USING KATSURADA'S DETERMINATION OF THE EISENSTEIN SERIES TO COMPUTE SIEGEL EIGENFORMS

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Abstract. We compute Hecke eigenform bases of spaces of level one, degree three Siegel modular forms and 2-Euler factors of the eigenforms through weight 22. Our method uses the Fourier coefficients of Siegel Eisenstein series, which are fully known and computationally tractable by the work of H. Katsurada; we also use P. Garrett's decomposition of the pullback of the Eisenstein series through the Witt map. Our results support I. Miyawaki's conjectural lift, and they give examples of eigenforms that are congruence neighbors.

1. Introduction

Eisenstein series are central in the theory of Siegel modular forms. The algorithmic computation of the Fourier coefficients of Siegel Eisenstein series began with C. L. Siegel [24] and was completed by H. Katsurada [14], whose work deserves to be widely known. For any positive integer degree n and even integer weight $k > n + 1$, the Siegel Eisenstein series of weight k and degree n is

$$E_k^{(n)}(z) = \sum_{\gamma \in P_{\mathbb{Z}} \backslash \Gamma_n} j(\gamma, z)^{-k}.$$

Here z lies in the Siegel upper half-space $\mathcal{H}_n$, and the summand $j(\gamma, z)^{-k}$ is 1 for the Siegel parabolic subgroup $P_{\mathbb{Z}} = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \right\}$ of the integral symplectic group $\Gamma_n = \mathrm{Sp}_n(\mathbb{Z})$. (Section 2 will review the background for this paper.) This Eisenstein series has the Fourier series representation

$$E_k^{(n)}(z) = \sum_{t \in \mathcal{X}_n^{\mathrm{semi}}} a(t; E_k^{(n)}) \, \mathrm{e}(\langle t, z \rangle),$$

where $\mathcal{X}_n^{\mathrm{semi}}$ denotes the set of semi-integral positive semidefinite n-by-n matrices. The Siegel Φ map takes Eisenstein series to Eisenstein series, $\Phi E_k^{(n)} = E_k^{(n-1)}$ and $\Phi E_k^{(1)} = 1$, so it suffices to compute the Fourier coefficients of Eisenstein series for definite indices t; the set of such matrices is denoted $\mathcal{X}_n$. The Fourier coefficient formula for definite indices (in which ζ is the Euler–Riemann zeta function, and $L(\chi_{D_t}, \cdot)$ is the Dirichlet L-function of the quadratic character of the fundamental discriminant D_t that will be explained below, and F_p is a polynomial to be discussed below) is

$$a(t; E_k^{(n)}) = \frac{2^{\lfloor \frac{n+1}{2} \rfloor} \prod_p F_p(t, p^{k-n-1})}{\zeta(1-k) \prod_{i=1}^{\lfloor n/2 \rfloor} \zeta(1-2k+2i)} \cdot \begin{cases} L(\chi_{D_t}, 1-k+n/2), & n \text{ even,} \\ 1, & n \text{ odd.} \end{cases}$$

The Fourier coefficient depends only on the genus of its index t. In fact the polynomial $F_p(t, X) \in \mathbb{Z}[X]$ depends only on the class of t over $\mathbb{Z}_p$. Algorithmic specification of these F_p polynomials was the last impediment to evaluating Siegel Eisenstein series Fourier coefficients, and it was overcome by recursion relations due to Katsurada [14]. This article explains how the Fourier coefficients of Siegel Eisenstein series are computationally accessible, and it makes computer programs to evaluate them publicly available.

For example, consider the Fourier coefficient index

$$t = \frac{1}{2} \begin{bmatrix} 2 & 1 & 1 & 0 & 1 & 2 \\ 1 & 4 & 2 & 2 & 0 & 1 \\ 1 & 2 & 4 & 2 & 0 & 0 \\ 0 & 2 & 2 & 4 & 2 & 2 \\ 1 & 0 & 0 & 2 & 4 & 2 \\ 2 & 1 & 0 & 2 & 2 & 8 \end{bmatrix} \in \mathcal{X}_6.$$

Our genus symbol program takes $2t$ as an input and returns the genus symbol $4_4^{-2}3^{-1}$. Our F_p polynomial program takes this genus symbol and the determinant $\det(2t) = 48$ as input and returns the $F_p(t, X)$ polynomials for all $p \mid 2\det(2t)$,

$$[F_2(t, X), F_3(t, X)] = \left[1 + 24X + 256X^2 + 3072X^3 + 16384X^4, 1\right].$$

These data make no reference to any particular Eisenstein series degree or weight. With these F_p polynomials and the weight $k = 16$ as input, our Eisenstein series Fourier coefficient program returns

$$a(t; E_{16}^{(6)}) = \frac{97801546544081473702552608817115200}{13912726954911229324966739363569}.$$

We hope that our programs [28] will be useful to researchers. The programs are in various computer languages, including Mathematica.

This article describes our computation of the 2-Euler factors of Hecke eigenform bases of the degree 3 cusp form spaces $\mathcal{S}_k(\Gamma_3)$ for even k up through 22. Along with Katsurada's completion of the Siegel Eisenstein series Fourier coefficient formula, which makes a coefficient readily computable from a genus symbol of its index, the second idea of our method is P. Garrett's decomposition of the pulled back Eisenstein series [8]. Thus we call our algorithm the pullback-genus method.

Our computation naturally continues the computational work in [20], where I. Miyawaki studied the weights $k = 12, 14$ and conjectured two kinds of lift in consequence of his results. The standard L-function part of Miyawaki's first lifting conjecture with an additional natural nonvanishing condition has been settled by T. Ikeda [13], who gave a construction in general degree that generalizes the degree 3 case; we call these lifts Ikeda–Miyawaki lifts. Miyawaki's second lifting conjecture is still open. Our computations in weights 16 through 22 support the conjecture, exhibiting eigenforms whose 2-Euler factors agree with all of Miyawaki's predicted lifts of the second type. Some of our results were announced in [12], which forward referenced this article.

Besides the Ikeda-Miyawaki lifts and the apparent Miyawaki lifts, we see apparent nonlift eigenforms that have unimodular Satake parameters at 2. So far, each nonlift eigenform has at least one lift eigenform as a *congruence neighbor*. Two eigenforms f and g are congruence neighbors if all their eigenvalues are congruent modulo some prime ideal $\mathfrak{p}$ in the ring of integers of the field of eigenvalues, and the

k	$\dim(\mathcal{S}_k(\Gamma_3))$	M I lifts	Apparent M II lifts	Apparent nonlifts
16	3	2		1
18	4	2	2	
20	6	3	2	1
22	9	3	3	3

FIGURE 1. Siegel cusp eigenforms in degree three

ideal $\mathfrak{p}$ is a *congruence prime*. Thus, although not all eigenforms are lifts, through weight 22 all eigenforms are lifts or congruence neighbors of lifts.

We summarize our results here, but the reader is encouraged to examine the database [28], which includes Fourier coefficients, Euler factors, noncusp forms, and more.

- $\dim(\mathcal{S}_{16}(\Gamma_3)) = 3$, with a conjugate pair $M_{16}^{I}(f_{28,\pm}, g_{16})$ of Ikeda–Miyawaki lifts over the quadratic number field of the $\mathcal{S}_{28}(\mathrm{SL}_2(\mathbb{Z}))$ Hecke eigenbasis, and one apparent nonlift over $\mathbb{Q}$ with unimodular Satake parameters at 2. The apparent nonlift is a congruence neighbor of the Ikeda–Miyawaki lifts modulo a prime over 107. This result was announced in [12].
- $\dim(\mathcal{S}_{18}(\Gamma_3)) = 4$, with a conjugate pair $M_{18}^{I}(f_{32,\pm}, g_{18})$ of Ikeda–Miyawaki lifts over the quadratic number field of the $\mathcal{S}_{32}(\mathrm{SL}_2(\mathbb{Z}))$ Hecke eigenbasis, and a conjugate pair $M_{18}^{II}(f_{34,\pm}, g_{16})$ of apparent Miyawaki II lifts over the quadratic number field of the $\mathcal{S}_{34}(\mathrm{SL}_2(\mathbb{Z}))$ Hecke eigenbasis. This space has no congruence neighbors for large primes.
- $\dim(\mathcal{S}_{20}(\Gamma_3)) = 6$, with a conjugate triple $M_{20}^{I}(f_{36,(1,2,3)}, g_{20})$ of Ikeda–Miyawaki lifts over the cubic number field of the $\mathcal{S}_{36}(\mathrm{SL}_2(\mathbb{Z}))$ Hecke eigenbasis, a conjugate pair $M_{20}^{II}(f_{38,\pm}, g_{18})$ of apparent Miyawaki II lifts over the quadratic number field of the $\mathcal{S}_{38}(\mathrm{SL}_2(\mathbb{Z}))$ Hecke eigenbasis, and one apparent nonlift over $\mathbb{Q}$ with unimodular Satake parameters at 2. The apparent nonlift is a congruence neighbor of the Ikeda–Miyawaki lifts modulo a prime over 157. This result was announced in [12].
- $\dim(\mathcal{S}_{22}(\Gamma_3)) = 9$, with a conjugate triple $M_{22}^{I}(f_{40,(1,2,3)}, g_{22})$ of Ikeda–Miyawaki lifts over the cubic number field of the $\mathcal{S}_{40}(\mathrm{SL}_2(\mathbb{Z}))$ Hecke eigenbasis, a conjugate triple of apparent Miyawaki II lifts $M_{22}^{II}(f_{42,(1,2,3)}, g_{20})$ over the cubic number field of the $\mathcal{S}_{42}(\mathrm{SL}_2(\mathbb{Z}))$ Hecke eigenbasis, and a conjugate triple of apparent nonlifts over a real cubic field with unimodular Satake parameters at 2. The apparent nonlifts are congruence neighbors of the Ikeda–Miyawaki lifts modulo a prime over 1753, and they are congruence neighbors of the apparent Miyawaki II lifts modulo two primes, one over 613 and the other over 677.

A table of these results is given in Figure 1. For computations and conjectures related to the work in this paper, see [3].

Heim [10] has raised the question of whether the twisted spinor L-function associated to a pair of eigenforms $G \in S_k(\Gamma_2)$ and $h \in S_{k-2}(\mathrm{SL}_2(\mathbb{Z}))$ occurs as the spinor L-function of an eigenform in $S_k(\Gamma_3)$. When G is a Saito–Kurokawa lift this is consistent with the Miyawaki II lift, but when G is not a Saito–Kurokawa lift our computations show that there is no corresponding spinor L-function from an eigenform in $\mathcal{S}_k(\Gamma_3)$ for $k \leq 22$.

Section 2 reviews the setting for our computations. Section 3 reviews the Eisenstein series Fourier coefficient formula. Section 4 describes the pullback-genus method, which gives a rational basis of $\mathcal{M}_k(\Gamma_n)$. Section 5 reviews how to compute Hecke eigenform Euler factors. Section 6 explains the pullback-genus method's implementation, which is essential for making these computations tractable.

2. Background

Let n be a positive integer. Let $\mathcal{V}_n$ denote the vector space $\mathrm{M}_n(\mathbb{R})^{\mathrm{sym}}$ of symmetric $n \times n$ real matrices, carrying the inner product $\langle t, u \rangle = \mathrm{tr}(tu)$. Let $\mathcal{C}_n$ denote the cone of positive definite elements of $\mathcal{V}_n$. The Siegel upper half-space of degree n is $\mathcal{H}_n = \{z = x + iy : x \in \mathcal{V}_n,\ y \in \mathcal{C}_n\}$. The symplectic group $\mathrm{Sp}(n)$ is defined by the $2n \times 2n$ matrix condition $J[g] = J$ where $J = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ with each block $n \times n$, and $J[g] = g'Jg$ with g' the transpose of g. The real symplectic group acts on the Siegel space, $g\langle z \rangle = (az + b)(cz + d)^{-1}$ where $g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. For any $g \in \mathrm{Sp}_n(\mathbb{R})$ and any $z \in \mathcal{H}_n$ the factor of automorphy $j(g, z)$ is $\det(cz + d)$. For any integer k, the weight k action $[\cdot]_k$ of $\mathrm{Sp}_n(\mathbb{R})$ on functions $f : \mathcal{H}_n \longrightarrow \mathbb{C}$ is $f[g]_k(z) = j(g, z)^{-k} f(g\langle z \rangle)$.

The integral symplectic group $\mathrm{Sp}_n(\mathbb{Z})$ is denoted Γ_n. For any integer k, a function $f : \mathcal{H}_n \longrightarrow \mathbb{C}$ is a Siegel modular form of weight k and degree n if (1) f is holomorphic, (2) $f[\gamma]_k = f$ for all $\gamma \in \Gamma_n$, (3) for any $y_o \in \mathcal{C}_n$, f is bounded on the set $\{z \in \mathcal{H}_n : \mathrm{Im}(z) > y_o\}$. The vector space of such Siegel modular forms is denoted $\mathcal{M}_k(\Gamma_n)$. For any positive integer n, Siegel's map $\Phi : \mathcal{M}_k(\Gamma_n) \longrightarrow \mathcal{M}_k(\Gamma_{n-1})$ is given by $(\Phi f)(z) = \lim_{\eta \to +\infty} f(\left[\begin{smallmatrix} z & 0 \\ 0 & i\eta \end{smallmatrix}\right])$. The kernel of Siegel's Φ-map on $\mathcal{M}_k(\Gamma_n)$ is the vector space of Siegel cusp forms of degree n and weight k, denoted $\mathcal{S}_k(\Gamma_n)$. By convention $\mathcal{M}_k(\Gamma_0) = \mathcal{S}_k(\Gamma_0) = \mathbb{C}$.

The Euclidean space $\mathcal{V}_n$ contains the integer lattice $\mathcal{V}_n(\mathbb{Z}) = \mathcal{V}_n \cap \mathrm{M}_n(\mathbb{Z}) = \mathrm{M}_n(\mathbb{Z})^{\mathrm{sym}}$, whose dual lattice $\mathcal{V}_n(\mathbb{Z})^*$ is all $v \in \mathcal{V}_n$ such that $v_{ii} \in \mathbb{Z}$ and $v_{ij} \in \frac{1}{2}\mathbb{Z}$ for all i, j. Let $\mathcal{X}_n = \mathcal{V}_n(\mathbb{Z})^* \cap \mathcal{C}_n$ and $\mathcal{X}_n^{\mathrm{semi}} = \mathcal{V}_n(\mathbb{Z})^* \cap \overline{\mathcal{C}}_n$ denote the positive definite elements and the positive semidefinite elements of the dual lattice. Any Siegel modular form has a Fourier series representation, setting $e(x) = e^{2\pi i x}$,

$$f(z) = \sum_{t \in \mathcal{X}_n^{\mathrm{semi}}} a(t; f)\, e(\langle t, z \rangle),$$

and f is a cusp form if and only its Fourier series is supported on $t \in \mathcal{X}_n$.

As already discussed, the Siegel parabolic subgroup of Γ_n-elements $\left[\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right]$ is denoted $P_{\mathbb{Z}}$, and the Siegel Eisenstein series of degree $n \geq 1$ and even weight $k > n + 1$ is $E_k^{(n)}(z) = \sum_{\gamma \in P_{\mathbb{Z}} \backslash \Gamma_n} j(\gamma, z)^{-k}$ for $z \in \mathcal{H}_n$, and the Fourier coefficients $a(t; E_k^{(n)})$ of the Eisenstein series have been fairly well understood since C. L. Siegel's work of 1939 [24] but their practical description for arbitrary n was completed by Katsurada only in 1999 [14]. In general the Fourier coefficients of a Siegel modular form f of even weight are $\mathrm{GL}_n(\mathbb{Z})$-equivalence class functions, meaning that $a(\cdot; f)$ is constant over each class $t[\mathrm{GL}_n(\mathbb{Z})]$. But a key computational point here is that, given any decomposition $t \sim u \oplus 0_{n-m}$ where $u \in \mathcal{X}_m$ is strictly positive, the Eisenstein series Fourier coefficient $a(t; E_k^{(n)})$ is determined by only the genus of u, i.e., by the set of matrices in $\mathcal{X}_m$ that lie in $u[\mathrm{GL}_m(\mathbb{Z}_p)]$ for every prime p. The genus is a coarser equivalence class than $t[\mathrm{GL}_n(\mathbb{Z})]$, and a symbol for it is much faster to compute than the $\mathrm{GL}_n(\mathbb{Z})$-class.

The symplectic similitude group $\mathrm{GSp}(n)$ is defined by the condition $J[g] = m(g)J$ for some invertible multiplier $m(g)$. The rational symplectic positive similitude group $\mathrm{GSp}_n^+(\mathbb{Q})$, carrying the condition $m(g) \in \mathbb{Q}_{>0}$, acts on $\mathcal{H}_n$ via $f[g]_k(z) = m(g)^e j(g,z)^{-k} f(g\langle z \rangle)$, where the classical choice of the multiplier power is $e = kn - \langle n \rangle$ with $\langle n \rangle = n(n+1)/2$. Any double coset in $\Gamma_n \backslash \mathrm{GSp}_n^+(\mathbb{Q})/\Gamma_n$ decomposes as finitely many right cosets, $\Gamma_n g \Gamma_n = \bigsqcup_{i=1}^d \Gamma_n g_i$, and it acts correspondingly on $\mathcal{M}_k(\Gamma_n)$ by $f[\Gamma_n g \Gamma_n]_k(z) = \sum_{i=1}^d f[g_i]_k(z)$. The double cosets generate a commutative algebra over $\mathbb{Q}$, the Hecke algebra $\mathcal{H}(\Gamma_n, \mathrm{GSp}_n^+(\mathbb{Q}))$. The space $\mathcal{M}_k(\Gamma_n)$ has a basis of Hecke eigenforms. Using standard generators of the Hecke algebra, any $f \in \mathcal{M}_k(\Gamma_n)$ is a Hecke eigenform if for each prime it is an eigenform of $T(p) = \Gamma_n \mathrm{diag}(1_n, p1_n)\Gamma_n$ and of $T_i(p^2) = \Gamma_n \mathrm{diag}(1_{n-i}, p1_i, p^2 1_{n-i}, p1_i)\Gamma_n$ for $i = 1, \cdots, n-1$.

Let $z_1 \oplus z_2 = \left[\begin{smallmatrix} z_1 & 0 \\ 0 & z_2 \end{smallmatrix}\right]$ for $z_1, z_2 \in \mathcal{H}_n$. The symplectic embedding $\iota(z_1 \times z_2) = z_1 \oplus z_2$ pulls back to a map of functions, $(\iota^* f)(z_1 \times z_2) = f(z_1 \oplus z_2)$. The pullback ι^* takes $\mathcal{M}_k(\Gamma_{2n})$ to $\mathcal{M}_k(\Gamma_n) \otimes \mathcal{M}_k(\Gamma_n)$ and takes $\mathcal{S}_k(\Gamma_{2n})$ to $\mathcal{S}_k(\Gamma_n) \otimes \mathcal{S}_k(\Gamma_n)$ by results of E. Witt [27]. These maps and variants of them are often called the Witt map. Garrett's formula [8] (originally appearing in a 1979 preprint by Garrett, then in S. Böcherer's Ph.D. directed by H. Klingen and in a paper of M. Harris) decomposes the pulled back Eisenstein series $\iota^* E_k^{(2n)}$ as a sum of nonzero multiples of the diagonal tensor products over a Hecke eigenform basis $\{f_1, \cdots, f_d\}$ of $\mathcal{M}_k(\Gamma_n)$,

$$\iota^* E_k^{(2n)} = \sum_{\ell=1}^d c_\ell \, f_\ell \otimes f_\ell, \quad \text{all } c_\ell \text{ nonzero.}$$

That is, $E_k^{(2n)}(z_1 \oplus z_2) = \sum_\ell c_\ell f_\ell(z_1) f_\ell(z_2)$ for $z_1, z_2 \in \mathcal{H}_n$. This connection between the Hecke eigenform basis and the Eisenstein series is what guarantees that the pullback-genus method works. However, when the dimension of $\mathcal{M}_k(\Gamma_n)$ is already known, the computations can be rigorously executed using only the existence of the Witt map.

Let $\mathbb{Q}[x^{\pm 1}]$ denote the algebra of rational Laurent polynomials in indeterminates $x_0, x_1, \cdots, x_n$. The Weyl group W of this algebra is generated by the permutations of $\{x_1, \cdots, x_n\}$ and the involutions τ_i for $i = 1, \cdots, n$ taking x_0 to $x_0 x_i$ and x_i to x_i^{-1}. Fix a prime p, and let $\mathcal{H}_p = \mathcal{H}(\Gamma_n, \mathrm{GSp}_n^+(\mathbb{Z}[1/p]))$. The Satake isomorphism $\Omega = \Omega_p$ from $\mathcal{H}_p$ to the subalgebra $\mathbb{Q}[x^{\pm 1}]^W$ of Laurent polynomials invariant under the Weyl group is defined on any double coset $\Gamma g \Gamma$ by taking each of its constituent cosets Γb, with Borel subgroup representative $b = \left[\begin{smallmatrix} p^{e_0} d^* & * \\ 0 & d \end{smallmatrix}\right]$ whose d-block has diagonal $(p^{e_1}, \cdots, p^{e_n})$, to $\Omega(\Gamma b) = x_0^{e_0}(x_1/p)^{e_1}(x_2/p^2)^{e_2} \cdots (x_n/p^n)^{e_n}$. For any Siegel Hecke eigenform f, the eigenvalue homomorphism $\lambda_f : \mathcal{H}_p \longrightarrow \mathbb{C}$ is defined by the condition $Tf = \lambda_f(T)f$ for $T \in \mathcal{H}_p$. There exists a Satake parameter $\alpha = \alpha_{f,p} \in \mathbb{C}^{n+1}$ such that $\lambda_f(T) = (\Omega(T))(\alpha_{f,p})$ for all $T \in \mathcal{H}_p$. The standard L-function of a Siegel Hecke eigenform f is the product of Euler factors defined in terms of the Satake parameters. Specifically, for each prime p let the pth Satake parameter be $\alpha_{p,f} = (\alpha_{0,p}, \alpha_{1,p}, \cdots, \alpha_{n,p})$; then $L^{\mathrm{st}}(f,s) = \prod_p Q_p^{\mathrm{st}}(f, p^{-s})^{-1}$ where $Q_p^{\mathrm{st}}(f, X) = (1-X)\prod_{i=1}^n (1 - \alpha_{i,p}X)(1 - \alpha_{i,p}^{-1}X)$ (see for example [2]).

Miyawaki [20] computed for the generator F_{12} of $\mathcal{S}_{12}(\Gamma_3)$ that $L^{\mathrm{st}}(F_{12}, s)$ has the same 2-Euler factor as $L(f_{20}, s+10)L(f_{20}, s+9)L^{\mathrm{st}}(g_{12}, s)$ where f_{20} and g_{12}, respectively, generate $\mathcal{S}_{20}(\mathrm{SL}_2(\mathbb{Z}))$ and $\mathcal{S}_{12}(\mathrm{SL}_2(\mathbb{Z}))$, and he computed for the generator F_{14} of $\mathcal{S}_{14}(\Gamma_3)$ that $L^{\mathrm{st}}(F_{14}, s)$ has the same 2-Euler factor as

$L(f_{26}, s+13)L(f_{26}, s+12)L^{st}(g_{12}, s)$. He conjectured that for any even weight k, and for each pair of elliptic Hecke eigenforms $f \in \mathcal{S}_{2k-4}(\mathrm{SL}_2(\mathbb{Z}))$ and $g \in \mathcal{S}_k(\mathrm{SL}_2(\mathbb{Z}))$, there exists a Siegel Hecke eigenform $F \in \mathcal{S}_k(\Gamma_3)$ whose standard L-function factors as $L^{st}(F, s) = L(f, s+k-2)L(f, s+k-3)L^{st}(g, s)$, and for each pair of elliptic Hecke eigenforms $f \in \mathcal{S}_{2k-2}(\mathrm{SL}_2(\mathbb{Z}))$ and $g \in \mathcal{S}_{k-2}(\mathrm{SL}_2(\mathbb{Z}))$ there exists a Siegel Hecke eigenform $F \in \mathcal{S}_k(\Gamma_3)$ whose standard L-function factors as $L^{st}(F, s) = L(f, s+k-1)L(f, s+k-2)L^{st}(g, s)$. Granting a nonvanishing condition and restricting attention to standard L-functions, Ikeda [13] established a general lift subsuming Miyawaki's first conjectured lift. The Hecke eigenfunction behavior of this lift was shown by Ikeda, Heim [11], and Hayashida [9].

Let $c_k^{(2n)} = 2^{-n}\zeta(1-k)\prod_{i=1}^n \zeta(1-2k+2i)$. The dilated Eisenstein series $\mathcal{E}_k^{(2n)} = c_k^{(2n)}E_k^{(2n)}$ has rational Fourier coefficients that are integral at all primes $p > 2k-1$. Garrett's formula with $\mathcal{E}_k^{(2n)}$ in place of $E_k^{(2n)}$ can quickly show congruences between Hecke eigenform basis elements. Specifically, the Fourier coefficients of a Hecke eigenform can be taken to lie in the integer ring $\mathcal{O}_{\mathbb{K}}$ of a number field $\mathbb{K}$, and the constants c_ℓ in the formula to lie in $\mathbb{K}^{\times}$. Call a maximal ideal $\mathfrak{p}$ of $\mathcal{O}_{\mathbb{K}}$ *big* if it lies over a rational prime $p > 2k-1$ ([12], and see [15,17]). Suppose that there exists a big prime $\mathfrak{p}$ of $\mathcal{O}_{\mathbb{K}}$ such that $\mathrm{ord}_{\mathfrak{p}}(c_1) = \mathrm{ord}_{\mathfrak{p}}(c_2) = -1$ and $\mathrm{ord}_{\mathfrak{p}}(c_\ell) \geq 0$ for $\ell = 3, \cdots, d$. Then for any index s such that $\mathrm{ord}_{\mathfrak{p}}(a(s; f_1)) = 0$, we deduce $f_1/a(s; f_1) = f_2/a(s; f_2) \bmod \mathfrak{p}$, and especially, if $a(s; f_1) = a(s; f_2) = 1$, then $f_1 = f_2 \bmod \mathfrak{p}$. Thus f_1 and f_2 are congruence neighbors and $\mathfrak{p}$ is a congruence prime.

3. Eisenstein series Fourier coefficients

3.1. F_p-polynomials.
Polynomials $F_p(u, X) \in \mathbb{Z}[X]$ for p prime and $u \in \mathcal{X}_m$ appear in the Siegel Eisenstein series Fourier coefficient formula. The first author of this paper wrote a program to compute these polynomials [18], which has since been modified to accept higher degree input. We refer to [14] for the definition of the F_p polynomials; there Katsurada proved a functional equation for these polynomials, which was an important step in his establishment of their recurrence relations. We review this functional equation because it serves as a check on computations. The functional equation makes reference to the Hilbert symbol and to the Hasse invariant. To review, for $a, b \in \mathbb{Q}_p^{\times}$ the Hilbert symbol $(a, b)_p$ is 1 if $aX^2 + bY^2 = Z^2$ has nontrivial solutions in $\mathbb{Q}_p^3$ and -1 if not. For $u \in \mathrm{GL}_m(\mathbb{Q}_p)^{\mathrm{sym}}$ the Hasse invariant of u is $h_p(u) = \prod_{i \leq j}(a_i, a_j)_p$ where u is $\mathrm{GL}_m(\mathbb{Q}_p)$-equivalent to the diagonal matrix having entries $a_1, \cdots, a_m$. If m is even, then $(-1)^{m/2}\det(2u)$ takes the form $D_u f_u^2$ where D_u is 1 or the fundamental discriminant of a quadratic number field and f_u is a positive integer; let χ_{D_u} denote the quadratic Dirichlet character of conductor $|D_u|$. For rank $m = 0$, the empty matrix has determinant 1 by convention and so $D_u = f_u = 1$.

Theorem 3.1 (Katsurada's functional equation). *Let $u \in \mathcal{X}_m$. Set*

$$e_p(u) = \begin{cases} 2(\lfloor \frac{\mathrm{ord}_p(\det(2u))-1-\delta_{p,2}}{2} \rfloor + \chi_{D_u}(p)^2) & \text{if } m \text{ is even,} \\ \mathrm{ord}_p(\det(2u)/2) & \text{if } m \text{ is odd.} \end{cases}$$

Here $\delta_{p,2}$ is the Kronecker delta. Then

$$F_p(u, p^{-m-1}X^{-1}) = \pm(p^{(m+1)/2}X)^{-e_p(u)}F_p(u, X),$$

where if m is even, then the "±" sign is positive, and if m is odd, then it is

$$\left(\det(u), (-1)^{(m-1)/2} \det(u) \right)_p (-1, -1)_p^{(m^2-1)/8} h_p(u)$$

with $(\cdot, \cdot)_p$ the Hilbert symbol and h_p the Hasse invariant as described above.

3.2. Fourier coefficient formula.

Let n be a positive integer. For any $t \in \mathcal{X}_n^{\mathrm{semi}}$ we have $t \sim u \oplus 0_{n-m}$ under $\mathrm{GL}_n(\mathbb{Z})$-equivalence, where $m = \mathrm{rank}(t) \in \mathbb{Z}_{\geq 0}$ and $u \in \mathcal{X}_m$. The following result may be found in [14, 16].

Theorem 3.2 (Siegel Eisenstein Fourier coefficient formula). *Let n be a positive integer and $k > n + 1$ an even integer. Let $t \in \mathcal{X}_n^{\mathrm{semi}}$, and let u, D_u, f_u, and χ_{D_u} be as above. Let $c_k^{(m)} = 2^{-\lfloor (m+1)/2 \rfloor} \zeta(1-k) \prod_{i=1}^{\lfloor m/2 \rfloor} \zeta(1 - 2k + 2i)$. Then*

$$a(t; E_k^{(n)}) = 1/c_k^{(m)} \cdot \begin{cases} L(\chi_{D_u}, 1 - k + m/2) \prod_{p|f_u} F_p(u, p^{k-m-1}), & m \text{ even}, \\ \prod_{p:\mathrm{ord}_p((1/2)\det(2u))>0} F_p(u, p^{k-m-1}), & m \text{ odd}. \end{cases}$$

The Riemann zeta values and the quadratic L value in the formula have the form $\zeta(1-j) = -B_j/j$ and $L(\chi, 1-j) = -B_j(\chi)/j$ with the B_j basic or quadratic Bernoulli numbers, and so they are known rational numbers: If f is the conductor of χ, then $\sum_{a=1}^{f} \chi(a) \frac{te^{at}}{e^{ft}-1} = \sum_{j=0}^{\infty} B_j(\chi) \frac{t^j}{j!}$ ([1], page 53). The genus symbol of any $u \in \mathcal{X}_m$ is easy to compute (section 4.2), and then our program gives $F_p(u, p^{k-m-1})$. Thus Siegel Eisenstein series Fourier coefficients are tractable.

The Clausen–von Staudt theorem for basic and quadratic Bernoulli numbers shows that the dilated Eisenstein series $\mathcal{E}_k^{(n)} = c_k^{(n)} E_k^{(n)}$ has rational Fourier coefficients that are integral at all primes $p > 2k - 1$. While the monic Eisenstein series has the computational advantage that its Fourier coefficients depend only on the nonsingular part of their indices, with no reference to the degree n, the integrality of the dilated Eisenstein lets us identify congruence neighbors.

For lower weights $\lceil (n+1)/2 \rceil \leq k \leq n + 1$, excluding a few cases, a Siegel Eisenstein series $E_k^{(n)}(z, s)$ with a complex parameter s can be continued leftward from its half-plane of absolute convergence $\mathrm{Re}(k + 2s) > n + 1$ to $s = 0$, where it is a Siegel modular form in z (see [23], or a summary in the introduction to [26]). Because our work here uses weights $k > 2n + 1$ we do not discuss these issues.

4. THE PULLBACK-GENUS METHOD

Recall Garrett's formula, $\iota^* E_k^{(2n)} = \sum_{\ell=1}^{d} c_\ell f_\ell \otimes f_\ell$ for even $k > 2n + 1$, where $\{f_1, \cdots, f_d\}$ is a Hecke eigenform basis of $\mathcal{M}_k(\Gamma_n)$. Garrett's conjecture that the c_ℓ are nonzero was proved by Garrett for $n = 2$ and for general degree n by Böcherer [4], and this result is important in the proof of our Proposition 4.1. For any two indices $t_1, t_2 \in \mathcal{X}_n^{\mathrm{semi}}$, equate the $t_1 \times t_2$ Fourier coefficients on the two sides of Garrett's formula to get a relation among Fourier coefficients and the c_ℓ,

$$(4.1) \qquad \sum_{r \in \mathcal{R}(t_1 \times t_2)} a\left(\begin{bmatrix} t_1 & r \\ r' & t_2 \end{bmatrix}; E_k^{(2n)} \right) = \sum_{\ell=1}^{d} c_\ell \, a(t_1; f_\ell) a(t_2; f_\ell),$$

summing the left side over $\mathcal{R}(t_1 \times t_2) = \{r \in \mathrm{M}_n(\frac{1}{2}\mathbb{Z}) : \begin{bmatrix} t_1 & r \\ r' & t_2 \end{bmatrix} \in \mathcal{X}_{2n}^{\mathrm{semi}}\}$. This set is finite because $\begin{bmatrix} t_{1ii} & r_{ij} \\ r_{ij} & t_{2jj} \end{bmatrix}$ is positive semidefinite for all i, j, bounding r_{ij}^2 by $t_{1ii} t_{2jj}$. The summand on the left side is tractable, and additionally the set of summation can be traversed quickly enough to make the left side computationally accessible

(section 4.1). Conceptually, for each fixed $t_1 \in \mathcal{X}_n^{\mathrm{semi}}$, equation (4.1) gives the Fourier coefficient at t_2 of an element in $\mathcal{M}_k(\Gamma_n)$ with rational Fourier coefficients. We use enough different t_1 to obtain a rational basis. If we additionally want eigenforms and Euler factors, we apply Hecke operators to this rational basis to obtain a basis of eigenforms f_ℓ, with some convenient choice of normalization. Sufficiently many Fourier coefficients of a eigenform basis of $\mathcal{S}_k(\Gamma_n)$ enable us to compute Euler factors of their L-functions (section 5). If we want to identify congruence neighbors among the f_ℓ, we then use equation (4.1) to solve for the coefficients c_ℓ.

4.1. **Index enumeration.** Given $t_1, t_2 \in \mathcal{X}_n^{\mathrm{semi}}$, the following algorithm traverses $\mathcal{R}(t_1 \times t_2)$ quickly enough for our programs to terminate. Immediately multiply by 2 to work with integers; that is, double the t_i for this algorithm and introduce the matrix $s = \begin{bmatrix} t_1 & r \\ r' & t_2 \end{bmatrix}$, find all integral r that make s positive semidefinite, and divide each such r by 2 before returning it. The algorithm builds matrices s by filling r columnwise. Thus, when determining possible provisional values for some $r_{i,j}$, a set of provisional values is already present for all other r-entries having row index at most i and column index at most j.

As noted, $|r_{i,j}| \leq m$ where $m = \sqrt{t_{1,i,i} t_{2,j,j}}$. A first version of the algorithm is therefore n^2 nested loops: For each $|r_{i_1,j_1}| \leq m_1$, for each $|r_{i_2,j_2}| \leq m_2$, ..., for each $|r_{i_{n^2},j_{n^2}}| \leq m_{n^2}$, test s for positive semidefiniteness. We refer to these m_i as the default loop-bounds. The algorithm admits two refinements that cut down the nested looping at the cost of further bounds-checking. The refinements are most easily explained by example. Let $\tilde{s}$ denote a submatrix of s as in the following diagram, in which $n = 3$:

$$ s = \begin{bmatrix} \circ & \circ & \cdot & * & * & \cdot \\ \circ & \circ & \cdot & * & r_{i,j} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ * & * & \cdot & \circ & \circ & \cdot \\ * & r_{i,j} & \cdot & \circ & \circ & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \qquad \tilde{s} = \begin{bmatrix} \circ & \circ & * & * \\ \circ & \circ & * & r_{i,j} \\ * & * & \circ & \circ \\ * & r_{i,j} & \circ & \circ \end{bmatrix}. $$

The circles are entries of t_1 and t_2 and the asterisks are some of the outer loop variables. First, the loop-bounds of $r_{i,j}$ can be improved. Introduce three auxiliary matrices a, b, and c by setting a to $\tilde{s}$ but with the row and column of the two $r_{i,j}$-entries deleted, b to $\tilde{s}$ but with 0 in place of the higher $r_{i,j}$ and with the row and column of the lower $r_{i,j}$ deleted, and c to $\tilde{s}$ but with 0 in place of both $r_{i,j}$ entries. Then $\det \tilde{s} = -\det a \cdot r_{i,j}^2 + (-1)^j 2 \det b \cdot r_{i,j} + \det c$. The algorithm needs to have checked the positive semidefiniteness of previous matrices to ensure that $\det a$ is nonnegative. In our implementation of columnwise traversal, the check needs to happen at the bottom of each column. When $\det a$ is positive, the condition $\det \tilde{s} \geq 0$ is quadratic and yields bounds of $r_{i,j}$, generally tighter than the default loop-bounds. As a second refinement, we may check whether a value of $r_{i,j}$ makes $\tilde{s}$ positive semidefinite before proceeding to more inner loops. In practice there are tradeoffs between naïve looping and checking to tighten loop-bounds or abort inner loops. Our current implementation is to improve the loop-bounds and to check $\tilde{s} \geq 0$ at the bottom of each column of r.

4.2. Genus symbol.

Consider any $t \in \mathcal{X}_n$. A symbol for the genus of t is described in chapter 15, section 7 of [6]. We summarize it briefly. As in the index-set traversal algorithm, immediately double t to ensure integral entries. The finitely many equivalence classes $\{t[\mathrm{GL}_n(\mathbb{Z}_p)] : p \mid 2 \det t\}$ determine every other class $t[\mathrm{GL}_n(\mathbb{Z}_p)]$ where $p \nmid 2 \det t$. Thus the genus symbol of t need only describe its local integral equivalence class for each $p \mid 2 \det t$.

For an odd prime divisor p of $\det t$, t is $\mathrm{GL}_n(\mathbb{Z}_p)$-equivalent to some $\bigoplus_{i=1}^{k} p^{e_i} \delta_i$ where $0 \le e_1 < \cdots < e_k$ and each δ_i is a diagonal matrix having p-adic units on the diagonal. Each $p^{e_i} \delta_i$ is a *constituent* of t, and p^{e_i} is the *scale* of the constituent. The $\mathrm{GL}_n(\mathbb{Z}_p)$-equivalence symbol of t is $q_1^{\epsilon_1 n_1} q_2^{\epsilon_2 n_2} \cdots q_k^{\epsilon_k n_k}$ where for $i = 1, \cdots, k$, $q_i = p^{e_i}$ and ϵ_i is the Legendre symbol $(\det \delta_i / p)$ and n_i is the size of δ_i. The $\mathrm{GL}_n(\mathbb{Z}_p)$-equivalence symbol of t is uniquely defined by t, and it completely characterizes $t[\mathrm{GL}_n(\mathbb{Z}_p)]$.

Also, t is $\mathrm{GL}_n(\mathbb{Z}_2)$-equivalent to some $\bigoplus_{i=1}^{k} 2^{e_i} d_i$ where $0 \le e_1 < \cdots < e_k$ and each d_i is either a diagonal matrix with units on the diagonal or a direct sum $d_i = \bigoplus_{j=1}^{\ell_i} \delta_{ij}$ where each δ is a 2×2 matrix $\begin{bmatrix} 2^{\alpha} a & b \\ b & 2^{\gamma} c \end{bmatrix}$. The $\mathrm{GL}_n(\mathbb{Z}_2)$-equivalence symbol of t is $(q_1)_{t_1}^{\epsilon_1 n_1} (q_2)_{t_2}^{\epsilon_2 n_2} \cdots (q_k)_{t_k}^{\epsilon_k n_r}$, where for $i = 1, \cdots, k$, $q_i = 2^{e_i}$ and ϵ_i is the Kronecker symbol $(\det d_i / 2)$ (1 if $\det d_i = \pm 1 \bmod 8$, -1 if $\det d_i = \pm 3 \bmod 8$) and n_i is the size of d_i and t_i is $\mathrm{tr}\, d_i \pmod 8$ if d_i is diagonal, while it is undefined or ∞ if d_i is a sum of 2×2 subblocks. The $\mathrm{GL}_n(\mathbb{Z}_2)$-equivalence symbol of t determines $t[\mathrm{GL}_n(\mathbb{Z}_2)]$, but not conversely. A unique symbol can be produced as explained in [6], but there are computational tradeoffs between actually computing a canonical 2-adic symbol and using a method to compute F_p polynomials that accepts different symbols for the same genus. Since our program accepts any 2-adic symbol, we do not discuss how to make the 2-part of the genus symbol unique.

4.3. Determining bases.

The next proposition shows how to find a rational basis for *any* level one space of Siegel modular forms, $\mathcal{M}_k(\Gamma_n)$, for even $k > 2n + 1$. It is an application of Garrett's formula and highlights the importance of the work of Katsurada, which makes the computation of the Fourier coefficients of Eisenstein series practical.

Proposition 4.1 (Pullback-genus method)**.** *Consider even $k > 2n + 1$. Let $\mathcal{T} = \{t_1, \cdots, t_m\}$ be a determining set of Fourier coefficient indices for $\mathcal{M}_k(\Gamma_n)$. Define an m-by-m matrix $M = [a(t_i \times t_j; \iota^* E_k^{(2n)})]_{m \times m}$. Then $\dim(\mathcal{M}_k(\Gamma_n)) = \mathrm{rank}(M)$.*

Column reduce M to get a matrix $\begin{bmatrix} * & | & 0 \end{bmatrix}$. The nonzero columns describe the $\mathcal{T}$-truncations of a basis of $\mathcal{M}_k(\Gamma_n)$, i.e., each nonzero column contains the $\mathcal{T}$th Fourier coefficients of a basis element. Further stipulating that $\mathcal{T}$ is ordered with the singular indices at the beginning, let the column reduction of M take the form

$$M \sim \left[\begin{array}{c|c|c} * & 0 & 0 \\ \hline * & * & 0 \end{array} \right] \quad (\text{column space equivalence}),$$

with the horizontal dividing line after the rows indexed by singular t. The columns between the vertical dividing lines describe the $\mathcal{T}$-truncations of a basis of $\mathcal{S}_k(\Gamma_n)$. In particular, $\dim(\mathcal{S}_k(\Gamma_n))$ is the number of columns between the vertical dividing lines.

Proof. Garrett's formula gives $M = \sum_{\ell=1}^{d} c_\ell [a(t_i; f_\ell) a(t_j; f_\ell)]_{m \times m}$, where $d = \dim(\mathcal{M}_k(\Gamma_n))$ is the desired dimension. Each summand matrix is an outer product $v_\ell v_\ell'$ where the column vector v_ℓ encodes a determining truncation of the Fourier series of f_ℓ, $M = \sum_{\ell=1}^{d} c_\ell v_\ell v_\ell'$ where $v_\ell = [a(t_1; f_\ell) \quad \cdot \quad a(t_m; f_\ell)]'$ for $\ell = 1, \cdots, d$. The matrix sum thus has the form $M = VCV'$ where $V = [v_1 \quad \cdots \quad v_d]_{m \times d}$ and $C = \operatorname{diag}(c_1, \cdots, c_d)$. Because $\mathcal{T}$ is a determining set, the d columns of V are linearly independent. Because each c_ℓ is nonzero, C is invertible. Therefore the column space of CV' is $\mathbb{C}^d$ and, consequently, $\operatorname{col sp}(VCV') = \operatorname{col sp}(V) = \operatorname{span}(v_1, \cdots, v_d)$, which has dimension d. This shows that $\dim(\mathcal{M}_k(\Gamma_n)) = \operatorname{rank}(M)$. For any $j \in \{1, \cdots, m\}$, the jth column of M is $\sum_{\ell=1}^{d} c_{j,\ell} [a(t_1; f_\ell) \cdot a(t_m; f_\ell)]'$ with each $c_{j,\ell} = c_\ell \, a(t_j; f_\ell)$. This is $[a(t_1; g_j) \quad \cdots \quad a(t_m; g_j)]'$ where $g_j = \sum_{\ell=1}^{d} c_{j,\ell} f_\ell$. After column reducing M to the form $\left[\begin{array}{c|c|c} * & 0 & 0 \\ \hline * & * & 0 \end{array}\right]$ described in the proposition, the desired statements about the reduced matrix are immediate since its rows are indexed by a determining set and its rank is $\dim(\mathcal{M}_k(\Gamma_n))$. $\qquad\square$

We make a few comments about Proposition 4.1. We have methods available to obtain a finite determining set $\mathcal{T}$ of indices for $\mathcal{M}_k(\Gamma_n)$ as needed by the proposition; see [21]. These determining sets are not needed when the dimension is known. In degree three, Tsuyumine gave the generating function for $\dim(\mathcal{M}_k(\Gamma_3))$ over all weights k ([25], p. 831, though the factor $(1 - T^{12})^3$ in the denominator at the top of p. 832 should be $(1-T^{12})^2$ instead). So for $n = 3$ we do not need the full strength of Proposition 4.1. Instead, we may simply grow the rows of M columnwise until we have a basis, or even take a subset of the rows of M and grow that matrix columnwise until getting a basis.

5. Computing Euler factors

This section condenses section 3 of [22]. The Hecke action on Fourier expansions is explained in [5]. Thus, given a basis of $\mathcal{M}_k(\Gamma_n)$ with sufficiently long Fourier expansions, we may compute a basis $\{f_\ell\}$ of eigenforms and their eigenvalues $T(p)f = \lambda_f(T(p))f$ and $T_j(p^2)f = \lambda_f(T_j(p^2))f$.

Letting square brackets connote Weyl group symmetrization, introduce elements of the invariant Laurent polynomial algebra $\mathbb{Q}[x^{\pm 1}]^W$, $g = [x_0] = x_0 \prod_{i=1}^{n}(x_i + 1)$ and $g_\ell = [x_0^2 x_1 \cdots x_{n-\ell}]$ for $\ell = 0, \cdots, n$. Altogether $g, g_0, g_1, \cdots, g_{n-1}, g_0^{-1}$ generate $\mathbb{Q}[x^{\pm 1}]^W$; $g_n = [x_0^2]$ is not needed as a generator but it will be used below. For any prime p, the Hecke algebra generators $T(p)$, $\{T_i(p^2)\}$ and the invariant polynomial algebra generators g, $\{g_\ell\}$ are related via the Satake isomorphism $\Omega : \mathcal{H}_p \longrightarrow \mathbb{Q}[x^{\pm 1}]^W$ and linear relations as follows (Hilfssatz 3.14 and Hilfssatz 3.17 in [7]).

Proposition 5.1 (Satake isomorphism on generators)**.** *Let $n \geq 2$ be an integer, and let p be prime. Then $\Omega(T(p)) = g$. Also, there exists an upper triangular matrix $K = K_n(p^2) \in \mathrm{M}_{n+1}(\mathbb{Z}[1/p])$, with positive entries on and above the diagonal, such that (applying Ω componentwise on the left side of the next equation)*

$$\Omega \begin{bmatrix} T_n(p^2) & \cdots & T_1(p^2) & T_0(p^2) \end{bmatrix} = \begin{bmatrix} g_0 & g_1 & \cdots & g_n \end{bmatrix} K_n(p^2).$$

A. Krieg [19] gave the entries of the matrix for $n \geq 2$, and a program that computes the matrix is at the author's website [28]. In particular,

$$K_3(p^2) = \begin{bmatrix} \dfrac{1}{p^6} & \dfrac{p^3-1}{p^6} & \dfrac{3p^3-p^2-p-1}{p^4} & \dfrac{(p-1)(3p^3-p^2-p-1)}{p^4} \\ 0 & \dfrac{1}{p^3} & \dfrac{p^2-1}{p^3} & \dfrac{2(p-1)}{p} \\ 0 & 0 & \dfrac{1}{p} & \dfrac{p-1}{p} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We introduce further Weyl-invariant polynomials $r_0, r_1, \cdots, r_{2n}$, defined by the relation $\prod_{i=1}^{n}(1-x_iX)(1-x_i^{-1}X) = \sum_{\ell=0}^{2n}(-1)^{\ell}r_{\ell}X^{\ell}$, with $r_{2n-\ell} = r_{\ell}$. Thus the standard p-Euler factor $Q_p^{\mathrm{st}}(f,X) = (1-X)\prod_{i=1}^{n}(1-\alpha_{i,p}X)(1-\alpha_{i,p}^{-1}X)$ of a Hecke eigenform $f \in \mathcal{M}_k(\Gamma_n)$ is $Q_p^{\mathrm{st}}(f,X) = (1-X)\sum_{\ell=0}^{2n}(-1)^{\ell}r_{\ell}(\alpha)X^{\ell}$. The r_{ℓ} and g_{ℓ} Laurent polynomials are related by the condition

$$\begin{bmatrix} r_0 & r_1 & \cdots & r_n \end{bmatrix} = p^{\langle n \rangle - kn}\begin{bmatrix} g_0 & g_1 & \cdots & g_n \end{bmatrix} P,$$

where $P = P_n$ is the Pascal-like upper triangular matrix whose nonzero entries are $\binom{n-i}{(j-i)/2}$ in the (i,j)th position if $j-i \in 2\mathbb{Z}_{\geq 0}$, the row and column indices starting at 0. In particular,

$$P_3 = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Given a Hecke eigenform, if we can compute its eigenvalues under $T(p)$ and under $T_i(p^2)$ for $i = 0, \cdots, n$, then we can produce the values $r_i(\alpha)$ that specify its standard p-Euler factor, as follows [22].

Theorem 5.2 (Standard Euler factor from eigenvalues). *Let $n \geq 2$ and k be positive integers, and let p be prime. Let $f \in \mathcal{M}_k(\Gamma_n)$ be a Hecke eigenform of $\mathcal{H}_p$. Let α be the Satake parameter of the eigenvalue function λ_f. Introduce the vector of eigenvalues and the vector of polynomial coefficients, $\vec{\lambda} = \begin{bmatrix} \lambda_f(T_n(p^2)) & \cdots & \lambda_f(T_0(p^2)) \end{bmatrix}$ and $\vec{r} = \begin{bmatrix} r_0(\alpha) & \cdots & r_n(\alpha) \end{bmatrix}$. Then $\vec{r} = p^{\langle n \rangle - kn}\vec{\lambda}K^{-1}P$, where the matrices P and K are as above.*

Indeed, introducing $\vec{g} = \begin{bmatrix} g_0(\alpha) & \cdots & g_n(\alpha) \end{bmatrix}$ we have $\vec{r} = p^{\langle n \rangle - kn}\vec{g}P$, and the Satake mapping property and Proposition 5.1 combine to give $\vec{g} = \vec{\lambda}K^{-1}$.

Computing the spinor Euler factor from the eigenvalues is similar. In degree 3, the spinor p-Euler factor is

$$Q_p^{\mathrm{spin}}(f,X) = (1-\alpha_0 X)\cdot(1-\alpha_0\alpha_1 X)(1-\alpha_0\alpha_2 X)(1-\alpha_0\alpha_3 X)$$
$$\cdot(1-\alpha_0\alpha_1\alpha_2 X)(1-\alpha_0\alpha_1\alpha_3 X)(1-\alpha_0\alpha_2\alpha_3 X)(1-\alpha_0\alpha_1\alpha_2\alpha_3 X).$$

Denote its expansion $\sum_{\ell=0}^{8}(-1)^{\ell}s_{\ell}X^{\ell}$. By direct computation, or [20], p. 310,

$$s_0 = 1, \qquad\qquad\qquad\qquad s_8 = g_0^4(\alpha)s_0,$$
$$s_1 = g(\alpha), \qquad\qquad\qquad\quad s_7 = g_0^3(\alpha)s_1,$$
$$s_2 = (4g_0 + 2g_1 + g_2)(\alpha), \qquad\quad s_6 = g_0^2(\alpha)s_2,$$
$$s_3 = g(\alpha)(g_0 + g_1)(\alpha), \qquad\quad s_5 = g_0(\alpha)s_3,$$
$$s_4 = (2g_0^2 + 4g_0g_1 + g_0g_3 + g_1^2)(\alpha).$$

By the Satake mapping property and Proposition 5.1, $g(\alpha) = \lambda_f(T(p))$ and (as above) $\vec{g} = \vec{\lambda}K^{-1}$. The spinor factor follows from the previous display.

6. IMPLEMENTATION

Implementing the pullback-genus method is not a purely mechanical matter.

For a given weight k, one wants to choose a small determining set of indices $\mathcal{T}$ that aptly comprises matrices of rank 1, 2, and 3 in light of the known dimensions of $\mathcal{M}_k(\Gamma_1)$, $\mathcal{M}_k(\Gamma_2)$, and $\mathcal{M}_k(\Gamma_3)$. These matrices should have small entries. But also the space that we are trying to determine can have an element that vanishes to high order, such as Igusa's $\chi_{18} \in \mathcal{S}_{18}(\Gamma_3)$, requiring a bigger index to "see" it in order for the method to succeed. Guessing a small determining set for a given degree k requires a combination of software experimentation and mathematical insight into the structure of $\mathcal{M}_k(\Gamma_3)$.

The index-enumeration algorithm of section 4.1 is a significant bottleneck, quickly growing expensive as the entries of the elements of the determining set grow. To carry out our computations through weight 22 we processed 1965 Fourier coefficient indices $t_1 \times t_2$ of the pullback $\iota^* E_k^{(6)}$, leading to $1\,561\,537\,201$ Fourier coefficient indices $\begin{bmatrix} t_1 & r \\ r' & t_2 \end{bmatrix}$ of $E_k^{(6)}$ itself. This multitude of indices gave rise to only $54\,314$ genus symbols, showing the crucial role of genus coarseness in the pullback-genus method. Indeed, the indices probably lie in considerably fewer genera, because we allow genus symbols that are not unique at 2 and the entries of our indices often are divisible by 2. The genera are recorded with multiplicity and the Fourier coefficient for each genus is computed only once.

Not only do large collections of indices t give rise to far fewer genus symbols than equivalence classes but, furthermore, the genus symbols are much faster and more space-efficient to compute because equivalence class computations require a sophisticated algorithm that uses lattice reduction and maintains an elaborate, memory-expensive tree structure. For example, the pair

$$t_1 \times t_2 = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 2 & 1 \\ \frac{1}{2} & 1 & 2 \end{bmatrix} \times \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

arose in weight 22 and gave rise to $6\,755\,849$ semidefinite indices $t = \begin{bmatrix} t_1 & r \\ r' & t_2 \end{bmatrix}$. Some 36 hours of computation on a typical server determined that these indices fell into 9132 equivalence classes. On the other hand, only five minutes of laptop computation produced 4238 distinct genus symbols from the indices, and, as in the previous paragraph, the actual number of genera is smaller. Another pair $t_1 \times t_2$ that we tested separately from our main computation took about 50 hours of computation

on the server, using over 1.3 gigabytes of space, to determine that the resulting $4\,002\,643$ indices t fell into $33\,440$ lattice classes, whereas the laptop computation to produce 9114 genus symbols from the indices took only several minutes and under 100 megabytes of space.

Separately from computing the pulled back Eisenstein series Fourier coefficients, which are rational, as the weight k grows so do the number fields underlying the Hecke eigenforms on the right side $\sum_\ell c_\ell f_\ell \otimes f_\ell$ of Garrett's formula, and this posed various programming challenges. The right side summand does not determine c_ℓ or f_ℓ individually, and considerable care was required to scale them in a way that allowed congruence primes to be diagnosed.

Acknowledgments

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