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COMPUTING HIGHLY OSCILLATORY INTEGRALS

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ABSTRACT. We develop two classes of composite moment-free numerical quadratures for computing highly oscillatory integrals having integrable singularities and stationary points. One class of the quadrature rules has a polynomial order of convergence and the other class has an exponential order of convergence. We first modify the moment-free Filon-type method for the oscillatory integrals without a singularity or a stationary point to accelerate its convergence. We then consider the oscillatory integrals without a singularity or a stationary point and then those with singularities and stationary points. The composite quadrature rules are developed based on partitioning the integration domain according to the wave number and the singularity of the integrand. The integral defined on the resulting subinterval has either a weak singularity without rapid oscillation or oscillation without a singularity. Classical quadrature rules for weakly singular integrals using graded points are employed for the singular integral without rapid oscillation and the modified moment-free Filon-type method is used for the oscillatory integrals without a singularity. Unlike the existing methods, the proposed methods do not have to compute the inverse of the oscillator which normally is a nontrivial task. Numerical experiments are presented to demonstrate the approximation accuracy and the computational efficiency of the proposed methods. Numerical results show that the proposed methods outperform methods published recently.

1. INTRODUCTION

We consider in this paper numerical computation of highly oscillatory integrals defined on a bounded interval whose integrands have the form $f e^{i\kappa g}$, where the wave number κ is large, the amplitude function f may have weak singularities, and the oscillator g has stationary points of a certain order. Computing highly oscillatory integrals is of importance in wide application areas ranging from quantum chemistry, computerized tomography, electrodynamics and fluid mechanics. For a large wave number κ, the integrands oscillate rapidly and cancel themselves over most of the range. The cancellation does not occur in neighborhoods of critical points of

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the integrand: the endpoints of the integration interval and the stationary points of the oscillator. Efficiency of a quadrature of highly oscillatory integrals depends on the behavior of functions f and g near the critical points. Traditional methods for evaluating oscillatory integrals become expensive when the wave number κ is large, since the number of the evaluations of the integrand used grows linearly with the wave number κ in order to obtain accuracy of a certain order. The calculation of integrals of this kind is widely perceived as a *challenge* issue. Calculating such oscillatory integrals requires special effort.

The interest in highly oscillatory integrals has led to much progress in developing numerical quadrature formulas for computing these integrals. In the literature, there are mainly four classes of methods for the computation: asymptotic methods, Filon-type methods, Levin-type methods and numerical steepest descent methods. The basis for convergence analysis of these quadrature rules is the asymptotic expansion of the oscillatory integral, an asymptotic expansion in negative powers of the wave number κ. The leading terms in the asymptotic expansion may be derived from integration by parts [16] for the case when the oscillator has no stationary point. For the case when the oscillator has stationary points, the main tool is the method of the stationary phase [24, 34]. For a fixed wave number, the convergence order of the asymptotic method is rather low. To overcome this weakness, the Filon-type methods [9, 10, 20] were proposed, which replace the amplitude function f by a suitable interpolating function. In many situations the convergence order of the Filon-type methods is significantly higher than that of the asymptotic methods. A thorough qualitative understanding of the Filon-type methods and the analysis of their convergence may be found in [13, 14, 16] for the univariate case and in [17] for the multivariate case. In these methods, interpolation at the Chebyshev points ensures convergence [23]. A drawback of the Filon-type methods is that they require to compute the moments, which themselves are oscillatory integrals. For the cases having nonlinear oscillators, it is not always possible to compute the moments exactly. In [6, 7, 27, 37], the moment-free Filon-type methods were developed. An entirely different approach without computing the moments is the Levin collocation method [21]. The Levin-type methods [22, 26, 28, 29] reduce computation of the oscillatory integral to a simple problem of finding the antiderivative F of the integrand, where F satisfies the differential equation $F' + \mathrm{i}\kappa g' F = f$. The Filon-type methods and the Levin-type methods with polynomial bases are identical for the cases having the linear oscillator but not for the cases having the nonlinear oscillator [21, 25, 38]. Numerical steepest descent methods [1, 8, 12] for removing the oscillation convert a real integration interval into a path in the complex plane, with a standard quadrature method used to calculate the resulting complex integral.

Although many methods were proposed in the literature for computing oscillatory integrals, there is still big room for improving their approximation accuracy and computational efficiency. The Filon-type and Levin-type methods require interpolating the derivatives of the amplitude f at critical points in order to achieve a higher convergence order. Even though computing derivatives can be avoided by allowing the interpolation points to approach the critical points as the wave number increases for the formula proposed in [15], the moments cannot always be explicitly computed. In particular, certain special functions were used for calculating the oscillatory integrals in the case when f has singularities and g has stationary points. The formulas proposed in [6, 7] do not require computing the special functions and the moments of these formulas can be computed exactly, at the expense of computing the inverse of the oscillator, which takes up much computing time. Numerical steepest descent methods also require computing the inverse of the oscillator or higher order derivatives of the integrand [1].

The purpose of this paper is to develop efficient composite quadrature rules for computing highly oscillatory integrals with singularities and stationary points. The methods to be described in this paper require neither inverting the oscillator nor calculating the derivatives of f or those of g'. The main idea used here is to divide the integration interval into subintervals according to the wave number κ and the singularity of the integrand. To avoid using the special functions, we first split the integration interval into the subintervals according to the singularity of f and the stationary points of g such that the integrand on the subintervals either has a weak singularity but no oscillation, or has oscillation but no singularity or stationary point. The weakly singular integrals are calculated by the classical quadratures using graded points [4, 19, 35]. To avoid using the derivatives of f and those of g' and avoid computing the inverse of the oscillator, we design composite quadrature formulas using a partition of the subinterval formed according to the wave number κ and the property of the oscillator g for the oscillatory integrals, where the modified moment-free Filon-type method is used to calculate the oscillatory integrals on the subintervals of the partition. These formulas can improve the approximation accuracy effectively, since the convergence order of the formulas computing the oscillatory integrals with smooth integrand and without stationary point may be increased by adding more internal interpolation nodes. Specifically, we develop two classes of composite moment-free quadrature formulas for highly oscillatory integrals. Class one uses a fixed number of quadrature nodes in each subinterval and has a polynomial order of convergence. This class of formulas is stable and easy to implement. Class two uses variate numbers of quadrature nodes in the subintervals and achieves an exponential order of convergence.

The quadrature formulas proposed in this paper have the following advantages. Compared with the existing formulas, the proposed formulas need not compute the inverse of the nonlinear oscillator, or utilize the incomplete Gamma function [2] for the oscillator with stationary points. These formulas not only reduce the computational complexity, but also enhance the approximation accuracy. The approximation accuracy of these formulas is higher than that of the existing formulas for the case when the oscillator has stationary points and when the oscillator is not easy to invert.

We organize this paper in seven sections. In Section 2, we present an improvement of the moment-free Filon-type method developed in [37]. In Section 3, we design a partition of the integration interval and propose composite moment-free Filon-type methods for the oscillatory integrals with smooth integrand and without a stationary point. In Sections 4 and 5, we develop the composite moment-free quadratures defined on a mesh according to the wave number κ and the properties of the integrand for the oscillatory integrals with both singularities and stationary points. The formulas proposed in Section 4 have a polynomial order of convergence, and those in Section 5 have an exponential order of convergence. Numerical results are presented in Section 6 to confirm the theoretical estimates on the accuracy of the proposed formulas. Moreover, we compare the numerical performance of the proposed formulas with that of those recently developed in [6, 7]. We summarize our conclusions in Section 7.

2. The Filon-type quadrature method

The goal of this paper is to develop quadrature methods for evaluating oscillatory integrals in the form

$$\mathcal{I} := \int_I f(x)\mathrm{e}^{\mathrm{i}\kappa g(x)}\mathrm{d}x, \tag{2.1}$$

where $I := [0,1]$, $\kappa \gg 1$ is the wave number, $f \in L^1(I)$ has weak singularities and the oscillator $g \in C^\infty(I)$ has stationary points. Our main idea to fulfill this may be described as follows. We first develop a basic quadrature formula for computing an oscillatory integral defined on a subinterval $[a,b]$ of I where the integrand has no singularity or stationary point. We then design an appropriate partition $0 = x_0 < x_1 < \ldots < x_{n-1} < x_n = 1$ of I according to the wave number κ, the singularities of f and the stationary points of g and employ the basic quadrature formula for each of the integrals defined on the subintervals $[x_j, x_{j+1}]$, for $j \in \mathbb{Z}_{n-1} := \{0, 1, \ldots, n-1\}$.

We recall the Filon-type quadrature method proposed in [37] for computing the integral

$$\mathcal{I}^{[a,b]} := \int_a^b f(x)\mathrm{e}^{\mathrm{i}\kappa g(x)}\mathrm{d}x, \tag{2.2}$$

where $[a,b] \subset I$, f is continuous on $[a,b]$ and the oscillator g is continuously differentiable on $[a,b]$ and has no stationary point in $[a,b]$. By a change of variables $g(x) \mapsto x$, the integral in (2.2) may be written as

$$\mathcal{I}^{[a,b]} = \int_\alpha^\beta \Psi(x)\mathrm{e}^{\mathrm{i}\kappa x}\mathrm{d}x, \tag{2.3}$$

where

$$\Psi(x) := \left((f/g') \circ g^{-1}\right)(x), \text{ for } x \in [\alpha, \beta], \quad \alpha := g(a), \ \beta := g(b). \tag{2.4}$$

For a fixed $m \in \mathbb{N} := \{1, 2, \ldots\}$, we approximate Ψ by its Lagrange interpolation polynomial of degree m. Since g is differentiable on $[a,b]$ and has no stationary point in the interval, g must be strictly monotone on the interval. Without loss of generality, we assume that g is strictly increasing since the other case can be similarly handled. Choosing $m+1$ points $a = t_0 < t_1 < \ldots < t_{m-1} < t_m = b$, we construct the Lagrange polynomial p_m which interpolates Ψ at the points $g(t_j)$, $j \in \mathbb{Z}_m$. Hence, its Newton form is given by $p_m = \sum_{j\in\mathbb{Z}_m} a_j w_j$ with $w_j(x) = \prod_{l\in\mathbb{Z}_{j-1}}(x - g(t_l))$ for $x \in [\alpha, \beta]$ and $j \in \mathbb{Z}_m$, where the coefficients a_j are the divided differences of Ψ, that is, $a_j := \Psi[g(t_0), g(t_1), \ldots, g(t_j)]$. When computing a_j, we are required to compute the values of Ψ at $g(t_j)$. Noting that $\left(g^{-1} \circ g\right)(t_j) = t_j$ for $j \in \mathbb{Z}_m$, we need only to evaluate the functional values of f and g' at the points t_j. Hence, there is no need to calculate the inverse of g. A Filon-type quadrature rule is then obtained by replacing Ψ in (2.3) with p_m. That is, we use

$$\mathcal{Q}_m^{[a,b]} := \sum_{j\in\mathbb{Z}_m} a_j W_j, \quad \text{where} \quad W_j := \int_\alpha^\beta w_j(x)\mathrm{e}^{\mathrm{i}\kappa x}\mathrm{d}x, \quad j \in \mathbb{Z}_m, \tag{2.5}$$

to approximate the integral (2.3). The integrals W_j appearing in formula (2.5) can be computed exactly. Specifically, to compute these integrals, we first represent w_j inductively on j. By the Lagrange interpolation, we see that $w_0(x) = b_{0,0} := 1$ for $x \in [\alpha, \beta]$. For $j \in \mathbb{Z}_{m-1}$, $w_{j+1}(x) = \sum_{l\in\mathbb{Z}_{j+1}} b_{j+1,l}x^l$ for $x \in [\alpha, \beta]$, where

$b_{j+1,0} := -g(t_j)b_{j,0}$, $b_{j+1,j+1} := 1$ and $b_{j+1,l} := b_{j,l-1} - g(t_j)b_{j,l}$ for $l \in \mathbb{Z}_j^+$. Define $X_j := \int_\alpha^\beta x^j e^{i\kappa x} dx$. The value of X_j can be computed recursively by

$$X_0 = (e^{i\kappa\beta} - e^{i\kappa\alpha})/(i\kappa), \quad X_j = (e^{i\kappa\beta}\beta^j - e^{i\kappa\alpha}\alpha^j - jX_{j-1})/(i\kappa), \quad j \in \mathbb{Z}_m^+.$$

We then obtain that $W_j = \sum_{l\in\mathbb{Z}_j} b_{j,l}X_l$ for $j \in \mathbb{Z}_m$.

We now turn our attention to error analysis of formula (2.5). For a function $\phi \in C(\Omega)$, let $\|\phi\|_\infty := \max_{x\in\Omega}\{|\phi(x)|\}$. We denote the error of (2.5) by $\mathcal{E}_m^{[a,b]} := \left|\mathcal{I}^{[a,b]} - \mathcal{Q}_m^{[a,b]}\right|$. For a quadrature formula Q that approximates integral (2.3), we use $\mathcal{N}(Q)$ to denote the number of evaluations of the integrand Ψ used in the formula. According to [37], we have the error estimate

$$\mathcal{E}_m^{[a,b]} \le \frac{3(m+1)}{m!\kappa^2}\left\|\Psi^{(m+1)}\right\|_\infty (\beta-\alpha)^m. \tag{2.6}$$

Moreover, according to (2.5), we have that $\mathcal{N}\big(\mathcal{Q}_m^{[a,b]}\big) \le m+1$.

Estimate (2.6) demonstrates that the decay of the error of the Filon-type method (2.5) is of order $\mathcal{O}(\kappa^{-2})$ when $\left\|\Psi^{(m+1)}\right\|_\infty$ is bounded. Note that the decay of the error of (2.5) for the integral having a linear oscillator is also $\mathcal{O}(\kappa^{-2})$ (see, [13, 17, 37]). Estimate (2.6) is consistent with the known estimate for the case having a linear oscillator.

From (2.6), we see that convergence of the quadrature rule $\mathcal{Q}_m^{[a,b]}$ is affected by $\left\|\Psi^{(m+1)}\right\|_\infty$ and $(\beta-\alpha)^m$, in addition to κ^{-2}. We first analyze the influence of $\left\|\Psi^{(m+1)}\right\|_\infty$. To this end, we compute $\Psi^{(m+1)}$ by using the chain rule of differentiation for a composite function. Let $\mathbb{Z}_{m+1}^+ := \{1, 2, \ldots, m+1\}$ and set

$$a_{0,0} := 1,\ a_{n,0} := g'a'_{n-1,0} - (2n-1)a_{n-1,0}g'',$$
$$a_{n,j} := g'(a_{n-1,j-1} + a'_{n-1,j}) - (2n-1)a_{n-1,j}g'', \quad j \in \mathbb{Z}_{n-1}^+,$$
$$a_{n,n} := g'a_{n-1,n-1} = (g')^n, \quad n \in \mathbb{Z}_{m+1}^+.$$

For $n \in \mathbb{Z}_{m+1}$, we define

$$G_n := \frac{1}{(g')^{2n+1}} \sum_{j\in\mathbb{Z}_n} a_{n,j} f^{(j)}.$$

With this notation, by the chain rule, we have that

$$\Psi^{(n)} = G_n \circ g^{-1}, \quad \text{for} \quad n \in \mathbb{Z}_{m+1}.$$

From the definition of $a_{n,j}$, it is clear that $a_{n,j}$ is a product of g' and its derivatives of orders up to n. If $f^{(j)}$, $j \in \mathbb{Z}_n$, g' and its derivatives of orders up to n are all bounded above and g' is bounded below from zero, then $\|\Psi^{(n)}\|_\infty$ is bounded. To avoid a complicated description of the condition, we shall assume in the rest of the paper that $\|\Psi^{(n)}\|_\infty$ is bounded.

We next discuss the influence of $(\beta-\alpha)^m$. Note that $(\beta-\alpha)^m \le \sigma^m(b-a)^m$ with $\sigma := \|g'\|_\infty$. In this paper $[a,b]$ will be a subinterval of $[0,1]$ and thus, $b-a < 1$. However, σ may be greater than 1, in which case, σ^m grows as m increases. To weaken the influence of σ, we shall subdivide $[a,b]$ into $N \in \mathbb{N}$ equal subintervals and approximate Ψ by its Lagrange interpolation polynomial p_m of degree $m \in \mathbb{N}$ on each of the subintervals. Specifically, we use $y_j := a + (b-a)j/N$ for $j \in \mathbb{Z}_N$ to denote the partition of $[a,b]$ and $\mathcal{Q}_m^{[y_{j-1},y_j]}$ defined in (2.5) to approximate $\mathcal{I}^{[y_{j-1},y_j]}$

for $j \in \mathbb{Z}_N^+$. This leads to the quadrature formula $\mathcal{Q}_{N,m}^{[a,b]}$ for computing (2.3), defined by

$$\mathcal{Q}_{N,m}^{[a,b]} := \sum_{j \in \mathbb{Z}_N^+} \mathcal{Q}_m^{[y_{j-1},y_j]}. \tag{2.7}$$

This quadrature formula will be used in the following sections for computing the oscillatory integral with the integrand without a singularity or a stationary point. The number N will be designed according to the property of g.

In the next theorem, we analyze the error $\mathcal{E}_{N,m}^{[a,b]} := \left| \mathcal{I}^{[a,b]} - \mathcal{Q}_{N,m}^{[a,b]} \right|$ of the quadrature formula.

Theorem 2.1. *If $\Psi \in C^{m+1}[\alpha, \beta]$, then for $N \in \mathbb{N}$*

$$\mathcal{E}_{N,m}^{[a,b]} \leq \frac{3(m+1)}{m!\kappa^2 N^{m-1}} \left\| \Psi^{(m+1)} \right\|_\infty \sigma^m (b-a)^m, \tag{2.8}$$

and $\mathcal{N}\left(\mathcal{Q}_{N,m}^{[a,b]}\right) \leq Nm + 1$. If $\sigma > 1$ and N is chosen as $\lceil \sigma \rceil$, the smallest integer not less than σ, then

$$\mathcal{E}_{N,m}^{[a,b]} \leq \frac{3(m+1)}{m!\kappa^2} \left\| \Psi^{(m+1)} \right\|_\infty \sigma (b-a)^m. \tag{2.9}$$

Proof. The bound (2.8) is proved by estimating the error $\left| \mathcal{I}^{[y_{j-1},y_j]} - \mathcal{Q}_m^{[y_{j-1},y_j]} \right|$ by first employing estimate (2.6) with $\beta - \alpha$ replaced by $\sigma(y_j - y_{j-1})$ and then summing up both sides of the resulting inequality over $j \in \mathbb{Z}_N^+$ and using $y_j - y_{j-1} = (b-a)/N$.

According to formula (2.7), the nodes y_j for $j \in \mathbb{Z}_{N-1}^+$ are used twice in it. Thus, we conclude that

$$\mathcal{N}\left(\mathcal{Q}_{N,m}^{[a,b]}\right) \leq N\mathcal{N}\left(\mathcal{Q}_{1,m}^{[a,b]}\right) - (N-1) \leq Nm + 1.$$

When $\sigma > 1$, we substitute $N = \lceil \sigma \rceil$ into estimate (2.8) to yield estimate (2.9). $\square$

Note that $\mathcal{Q}_{1,m}^{[a,b]} = \mathcal{Q}_m^{[a,b]}$. Comparing estimate (2.8) in Theorem 2.1 for $N > 1$ and $N = 1$, we see that formula (2.7) uses $Nm + 1$ number of the functional evaluations of Ψ, which is about N times of that used in the traditional Filon method to reach the order of error estimate $N^{-m+1}\mathcal{E}_m^{[a,b]}$. Formula (2.7) will serve as a basic quadrature formula in this paper for developing sophisticated formulas for computing singular oscillatory integrals.

In the remaining sections of this paper, we shall consider the following three cases:

(i) When f and g are smooth and g has no stationary point or inflection point in I, according to the wave number κ we design a partition $0 = x_0 < x_1 < \ldots < x_n = 1$, and write $\mathcal{I} = \sum_{j \in \mathbb{Z}_n^+} \mathcal{I}^{[x_{j-1},x_j]}$. Formula (2.7) is then used to compute integrals $\mathcal{I}^{[x_{j-1},x_j]}$ for $j \in \mathbb{Z}_n^+$.

(ii) When f has a weak singularity only at the origin and g is smooth without a stationary point or an inflection point in I, we first divide I into two subintervals $[0, b]$ and $\Lambda := [b, 1]$ such that the integrand of $\mathcal{I}^{[0,b]}$ does not rapidly oscillate and that of $\mathcal{I}^{\Lambda}$ does not have singularity. The integral $\mathcal{I}^{[0,b]}$ is calculated by a quadrature rule for weakly singular integrals. The integral $\mathcal{I}^{\Lambda}$ is computed by the method described in item (i).

(iii) When f has a weak singularity only at the origin and g is smooth with one stationary point at the origin and has no inflection point in the interior of I, we first divide I into two subintervals $[0,b]$ and Λ such that the integrand of $\mathcal{I}^{[0,b]}$ does not rapidly oscillate and that of $\mathcal{I}^{\Lambda}$ does not have singularity or g has no stationary point in Λ. The integrals $\mathcal{I}^{[0,b]}$ and $\mathcal{I}^{\Lambda}$ are handled in the way described in (ii).

The case of f having a finite number of singularities in I, g having a finite number of stationary points or inflection points in I can be treated by splitting I into subintervals, on each of which f has only one singular point or g has only one stationary point at an endpoint and without an inflection point in the interior of the subinterval. For simplicity of representation, we shall not provide details for this general case. Extension of the proposed quadrature method to this general case is straightforward.

3. A composite Filon-type quadrature method

In this section, we develop a composite Filon-type quadrature method for computing the oscillatory integrals (2.1).

For reasons explained at the end of the last section, without loss of generality we consider only the special case when $f,\, g \in C^2(I)$, g is increasing monotonically, $g'(x) \neq 0$ for $x \in I$ and $g''(x) \neq 0$ for $x \in (0,1)$. The last condition is imposed to ensure that g has no inflection points in $(0,1)$. We shall partition the interval I into n subintervals according to the wave number κ, and propose a composite quadrature rule, where we approximate Ψ by its Lagrange interpolation polynomial of variable degrees in the subintervals, aiming at the asymptotic error order $\mathcal{O}(\kappa^{-n-1})$.

We first motivate the construction of a κ-dependent partition of I. By a change of variables of $\kappa x \mapsto x$, the integral (2.1) becomes

$$\mathcal{I} = 1/\kappa \int_0^1 f(x/\kappa)\mathrm{e}^{\mathrm{i}\kappa g(x/\kappa)}\mathrm{d}x + 1/\kappa \int_1^\kappa f(x/\kappa)\mathrm{e}^{\mathrm{i}\kappa g(x/\kappa)}\mathrm{d}x. \tag{3.1}$$

The integrals on the right hand side do not oscillate rapidly since $\left|(\kappa g(x/\kappa))'\right| \leq \sigma$ for $x \in [0,\kappa]$. However, for a large κ, standard quadratures for computing the second integral on the right hand side of (3.1) lead to prohibitive computational costs. Inspired by the quadratures for singular integrals using graded points proposed in [19], for $n \in \mathbb{N}$ with $n > 1$ we suggest the partition of $[1,\kappa]$ with graded points $\kappa^{(j-1)/(n-1)}$, for $j \in \mathbb{Z}_n^+$. The graded points are chosen so that the resulting quadrature formula has equal-errors on all the resulting subintervals. Using the change of variables $x/\kappa \mapsto x$, we obtain the desired partition of I.

We now describe the κ-dependent partition of I. Throughout this section, we assume that $n \in \mathbb{N}$ with $n > 1$. For fixed κ, let Π_κ denote the partition of I with nodes

$$x_0 := 0,\ x_j := \kappa^{(j-1)/(n-1)-1},\ \text{for } j \in \mathbb{Z}_n^+. \tag{3.2}$$

According to the partition Π_κ, the integral (2.1) is written as $\mathcal{I} = \sum_{j\in\mathbb{Z}_n^+} \mathcal{I}^{[x_{j-1},x_j]}$. We then use formula (2.7) with $[a,b] := [x_{j-1},x_j]$ to calculate the integrals $\mathcal{I}^{[x_{j-1},x_j]}$ for $j \in \mathbb{Z}_n^+$. We shall develop two quadrature methods. Method one uses a fixed number of quadrature nodes in each of the subintervals and has a polynomial order (in terms of the wave number) of convergence. Method two uses variable numbers of quadrature points in the subintervals and achieves an exponential order (in terms of the wave number) of convergence. The number of quadrature nodes used depends on the behavior of g'. Specifically, we define the quantities

$$M_j := \max\left\{|g'(x_{j-1})|, |g'(x_j)|\right\} \text{ and } N_j := \lceil M_j \rceil, \text{ for } j \in \mathbb{Z}_n^+. \tag{3.3}$$

Then, $M_j \le \sigma$. Since $g'(x) \ne 0$ for $x \in I$, $N_j > 0$ for $j \in \mathbb{Z}_n^+$. By the hypothesis $g''(x) \ne 0$ for $x \in (0,1)$, we have for $j \in \mathbb{Z}_n^+$ that

$$M_j = \max\left\{|g'(x)| : x \in [x_{j-1}, x_j]\right\}. \tag{3.4}$$

We first describe the method having a polynomial order of convergence. We choose a fixed positive integer m. For each $j \in \mathbb{Z}_n^+$, we use $\mathcal{Q}_{N_j,m}^{[x_{j-1},x_j]}$ defined as in (2.7) to approximate $\mathcal{I}^{[x_{j-1},x_j]}$. Integral $\mathcal{I}$ defined by (2.1) is then approximated by the quadrature formula

$$\mathcal{Q}_{n,m} := \sum_{j\in\mathbb{Z}_n^+} \mathcal{Q}_{N_j,m}^{[x_{j-1},x_j]}. \tag{3.5}$$

Below, we estimate the error $\mathcal{E}_{n,m} := |\mathcal{I} - \mathcal{Q}_{n,m}|$ and $\mathcal{N}(\mathcal{Q}_{n,m})$. We let $\eta := \max\left\{1/\kappa, 1-\kappa^{-1/(n-1)}\right\}$.

Proposition 3.1. *If $\Psi \in C^{m+1}[g(0), g(1)]$ for some $m \in \mathbb{N}$, then*

$$\mathcal{E}_{n,m} \le \frac{3(m+1)}{m!\kappa^2} \left\|\Psi^{(m+1)}\right\|_\infty \sigma\eta^{m-1},$$

and $\mathcal{N}(\mathcal{Q}_{n,m}) \le \lceil\sigma\rceil nm + 1$.

Proof. The proof is done by applying Theorem 2.1 on each of the subintervals $[x_{j-1}, x_j]$. Specifically, for $j \in \mathbb{Z}_n^+$, we use Theorem 2.1 to estimate

$$e_j := \left|\mathcal{I}^{[x_{j-1},x_j]} - \mathcal{Q}_{N_j,m}^{[x_{j-1},x_j]}\right|.$$

For $j \in \mathbb{Z}_n^+$, we apply (2.9) with σ being replaced by M_j and $b-a$ by $h_j := x_j - x_{j-1}$ to conclude that

$$e_j \le \frac{3(m+1)}{m!\kappa^2} \left\|\Psi^{(m+1)}\right\|_\infty M_j h_j^m.$$

Note that $h_1 = 1/\kappa \le \eta$, and for $j \in \mathbb{Z}_n^+$ with $j > 1$, $h_j = x_j(1-\kappa^{-1/(n-1)}) \le x_j\eta \le \eta$, and $M_j \le \sigma$. Substituting these bounds into the inequalities above and summing the resulting inequalities over $j \in \mathbb{Z}_n^+$, we obtain the desired estimate for $\mathcal{E}_{n,m}$. Using Theorem 2.1 again yields that

$$\mathcal{N}(\mathcal{Q}_{n,m}) \le \sum_{j\in\mathbb{Z}_n^+} (N_j m + 1) - (n-1) \le \lceil\sigma\rceil nm + 1.$$

$\square$

As a direct consequence of Proposition 3.1, we have the next estimate for the case having the linear oscillator $g(x) = x$, $x \in I$, where $\sigma = 1$.

Corollary 3.2. *If $f \in C^{m+1}(I)$ for some $m \in \mathbb{N}$ and $g(x) = x$ for $x \in I$, then*

$$\mathcal{E}_{n,m} \le \frac{3(m+1)}{m!\kappa^2} \left\|f^{(m+1)}\right\|_\infty \eta^{m-1}$$

and $\mathcal{N}(\mathcal{Q}_{n,m}) \le nm + 1$.

We now turn to developing the quadrature formula having an exponential order of convergence. This is done by choosing variable numbers of quadrature nodes in the subintervals of I. Specifically, for the partition Π_κ of I chosen as (3.2), we let

$$m_j := \lceil n(n-1)/(n+1-j) \rceil\,, \text{ for } j \in \mathbb{Z}_n^+. \tag{3.6}$$

For each $j \in \mathbb{Z}_n^+$, we use $\mathcal{Q}_{N_j,m_j}^{[x_{j-1},x_j]}$ to approximate $\mathcal{I}^{[x_{j-1},x_j]}$, where N_j is defined by (3.3). Integral $\mathcal{I}$ defined by (2.1) is then approximated by the quadrature formula

$$\mathcal{Q}_n := \sum_{j\in\mathbb{Z}_n^+} \mathcal{Q}_{N_j,m_j}^{[x_{j-1},x_j]}. \tag{3.7}$$

We next study the error $\mathcal{E}_n := |\mathcal{I} - \mathcal{Q}_n|$. To this end, we first establish two technical lemmas.

Lemma 3.3. *There exists a positive constant c such that for all $n \in \mathbb{N}$ with $n > 2$, $\sum_{j\in\mathbb{Z}_n^+} 1/(m_j-1) \le c$.*

Proof. We prove this result by estimating the lower bound of the set $\{m_j - 1 : j \in \mathbb{Z}_n^+\}$. For $j \in \mathbb{Z}_n^+$, by the choice (3.6) of m_j, we have that $m_j - 1 \ge n-2$. Thus, we obtain that $\sum_{j\in\mathbb{Z}_n^+} 1/(m_j-1) \le n/(n-2)$, which is bounded by a constant c. ☐

Lemma 3.4. *There exists a positive constant c such that for all $\kappa > 1$, $n \in \mathbb{N}$ that satisfy*

$$(n-1)(\ln(n-2+\mathrm{e}) - 1) \ge \ln\kappa, \tag{3.8}$$

and $j \in \mathbb{Z}_n^+$ with $j > 1$,

$$(\kappa^{1/(n-1)} - 1)^{m_j-2}/(m_j-2)! \le c(n-2)^{-1/2}.$$

Proof. By the choice (3.6) of m_j, we see that $m_j - 2 \ge n-2$ for $j > 1$. Condition (3.8) implies that $n > 2$ since $\kappa > 1$. We observe from the Stirling formula [2] for $n \in \mathbb{N}$ that

$$n! \ge \sqrt{2\pi n}\,(n/\mathrm{e})^n\,. \tag{3.9}$$

Using inequality (3.9) with $n := m_j - 2$, we conclude that there exists a positive constant c such that for all $n \in \mathbb{N}$ with $n > 2$ and $j \in \mathbb{Z}_n^+$ with $j > 1$,

$$1/(m_j-2)! \le c(m_j-2)^{-1/2}\left(\mathrm{e}/(m_j-2)\right)^{m_j-2} \le c(n-2)^{-1/2}\left(\mathrm{e}/(n-2)\right)^{m_j-2}.$$

Moreover, (3.8) implies that $\mathrm{e}(\kappa^{1/(n-1)} - 1) \le n-2$. This together with the above inequality ensures that

$$\begin{aligned}(\kappa^{1/(n-1)} - 1)^{m_j-2}/(m_j-2)! &\le c(n-2)^{-1/2}\left(\mathrm{e}(\kappa^{1/(n-1)}-1)/(n-2)\right)^{m_j-2}\\ &\le c(n-2)^{-1/2},\end{aligned}$$

proving the desired result. ☐

For a function $\phi \in C^n(\Omega)$, we let $\|\phi\|_n := \max\left\{\left\|\phi^{(j)}\right\|_\infty : j \in \mathbb{Z}_n\right\}$. We next estimate $\mathcal{E}_n$.

Theorem 3.5. *If $\Psi \in C^\infty[g(0), g(1)]$, then there exists a positive constant c such that for all $\kappa > 1$ and $n \in \mathbb{N}$ satisfying (3.8),*

$$\mathcal{E}_n \le c\sigma(n-2)^{-1/2}\kappa^{-n-1}\,\|\Psi\|_{(n-1)n+1}\,.$$

For $n \in \mathbb{N}$ with $n > 2$, there holds the estimate $\mathcal{N}(\mathcal{Q}_n) \le \lceil\sigma\rceil\left(n(n-1)\ln n + n^2\right) + 1$.

Proof. We establish the error bound by estimating errors

$$e_j := \left|\mathcal{I}^{[x_{j-1},x_j]} - \mathcal{Q}^{[x_{j-1},x_j]}_{N_j,m_j}\right|$$

for $j \in \mathbb{Z}_n^+$, and then sum them over j. By using (2.9) with $b-a$ being replaced by h_j and m by m_j and (3.4), we obtain that

$$e_j \le \frac{3\sigma(m_j+1)}{m_j!\kappa^2} h_j^{m_j} \left\|\Psi^{(m_j+1)}\right\|_\infty, \quad j \in \mathbb{Z}_n^+. \tag{3.10}$$

For $j = 1$, we have that $e_1 \le \dfrac{6\sigma\kappa^{-n-1}}{(n-2)!} \left\|\Psi^{(n)}\right\|_\infty$. For $j > 1$, by substituting $h_j = \kappa^{\frac{j-n-1}{n-1}}(\kappa^{1/(n-1)} - 1)$ into (3.10) we obtain that

$$e_j \le \frac{6\sigma}{m_j-1} \frac{(\kappa^{1/(n-1)}-1)^{m_j-2}}{(m_j-2)!} \left(\kappa^{1/(n-1)}-1\right)^2 \kappa^{\frac{j-n-1}{n-1}m_j-2} \left\|\Psi^{(m_j+1)}\right\|_\infty. \tag{3.11}$$

In addition, (3.8) implies that $n > 2$. Thus, $\left(\kappa^{1/(n-1)}-1\right)^2 < \kappa$. Applying Lemma 3.4 to (3.11) yields a positive constant c such that for all $\kappa > 1$, $n \in \mathbb{N}$ satisfying (3.8), and $j \in \mathbb{Z}_n^+$ with $j > 1$,

$$e_j \le c\sigma \frac{(n-2)^{-1/2}}{m_j-1} \kappa^{\frac{j-n-1}{n-1}m_j-1} \left\|\Psi^{(m_j+1)}\right\|_\infty.$$

Using the choice (3.6) of m_j, we have that $\dfrac{j-n-1}{n-1}m_j - 1 \le -n-1$, for $j \in \mathbb{Z}_n^+$ with $j > 1$. Thus, the above inequality becomes

$$e_j \le c\sigma \frac{(n-2)^{-1/2}}{m_j-1} \kappa^{-n-1} \left\|\Psi^{(m_j+1)}\right\|_\infty.$$

Summing up the bound of errors e_j over $j \in \mathbb{Z}_n^+$, we observe that

$$\mathcal{E}_n \le \frac{6\sigma\kappa^{-n-1}}{(n-2)!} \left\|\Psi^{(n)}\right\|_\infty + c\sigma\kappa^{-n-1} \left\{\sum_{j=2}^{n} \frac{(n-2)^{-1/2}}{m_j-1}\right\} \|\Psi\|_{(n-1)n+1}.$$

Applying Lemma 3.3 to the second term leads to the desired estimate of $\mathcal{E}_n$.

It remains to estimate the number of functional evaluations used in the quadrature formula. To this end, we note that Theorem 2.1 yields

$$\mathcal{N}\left(\mathcal{Q}_n\right) \le \lceil\sigma\rceil \left\{n + n(n-1)\sum_{j\in\mathbb{Z}_n^+} 1/(n+1-j)\right\} + 1.$$

For $n \in \mathbb{N}$ by using $\sum_{j\in\mathbb{Z}_n^+} 1/j \le \ln n + 1$ we have that

$$\mathcal{N}(\mathcal{Q}_n) \le \lceil\sigma\rceil \left\{n + n(n-1)(\ln n + 1)\right\} + 1,$$

which leads to the desired estimate of $\mathcal{N}(\mathcal{Q}_n)$. □

As a direct consequence of Theorem 3.5, we have the next estimate for the case having the linear oscillator.

Corollary 3.6. *If $f \in C^\infty(I)$, then there exists a positive constant c such that for all $\kappa > 1$ and $n \in \mathbb{N}$ satisfying* (3.8),

$$\mathcal{E}_n \leq c(n-2)^{-1/2}\kappa^{-n-1}\|f\|_{(n-1)n+1}.$$

For $n \in \mathbb{N}$ with $n > 2$, there holds the estimate $\mathcal{N}(\mathcal{Q}_n) \leq n(n-1)\ln n + n^2 + 1$.

Quadrature methods developed in this section require neither computing derivatives of f or those of g', nor evaluating g^{-1}.

4. Quadratures with a polynomial order of convergence

In this section, we consider computing the oscillatory integral (2.1), where f is allowed to have weak singularities, g has stationary points and has no inflection point in I. We develop quadrature formulas having a polynomial order of convergence.

The key idea to be employed is to split the interval I into two subintervals such that on one subinterval the integrand has a weak singularity but no oscillation and on the other subinterval it has oscillation but no singularity or stationary point. Then, for the singular integral we employ quadratures using graded points and for the oscillatory integral we design a composite quadrature rule using a partition of the subinterval which is formed according to the wave number κ and the property of g.

We first describe the weak singularity of a function at the origin defined on I according to [19, 31]. For some $\mu \in (-1, 1)$ and a nonnegative integer m, a real-valued function $f \in C^m(0, 1]$ is said to be of $\mathcal{C}^m_\mu$ if there exists a positive constant c such that

$$\left|f^{(m)}(x)\right| \leq cx^{\mu-m}, \quad \text{for all} \quad x \in (0, 1]. \tag{4.1}$$

When we say that f is of $\mathcal{C}^\infty_\mu$, we mean that for all $m \in \mathbb{N}$, $f \in C^\infty(0, 1]$ satisfies (4.1). A function of $\mathcal{C}^\infty_\mu$ may have an integrable singularity at the origin. The parameter μ is called the index of singularity.

In this section, f is allowed to have a single weakly singular point at zero with index $\mu \in (-1, 1)$ and g satisfies the following assumption:

Assumption 4.1. *For a nonnegative integer r, the function $g \in C^{r+1}(I)$ has a single stationary point at zero satisfying $g^{(j)}(0) = 0$ for $j \in \mathbb{Z}_r$, $g^{(r+1)}(x) \neq 0$ for $x \in I$, and*

$$\sigma_r := \left\|g^{(r+1)}\right\|_\infty/(r+1)! \ll \kappa, \tag{4.2}$$

and g is increasing monotonically without an inflection point in $(0, 1)$.

The requirement that $g(0) = 0$ in Assumption 4.1 is without loss of generality, since if $g(0) \neq 0$, we compute instead the integral $\int_I f(x)\mathrm{e}^{\mathrm{i}\kappa(g(x)-g(0))}\mathrm{d}x + \exp\{\mathrm{i}\kappa g(0)\}\int_I f(x)\mathrm{d}x$. An oscillator g that satisfies Assumption 4.1 with $r = 0$ does not have a stationary point.

We now write the integral (2.1) as the sum of two integrals: a weakly singular integral without rapid oscillation and an oscillatory integral without a singularity or a stationary point. According to the assumption on g, by the Taylor theorem, for each $x \in I$ there exists a constant $\xi_x \in [0, (\kappa\sigma_r)^{-1/(r+1)}x]$ such that

$$\kappa\left|g\left((\kappa\sigma_r)^{-1/(r+1)}x\right)\right| = \left|g^{(r+1)}(\xi_x)x^{r+1}/\sigma_r\right|/(r+1)!.$$

Hence, we see that $\kappa|g(x)| \leq 1$ for $0 \leq x \leq (\kappa\sigma_r)^{-1/(r+1)}$. Thus, for such an x the function $\mathrm{e}^{\mathrm{i}\kappa g(x)}$ does not oscillate rapidly. Motivated from the above discussion, we introduce

$$\kappa_r := \begin{cases} \kappa, & \sigma_r \leq 1, \\ \kappa\sigma_r, & \sigma_r > 1, \end{cases}$$

and define

$$\lambda_r := \kappa_r^{-1/(r+1)}. \tag{4.3}$$

We split the interval I into two subintervals $[0, \lambda_r]$ and $\Lambda := [\lambda_r, 1]$. Correspondingly, integral (2.1) may be written as the sum of integrals on these two subintervals. For $\phi \in L^1(I)$ we set $\mathcal{I}[\phi] := \int_I \phi(x)\mathrm{d}x$. Using a change of variables: $y = \lambda_r^{-1}x$ for $x \in [0, \lambda_r]$, the integral (2.1) is rewritten as $\mathcal{I} = \mathcal{I}^0 + \mathcal{I}^\Lambda$, where

$$\mathcal{I}^0 := \lambda_r \mathcal{I}[\varphi_\kappa], \tag{4.4}$$

with

$$\varphi_\kappa(x) := f(\lambda_r x)\, \mathrm{e}^{\mathrm{i}\kappa g(\lambda_r x)}, \text{ for } x \in I, \tag{4.5}$$

and

$$\mathcal{I}^\Lambda := \int_\Lambda f(x)\mathrm{e}^{\mathrm{i}\kappa g(x)}\mathrm{d}x. \tag{4.6}$$

Note that the function φ_κ defined as in (4.5) has a singularity at the origin but has no oscillation and the integrand in (4.6) has no singularity or stationary point but has oscillation. We shall treat these two integrals separately.

4.1. **Quadrature formulas for integrals with a singularity.** We consider in this subsection the integral $\mathcal{I}[\varphi_\kappa]$ that appears in (4.4) and develop quadrature rules for computing the singular integral (4.4) having a polynomial order (in terms of the number of nodes used) of convergence. The integrand φ_κ defined by (4.5) does not oscillate rapidly. The classical quadrature rules for weakly singular integrals developed in [19] can then be used to treat the singularity. Below, we briefly review the quadrature rules.

We begin with describing the Gauss-Legendre quadrature rule for integral $\mathcal{I}^{[a,b]}[\psi] := \int_a^b \psi(x)\mathrm{d}x$, where ψ is a smooth function defined on $[a, b]$. Given $m \in \mathbb{N}$, we denote by $-1 < t_1 < t_2 < \ldots < t_m < 1$ the zeros of the Legendre polynomial P_m of degree m and by $\omega_j := 2(1 - t_j^2)[mP_{m-1}(t_j)]^{-2}$, $j \in \mathbb{Z}_m^+$, the weights of the Gauss-Legendre quadrature rule which has the form

$$\mathcal{Q}_m^{[a,b]}[\psi] := (b-a)/2 \sum_{j\in\mathbb{Z}_m^+} \omega_j \psi\left([(b-a)t_j + (b+a)]/2\right). \tag{4.7}$$

There is a constant $\xi \in [a, b]$ such that the error of the approximation $\mathcal{Q}_m^{[a,b]}[\psi]$ to $\mathcal{I}^{[a,b]}[\psi]$ is given by

$$\mathcal{R}_m^{[a,b]}[\psi] := (b-a)^{2m+1}\psi^{(2m)}(\xi)/(2^{2m}(2m+1)!). \tag{4.8}$$

We now recall the integration method proposed in [19]. Given $m \in \mathbb{N}$, let $p := (2m+1)/(1+\mu)$, where $\mu \in (-1, 1)$ is the index of singularity of f. For $s \in \mathbb{N}$ with $s > 1$, according to the parameter p we choose $s + 1$ points given by $x_j := s^{-p}j^p$ for $j \in \mathbb{Z}_s$. The quadrature rule for $\mathcal{I}[\varphi_\kappa]$ is obtained by replacing φ_κ on $[x_0, x_1]$ by zero and using $\mathcal{Q}_m^{[x_j,x_{j+1}]}[\varphi_\kappa]$ for computing the integrals $\mathcal{I}^{[x_j,x_{j+1}]}[\varphi_\kappa]$

for $j \in \mathbb{Z}_{s-1}^+$. Integral $\mathcal{I}^0$ defined by (4.4) is then approximated by the quadrature formula

$$\mathcal{Q}_{\mu,m}^s := \lambda_r \sum_{j \in \mathbb{Z}_{s-1}^+} \mathcal{Q}_m^{[x_j, x_{j+1}]}[\varphi_\kappa], \tag{4.9}$$

where λ_r is defined in (4.3).

We next estimate the error $\mathcal{E}_{\mu,m}^s := \left|\mathcal{Q}_{\mu,m}^s - \mathcal{I}^0\right|$. We need a lemma that estimates the norm of

$$w_\kappa(x) := \exp\{u_\kappa(x)\} \text{ with } u_\kappa(x) := \mathrm{i}\kappa g(\lambda_r x), \text{ for } x \in I. \tag{4.10}$$

This requires the use of the Faà di Bruno formula [18, 30, 32] for derivatives of the composition of two functions. For a fixed $n \in \mathbb{N}$, if the upto n-th order derivatives of two functions ϕ and ψ exist, then

$$(\phi \circ \psi)^{(n)} = \sum_{j \in \mathbb{Z}_n^+} \phi^{(j)}(\psi) B_{n,j}\left(\psi^{(1)}, \psi^{(2)}, \ldots, \psi^{(n-j+1)}\right), \tag{4.11}$$

where for $j \in \mathbb{Z}_n^+$,

$$B_{n,j}(x_1, x_2, \ldots, x_{n-j+1}) = \sum \frac{n!}{m_1! m_2! \cdots m_{n-j+1}!} \prod_{l \in \mathbb{Z}_{n-j+1}^+} \left(\frac{x_l}{l!}\right)^{m_l}, \tag{4.12}$$

with the sum being taken over all $(n-j+1)$-tuples $(m_1, \ldots, m_{n-j+1})$ satisfying the constraints

$$\sum_{l \in \mathbb{Z}_{n-j+1}^+} m_l = j \quad \text{and} \quad \sum_{l \in \mathbb{Z}_{n-j+1}^+} l m_l = n.$$

For $j \in \mathbb{Z}_n^+$, we let $B_{n,j} := B_{n,j}(1, 1, \ldots, 1)$ for brevity. Note that

$$B_{n,j} = 1/j! \sum_{l \in \mathbb{Z}_j} (-1)^{j-l} C_j^l l^n$$

are the Stirling numbers of the second kind [2, 33]. The Bell number has the bound [3] that

$$B_n := \sum_{j \in \mathbb{Z}_n} B_{n,j} \le \left(\frac{0.792n}{\ln(n+1)}\right)^n. \tag{4.13}$$

Lemma 4.2. *If $g \in C^{2m}(I)$ for some $m \in \mathbb{N}$ satisfying Assumption 4.1, then there exists a positive constant c such that for all $\kappa > 1$, $\|w_\kappa\|_{2m} \le c$.*

Proof. For $n \in \mathbb{Z}_{2m}^+$, by using (4.11), we obtain that

$$w_\kappa^{(n)} = \sum_{j \in \mathbb{Z}_n^+} w_\kappa B_{n,j}\left(u_\kappa^{(1)}, u_\kappa^{(2)}, \ldots, u_\kappa^{(n-j+1)}\right). \tag{4.14}$$

From the definition (4.10) of u_κ, for $j \in \mathbb{Z}_n^+$ with $j \ge r+1$, we have that

$$\left|u_\kappa^{(j)}(x)\right| \le \kappa \lambda_r^j \left|g^{(j)}(\lambda_r x)\right| \le \|g\|_n, \text{ for } x \in I. \tag{4.15}$$

For $j \in \mathbb{Z}_n^+$ with $j < r+1$, by Assumption 4.1 there exists a constant $\xi_x \in [0, \lambda_r x]$ such that

$$g^{(j)}(\lambda_r x) = g^{(r+1)}(\xi_x)/(r+1-j)!\,(\lambda_r x)^{r+1-j}, \text{ for } x \in I,$$

which implies that for all $j \in \mathbb{Z}_n^+$ with $j < r+1$ and $x \in I$,

$$\left|u_\kappa^{(j)}(x)\right| \le \kappa\lambda_r^j \left|g^{(j)}\left(\lambda_r x\right)\right| \le \|g\|_{r+1}. \tag{4.16}$$

Letting $M := \max\left\{1, \|g\|_{r+1}, \|g\|_{2m}\right\}$ and applying (4.12) with (4.15) and (4.16), we observe for all $\kappa > 1$ and $x \in I$ that

$$\left|B_{n,j}\left(u_\kappa^{(1)}, u_\kappa^{(2)}, \ldots, u_\kappa^{(n-j+1)}\right)\right| \le B_{n,j} M^j.$$

This together with (4.14) yields for all $\kappa > 1$ and $n \in \mathbb{Z}_{2m}^+$ that

$$\left\|w_\kappa^{(n)}\right\|_\infty \le \|w_\kappa\|_\infty \sum_{j \in \mathbb{Z}_n^+} B_{n,j} M^j \le \|w_\kappa\|_\infty B_n M^n.$$

According to $\|w_\kappa\|_\infty \le 1$, we obtain the desired estimate, since B_n and M^n are constants independent of κ from equation (4.13) and the assumption on g. ☐

We next estimate the derivatives of the function φ_κ defined by (4.5). We recall the Leibniz formula for the n-th derivative of the product of two functions for $n \in \mathbb{N}$,

$$(\phi\psi)^{(n)} = \sum_{m \in \mathbb{Z}_n} C_n^m \phi^{(m)} \psi^{(n-m)}, \tag{4.17}$$

where $C_n^m := \dfrac{n!}{m!(n-m)!}$ are the binomial coefficients that satisfy

$$\sum_{m \in \mathbb{Z}_n} C_n^m = 2^n. \tag{4.18}$$

Lemma 4.3. *Let $m \in \mathbb{N}$. If f is of $\mathcal{C}_\mu^{2m}$ for some $\mu \in (-1,1)$ and $g \in C^{2m}(I)$ satisfies Assumption 4.1, then there exists a positive constant c such that for all $\kappa > 1$ and $x \in (0,1]$, $\left|\varphi_\kappa^{(2m)}(x)\right| \le c\lambda_r^\mu x^{\mu-2m}$.*

Proof. Applying the Leibniz formula (4.17) to the function φ_κ yields

$$\varphi_\kappa^{(2m)}(x) = \sum_{j \in \mathbb{Z}_{2m}} C_{2m}^j \left(f\left(\lambda_r x\right)\right)^{(j)} w_\kappa^{(2m-j)}(x), \text{ for } x \in (0,1]. \tag{4.19}$$

From the assumption on f, there exists a positive constant c such that for all $\kappa > 1$, $j \in \mathbb{Z}_{2m}$ and $x \in (0,1]$,

$$\left|\left(f\left(\lambda_r x\right)\right)^{(j)}\right| \le \lambda_r^j \left|f^{(j)}\left(\lambda_r x\right)\right| \le c\lambda_r^\mu x^{\mu-j}. \tag{4.20}$$

This together with (4.19) yields a positive constant c such that for all $\kappa > 1$ and $x \in (0,1]$,

$$\left|\varphi_\kappa^{(2m)}(x)\right| \le c \sum_{j \in \mathbb{Z}_{2m}} C_{2m}^j \lambda_r^\mu x^{\mu-j} \left|w_\kappa^{(2m-j)}(x)\right| \le c\lambda_r^\mu x^{\mu-2m} \|w_\kappa\|_{2m} \sum_{j \in \mathbb{Z}_{2m}} C_{2m}^j.$$

Using Lemma 4.2 and formula (4.18) in the inequality above, we obtain the desired estimate. ☐

We need a technical result for the integral of a function of $\mathcal{C}_\mu^0$ for some $\mu \in (-1,1)$.

Lemma 4.4. *If ϕ is of $\mathcal{C}_\mu^0$ for some $\mu \in (-1,1)$, then there exists a positive constant c such that for all $0 < b < 1$ and $\varrho > 0$, $\int_0^b |\phi(\varrho x)|\,\mathrm{d}x \le c\varrho^\mu b^{1+\mu}$.*

Proof. We prove the result of this lemma by bounding $|\phi(\varrho x)|$ by $c\varrho^\mu x^\mu$ and computing the resulting integral exactly. $\square$

With the above preparation, we estimate the error $\mathcal{E}^s_{\mu,m}$ and $\mathcal{N}\left(\mathcal{Q}^s_{\mu,m}\right)$ in the following theorem.

Theorem 4.5. *Let $m \in \mathbb{N}$. If f is of $\mathcal{C}^{2m}_\mu$ for some $\mu \in (-1,1)$ and $g \in C^{2m}(I)$ satisfies Assumption* 4.1*, then there exists a positive constant c such that for all $\kappa > 1$, $s \in \mathbb{N}$ with $s > 1$*

$$\mathcal{E}^s_{\mu,m} \leq c\kappa_r^{-(1+\mu)/(r+1)} s^{-2m}, \tag{4.21}$$

and $\mathcal{N}\left(\mathcal{Q}^s_{\mu,m}\right) \leq (s-1)m$. If $g(x) = x$ for $x \in I$, then the upper bound in (4.21) *reduces to $c\kappa^{-1-\mu}s^{-2m}$.*

Proof. Let $\mathcal{E}_0[\varphi_\kappa] := \left|\int_{x_0}^{x_1} \varphi_\kappa(x)\mathrm{d}x\right|$ and $\mathcal{E}_j[\varphi_\kappa] := \left|\mathcal{Q}_m^{[x_j,x_{j+1}]}[\varphi_\kappa] - \mathcal{I}^{[x_j,x_{j+1}]}[\varphi_\kappa]\right|$ for $j \in \mathbb{Z}^+_{s-1}$. We prove (4.21) by estimating $\mathcal{E}_j[\varphi_\kappa]$ and then sum them over $j \in \mathbb{Z}_{s-1}$.

We first consider $\mathcal{E}_0[\varphi_\kappa]$. By applying Lemma 4.4 with $\varrho := \lambda_r$ and $b := x_1 = s^{-(2m+1)/(1+\mu)}$ we conclude that there exists a positive constant c such that for all $\kappa > 1$ and $s \in \mathbb{N}$ with $s > 1$

$$\mathcal{E}_0[\varphi_\kappa] \leq \int_0^{x_1} |f(\lambda_r x)|\,\mathrm{d}x \leq c\lambda_r^\mu s^{-2m-1},$$

where λ_r is defined by (4.3). For $j \in \mathbb{Z}^+_{s-1}$, by the error bound (4.8) of the Gauss-Legendre quadrature, there exists a constant $\xi_j \in [x_j, x_{j+1}]$ such that

$$\mathcal{E}_j[\varphi_\kappa] \leq h_j^{2m+1}\left|\varphi_\kappa^{(2m)}(\xi_j)\right| / (2^{2m}(2m+1)!),$$

where $h_j := x_{j+1} - x_j$. Using Lemma 4.3 with $\xi_j \geq x_j$ for $j \in \mathbb{Z}^+_{s-1}$, there exists a positive constant c such that for all $\kappa > 1$, $s \in \mathbb{N}$ with $s > 1$ and $j \in \mathbb{Z}^+_{s-1}$, $\left|\varphi_\kappa^{(2m)}(\xi_j)\right| \leq c\lambda_r^\mu x_j^{\mu-2m}$. Substituting this estimate into the inequality above and employing the estimate $h_j^{2m+1}x_j^{\mu-2m} \leq cs^{-2m-1}$, we see that the upper bound of $\mathcal{E}_j[\varphi_\kappa]$ becomes $c\lambda_r^\mu s^{-2m-1}$. Summing up the bound of errors $\mathcal{E}_j[\varphi_\kappa]$ over $j \in \mathbb{Z}_{s-1}$ and using the definition (4.3) of λ_r we obtain the estimate (4.21).

The bound on the number of functional evaluations used in the algorithm (4.9) may be obtained directly.

When $g(x) = x$, we substitute $r = 0$ and $\kappa_0 = \kappa$ into estimate (4.21) to yield the special result. $\square$

4.2. **Quadrature formulas for integrals with oscillation.** We develop in this subsection composite quadrature formulas for computing the oscillatory integral (4.6) having a polynomial order (in terms of the wave number) of convergence.

We first present a partition of Λ. For $n \in \mathbb{N}$, we choose the partition $\Pi_{\kappa,r}$ of Λ with nodes defined by

$$x_j := \kappa_r^{(\frac{j}{n}-1)/(r+1)}, \quad j \in \mathbb{Z}_n. \tag{4.22}$$

According to this partition, the integral (4.6) is rewritten as $\mathcal{I}^\Lambda = \sum_{j\in\mathbb{Z}^+_n} \mathcal{I}^{[x_{j-1},x_j]}$. For each $j \in \mathbb{Z}^+_n$, we use quadrature formula (2.7) to compute the approximation $\mathcal{Q}^{[x_{j-1},x_j]}_{N_j,m}$ of the integral $\mathcal{I}^{[x_{j-1},x_j]}$, where N_j is a positive integer to be specified later. We estimate in the following lemma the error $e_j := \left|\mathcal{I}^{[x_{j-1},x_j]} - \mathcal{Q}_{N_j,m}^{[x_{j-1},x_j]}\right|$ for $j \in \mathbb{Z}_n^+$. For this purpose, we let

$$\Theta(y) := \Psi(g(1)y), \text{ for } y \in I, \tag{4.23}$$

with Ψ defined by (2.4), and

$$\delta_r := \min\left\{\left|g^{(r+1)}(x)\right|/(r+1)! : x \in I\right\}. \tag{4.24}$$

Recalling $M_j = \max\{|g'(x_{j-1})|, |g'(x_j)|\}$ and λ_r defined by (4.3), we define

$$q_j := M_j x_{j-1}/g(x_{j-1}), \text{ for } j \in \mathbb{Z}_n^+, \tag{4.25}$$

$$\theta_{n,r} := \lambda_r^{-1/n} - 1. \tag{4.26}$$

Lemma 4.6. *If Θ is of $\mathcal{C}_\nu^{m+1}$ for some $\nu \in (-1,1)$ and some $m \in \mathbb{N}$, and $g \in C^{m+2}(I)$ satisfies Assumption 4.1, then there exists a positive constant c such that for all $\kappa > 1$, $n \in \mathbb{N}$ and $j \in \mathbb{Z}_n^+$*

$$e_j \le c\frac{|g(1)|^{-\nu}\,\delta_r^{\nu-1}}{(m-1)!\kappa^2 N_j^{m-1}} q_j^m x_{j-1}^{(\nu-1)(r+1)}\theta_{n,r}^m.$$

Proof. We prove this result by applying (2.8) in Theorem 2.1 to e_j for $j \in \mathbb{Z}_n^+$. From (2.8), we have for all $j \in \mathbb{Z}_n^+$ that

$$e_j \le \frac{3(m+1)}{m!\kappa^2 N_j^{m-1}}\left\|\Psi^{(m+1)}\right\|_\infty M_j^m h_j^m,$$

where $h_j := x_j - x_{j-1}$. In the estimate above, using the relation

$$\Psi^{(m+1)}(g(1)y) = (g(1))^{-m-1}\Theta^{(m+1)}(y) \text{ for } y \in (0,1],$$

with the assumption on Θ, we observe that there exists a positive constant c such that for all $\kappa > 1$, $n \in \mathbb{N}$ and $j \in \mathbb{Z}_n^+$,

$$e_j \le c\frac{|g(1)|^{-\nu}}{(m-1)!\kappa^2 N_j^{m-1}} q_j^m (g(x_{j-1}))^{\nu-1} x_{j-1}^{-m} h_j^m.$$

Note that $(g(x_{j-1}))^{\nu-1} \le \delta_r^{\nu-1} x_{j-1}^{(r+1)(\nu-1)}$ and $x_{j-1}^{-m}h_j^m = \theta_{n,r}^m$ for $j \in \mathbb{Z}_n^+$. Substituting these relations into the inequality above yields the desired result. $\square$

We estimate an upper bound of the quantity $\mathbf{Q}_n := \max\{q_j : j \in \mathbb{Z}_n^+\}$.

Lemma 4.7. *If g satisfies Assumption 4.1, then for all $n \in \mathbb{N}$ and $\kappa > 1$, $\mathbf{Q}_n \le (r+1)\kappa_r^{r/(n(r+1))}\sigma_r/\delta_r$.*

Proof. Using the definitions (4.2) and (4.24) of σ_r and δ_r, we have that $g(x) \ge \delta_r x^{r+1}$ and $|g'(x)| \le (r+1)\sigma_r x^r$ for $x \in I$. Thus, $|g(x_{j-1})| \ge \delta_r x_{j-1}^{r+1}$ and $M_j \le (r+1)\sigma_r x_j^r$. For $n \in \mathbb{N}$, we obtain for $\kappa > 1$ and $j \in \mathbb{Z}_n^+$ that $q_j \le (r+1)\kappa_r^{r/(n(r+1))}\sigma_r/\delta_r$. $\square$

We now discuss the choice of N_j. Lemma 4.7 demonstrates that the upper bound of $\mathbf{Q}_n$ is independent of κ for the case $r = 0$ and it depends on κ for the case $r > 0$. Therefore, we need to consider these two cases separately. For the case $r = 0$, we choose

$$N_j := \lceil q_j \rceil \text{ for } j \in \mathbb{Z}_n^+, \tag{4.27}$$

where q_j are defined by (4.25). For the case $r > 0$, we choose

$$N_j := \left\lceil q_j^{m/(m-1)} \right\rceil \text{ for } j \in \mathbb{Z}_n^+,$$

so that $N_j^{-(m-1)} q_j^m$ for all $j \in \mathbb{Z}_n^+$ are independent of κ. With this choice of N_j, we use $\mathcal{Q}_{N_j,m}^{[x_{j-1},x_j]}$ to approximate the integral $\mathcal{I}^{[x_{j-1},x_j]}$ for $j \in \mathbb{Z}_n^+$. Thus, integral $\mathcal{I}^\Lambda$ defined by (4.6) is approximated by the quadrature formula

$$\mathcal{Q}_{n,m}^\Lambda := \sum_{j \in \mathbb{Z}_n^+} \mathcal{Q}_{N_j,m}^{[x_{j-1},x_j]}. \tag{4.28}$$

We next estimate the error $\mathcal{E}_{n,m}^\Lambda := \left| \mathcal{I}^\Lambda - \mathcal{Q}_{n,m}^\Lambda \right|$ and $\mathcal{N}\left(\mathcal{Q}_{n,m}^\Lambda\right)$. We first consider the case $r = 0$.

Theorem 4.8. *If Θ is of $\mathcal{C}_\nu^{m+1}$ for some $\nu \in (-1,1)$, $m \in \mathbb{N}$ and $g \in C^{m+2}(I)$ satisfies Assumption 4.1 with $r = 0$, then there exists a positive constant c such that for all $\kappa > 1$ and $n \in \mathbb{N}$ satisfying $n \geq \ln \kappa_0 / \ln 2$,*

$$\mathcal{E}_{n,m}^\Lambda \leq c\, |g(1)|^{-\nu}\, \delta_0^{\nu-2} \sigma_0 \kappa^{-2} \kappa_0^{1-\nu} \ln \kappa_0 \theta_{n,0}^{m-1}. \tag{4.29}$$

There holds the estimate $\mathcal{N}\left(\mathcal{Q}_{n,m}^\Lambda\right) \leq \lceil \sigma_0/\delta_0 \rceil nm + 1$.

Proof. We prove (4.29) by estimating e_j for $j \in \mathbb{Z}_n^+$, and then summing them over j. By employing Lemmas 4.6 and 4.7 with $r = 0$ and the choice (4.27) of N_j, there exists a positive constant c such that for all $\kappa > 1$, $n \in \mathbb{N}$ and $j \in \mathbb{Z}_n^+$,

$$e_j \leq c\, |g(1)|^{-\nu}\, \delta_0^{\nu-2} \sigma_0 \kappa^{-2} x_{j-1}^{\nu-1} \theta_{n,0}^m.$$

Since $-1 < \nu < 1$, we see that $x_{j-1}^{\nu-1} \leq x_0^{\nu-1} = \kappa_0^{1-\nu}$. Thus, the inequalities above become

$$e_j \leq c\, |g(1)|^{-\nu}\, \delta_0^{\nu-2} \sigma_0 \kappa^{-2} \kappa_0^{1-\nu} \theta_{n,0}^m.$$

Moreover, condition $n \geq \ln \kappa_0 / \ln 2$ implies that $n\theta_{n,0} \leq 2 \ln \kappa_0$. Using this inequality and summing the inequalities above over j leads to the desired estimate (4.29).

The estimate of $\mathcal{N}\left(\mathcal{Q}_{n,m}^\Lambda\right)$ is obtained by using formula (4.28), the choice of N_j and Lemma 4.7 with $r = 0$. $\square$

In passing, we comment on the hypothesis imposed to Θ in the last theorem. If $f \in C^m(I)$ for some $m \in \mathbb{N}$ and $g \in C^m(I)$ satisfies Assumption 4.1, then Θ is of $\mathcal{C}_\nu^m$ with $\nu := -r/(r+1)$. A proof of this conclusion may be found in [6]. As a direct consequence of Theorem 4.8, we have the next estimate for the case having the linear oscillator $g(x) = x$ for $x \in I$, where we have that $g(1) = \delta_0 = \sigma_0 = 1$ and $\kappa_0 = \kappa$.

Corollary 4.9. *If f is of $\mathcal{C}_\nu^{m+1}$ for some $\nu \in (-1,1)$, $m \in \mathbb{N}$, then there exists a positive constant c such that for all $\kappa > 1$ and $n \in \mathbb{N}$ satisfying $n \geq \ln \kappa / \ln 2$,*

$$\mathcal{E}_{n,m}^\Lambda \leq c \kappa^{-\nu-1} \ln \kappa (\kappa^{1/n} - 1)^{m-1}.$$

There holds the estimate $\mathcal{N}\left(\mathcal{Q}_{n,m}^\Lambda\right) \leq nm + 1$.

We now consider the case $r > 0$.

Theorem 4.10. *If Θ is of $\mathcal{C}_\nu^{m+1}$ for some $\nu \in (-1,1)$, $m \in \mathbb{N}$ and $g \in C^{m+2}(I)$ satisfies Assumption 4.1 with $r > 0$, then there exists a positive constant c such that for all $\kappa > 1$ and $n \in \mathbb{N}$ satisfying $n \geq \ln \kappa_r / \ln 2$,*

$$\mathcal{E}_{n,m}^{\Lambda} \leq c(r+1) \left|g(1)\right|^{-\nu} \delta_r^{\nu-1} \kappa^{-2} \kappa_r^{1-\nu} \ln \kappa_r \theta_{n,r}^{m-1}. \tag{4.30}$$

There holds $\mathcal{N}\left(\mathcal{Q}_{n,m}^{\Lambda}\right) \leq \left\lceil [2(r+1)\sigma_r/\delta_r]^{m/(m-1)} \right\rceil nm + 1$.

Proof. Estimate (4.30) is proved in a similar way to that of estimate (4.29) in Theorem 4.8. The only difference is that instead of using the hypotheses and results for the special case $r = 0$, we use those for the case $r > 0$. We leave the details to the interested reader.

It remains to estimate $\mathcal{N}\left(\mathcal{Q}_{n,m}^{\Lambda}\right)$. According to formula (4.28) and Lemma 4.7, we obtain that

$$\mathcal{N}\left(\mathcal{Q}_{n,m}^{\Lambda}\right) \leq \sum_{j \in \mathbb{Z}_n^+} \{N_j m + 1\} - (n-1) \leq \left\lceil \left((r+1)\kappa_r^{1/n} \sigma_r/\delta_r \right)^{m/(m-1)} \right\rceil nm + 1.$$

Note that condition $n \geq \ln \kappa_r / \ln 2$ for $n \in \mathbb{N}$ implies that $\kappa_r^{1/n} \leq 2$. This concludes the last result. □

To close this section, we point out that the oscillatory integral (2.1) may be computed according to the formula $\mathcal{Q}_{n,m}^{s,\widetilde{m}} := \mathcal{Q}_{\mu,\widetilde{m}}^{s} + \mathcal{Q}_{n,m}^{\Lambda}$, with two fixed positive integers $\widetilde{m}$ and m.

5. Quadratures with an exponential order of convergence

In this section, we develop quadrature methods for computing the oscillatory integral (2.1) having an exponential order of convergence. As in Section 4, we shall write integral (2.1) as the sum of a weakly singular integral (4.4) and an oscillatory integral (4.6), and treat them separately. Specifically, we shall develop a quadrature rule for computing (4.4) having an exponential order (in terms of a constant $\gamma \in (0,1)$) of convergence and a quadrature rule for computing (4.6) having an exponential order (in terms of the wave number) of convergence.

5.1. **Quadrature formulas for integrals with a singularity.**

We describe in this subsection quadrature rules for the integral $\mathcal{I}[\varphi_\kappa]$ that appears in (4.4) by borrowing an idea from [35] (see also [4]).

We begin with describing a partition of I. For $\gamma \in (0,1)$ and $s \in \mathbb{N}$ with $s > 1$, let Π^γ be the partition of I with the nodes $x_0 := 0$ and $x_j := \gamma^{s-j}$, for $j \in \mathbb{Z}_s^+$. For $\varepsilon \geq 1$, we choose

$$m_j := \lceil j\varepsilon \rceil, \text{ for } j \in \mathbb{Z}_{s-1}^+. \tag{5.1}$$

The quadrature rule for computing $\mathcal{I}[\varphi_\kappa]$ is formed by replacing φ_κ on $[x_0, x_1]$ by 0 and using $\mathcal{Q}_{m_j}^{[x_j,x_{j+1}]}[\varphi_\kappa]$ defined by (4.7) for computing the integrals $\mathcal{I}^{[x_j,x_{j+1}]}[\varphi_\kappa]$, for $j \in \mathbb{Z}_{s-1}^+$. Integral $\mathcal{I}^0$ defined by (4.4) is then approximated by the quadrature formula

$$\mathcal{Q}_\gamma^s := \lambda_r \sum_{j \in \mathbb{Z}_{s-1}^+} \mathcal{Q}_{m_j}^{[x_j,x_{j+1}]}[\varphi_\kappa], \tag{5.2}$$

where λ_r is defined by (4.3). We next estimate the error $\mathcal{E}_\gamma^s := \left|\mathcal{I}^0 - \mathcal{Q}_\gamma^s\right|$. To this end, we impose the following hypothesis on g.

Assumption 5.1. *The function $g \in C^\infty(I)$ and there exists a positive constant $\zeta > 1$ such that for all $n \in \mathbb{N}$, $\|g\|_n \leq \zeta$.*

We first estimate the norm of $w_\kappa(x) = \exp\{u_\kappa(x)\}$ with $u_\kappa(x) = \mathrm{i}\kappa g(\lambda_r x)$ for $x \in I$ defined by (4.10).

Lemma 5.2. *If g satisfies Assumptions 4.1 and 5.1, then for all $\kappa > 1$ and $n \in \mathbb{N}$, $\|w_\kappa\|_n \leq B_n \zeta^n$.*

Proof. The proof of this lemma is similar to that of Lemma 4.2. Since g satisfies Assumptions 4.1 and 5.1, we apply (4.12) with (4.15) and (4.16) to get that for all $n \in \mathbb{N}$, $\kappa > 1$, $j \in \mathbb{Z}_n^+$, $l \in \mathbb{Z}_j^+$ and $x \in I$,

$$\left|B_{j,l}\left(u_\kappa^{(1)}, u_\kappa^{(2)}, \ldots, u_\kappa^{(j-l+1)}\right)\right| \leq B_{j,l}\zeta^l.$$

This together with (4.14) yields that for all $n \in \mathbb{N}$, $j \in \mathbb{Z}_n^+$ and $\kappa > 1$

$$\left\|w_\kappa^{(j)}\right\|_\infty \leq \|w_\kappa\|_\infty \sum_{l \in \mathbb{Z}_j^+} B_{j,l}\zeta^l \leq B_j \zeta^j,$$

since $\|w_\kappa\|_\infty \leq 1$. Note that $B_n < B_{n+1}$ for $n \in \mathbb{N}$ from the recurrence relation of the Bell number involving binomial coefficients [36]. This together with the inequalities above yields the conclusion. □

In the next lemma, we estimate the derivatives of φ_κ defined as in (4.5).

Lemma 5.3. *If f is of $\mathcal{C}_\mu^\infty$ for some $\mu \in (-1, 1)$ and g satisfies Assumptions 4.1 and 5.1, then there exist positive constants c and $\zeta > 1$ such that for all $\kappa > 1$, $n \in \mathbb{N}$ and $x \in (0, 1]$, $\left|\varphi_\kappa^{(n)}(x)\right| \leq c2^n B_n \zeta^n \lambda_r^\mu x^{\mu-n}$.*

Proof. Applying the Leibniz formula (4.17) to φ_κ yields

$$\varphi_\kappa^{(n)}(x) = \sum_{j \in \mathbb{Z}_n} C_n^j \left(f(\lambda_r x)\right)^{(j)} w_\kappa^{(n-j)}(x),$$

for $x \in (0, 1]$ and $n \in \mathbb{N}$. Using Lemma 5.2 with the inequality (4.20), we conclude that there exist constants $c > 0$ and $\zeta > 1$ such that for all $\kappa > 1$, $n \in \mathbb{N}$ and $x \in (0, 1]$,

$$\left|\varphi_\kappa^{(n)}(x)\right| \leq c B_n \zeta^n \lambda_r^\mu x^{\mu-n} \sum_{j \in \mathbb{Z}_n} C_n^j.$$

The desired estimate of this lemma is then obtained from this inequality with (4.18). □

We need two lemmas regarding the parameters m_j, $j \in \mathbb{Z}_{s-1}^+$, defined by (5.1).

Lemma 5.4. *Let $s \in \mathbb{N}$ with $s > 1$ and $\varepsilon \geq 1$. If $j, l \in \mathbb{Z}_{s-1}^+$, $j \neq l$, then $m_j \neq m_l$. Moreover, there holds $(1+\mu)(j-1) < 2m_j - 2$ for $j \in \mathbb{Z}_{s-1}^+$.*

Proof. The proof of the first inequality is straightforward and thus we omit it. The second inequality follows from the assumptions that $-1 < \mu < 1$ and $\varepsilon \geq 1$, from which we conclude that

$$(1+\mu)(j-1) < 2(j-1) \leq 2j\varepsilon - 2 \leq 2m_j - 2. \qquad \square$$

Lemma 5.5. *If $\gamma \in (0,1)$, $\varepsilon \geq 1$ and $\zeta > 1$, then there exists a positive constant c such that for all $s \in \mathbb{N}$ with $s > 1$ and $j \in \mathbb{Z}_{s-1}^+$,*

$$B_{2m_j+1}(\zeta/\gamma)^{(1+\mu)(j-1)+(2m_j+1)}/(2m_j+1)! \leq c.$$

Proof. The proof is similar to that of Lemma 4.1 in [4]. Let $\iota := \zeta/\gamma$. Using the second result of Lemma 5.4 and the fact $\zeta > \gamma$, we obtain that

$$\iota^{(1+\mu)(j-1)+(2m_j+1)} < \iota^{4m_j-1}.$$

Applying inequality (3.9) with $n := 2m_j + 1$, we find that

$$1/(2m_j+1)! \leq 1/\sqrt{2\pi \mathrm{e}}\left(\mathrm{e}/(2m_j+1)\right)^{2m_j+3/2}.$$

Using the bound of the Bell number $B_n \leq (0.792n/\ln(n+1))^n$ with $n := 2m_j+1$, we have that

$$B_{2m_j+1} \leq (0.792(2m_j+1)/\ln(2m_j+2))^{2m_j+1}.$$

Combining the three inequalities above yields

$$B_{2m_j+1}\iota^{(1+\mu)(j-1)+(2m_j+1)}/(2m_j+1)! \leq d_j^{2m_j+1}/\left(\sqrt{2\pi(2m_j+1)}\iota^3\right),$$

where $d_j := 0.792\iota^2\mathrm{e}/\ln(2m_j+2)$. It suffices to prove that there exists a positive constant c such that for all $s \in \mathbb{N}$ with $s > 1$ and $j \in \mathbb{Z}_{s-1}^+$, $d_j^{2m_j+1} \leq c$. According to the definition of d_j and m_j, we observe that $d_j \leq 0.792\iota^2\mathrm{e}/\ln(2j\varepsilon+2)$. It follows for $j \in \mathbb{N}$ satisfying $j \geq (2\varepsilon)^{-1}(\mathrm{e}^{0.792\iota^2\mathrm{e}} - 2)$ that $d_j^{2m_j+1} \leq 1$. On the other hand, for $j \in \mathbb{N}$ satisfying $j < (2\varepsilon)^{-1}(\mathrm{e}^{0.792\iota^2\mathrm{e}} - 2)$, we have that

$$d_j^{2m_j+1} \leq \max\left\{d_j^{2m_j+1} : j \in \mathbb{N}, j < (2\varepsilon)^{-1}(\mathrm{e}^{0.792\iota^2\mathrm{e}} - 2)\right\},$$

which is a constant. This proves the desired estimate. □

We are now ready to estimate the error $\mathcal{E}_\gamma^s$ and $\mathcal{N}\left(\mathcal{Q}_\gamma^s\right)$ of the quadrature rule $\mathcal{Q}_\gamma^s$.

Theorem 5.6. *Let $\gamma \in (0,1)$ and $\varepsilon \geq 1$. If f is of $\mathcal{C}_\mu^\infty$ for some $\mu \in (-1,1)$ and g satisfies Assumptions 4.1 and 5.1, then there exists a positive constant c such that for all $\kappa > 1$ and integers $s > 1$*

$$\mathcal{E}_\gamma^s \leq c\kappa_r^{-(1+\mu)/(r+1)}\gamma^{(1+\mu)(s-1)-1}, \tag{5.3}$$

and $\mathcal{N}\left(\mathcal{Q}_\gamma^s\right) \leq \lceil\varepsilon\rceil(s^2-s)/2$. In particular, if $g(x) = x$ for $x \in I$, then the upper bound in (5.3) reduces to $c\kappa^{-1-\mu}\gamma^{(1+\mu)(s-1)-1}$.

Proof. We prove (5.3) by estimating $\mathcal{E}_0[\varphi_\kappa]$ and

$$\mathcal{E}_j[\varphi_\kappa] := \left|\mathcal{I}^{[x_j,x_{j+1}]}[\varphi_\kappa] - \mathcal{Q}_{m_j}^{[x_j,x_{j+1}]}[\varphi_\kappa]\right|, \quad \text{for } j \in \mathbb{Z}_{s-1}^+,$$

and then summing them over j.

We first consider $\mathcal{E}_0[\varphi_\kappa]$. By applying Lemma 4.4 with $\varrho := \lambda_r$ and $b := x_1 = \gamma^{s-1}$, there exists a positive constant c such that for all $\kappa > 1$ and $s \in \mathbb{N}$ with $s > 1$, $\mathcal{E}_0[\varphi_\kappa] \leq c\lambda_r^\mu\gamma^{(1+\mu)(s-1)}$. For $j \in \mathbb{Z}_{s-1}^+$, by the error bound (4.8) of the Gauss-Legendre quadrature, there exists a constant $\xi_j \in [x_j, x_{j+1}]$ such that

$$\mathcal{E}_j[\varphi_\kappa] \leq h_j^{2m_j+1}/(2^{2m_j}(2m_j+1)!)\left|\varphi_\kappa^{(2m_j)}(\xi_j)\right|,$$

where $h_j := x_{j+1} - x_j$. Applying Lemma 5.3 with $\xi_j \geq x_j$ for $j \in \mathbb{Z}_{s-1}^+$, there exist two positive constants c and $\zeta > 1$ such that for all $\kappa > 1$, $s \in \mathbb{N}$ with $s > 1$ and $j \in \mathbb{Z}_{s-1}^+$,

$$\left|\varphi_\kappa^{(2m_j)}(\xi_j)\right| \leq c2^{2m_j} B_{2m_j} \zeta^{2m_j} \lambda_r^\mu x_j^{\mu-2m_j}.$$

Combining these two inequalities yields two positive constants c and $\zeta > 1$ such that for all $\kappa > 1$, $s \in \mathbb{N}$ with $s > 1$ and $j \in \mathbb{Z}_{s-1}^+$

$$\begin{aligned}\mathcal{E}_j[\varphi_\kappa] &\leq c\lambda_r^\mu \frac{B_{2m_j}\zeta^{2m_j} h_j^{2m_j+1} x_j^{\mu-2m_j}}{(2m_j+1)!} \\ &\leq c\frac{B_{2m_j+1}\iota^{(1+\mu)(j-1)+(2m_j+1)}}{(2m_j+1)!}\lambda_r^\mu \gamma^{(1+\mu)(s-1)}(1-\gamma)^{2m_j+1}.\end{aligned}$$

Using Lemma 5.5, the above inequality reduces to

$$\mathcal{E}_j[\varphi_\kappa] \leq c\lambda_r^\mu \gamma^{(1+\mu)(s-1)}(1-\gamma)^{2m_j+1}.$$

Summing up the bound of errors $\mathcal{E}_j[\varphi_\kappa]$ over $j \in \mathbb{Z}_{s-1}$, we conclude that there exists a positive constant c such that for all $\kappa > 1$ and $s \in \mathbb{N}$ with $s > 1$,

$$\mathcal{E}_\gamma^s \leq c\lambda_r^{1+\mu}\gamma^{(1+\mu)(s-1)}\Big(1 + \sum_{j\in\mathbb{Z}_{s-1}^+}(1-\gamma)^{2m_j+1}\Big). \tag{5.4}$$

According to Lemma 5.4, we observe that $\left\{2m_j + 1 : j \in \mathbb{Z}_{s-1}^+\right\} \cup \{0\} \subset \mathbb{Z} := \{0, 1, \ldots\}$. It follows that

$$1 + \sum_{j\in\mathbb{Z}_{s-1}^+}(1-\gamma)^{2m_j+1} < \sum_{j\in\mathbb{Z}}(1-\gamma)^j = \gamma^{-1}.$$

Substituting this result and the definition (4.3) of λ_r into the right hand side of inequality (5.4) yields the estimate (5.3).

The bound of the number of functional evaluations used in quadrature (5.2) may be obtained directly.

When $g(x) = x$, we substitute $r = 0$ and $\kappa_0 = \kappa$ into estimate (5.3) to yield the special result. □

5.2. **Quadrature formulas for integrals with oscillation.** We develop in this subsection a quadrature rule for the oscillatory integral (4.6) having an exponential order (in terms of the wave number) of convergence. This is done by choosing variable numbers of quadrature nodes in the subintervals of Λ. Specifically, for $n \in \mathbb{N}$ and for the partition $\Pi_{\kappa,r}$ of Λ with nodes defined by (4.22), we let

$$m_j := n + \lceil (n+1-j)(1-\nu)\rceil \text{ and } N_j := \lceil q_j \rceil \text{ for } j \in \mathbb{Z}_n^+, \tag{5.5}$$

where q_j are defined by (4.25), and $\nu \in (-1, 1)$ is the index of singularity of Θ. Note that although m_j defined here differ from those defined in (5.1), the reader should be able to distinguish them from the context. For each $j \in \mathbb{Z}_n^+$, we use $\mathcal{Q}_{N_j,m_j}^{[x_{j-1},x_j]}$ to approximate the integral $\mathcal{I}^{[x_{j-1},x_j]}$. Integral $\mathcal{I}^\Lambda$ defined by (4.6) is then approximated by the quadrature formula

$$\mathcal{Q}_n^\Lambda := \sum_{j\in\mathbb{Z}_n^+}\mathcal{Q}_{N_j,m_j}^{[x_{j-1},x_j]}. \tag{5.6}$$

We first provide an estimate of the errors $e_j := \left|\mathcal{I}^{[x_{j-1},x_j]} - \mathcal{Q}^{[x_{j-1},x_j]}_{N_j,m_j}\right|$ for $j \in \mathbb{Z}_n^+$ in the next lemma. We recall that Θ, δ_r and $\theta_{n,r}$ are defined by (4.2), (4.23) and (4.26).

Lemma 5.7. *If $g \in C^\infty(I)$ satisfies Assumption 4.1 and Θ is of $\mathcal{C}_\nu^\infty$ for some $\nu \in (-1,1)$, then there exists a positive constant c such that for all $\kappa > 1$, $n \in \mathbb{N}$ and $j \in \mathbb{Z}_n^+$*

$$e_j \le c\frac{|g(1)|^{-\nu}\,\delta_r^{\nu-1}}{(m_j-1)!\kappa^2 N_j^{m_j-1}} q_j^{m_j} x_{j-1}^{(\nu-1)(r+1)} \theta_{n,r}^{m_j}.$$

This lemma may be proved in the same way as Lemma 4.6 with $m = m_j$. We next estimate the error $\mathcal{E}_n^\Lambda := \left|\mathcal{I}^\Lambda - \mathcal{Q}_n^\Lambda\right|$. To this end, we establish two technical lemmas.

Lemma 5.8. *For all $n \in \mathbb{N}$ with $n > 1$, $\sum_{j\in\mathbb{Z}_n^+} 1/(m_j - 1) \le 1$.*

Proof. The proof is similar to that of Lemma 3.3. For $n \in \mathbb{N}$ with $n > 1$, we need to estimate the lower bound of the set $\{m_j - 1 : j \in \mathbb{Z}_n^+\}$. By $\nu < 1$, we have that $m_j - 1 \ge n$ for $j \in \mathbb{Z}_n^+$. It follows that $\sum_{j\in\mathbb{Z}_n^+} 1/(m_j-1) \le \sum_{j\in\mathbb{Z}_n^+} 1/n = 1$. □

For simple notation, we let $\tau_{n,r} := \kappa_r^{1/n}(\kappa_r^{1/n} - 1)$.

Lemma 5.9. *If there exists a positive constant c such that for all $\kappa > 1$, $j \in \mathbb{Z}_n^+$ and $n \in \mathbb{N}$ satisfying*

$$\tau_{n,r} \le (n-1)/\mathrm{e}, \tag{5.7}$$

then $\tau_{n,r}^{m_j-n}/(m_j-2)! \le c(n-1)^{-1/2}$. Moreover, $\kappa_r^{1/n} \le 2(n-1)^{1/2}$.

Proof. By $\nu < 1$ we see that $m_j - 2 \ge n - 1$ for $j \in \mathbb{Z}_n^+$. Condition (5.7) implies $n \ge 2$ since $\tau_{1,r} > 0$ for $\kappa > 1$. It follows for $n \in \mathbb{N}$ satisfying (5.7) with $\tau_{n,r} \le 1$ that

$$\tau_{n,r}^{m_j-n}/(m_j-2)! \le 1/(m_j-2)! \le (n-1)^{-1/2}.$$

On the other hand, for $n \in \mathbb{N}$ satisfying (5.7) with $\tau_{n,r} > 1$, we have that

$$\tau_{n,r}^{m_j-n}/(m_j-2)! \le \tau_{n,r}^{m_j-2}/(m_j-2)!.$$

According to inequality (3.9) with $n := m_j - 2$, there exists a positive constant c such that for all $n \in \mathbb{N}$ with $n > 1$ and $j \in \mathbb{Z}_n^+$,

$$1/(m_j-2)! \le c(m_j-2)^{-1/2}\left(\mathrm{e}/(m_j-2)\right)^{m_j-2} \le c(n-1)^{-1/2}\left(\mathrm{e}/(n-1)\right)^{m_j-2}.$$

This together with the inequality above ensures that there exists a positive constant c such that for all $\kappa > 1$, $j \in \mathbb{Z}_n^+$ and $n \in \mathbb{N}$ satisfying (5.7) with $\tau_{n,r} > 1$,

$$\tau_{n,r}^{m_j-n}/(m_j-2)! \le c(n-1)^{-1/2}\left(\mathrm{e}\tau_{n,r}/(n-1)\right)^{m_j-2} \le c(n-1)^{-1/2}.$$

This concludes the desired result.

We next prove the second inequality. From condition (5.7) we observe that

$$\left(\kappa_r^{1/n} - 1\right)^2 \le \tau_{n,r} \le (n-1)/\mathrm{e}.$$

Using the inequality $((n-1)/\mathrm{e})^{1/2} + 1 \le 2(n-1)^{1/2}$ for $n > 1$ yields that for $n \in \mathbb{N}$ satisfying (5.7) with $\kappa_r^{1/n} > 2$,

$$\kappa_r^{1/n} \le ((n-1)/\mathrm{e})^{1/2} + 1 \le 2(n-1)^{1/2}.$$

Moreover, it follows for $n \in \mathbb{N}$ satisfying (5.7) with $\kappa_r^{1/n} \le 2$ that $\kappa_r^{1/n} \le 2(n-1)^{1/2}$. Combining these two results we obtain the second conclusion. □

We are now ready to estimate the error $\mathcal{E}_n^\Lambda$ and $\mathcal{N}\left(\mathcal{Q}_n^\Lambda\right)$. Let $\rho_{n,r} := 1 - \lambda_r^{1/n}$, where λ_r is defined by (4.3). We first consider the case $r = 0$.

Theorem 5.10. *If $g \in C^\infty(I)$ satisfies Assumption 4.1 with $r = 0$ and Θ is of $\mathcal{C}_\nu^\infty$ for some $\nu \in (-1, 1)$, then there exists a positive constant c such that for all $\kappa > 1$ and $n \in \mathbb{N}$ satisfying (5.7),*

$$\mathcal{E}_n^\Lambda \le c\,|g(1)|^{-\nu}\,\sigma_0\delta_0^{\nu-2}(n-1)^{-1/2}\kappa^{-2}\kappa_0\rho_{n,0}^n. \tag{5.8}$$

For $n \in \mathbb{N}$, there holds the estimate $\mathcal{N}\left(\mathcal{Q}_n^\Lambda\right) \le \lceil\sigma_0/\delta_0\rceil\left(2n^2 + \lceil 1-\nu\rceil(n^2+n)\right)/2 + 1$.

Proof. We prove (5.8) by estimating e_j and then summing them over $j \in \mathbb{Z}_n^+$. By employing Lemmas 4.7 and 5.7 with $r = 0$, and the choice of N_j defined by (5.5), there exists a positive constant c such that for all $\kappa > 1$, $j \in \mathbb{Z}_n^+$ and $n \in \mathbb{N}$,

$$e_j \le c\frac{|g(1)|^{-\nu}\,\sigma_0\delta_0^{\nu-2}}{(m_j-1)!\kappa^2}x_{j-1}^{\nu-1}\theta_{n,0}^{m_j}.$$

Applying $\theta_{n,0} = \lambda_0^{-1/n}\rho_{n,0}$ with the definition of x_{j-1}, we observe that

$$x_{j-1}^{\nu-1}\theta_{n,0}^{m_j} \le \tau_{n,0}^{m_j-n}\lambda_0^{-1}\rho_{n,0}^n.$$

Combining these two inequalities yields a positive constant c such that for all $\kappa > 1$, $j \in \mathbb{Z}_n^+$ and $n \in \mathbb{N}$,

$$e_j \le c\frac{|g(1)|^{-\nu}\,\sigma_0\delta_0^{\nu-2}}{(m_j-1)\kappa^2}\frac{\tau_{n,0}^{m_j-n}}{(m_j-2)!}\lambda_0^{-1}\rho_{n,0}^n.$$

This together with the first result of Lemma 5.9 with $r = 0$ ensures that

$$e_j \le c(m_j-1)^{-1}\,|g(1)|^{-\nu}\,\sigma_0\delta_0^{\nu-2}(n-1)^{-1/2}\kappa^{-2}\lambda_0^{-1}\rho_{n,0}^n.$$

Summing up the inequality above over $j \in \mathbb{Z}_n^+$, we obtain that there exists a positive constant c such that for all $\kappa > 1$ and $n \in \mathbb{N}$ satisfying (5.7),

$$\mathcal{E}_n^\Lambda \le c\,|g(1)|^{-\nu}\,\sigma_0\delta_0^{\nu-2}(n-1)^{-1/2}\kappa^{-2}\lambda_0^{-1}\rho_{n,0}^n\sum_{j\in\mathbb{Z}_n^+}(m_j-1)^{-1}.$$

This together with Lemma 5.8 and $\lambda_0^{-1} = \kappa_0$ leads to the estimate (5.8).

According to formula (5.6) and Lemma 4.7 with $r = 0$, we obtain for $n \in \mathbb{N}$ that

$$\mathcal{N}\left(\mathcal{Q}_n^\Lambda\right) \le \sum_{j\in\mathbb{Z}_n^+}\left\{\lceil\sigma_0/\delta_0\rceil\,m_j + 1\right\} - (n-1) \le \lceil\sigma_0/\delta_0\rceil\left(2n^2 + \lceil 1-\nu\rceil(n^2+n)\right)/2 + 1.$$

□

Following Theorem 5.10, we have the following special result for the case $g(x) = x$ for $x \in I$.

Corollary 5.11. *If f is of $\mathcal{C}_\mu^\infty$ for some $\mu \in (-1, 1)$, then there exists a positive constant c such that for all $\kappa > 1$ and $n \in \mathbb{N}$ that satisfy (5.7) with $r = 0$,*

$$\mathcal{E}_n^\Lambda \le c(n-1)^{-1/2}\kappa^{-1}(1-\kappa^{-1/n})^n.$$

There holds the estimate $\mathcal{N}\left(\mathcal{Q}_n^\Lambda\right) \le \left(2n^2 + \lceil 1-\mu\rceil(n^2+n)\right)/2 + 1$.

Next, we consider the case $r > 0$.

Theorem 5.12. *If $g \in C^\infty(I)$ satisfies Assumption 4.1 with $r > 0$ and Θ is of $\mathcal{C}_\nu^\infty$ for some $\nu \in (-1,1)$, then there exists a positive constant c such that for all $\kappa > 1$ and $n \in \mathbb{N}$ satisfying (5.7),*

$$\mathcal{E}_n^\Lambda \le c(r+1)\,|g(1)|^{-\nu}\,\sigma_r \delta_r^{\nu-2}\kappa^{-2}\kappa_r^{1/(r+1)}\rho_{n,r}^n. \tag{5.9}$$

There holds

$$\mathcal{N}\left(\mathcal{Q}_n^\Lambda\right) \le \left\lceil (r+1)\kappa_r^{r/(n(r+1))}\sigma_r/\delta_r \right\rceil \left(2n^2 + \lceil 1-\nu \rceil (n^2+n)\right)/2 + 1.$$

Proof. The proof is handled in the same way as that of Theorem 5.10. Using $\theta_{n,r} = \lambda_r^{-1/n}\rho_{n,r}$ and the definition of x_{j-1}, we obtain that

$$x_{j-1}^{(\nu-1)(r+1)}\theta_{n,r}^{m_j} \le \tau_{n,r}^{m_j-n}\lambda_r^{-1}\rho_{n,r}^n, \text{ for } j \in \mathbb{Z}_n^+.$$

By employing Lemmas 4.7 and 5.7, the choice of N_j defined by (5.5) and the inequality above, there exists a positive constant c such that for all $\kappa > 1$, $j \in \mathbb{Z}_n^+$ and $n \in \mathbb{N}$,

$$e_j \le c\frac{(r+1)\,|g(1)|^{-\nu}\,\sigma_r\delta_r^{\nu-2}\kappa_r^{1/n}}{(m_j-1)\kappa^2}\frac{\tau_{n,r}^{m_j-n}}{(m_j-2)!}\lambda_r^{-1}\rho_{n,r}^n.$$

Using the first result of Lemma 5.9, we obtain that there exists a positive constant c such that for all $\kappa > 1$, $j \in \mathbb{Z}_n^+$ and $n \in \mathbb{N}$ satisfying (5.7),

$$e_j \le c(r+1)\,|g(1)|^{-\nu}\,\sigma_r\delta_r^{\nu-2}\kappa_r^{1/n}(n-1)^{-1/2}\kappa^{-2}\lambda_r^{-1}\rho_{n,r}^n/(m_j-1).$$

This together with the second result of Lemma 5.9 and $\lambda_r^{-1} = \kappa_r^{1/(r+1)}$ ensures that

$$e_j \le c(r+1)\,|g(1)|^{-\nu}\,\sigma_r\delta_r^{\nu-2}\kappa^{-2}\kappa_r^{1/(r+1)}\rho_{n,r}^n/(m_j-1).$$

Summing up the inequality above over $j \in \mathbb{Z}_n^+$ and applying Lemma 5.8, we obtain the estimate (5.9).

It remains to estimate $\mathcal{N}\left(\mathcal{Q}_n^\Lambda\right)$. According to formula (5.6) and Lemma 4.7, we have for $n \in \mathbb{N}$ that

$$\begin{aligned}\mathcal{N}\left(\mathcal{Q}_n^\Lambda\right) &\le \sum_{j\in\mathbb{Z}_n^+}\left\{\lceil q_j \rceil m_j + 1\right\} - (n-1)\\ &\le \left\lceil (r+1)\kappa_r^{r/(n(r+1))}\sigma_r/\delta_r \right\rceil \left(2n^2 + \lceil 1-\nu \rceil (n^2+n)\right)/2 + 1.\end{aligned}$$

This yields the last result. ☐

Note that the decay of the bound of $\mathcal{E}_n^\Lambda$ given in Theorems 5.10 and 5.12 is faster than the exponential decay with the base $\rho_{n_0,r}$, where $n_0 := \min\{n : n \in \mathbb{N}$ satisfy condition (5.7)$\}$. For $\kappa > 1$, we conclude that $0 < \rho_{n_0,r} < 1$ and $\lim_{n\to\infty}\rho_{n,r}^n/\rho_{n_0,r}^n = 0$.

The oscillatory integral (2.1) may be computed by $\mathcal{Q}_n^{s,\gamma} := \mathcal{Q}_\gamma^s + \mathcal{Q}_n^\Lambda$. We next estimate the error $\mathcal{E}_n^{\gamma,s}$ of the quadrature formula $\mathcal{Q}_n^{s,\gamma}$ under reasonable hypotheses, where f is of $\mathcal{C}_\mu^\infty$ for some $\mu \in (-1,1)$, $g \in C^\infty(I)$ satisfies Assumptions 4.1 and 5.1, and Θ is of $\mathcal{C}_\nu^\infty$ for some $\nu \in (-1,1)$. Note that $-n\ln\left(1-\gamma^{1+\mu}\right) \ge \ln\kappa_r$ for $n \in \mathbb{N}$ implies that $\rho_{n,r} \le \gamma^{1+\mu}$. If $r > 0$ and $n = s$, then there exists a positive constant c such that for all $\kappa > 1$ and $n \in \mathbb{N}$ that satisfy (5.7) and $-n\ln\left(1-\gamma^{1+\mu}\right) \ge \ln\kappa_r$, $\mathcal{E}_n^{\gamma,s} \le c\kappa^{-(1+\mu)/(r+1)}\gamma^{(1+\mu)(n-1)}$. If $r = 0$ and $n = s$, we consider two cases. In the case $-1 < \mu < 0$, there exists a positive constant c such that for all $\kappa > 1$ and

$n \in \mathbb{N}$ satisfying (5.7) and $-n \ln\left(1-\gamma^{1+\mu}\right) \geq \ln \kappa_0$, $\mathcal{E}_n^{\gamma,s} \leq c\kappa^{-1-\mu}\gamma^{(1+\mu)(n-1)}$. In the case $0 \leq \mu < 1$, there exists a positive constant c such that for all $\kappa > 1$ and $n \in \mathbb{N}$ satisfying (5.7) and $-n \ln\left(1-\gamma^{1+\mu}\right) \geq \ln \kappa_0$, $\mathcal{E}_n^{\gamma,s} \leq c\kappa^{-1}\gamma^{(1+\mu)(n-1)}$.

To close this section, we comment on computing the integrals (2.2) for the case when f has a logarithmic singularity. Note that $\ln x \in \mathcal{C}_\mu^\infty$ for $\mu \in (-1, 0)$. Then the formulas (4.9) and (5.2) can be used for this case by choosing $\mu \in (-1, 0)$. In the numerical examples to be presented in the next section, we calculate the integrals of this kind by choosing $\mu = 0$.

6. Numerical experiments

In this section, we present numerical results of seven computational experiments to verify the approximation accuracy and computational efficiency of the proposed composite moment-free Filon-type (CMF) quadrature formulas. The CMF quadrature formulas with a polynomial (resp. an exponential) order of convergence will be denoted by CMFP (resp. CMFE). The numerical experiments to be presented are divided into two groups. The first three examples are about the CMFP formulas and the last four examples concern the CMFE formulas. We also compare the computational performance of the proposed quadrature formulas with that of the quadrature rules proposed in [6,7]. The numerical results presented below were all obtained by using Matlab in a desktop that has a Core 2 Quad with 4GB of Ram memory.

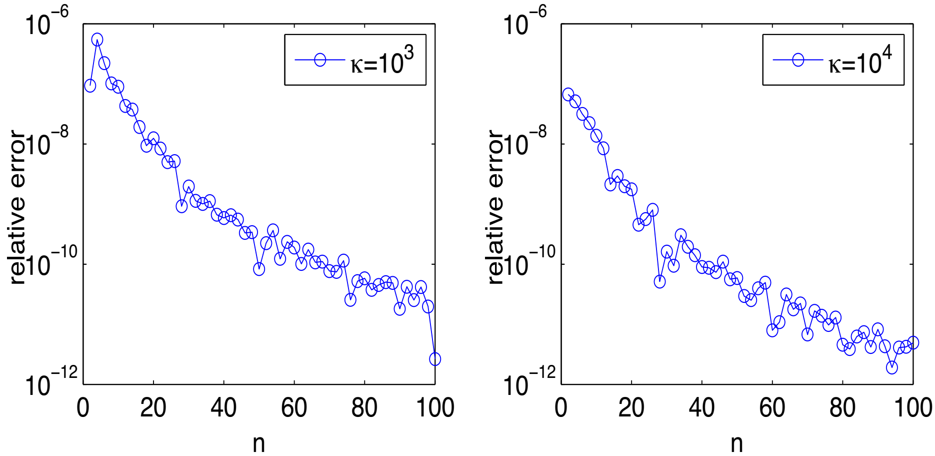
In the first example, we test the CMFP formula (3.5) for computing an oscillatory integral with a smooth integrand and a linear oscillator.

Example 6.1. The purpose of this example is to confirm the estimate presented in Proposition 3.1 for the CMFP formula (3.5). In this example, we consider the case $f(x) := \mathrm{e}^x$ and $g(x) := x$ for $x \in I$. The exact value of the corresponding integral can be computed exactly.

We present in Table 6.1 the relative error (RE) of $\mathcal{Q}_{n,4}$ and the number $\mathcal{N}$ of functional evaluations used in the formula for different values of κ and different choices of n. We plot in Figure 6.1 the RE values of the quadrature formula for (A) $\kappa = 10^3$ and for (B) $\kappa = 10^4$. Figure 6.1 and Table 6.1 confirm that for a fixed κ the approximation accuracy of the quadrature formula increases as n grows and for a fixed n it increases as κ grows.

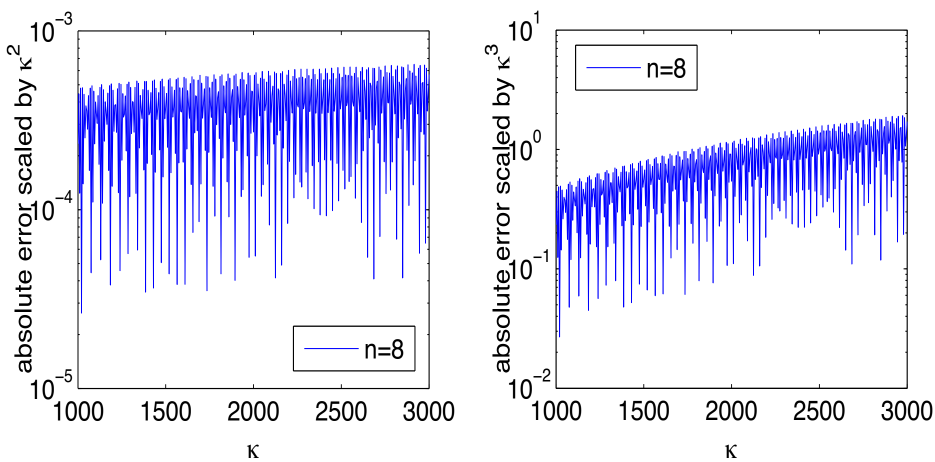
Table 6.1. Numerical results of CMFP for $f(x) := \mathrm{e}^x$ and $g(x) := x$

κ	$n=5$		$n=15$		$n=20$		$n=30$	
	RE	$\mathcal{N}$	RE	$\mathcal{N}$	RE	$\mathcal{N}$	RE	$\mathcal{N}$
10^2	4.68e-07	21	8.93e-08	61	4.13e-08	81	1.15e-08	121
10^3	2.43e-07	21	2.21e-09	61	1.24e-08	81	1.95e-09	121
10^4	4.62e-08	21	2.99e-09	61	1.77e-09	81	1.62e-10	121
10^5	5.72e-09	21	6.38e-10	61	3.47e-10	81	1.65e-11	121
10^6	1.36e-10	21	2.62e-10	61	1.26e-10	81	2.67e-12	121
10^7	4.54e-11	21	1.51e-11	61	5.93e-12	81	2.79e-12	121



A: RE of $\mathcal{Q}_{n,4}$ for $\kappa = 10^3$ B: RE of $\mathcal{Q}_{n,4}$ for $\kappa = 10^4$

FIGURE 6.1. RE of $\mathcal{Q}_{n,4}$ for $f(x) := \mathrm{e}^x$ and $g(x) := x$



A: $\kappa^2\mathcal{E}_{8,4}$ B: $\kappa^3\mathcal{E}_{8,4}$

FIGURE 6.2. Error $\mathcal{E}_{8,4}$ scaled by κ^2 and κ^3 for $f(x) := \mathrm{e}^x$ and $g(x) := x$

We verify in Figure 6.2 the order in $1/\kappa$ of the error $\mathcal{E}_{8,4}$. We plot the error $\mathcal{E}_{8,4}$ scaled by κ^2 in (A) and that scaled by κ^3 in (B), for $n = 8$ with κ changing from 1000 to 3000. Comparing (A) and (B) of Figure 6.2, we observe that the decay of the error $\mathcal{E}_{8,4}$ is faster than $\mathcal{O}(\kappa^{-2})$, but is slower than $\mathcal{O}(\kappa^{-4})$.

We next compare the performance of the proposed quadrature formulas with that of the existing formulas described in [6, 7]. To this end, we recall the quadrature formulas of [6, 7]. By $\mu \in (-1, 1)$ we denote the index of singularity of f, by $N \in \mathbb{N}$ we denote the degree of the polynomial interpolant, by $q > 0$ we denote the grading parameter and by $M \in \mathbb{N}$ we denote the number of the subintervals. The Filon-Clenshaw-Curtis (FCC) rule proposed in [7] for the integral $\mathcal{I}^{[-1,1]}$ with a smooth integrand f and a linear oscillator g was designed by approximating f by its Lagrange interpolation $\mathcal{P}_N(f)$ of degree N at the Clenshaw-Curtis points $t_j := \cos(j\pi/N)$ for $j \in \mathbb{Z}_N$. The FCC rule for the integral (2.1) with a smooth integrand and a linear oscillator $g(x) = x$ is given by

$$\tilde{\mathcal{Q}}_N := \mathrm{e}^{\mathrm{i}\kappa/2}/2 \int_I \mathcal{P}_N(\tilde{f})(x)\mathrm{e}^{\mathrm{i}\kappa x/2}\mathrm{d}x,$$

where $\tilde{f}(x) := f((x+1)/2)$ for $x \in [-1, 1]$. When f has an integrable singularity at the origin, a composite Filon-Clenshaw-Curtis (CFCC) quadrature rule was proposed in [6]. Specifically, a positive integer M is chosen, a number q is chosen to satisfy the condition $q > (N + 1 - r)/(1 + \mu - r)$ for some $0 \le r < 1 + \mu$, and a partition of the interval I is formed with the nodes $x_j := M^{-q}j^q$ for $j \in \mathbb{Z}_M$. On all (but the first) subintervals of the partition, the FCC rule is applied. The treatment of the first subinterval depends on the value of the parameter μ. For $\mu \in (-1, 0]$, the integral $\mathcal{I}^{[x_0,x_1]}$ is approximated by zero. For $\mu \in (0, 1)$ with $x_1\kappa \ge 1$, the linear function interpolating the two points $(x_0, f(x_0))$ and $(x_1, f(x_1))$ is used to approximate the function f on $[x_0, x_1]$, while for $\mu \in (0, 1)$ with $x_1\kappa < 1$, the integrand of (2.1) on $[x_0, x_1]$ is approximated by the linear function interpolating $(x_0, f(x_0)\mathrm{e}^{\mathrm{i}\kappa g(x_0)})$ and $(x_1, f(x_1)\mathrm{e}^{\mathrm{i}\kappa g(x_1)})$. The FCC rule with the same number $N+1$ of points is used to calculate the integrals defined on the subintervals $[x_j, x_{j+1}]$ for $j \in \mathbb{Z}^+_{M-1}$. When g is a strictly increasing nonlinear function and has a single stationary point at zero, we calculate the integral (2.3) instead by a change of variables $g(x) \mapsto x$ and mapping $[g(0), g(1)]$ onto I. We denote the CFCC rule by $\tilde{\mathcal{Q}}_{N,M,q}$. The FCC and CFCC rules were implemented by using the Matlab code available in [5].

To compute the errors to be presented in Examples 6.2, 6.3, 6.5 and 6.6, we evaluate the true values of the integrals by using the Matlab symbolic toolbox. We then use these true values in computing the errors of the quadrature rules.

In the next two examples, we test the efficiency of the quadrature rule proposed in Section 4 for calculating the oscillatory integrals with singular f and with/without a stationary point of g.

Example 6.2. This example is designed to verify the estimates given in Theorems 4.5 and 4.8 for the CMFP formula $\mathcal{Q}^{s,\widetilde{m}}_{n,m}$. We consider the cases $f_1(x) := x^{1/2}$, $f_2(x) := \ln x$, $f_3(x) := x^{-1/2}$ and $g(x) := x$ for $x \in I$.

Numerical results of this example are presented in Tables 6.2, 6.3 and 6.4. We compare the RE values of the formula $\mathcal{Q}^{n,4}_{n,4}$ and the CFCC formula $\tilde{\mathcal{Q}}_{6,M,8}$ for f_1 in Table 6.2, those of the formula $\mathcal{Q}^{n,4}_{n,4}$ and the CFCC formula $\tilde{\mathcal{Q}}_{6,M,12}$ for f_2 in Table 6.3, and those of the formula $\mathcal{Q}^{n,4}_{n,4}$ and the CFCC formula $\tilde{\mathcal{Q}}_{6,M,16}$ for f_3 in Table 6.4. These numerical results show that overall the CMFP formula has a better approximation accuracy than the CFCC rule.

TABLE 6.2. Comparison of CMFP and CFCC for $f_1(x) := x^{1/2}$ and $g(x) := x$

κ	CMFP				CFCC			
	$n=5$		$n=10$		$M=7$		$M=14$	
	RE	$\mathcal{N}$	RE	$\mathcal{N}$	RE	$\mathcal{N}$	RE	$\mathcal{N}$
10^2	6.25e-05	37	4.33e-06	77	3.84e-05	38	7.21e-07	80
10^3	1.78e-04	37	3.82e-06	77	9.93e-05	38	1.79e-06	80
10^4	1.05e-04	37	5.68e-06	77	1.60e-04	38	1.83e-06	80
10^5	8.28e-05	37	1.97e-06	77	7.26e-05	38	3.90e-06	80
10^6	3.21e-05	37	5.68e-06	77	2.17e-04	38	3.50e-06	80
10^7	3.02e-05	37	2.55e-06	77	3.84e-04	38	8.62e-06	80

TABLE 6.3. Comparison of CMFP and CFCC for $f_2(x) := \ln x$ and $g(x) := x$

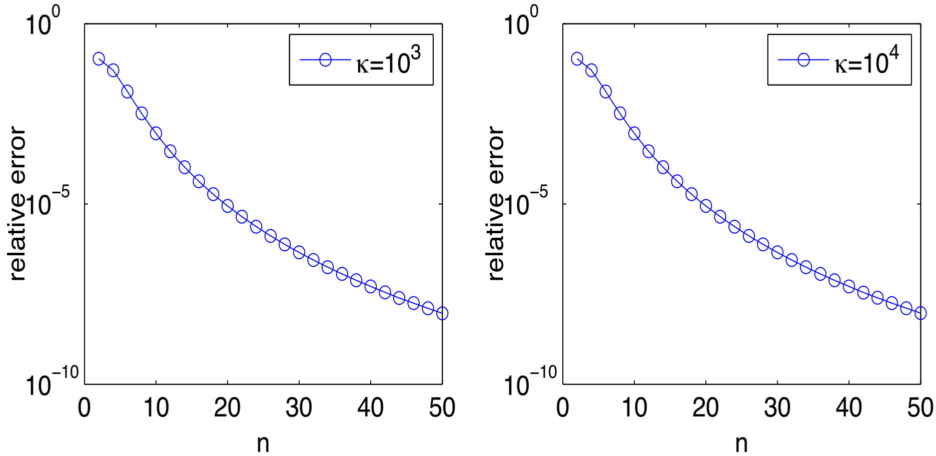
κ	CMFP				CFCC			
	$n=5$		$n=10$		$M=7$		$M=14$	
	RE	$\mathcal{N}$	RE	$\mathcal{N}$	RE	$\mathcal{N}$	RE	$\mathcal{N}$
10^2	1.09e-04	37	1.09e-05	77	4.10e-03	37	2.16e-05	79
10^3	1.68e-03	37	1.13e-05	77	8.81e-03	37	7.40e-05	79
10^4	2.90e-03	37	1.05e-04	77	1.59e-02	37	1.55e-04	79
10^5	6.23e-03	37	1.42e-04	77	5.14e-03	37	2.46e-04	79
10^6	1.08e-02	37	9.17e-04	77	4.76e-02	37	3.99e-04	79
10^7	1.60e-02	37	1.33e-03	77	1.53e-02	37	1.56e-03	79

TABLE 6.4. Comparison of CMFP and CFCC for $f_3(x) := x^{-1/2}$ and $g(x) := x$

κ	CMFP				CFCC			
	$n=5$		$n=10$		$M=7$		$M=14$	
	RE	$\mathcal{N}$	RE	$\mathcal{N}$	RE	$\mathcal{N}$	RE	$\mathcal{N}$
10^2	2.86e-02	37	9.88e-04	77	1.17e-02	37	1.24e-04	79
10^3	2.50e-02	37	9.18e-04	77	1.01e-02	37	6.54e-04	79
10^4	3.14e-02	37	7.49e-04	77	2.57e-02	37	1.05e-03	79
10^5	3.38e-02	37	5.54e-04	77	4.49e-01	37	4.82e-03	79
10^6	8.55e-02	37	2.22e-03	77	1.12e-01	37	1.70e-02	79
10^7	1.12e-01	37	5.93e-03	77	2.06e+00	37	4.02e-02	79

Example 6.3. This example is to verify the estimates proved in Theorems 4.5 and 4.10 for the CMFP formula $\mathcal{Q}^{s,\widetilde{m}}_{n,m}$. We consider the case $f(x) := x^{-1/2}$ and $g(x) := x^2$ for $x \in I$.

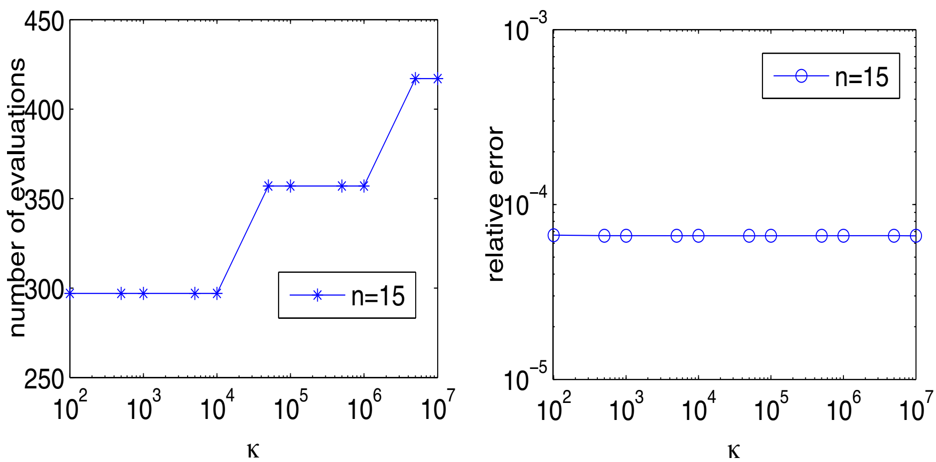
Numerical results of this example are presented in Figures 6.3–6.6 and Table 6.5. We plot in Figure 6.3 the RE values of the formula $\mathcal{Q}^{n,4}_{n,4}$ for (A) $\kappa = 10^3$ and for (B) $\kappa = 10^4$. We present in Figure 6.4 the number $\mathcal{N}$ of functional evaluations used in the formula $\mathcal{Q}^{15,4}_{15,4}$ (A) and the RE values (B), and those of the formula $\mathcal{Q}^{30,4}_{30,4}$ in Figure 6.5. We plot in Figure 6.6 the RE values obtained from the CMFP



A: RE of $\mathcal{Q}_{n,4}^{n,4}$ for $\kappa = 10^3$

B: RE of $\mathcal{Q}_{n,4}^{n,4}$ for $\kappa = 10^4$

FIGURE 6.3. RE of $\mathcal{Q}_{n,4}^{n,4}$ for $f(x) := x^{-1/2}$ and $g(x) := x^2$



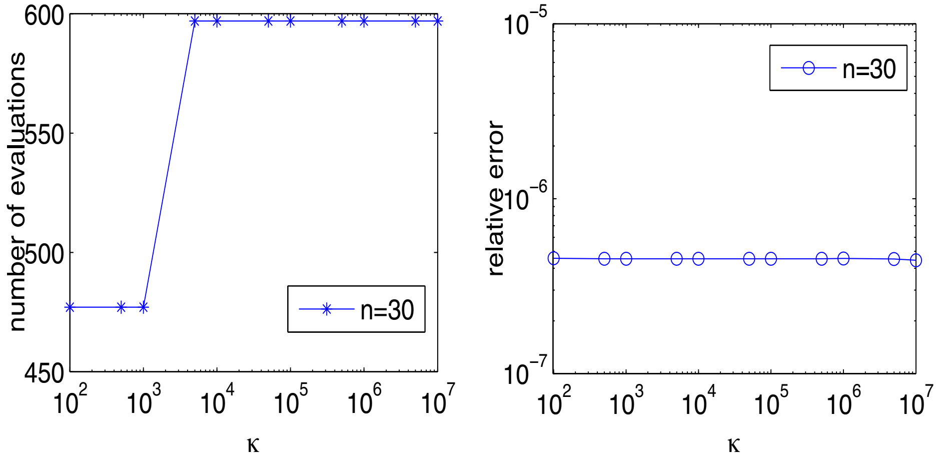
A: $\mathcal{N}(\mathcal{Q}_{n,4}^{n,4})$

B: RE of $\mathcal{Q}_{n,4}^{n,4}$

FIGURE 6.4. $\mathcal{N}(\mathcal{Q}_{n,4}^{n,4})$ and RE of $\mathcal{Q}_{n,4}^{n,4}$ for $f(x) := x^{-1/2}$ and $g(x) := x^2$

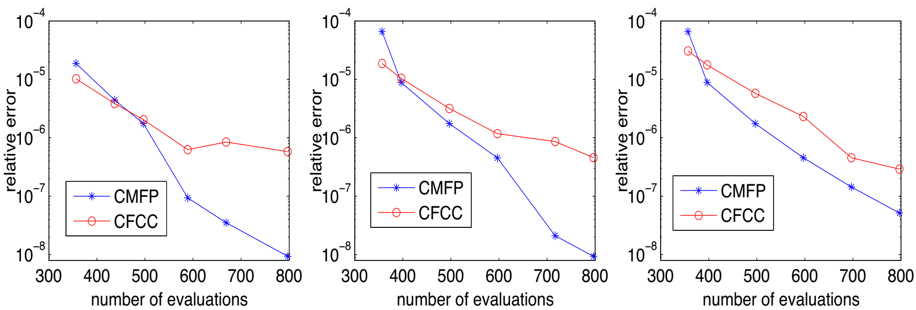
and CFCC formulas by using the same number of functional evaluations for (A) $\kappa = 10^4$, for (B) $\kappa = 10^5$ and for (C) $\kappa = 10^6$. The RE values of the formula $\mathcal{Q}_{n,4}^{n,4}$ and those of the CFCC rule $\tilde{\mathcal{Q}}_{4,M,21}$ are reported in Table 6.5.

From these numerical results, we have the following conclusions. For a fixed κ the approximation accuracy of the CMFP formula increases as n grows. The CMFP formula achieves an accuracy of an order as higher as the CFCC formula and is more effective and convenient for implementation.



A: $\mathcal{N}(\mathcal{Q}_{n,4}^{n,4})$ B: RE of $\mathcal{Q}_{n,4}^{n,4}$

FIGURE 6.5. $\mathcal{N}(\mathcal{Q}_{n,4}^{n,4})$ and RE of $\mathcal{Q}_{n,4}^{n,4}$ for $f(x) := x^{-1/2}$ and $g(x) := x^2$



A: $\kappa = 10^4$ B: $\kappa = 10^5$ C: $\kappa = 10^6$

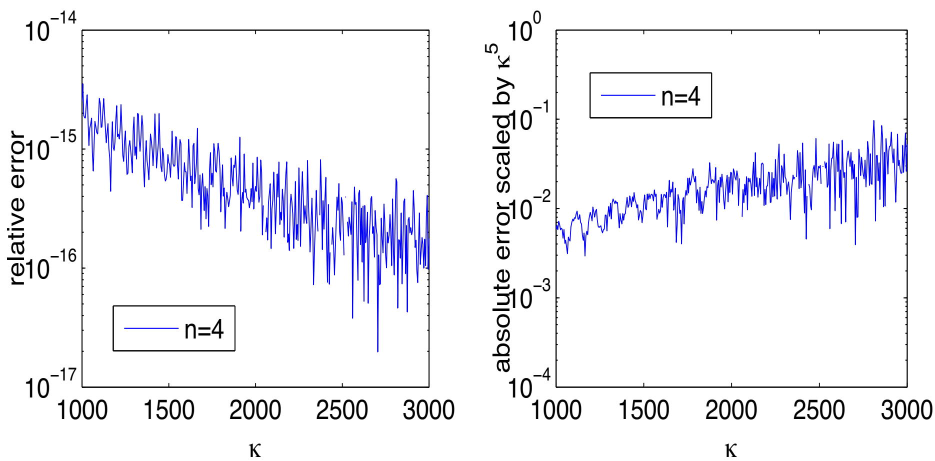
FIGURE 6.6. Comparison of CMFP and CFCC for $f(x) := x^{-1/2}$ and $g(x) := x^2$

TABLE 6.5. Comparison of CMFP and CFCC for $f(x) := x^{-1/2}$ and $g(x) := x^2$

κ	CMFP				CFCC			
	$n = 30$		$n = 50$		$M = 200$		$M = 400$	
	RE	$\mathcal{N}$	RE	$\mathcal{N}$	RE	$\mathcal{N}$	RE	$\mathcal{N}$
10^2	4.56e-07	477	9.39e-09	797	4.03e-07	797	2.71e-07	1597
10^3	4.52e-07	477	9.32e-09	797	4.53e-07	797	3.39e-07	1597
10^4	4.53e-07	597	9.37e-09	797	5.81e-07	797	3.59e-07	1597
10^5	4.53e-07	597	9.32e-09	797	4.57e-07	797	3.91e-07	1597
10^6	4.55e-07	597	9.30e-09	997	2.91e-07	797	3.07e-07	1597
10^7	4.45e-07	597	8.88e-09	997	5.64e-07	797	3.72e-07	1597

TABLE 6.6. Numerical results of CMFE for $f(x) := \mathrm{e}^x$ and $g(x) := x$

κ	$n=4$		$n=5$	
	RE	$\mathcal{N}$	RE	$\mathcal{N}$
10^2	9.65e-14	26	2.24e-13	47
10^3	2.17e-15	26	6.39e-15	47
10^4	7.36e-17	26	1.62e-15	47
10^5	1.14e-17	26	5.70e-17	47
10^6	1.30e-16	26	2.61e-16	47
10^7	1.50e-16	26	2.32e-16	47



A: RE of $\mathcal{Q}_n$ B: $\kappa^{n+1}\mathcal{E}_n$

FIGURE 6.7. RE and error $\mathcal{E}_n$ scaled by κ^{n+1} of Example 6.4

The numerical results of these three examples indicate that the CMFP formulas are stable, easy to implement and computationally inexpensive. For the irregular oscillator, the performance of the CMFP formulas is superior to that of the FCC and CFCC formulas. The advantage of the proposed formulas is especially obvious for the oscillator with stationary points. We next show the numerical results calculated by the CMFE formulas.

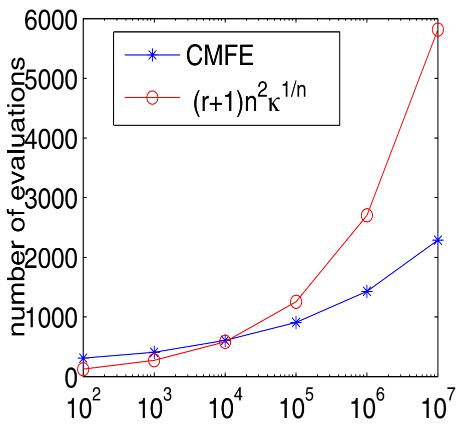
In Example 6.4, we test the CMFE formula (3.7) for computing the oscillatory integral considered in Example 6.1 and compare it with the CMFP formula (3.5).

Example 6.4. This example is to confirm the estimate shown in Theorem 3.5 for the CMFE formula (3.7). We consider the case $f(x) := \mathrm{e}^x$ and $g(x) := x$ for $x \in I$, the same as that in Example 6.1.

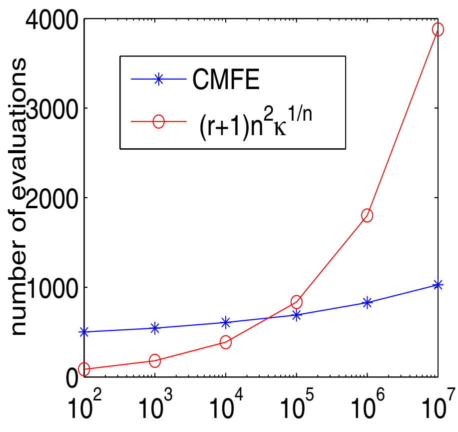
Numerical results of this example are reported in Table 6.6 and Figure 6.7. We list in Table 6.6 the RE values of the formula $\mathcal{Q}_n$. We depict the RE values of the formula $\mathcal{Q}_n$ in Figure 6.7 (A) and the error $\mathcal{E}_n$ scaled by κ^{n+1} in Figure 6.7 (B) for $n = 4$ with κ changing from 1000 to 3000. From Table 6.6 and Figure 6.7, we observe that the approximation accuracy of the formula $\mathcal{Q}_n$ increases as κ grows for a fixed n and the asymptotic order of convergence is $\mathcal{O}(\kappa^{-n-1})$ for $n = 4$. It concurs with the theoretical estimate. We see in Figure 6.7 (B) that the values

TABLE 6.7. Comparison of CMFE and CFCC for $f(x) := 1$ and $g(x) := x^3$

κ	CMFE				CFCC			
	$n = 3$		$n = 4$		$\tilde{\mathcal{Q}}_{4,400,16}$		$\tilde{\mathcal{Q}}_{12,50,36}$	
	RE	$\mathcal{N}$	RE	$\mathcal{N}$	RE	$\mathcal{N}$	RE	$\mathcal{N}$
10^2	3.52e-09	307	9.51e-11	365	4.46e-07	1597	5.53e-11	589
10^3	1.24e-09	407	1.32e-10	467	5.31e-07	1597	1.53e-10	589
10^4	1.56e-10	607	9.31e-11	603	5.32e-07	1597	6.06e-10	589
10^5	7.94e-11	907	2.25e-10	841	5.83e-07	1597	1.94e-10	589
10^6	7.08e-11	1427	3.86e-10	1156	7.22e-07	1597	1.29e-09	589
10^7	7.18e-11	2287	7.21e-10	1657	6.10e-07	1597	2.36e-09	589



A: $\mathcal{N}(\mathcal{Q}_3^{7,0.02})$



B: $\mathcal{N}(\mathcal{Q}_3^{12,0.02})$

FIGURE 6.8. The number of evaluations $\mathcal{N}$ for Examples 6.5 and 6.6

$\kappa^{n+1}\mathcal{E}_n$ for $n = 4$ have a slight increase as κ grows with 1 as their upper bound. A reason for the increase of these values as κ grows shown in Figures 6.2 and 6.7 (B) is that the approximation accuracy of the formulas when κ is large enough is close to the machine accuracy. Nevertheless, this indicates that the decay of the error $\mathcal{E}_n$ for $n = 4$ will not be faster than $\mathcal{O}(\kappa^{-n-1})$.

In the following two examples, we validate the efficiency of the quadrature rule proposed in Section 5 for calculating the oscillatory integrals with a stationary point, without/with singular f.

Example 6.5. This example is to verify the estimates established in Theorems 5.6 and 5.12 for the CMFE formula $\mathcal{Q}_n^{s,\gamma}$. We consider the case $f(x) := 1$ and $g(x) := x^3$ for $x \in I$.

Numerical results of this example are presented in Table 6.7 and Figure 6.8 (A). We list in Table 6.7 the RE values of the formula $\mathcal{Q}_n^{7,0.02}$ and those of the CFCC formulas $\tilde{\mathcal{Q}}_{4,400,16}$ and $\tilde{\mathcal{Q}}_{12,50,36}$. In (A) of Figure 6.8, we compare $\mathcal{N}(\mathcal{Q}_3^{7,0.02})$ with $(r+1)n^2\kappa^{1/n}$ for $r = 2$ and $n = 3$.

TABLE 6.8. Comparison of CMFE and CFCC for $f(x) := x^{-1/2}$ and $g(x) := x^2$

κ	CMFE				CFCC			
	$n=3$		$n=4$		$M=50$		$M=100$	
	RE	$\mathcal{N}$	RE	$\mathcal{N}$	RE	$\mathcal{N}$	RE	$\mathcal{N}$
10^2	9.26e-08	502	3.42e-09	537	2.22e-07	393	3.40e-10	793
10^3	2.94e-08	544	2.19e-09	572	3.52e-07	393	7.67e-10	793
10^4	5.47e-08	607	3.58e-09	642	5.95e-07	393	1.53e-09	793
10^5	4.54e-08	691	3.10e-09	712	1.49e-06	393	2.42e-09	793
10^6	1.90e-08	829	2.12e-09	817	1.61e-06	393	3.92e-09	793
10^7	1.09e-08	1027	4.12e-09	922	5.06e-06	393	8.26e-09	793

We conclude from the results in Table 6.7 and Figure 6.8 (A) that for a fixed n the approximation accuracy of the formula $\mathcal{Q}_n^{7,0.02}$ increases as κ grows and $\mathcal{N}(\mathcal{Q}_n^{7,0.02})$ increases slowly as κ grows. Furthermore, from the results presented in Table 6.7, we observe that for $n = 3$, 4, quadrature formula $\mathcal{Q}_n^{7,0.02}$ has better approximation accuracy than the low order CFCC rule $\tilde{\mathcal{Q}}_{4,400,16}$, and has the same approximation accuracy as the higher order CFCC rule $\tilde{\mathcal{Q}}_{12,50,36}$. In fact, from Figure 6.8 (A), we see that the number of functional evaluations used in formula $\mathcal{Q}_3^{7,0.02}$ is significantly less than $27\kappa^{1/3}$ for large κ.

Example 6.6. The purpose of this example is to test the estimates proved in Theorems 5.6 and 5.12 for the CMFE formula $\mathcal{Q}_n^{s,\gamma}$. We consider the case $f(x) := x^{-1/2}$ and $g(x) := x^2$ for $x \in I$, the same as that in Example 6.3.

We list in Table 6.8 the RE values of the formula $\mathcal{Q}_n^{12,0.02}$ and those of the formula $\tilde{\mathcal{Q}}_{8,M,36}$. In (B) of Figure 6.8, we compare $\mathcal{N}(\mathcal{Q}_3^{12,0.02})$ with $(r+1)n^2\kappa^{1/n}$ for $r = 1$ and $n = 3$.

From Table 6.8, we conclude that the accuracy of $\mathcal{Q}_n^{12,0.02}$ is as high as that of the CFCC rule. From Figure 6.8, we see that the number of functional evaluations used in $Q_n^{12,0.02}$ remains almost constant when κ increases from 10^2 to 10^7.

From Examples 6.5 and 6.6, we observe that the CMFE formula proposed in Section 5 has higher order of approximation accuracy than the low order CFCC formula when the integrand has a singularity with the index $\mu < 0$ and a stationary point of the order $r > 0$. Moreover, the CMFE formula does not have to compute g^{-1} for the nonlinear oscillator g (which is required by the FCC and CFCC rules) and as a result, they can save enormous amount of computing time in comparing with the FCC and CFCC formulas.

In the next example, we compare the CPU time spent using the CMFE formulas, the FCC and CFCC formulas when computing oscillatory integrals with a nonlinear oscillator. The CPU time is monitored by using *tic* and *toc* of Matlab. The computation of the change of variables $g(x) \mapsto x$ is carried out using *fzero* of Matlab.

Example 6.7. This example compares the CPU time spent when computing the integral $\mathcal{I}$ by using the CMFE formulas, the FCC and CFCC rules. We consider the cases $f_1(x) := 1$, $f_2(x) := \ln x$ and $g(x) := (\sin(\pi x/2) + 2x)/3$ for $x \in I$.

TABLE 6.9. Comparison of CPU time for $f_1(x) := 1$ and $g(x) := (\sin(\pi x/2) + 2x)/3$

κ	CMFE			FCC		
	Time (sec.)	Approximation	$\mathcal{N}$	Time (sec.)	Approximation	$\mathcal{N}$
10^2	3.34e-02	-7.35e-03-(4.66e-03)i	12	2.84e-01	-7.35e-03-(4.66e-03)i	9
10^3	3.35e-02	1.24e-03-(1.12e-06)i	12	2.97e-01	1.24e-03-(1.12e-06)i	9
10^4	3.31e-02	-4.59e-05+(2.27e-04)i	12	2.85e-01	-4.59e-05+(2.27e-04)i	9
10^5	3.33e-02	5.36e-07+(2.34e-05)i	12	2.83e-01	5.36e-07+(2.34e-05)i	9
10^6	3.31e-02	-5.25e-07-(5.65e-07)i	12	2.84e-01	-5.25e-07-(5.65e-07)i	9
10^7	3.35e-02	6.31e-08+(2.20e-07)i	12	2.85e-01	6.31e-08+(2.20e-07)i	9

TABLE 6.10. Comparison of CPU time for $f_2(x) := \ln x$ and $g(x) := (\sin(\pi x/2) + 2x)/3$

κ	CMFE			CFCC		
	Time (sec.)	Approximation	$\mathcal{N}$	Time (sec.)	Approximation	$\mathcal{N}$
10^2	4.09e-02	-1.30e-02-(4.51e-02)i	88	3.90e-01	-1.30e-02-(4.51e-02)i	73
10^3	4.10e-02	-1.31e-03-(6.43e-03)i	88	3.95e-01	-1.32e-03-(6.43e-03)i	73
10^4	4.10e-02	-1.32e-04-(8.37e-04)i	88	3.96e-01	-1.32e-04-(8.37e-04)i	73
10^5	4.10e-02	-1.32e-05-(1.03e-04)i	88	3.63e-01	-1.32e-05-(1.03e-04)i	73
10^6	4.15e-02	-1.32e-06-(1.22e-05)i	88	3.64e-01	-1.29e-06-(1.22e-05)i	73
10^7	4.16e-02	-1.32e-07-(1.42e-06)i	88	3.94e-01	-1.32e-07-(1.42e-06)i	73

Table 6.9 lists approximation values produced by the CMFE formula $\mathcal{Q}_3$ and the FCC formula $\tilde{\mathcal{Q}}_8$ for f_1 and computing time they consume, while Table 6.10 lists approximation values produced by the CMFE formula $\mathcal{Q}_6^{6,0.02}$ and the CFCC formula $\tilde{\mathcal{Q}}_{8,10,12}$ for f_2 and the computing time that they consume. Both tables show that the CMFE formula consumes significantly less CPU time than the FCC, CFCC formulas when they produce comparable approximation results.

7. Concluding remarks

We develop in this paper composite quadrature formulas for computing highly oscillatory integrals defined on a finite interval with both singularities and stationary points. The partitions of the integration interval used in the composite quadrature formulas are designed according to the degree of oscillation and the singularity. We subdivide the integration interval in the way that the resulting composite quadrature formulas have "equal-errors" on the subintervals. In each of the subintervals, we use polynomial interpolants to approximate the integrand to form two classes of formulas having polynomial (resp. exponential) order of convergence by using fixed (resp. variable) numbers of interpolation nodes. Numerical results show that the proposed formulas outperform the existing methods in both approximation accuracy and computational efficiency.

Finally, for easy reference, we summarize in Table 7.1 the quadrature formulas proposed in this paper.

Table 7.1. Quadrature formulas

<table>
<tr><th>$\mathcal{I}$</th><th colspan="2">Quadrature</th><th>Partition on I or Λ</th><th># of nodes in each subinterval</th><th>Parameters $M_j = \max\{\lvert g'(x_{j-1})\rvert, \lvert g'(x_j)\rvert\}$, for $j \in \mathbb{Z}_n^+$</th></tr>
<tr><td rowspan="2">no singularity, no inflection point, $r = 0$</td><td colspan="2">$\mathcal{Q}_{n,m}$</td><td>$x_0 = 0$, $x_j = \kappa^{(j-1)/(n-1)-1}$, $j \in \mathbb{Z}_n^+$</td><td>$m+1$</td><td>$N_j = \lceil M_j \rceil$, $j \in \mathbb{Z}_n^+$, $m \in \mathbb{N}$</td></tr>
<tr><td colspan="2">$\mathcal{Q}_n$</td><td>$x_0 = 0$, $x_j = \kappa^{(j-1)/(n-1)-1}$, $j \in \mathbb{Z}_n^+$</td><td>$m_j + 1$</td><td>$N_j = \lceil M_j \rceil$, $m_j = \left\lceil \frac{n(n-1)}{n+1-j} \right\rceil$, $j \in \mathbb{Z}_n^+$</td></tr>
<tr><td rowspan="4">no inflection point, $\mu \in (-1,1)$, $r \in \mathbb{N}_0$</td><td rowspan="2">$\mathcal{Q}_{n,m}^{s,\widetilde{m}}$</td><td>$\mathcal{Q}_{\mu,\widetilde{m}}^{s}$</td><td>$x_j = s^{-p} j^p$, $j \in \mathbb{Z}_s$</td><td>$\widetilde{m}$</td><td>$p = \frac{2\widetilde{m}+1}{1+\mu}$, $\widetilde{m} \in \mathbb{N}$</td></tr>
<tr><td>$\mathcal{Q}_{n,m}^{\Lambda}$</td><td>$x_j = \kappa_r^{(j/n-1)/(r+1)}$, $j \in \mathbb{Z}_n$</td><td>$m+1$</td><td>$N_j = \lfloor q_j \rfloor$, for $r = 0$, $N_j = \left\lceil q_j^{m/(m-1)} \right\rceil$, for $r > 0$, $q_j = M_j x_{j-1}/g(x_{j-1})$, $j \in \mathbb{Z}_n^+$, $m \in \mathbb{N}$</td></tr>
<tr><td rowspan="2">$\mathcal{Q}_n^{s,\gamma}$</td><td>$\mathcal{Q}_{\gamma}^{s}$</td><td>$x_0 = 0$, $x_j = \gamma^{s-j}$, $j \in \mathbb{Z}_s^+$</td><td>m_j</td><td>$\gamma \in (0,1)$, $m_j := \lceil j\varepsilon \rceil$, $j \in \mathbb{Z}_{s-1}^+$, $\varepsilon > 1$</td></tr>
<tr><td>$\mathcal{Q}_n^{\Lambda}$</td><td>$x_j = \kappa_r^{(j/n-1)/(r+1)}$, $j \in \mathbb{Z}_n$</td><td>$m_j + 1$</td><td>$N_j = \lfloor q_j \rfloor$, $q_j = M_j x_{j-1}/g(x_{j-1})$, $m_j = n + \lceil \frac{n+1-j}{1-\nu} \rceil$, $j \in \mathbb{Z}_n^+$, ν is the index of singularity of Θ</td></tr>
</table>

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