2}\left(I_{n}\right)}^{2} \leq c h_{n}^{2-2 \mu}\left\|e_{n}\right\|_{L^{2}\left(I_{n}\right)}^{2}+c h_{n}^{4-2 \mu}\left\|e_{n}^{\prime}\right\|_{L^{2}\left(I_{n}\right)}^{2}+c h_{n}^{2-\mu} e^{-b_{0} N} . \tag{4.65}
\end{equation*}
$$

Thus, by (4.4), (4.5), (4.7), (4.57), (4.60) and (4.65), we deduce that

$$
\begin{align*}
\left\|e_{n}^{\prime}\right\|_{L^{2}\left(I_{n}\right)}^{2} \leq & c\left\|e_{n}\right\|_{L^{2}\left(I_{n}\right)}^{2}+c h_{n}^{2}\left(M_{n}+1\right)^{-2}\left\|e_{n}^{\prime}\right\|_{L^{2}\left(I_{n}\right)}^{2}+c e^{-b_{0} N} \\
& +c h_{n} T^{3-2 \mu} \sum_{k=1}^{n-1}\left\|e_{k}\right\|_{H^{1}\left(I_{k}\right)}^{2}+c h_{n}^{2-2 \mu}\left\|e_{n}\right\|_{L^{2}\left(I_{n}\right)}^{2}  \tag{4.66}\\
& +c h_{n}^{4-2 \mu}\left\|e_{n}^{\prime}\right\|_{L^{2}\left(I_{n}\right)}^{2}
\end{align*}
$$

Assume that $h_{n}$ is sufficiently small such that

$$
\begin{equation*}
c h_{n}^{2}+c h_{n}^{4-2 \mu} \leq \eta<1 \tag{4.67}
\end{equation*}
$$

Then we may rewrite (4.66) as

$$
\begin{equation*}
\left\|e_{n}^{\prime}\right\|_{L^{2}\left(I_{n}\right)}^{2} \leq c\left\|e_{n}\right\|_{L^{2}\left(I_{n}\right)}^{2}+c h_{n} T^{3-2 \mu} \sum_{k=1}^{n-1}\left\|e_{k}\right\|_{H^{1}\left(I_{k}\right)}^{2}+c e^{-b_{0} N} \tag{4.68}
\end{equation*}
$$

This, along with (4.30) and (4.67), yields

$$
\begin{equation*}
\left\|e_{n}\right\|_{H^{1}\left(I_{n}\right)}^{2} \quad \leq c h_{n} T^{3-2 \mu} \sum_{k=1}^{n-1}\left\|e_{k}\right\|_{H^{1}\left(I_{k}\right)}^{2}+c e^{-b_{0} N} \tag{4.69}
\end{equation*}
$$

By taking $\varepsilon_{k}=h_{k}^{-1}\left\|e_{k}\right\|_{H^{1}\left(I_{k}\right)}^{2}$ in Lemma 4.2, we get

$$
\left\|e_{n}\right\|_{H^{1}\left(I_{n}\right)}^{2} \leq c e^{-b_{0} N}+c h_{n} \exp \left(c T^{4-2 \mu}\right) N e^{-b_{0} N} \leq c e^{-b_{0} N}
$$

where we have absorbed the terms $h_{n} \exp \left(c T^{4-2 \mu}\right) N$ into the constants $c$ and $b_{0}$. Thereby,

$$
\begin{equation*}
\|u-U\|_{H^{1}(I)}^{2} \quad \leq c e^{-b_{0} N} \tag{4.70}
\end{equation*}
$$

Since DOF $\leq c N^{2}$ for $N$ sufficiently large, we obtain the desired result.

## 5. Numerical results

In this section, we present some numerical results to illustrate the efficiency of the Legendre-Jacobi spectral collocation method. Consider the linear VIDEs (cf. [4):

$$
\left\{\begin{array}{l}
u^{\prime}(t)+u(t)+\int_{0}^{t}(t-s)^{-\mu} e^{s} u(s) d s=f(t), \quad t \in(0, T]  \tag{5.1}\\
u(0)=0
\end{array}\right.
$$

We choose the right-hand side $f$ such that the solution $u$ of (5.1) is given by $u(t)=t^{2-\mu} e^{-t}$.
5.1. Smooth solution. We start by considering the case $\mu=-1$ so that the solution $u$ in (5.1) is analytic on [0,T].

In Figure 5.1. we list the discrete $H^{1}$-errors of the $h$-version of the LegendreJacobi spectral collocation method with $T=1$. The uniform time partitions are refined by bisection of each time step at a fixed uniform mode $M=0,1,2,3$. In Table 5.1. we also list the convergence rates of the discrete $H^{1}$-errors of the $h$-version, which are algebraic and in accordance with the $h$-version results as predicted by Theorem 4.2

In Figure 5.2, we show the discrete $H^{1}$-errors of the $p$-version of the LegendreJacobi spectral collocation method with $T=1$. The mode $M$ is increased for each fixed time-step $h=1,1 / 2,1 / 4,1 / 8$. The results show that exponential rates of convergence are achieved, which are in agreement with the comments in Remark 4.2.


Figure 5.1. $h$-version: $\mu=-1$.


Figure 5.2. $p$-version: $\mu=-1$.

TABLE 5.1. $h$-version: $\mu=-1$.

| $h$ | $M$ | $H^{1}$-errors | order | $M$ | $H^{1}$-errors | order | $M$ | $H^{1}$-errors | order |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $1 / 64$ |  | $3.50 \mathrm{E}-03$ | 1.00 |  | 1.93 E-05 | 2.00 |  | $1.22 \mathrm{E}-07$ | 3.00 |
| $1 / 128$ |  | $1.75 \mathrm{E}-03$ | 1.00 |  | $4.82 \mathrm{E}-06$ | 2.00 |  | $1.52 \mathrm{E}-08$ | 3.00 |
| $1 / 256$ | 0 | $8.74 \mathrm{E}-04$ | 1.00 | 1 | $1.21 \mathrm{E}-06$ | 2.00 | 2 | $1.90 \mathrm{E}-09$ | 3.00 |
| $1 / 512$ |  | $4.37 \mathrm{E}-04$ | 1.00 |  | $3.01 \mathrm{E}-07$ | 2.00 |  | $2.38 \mathrm{E}-10$ | 3.00 |
| $1 / 1024$ |  | $2.18 \mathrm{E}-04$ | 1.00 |  | $7.53 \mathrm{E}-08$ | 2.00 |  | $2.97 \mathrm{E}-11$ | 3.00 |



Figure 5.3. $h$-version: $\mu=0.6$.


Figure 5.4. $p$-version: $\mu=0.6$.

TABLE 5.2. $h$-version: $\mu=0.6$.

| $h$ | $M$ | $H^{1}$-errors | order | $M$ | $H^{1}$-errors | order | $M$ | $H^{1}$-errors | order |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $1 / 256$ |  | $2.57 \mathrm{E}-03$ | 0.84 |  | $5.53 \mathrm{E}-04$ | 0.91 |  | $2.70 \mathrm{E}-04$ | 0.90 |
| $1 / 512$ | 0 | $1.43 \mathrm{E}-03$ | 0.85 | 1 | $2.96 \mathrm{E}-04$ | 0.90 | 2 | $1.45 \mathrm{E}-04$ | 0.90 |
| $1 / 1024$ |  | $7.87 \mathrm{E}-04$ | 0.86 |  | $1.58 \mathrm{E}-04$ | 0.90 |  | $7.75 \mathrm{E}-05$ | 0.90 |
| $1 / 2048$ |  | $4.32 \mathrm{E}-04$ | 0.87 |  | $8.48 \mathrm{E}-05$ | 0.90 |  | $4.15 \mathrm{E}-05$ | 0.90 |

5.2. Nonsmooth solution. We next consider the case $\mu=0.6$ so that the solution $u$ in (5.1) has a weak singularity at $t=0$.

We first test the $h$-version of the Legendre-Jacobi spectral collocation method on the uniform partitions as adopted in Subsection 5.1. In Figure 5.3, we present the discrete $H^{1}$-errors of the $h$-version with $T=1$ and the fixed uniform mode $M=0,1,2,3$. In Table 5.2, we also list the convergence rates of the discrete $H^{1}$-errors of the $h$-version. It can be observed that almost the same algebraic convergence rate of order 0.9 for the $H^{1}$-errors is achieved, which coincides well with the $h$-version results as predicted by Theorem 4.3.

In Figure 5.4, we show the discrete $H^{1}$-errors of the $p$-version of the LegendreJacobi spectral collocation method with $T=1$. The mode $M$ is increased for each fixed time step $h=1,1 / 2,1 / 4,1 / 8$. In Table 5.3, we also list the convergence order of the discrete $H^{1}$-errors of the $p$-version. The order of convergence for the $H^{1}$ errors is about 1.8 , which is twice as fast as the $h$-version results. This confirms the doubling convergence rates for the $p$-version in Theorem 4.3,

TABLE 5.3. $p$-version: $\mu=0.6$.

| $M$ | $h$ | $H^{1}$-errors | order | $h$ | $H^{1}$-errors | order |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 21 |  | $1.21 \mathrm{E}-03$ | 1.77 |  | $3.21 \mathrm{E}-04$ | 1.78 |
| 22 |  | $1.12 \mathrm{E}-03$ | 1.79 |  | $2.96 \mathrm{E}-04$ | 1.80 |
| 23 | 1 | $1.03 \mathrm{E}-03$ | 1.80 | $1 / 4$ | $2.74 \mathrm{E}-04$ | 1.82 |
| 24 |  | $9.56 \mathrm{E}-04$ | 1.82 |  | $2.55 \mathrm{E}-04$ | 1.83 |
| 25 |  | $8.88 \mathrm{E}-04$ | 1.84 |  | $2.37 \mathrm{E}-04$ | 1.85 |



Figure 5.5. $h p$-version: $\mu=0.6$.


Figure 5.6. $h p$-version: $\mu=0.6$.

To resolve the singular behavior of the solution more efficiently, we shall make use of geometrically refined steps and linearly increasing degree vectors as described in Subsection 4.3.

In Figure 5.5, we plot the discrete $H^{1}(0,1)$-errors against the square root of the number of degrees of freedom, with $\mu=0.6, \theta=1.5$ and various values of $\rho$. It can be seen that the exponential convergence is achieved for each $\rho$. In Figure 5.6, we present the discrete $H^{1}(0,1)$-errors with $\rho=0.1$ and various values of $\theta$. The near straight lines also indicate exponential convergence for each $\theta$. These experimental results agree well with the theory in Theorem 4.4.

## 6. Concluding remarks

In this paper we introduced an $h p$-version Legendre-Jacobi spectral collocation method for Volterra integro-differential equations with smooth and weakly singular kernels. We derived a priori error estimates under the $H^{1}$-norm that are explicit with respect to the time steps and the approximation orders. These theoretical results were confirmed by some numerical examples. Furthermore, as shown in Theorem 4.4 and the numerical examples, for problems with start-up singularities, the $h p$-version Legendre-Jacobi spectral collocation method with geometric mesh refinement and linearly increasing polynomial degrees in the discretization can yield exponential rate of convergence.

## References

[1] H. Brunner, Implicit Runge-Kutta methods of optimal order for Volterra integro-differential equations, Math. Comp. 42 (1984), no. 165, 95-109, DOI 10.2307/2007561. MR725986
[2] H. Brunner, Polynomial spline collocation methods for Volterra integrodifferential equations with weakly singular kernels, IMA J. Numer. Anal. 6 (1986), no. 2, 221-239, DOI 10.1093/imanum/6.2.221. MR967664
[3] H. Brunner, Collocation Methods for Volterra Integral and Related Functional Differential Equations, Cambridge Monographs on Applied and Computational Mathematics, vol. 15, Cambridge University Press, Cambridge, 2004. MR 2128285
[4] H. Brunner and D. Schötzau, hp-discontinuous Galerkin time-stepping for Volterra integrodifferential equations, SIAM J. Numer. Anal. 44 (2006), no. 1, 224-245 (electronic), DOI 10.1137/040619314. MR 2217380
[5] H. Brunner and P. J. van der Houwen, The Numerical Solution of Volterra Equations, CWI Monographs, vol. 3, North-Holland Publishing Co., Amsterdam, 1986. MR871871
[6] Y. Chen and T. Tang, Convergence analysis of the Jacobi spectral-collocation methods for Volterra integral equations with a weakly singular kernel, Math. Comp. 79 (2010), no. 269, 147-167, DOI 10.1090/S0025-5718-09-02269-8. MR 2552221
[7] W. Gui and I. Babuška, The h, $p$ and $h-p$ versions of the finite element method in 1 dimension. I. The error analysis of the p-version, Numer. Math. 49 (1986), no. 6, 577-612, DOI 10.1007/BF01389733. MR861522
[8] C. Huang, T. Tang, and Z. Zhang, Supergeometric convergence of spectral collocation methods for weakly singular Volterra and Fredholm integral equations with smooth solutions, J. Comput. Math. 29 (2011), no. 6, 698-719, DOI 10.4208/jcm.1110-m11si06. MR 2869428
[9] Y. Jiang and J. Ma, Spectral collocation methods for Volterra-integro differential equations with noncompact kernels, J. Comput. Appl. Math. 244 (2013), 115-124, DOI 10.1016/j.cam.2012.10.033. MR3005763
[10] X. Li, T. Tang, and C. Xu, Parallel in time algorithm with spectral-subdomain enhancement for Volterra integral equations, SIAM J. Numer. Anal. 51 (2013), no. 3, 1735-1756, DOI 10.1137/120876241. MR3066805
[11] T. Lin, Y. Lin, M. Rao, and S. Zhang, Petrov-Galerkin methods for linear Volterra integro-differential equations, SIAM J. Numer. Anal. 38 (2000), no. 3, 937-963, DOI 10.1137/S0036142999336145. MR 1781210
[12] K. Mustapha, A superconvergent discontinuous Galerkin method for Volterra integrodifferential equations, smooth and non-smooth kernels, Math. Comp. 82 (2013), no. 284, 1987-2005, DOI 10.1090/S0025-5718-2013-02689-0. MR3073189
[13] K. Mustapha, H. Brunner, H. Mustapha, and D. Schötzau, An hp-version discontinuous Galerkin method for integro-differential equations of parabolic type, SIAM J. Numer. Anal. 49 (2011), no. 4, 1369-1396, DOI 10.1137/100797114. MR2817543
[14] D. Schötzau and C. Schwab, An hp a priori error analysis of the DG time-stepping method for initial value problems, Calcolo 37 (2000), no. 4, 207-232, DOI 10.1007/s100920070002. MR1812787
[15] D. Schötzau and C. Schwab, Time discretization of parabolic problems by the hp-version of the discontinuous Galerkin finite element method, SIAM J. Numer. Anal. 38 (2000), no. 3, 837-875, DOI 10.1137/S0036142999352394. MR1781206
[16] Ch. Schwab, p-and hp-Finite Element Methods: Theory and Applications in Solid and Fluid Mechanics, Numerical Mathematics and Scientific Computation, The Clarendon Press, Oxford University Press, New York, 1998. MR 1695813
[17] J. Shen, T. Tang, and L.-L. Wang, Spectral Methods: Algorithms, Analysis and Applications, Springer Series in Computational Mathematics, vol. 41, Springer, Heidelberg, 2011. MR2867779
[18] C.-T. Sheng, Z.-Q. Wang, and B.-Y. Guo, A multistep Legendre-Gauss spectral collocation method for nonlinear Volterra integral equations, SIAM J. Numer. Anal. 52 (2014), no. 4, 1953-1980, DOI 10.1137/130915200. MR3246901
[19] T. Tang, A note on collocation methods for Volterra integro-differential equations with weakly singular kernels, IMA J. Numer. Anal. 13 (1993), no. 1, 93-99, DOI 10.1093/imanum/13.1.93. MR1199031
[20] T. Tang, X. Xu, and J. Cheng, On spectral methods for Volterra integral equations and the convergence analysis, J. Comput. Math. 26 (2008), no. 6, 825-837. MR2464738
[21] X. Tao, Z. Xie, and X. Zhou, Spectral Petrov-Galerkin methods for the second kind Volterra type integro-differential equations, Numer. Math. Theory Methods Appl. 4 (2011), no. 2, 216-236. MR2862668
[22] Z.-Q. Wang and C.-T. Sheng, An hp-spectral collocation method for nonlinear Volterra integral equations with vanishing variable delays, Math. Comp. 85 (2016), no. 298, 635-666, DOI $10.1090 / \mathrm{mcom} / 3023$. MR3434874
[23] L. Yi, An h-p version of the continuous Petrov-Galerkin finite element method for nonlinear Volterra integro-differential equations, J. Sci. Comput. 65 (2015), no. 2, 715-734, DOI 10.1007/s10915-015-9983-9. MR3411285
[24] L. Yi and B. Guo, An h-p Petrov-Galerkin finite element method for linear Volterra integrodifferential equations, Sci. China Math. 57 (2014), no. 11, 2285-2300, DOI 10.1007/s11425-014-4805-3. MR 3266490
[25] W. Yuan and T. Tang, The numerical analysis of implicit Runge-Kutta methods for a certain nonlinear integro-differential equation, Math. Comp. 54 (1990), no. 189, 155-168, DOI 10.2307/2008687. MR979942

College of Science, University of Shanghai for Science and Technology, Shanghai 200093, People's Republic of China

E-mail address: zqwang@usst.edu.cn
Department of Mathematics, Shanghai Jiao Tong University, Shanghai 200240, People's Republic of China

Department of Mathematics, Shanghai Normal University, Shanghai 200234, People's Republic of China


[^0]:    Received by the editor March 16, 2015 and, in revised form, March 28, 2016.
    2010 Mathematics Subject Classification. Primary 65L60, 45D05, 41A10, 65L70.
    Key words and phrases. hp-version Legendre-Jacobi spectral collocation method, Volterra integro-differential equations, smooth and weakly singular kernels, convergence analysis.

    This work was supported in part by the National Natural Science Foundation of China (Nos. 11571238 and 11301343), the Research Fund for Doctoral Program of Higher Education of China (Nos. 20133127110006 and 20113127120002), and the Natural Science Foundation of Shanghai (No. 15ZR1430900).

