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AN ANALOG OF THE PRIME NUMBER THEOREM FOR FINITE FIELDS VIA TRUNCATED POLYLOGARITHM EXPANSIONS

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Abstract. An exponentially accurate asymptotic expansion of the truncated polylogarithm function is derived that leads to an asymptotic formula for enumerating monic irreducible polynomials over finite fields. This formula is analogous to the asymptotic expansion formula of the classical prime counting function. Results are presented which show that it is more accurate than previous results in the literature while requiring very little computational effort. Asymptotic expansions of the Lerch transcendent, Eulerian polynomials, and polylogarithms of negative integer order are also given. The accuracy of the proposed approach is verified via numerical results.

1. Introduction

In this paper, a precise asymptotic expansion for the finite field analog of the classical prime counting function from number theory is derived. The prime counting function enumerates the prime numbers less than or equal to x and is given by

$$\pi(x) = \sum_{\substack{p \leqslant x \\ p \text{ prime}}} 1 \,. \tag{1.1}$$

A generalization of (1.1) is the prime ideal counting function $\pi_K(x)$ that counts the nonzero prime ideals of O_K, a Dedekind ring whose field of fractions K is a global field. Every nonzero prime ideal $\mathfrak{p}$ of O_K is maximal and its absolute norm $\mathcal{N}(\mathfrak{p})$ is defined as the (finite) number of elements in the residue class field $O_K/\mathfrak{p}$. Hence, $\pi_K(x) = |\{\mathfrak{p} \subset O_K \,|\, \mathcal{N}(\mathfrak{p}) \leqslant x\}|$ [26, p. 413]. Here K is either a number field, i.e., a finite extension of $\mathbb{Q}$, or a global function field with full constant field $\mathbb{F}_q$, i.e., a finite extension of the field of rational functions $\mathbb{F}_q(T)$ in one variable over the finite field $\mathbb{F}_q$, such that $\mathbb{F}_q$ is algebraically closed in K.

While a rigorous direct connection between number fields and global function fields has not yet been established, there are many fundamental analogies (see Rosen [32], Stichtenoth [35], and Iwaniec et al. [18] for a detailed discussion). If K is a number field, O_K is the integral closure of the ring $\mathbb{Z}$ in K, and every

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nonzero prime ideal in O_K is generated by a positive prime number. Similarly, if K is a global function field as defined above, O_K is the integral closure of the ring of polynomials $\mathbb{F}_q[T]$ in K, and every nonzero prime ideal $\mathfrak{p} = (f)$ is generated by a monic irreducible polynomial over $\mathbb{F}_q$ also known as a prime polynomial and denoted by f. Each prime polynomial gives rise to a residue class field $\mathbb{F}_q[T]/(f)$ of cardinality $\mathcal{N}(\mathfrak{p}) = q^m$, where $m = \deg(f)$ [5, p. 198]. Hence, counting prime polynomials in $\mathbb{F}_q[T]$ of degree less than m is analogous to counting positive prime numbers in the integers less than x, with the finite field analog to (1.1) given as

$$\pi_q(m) = \sum_{\substack{\deg f \leqslant m \\ f \text{ monic, irreducible}}} 1, \tag{1.2}$$

where $m \geqslant 1$. Multiplying (1.2) by $q-1$ yields the number of irreducible polynomials (not necessarily monic). This is due to the fact that multiplying a monic irreducible polynomial by any element of the multiplicative group $\mathbb{F}_q^*$ does not change its degree and results in an irreducible polynomial.

Conditional on the still unproven Riemann hypothesis, an approximation and error bound for the prime counting function in (1.1) was given by von Koch in [20] as

$$\pi(x) = \int_2^x \frac{dt}{\log t} + O(\sqrt{x}\log x) \tag{1.3}$$

$$= \mathrm{li}(x) - \mathrm{li}(2) + O(\sqrt{x}\log x) \tag{1.4}$$

$$= \mathrm{Li}(x) + O(\sqrt{x}\log x), \tag{1.5}$$

where $\mathrm{li}(x)$ and $\mathrm{Li}(x)$ denote the logarithmic integral and the offset logarithmic integral, respectively. The latter notation is an historic artifact and should not be confused with $\mathrm{Li}_s(x)$ which denotes the polylogarithm function and is used subsequently in this paper.

The logarithmic and exponential integral are related via $\mathrm{li}(x) = \mathrm{Ei}(\log x)$. Analytic continuation of the exponential integral and repeated integration by parts yields the well-known Poincaré-type expansion formula for $\pi(x)$ (see Lebedev [23, pp. 32–38]),

$$\pi_N(x) \sim \frac{x}{\log x}\left[\sum_{n=0}^{N-1} \frac{n!}{(\log x)^n} + R_N(x)\right], \tag{1.6}$$

where

$$R_N(x) \leqslant C_N \frac{N!}{(\log x)^N}, \tag{1.7}$$

$x \in \mathbb{R}$ with $x \geqslant 2$, $N \in \mathbb{N}^+$ is the truncation order (subsequently also referred to as order) of the expansion, and C_N is a constant. For $N \to \infty$, the expansion in (1.6) eventually diverges for any finite value of x because $R_N(x)$ is unbounded. Therefore, this expansion can provide a reasonable estimate only if the series is truncated at a finite number of terms since $R_N(x)$ is then of order $O(x^{-N})$ and approaches zero as $x \to \infty$.

In what follows we provide a Poincaré-type expansion formula analogous to (1.6) for the prime polynomial counting function in (1.2). This expansion is obtained via an exponentially accurate asymptotic expansion of the truncated polylogarithm function which requires very little computational effort. The expansion formulas developed are general and have applications in numerous areas other than the enumeration of irreducible polynomials.

A well-known result from combinatorics (see for example Rosen [32, p. 13] and Berlekamp [3, p. 84]), gives the number of monic irreducible polynomials over the finite field $\mathbb{F}_q$ of degree n as

$$N_q(n) = \frac{1}{n}\sum_{d|n}\mu(d)q^{\frac{n}{d}}, \tag{1.8}$$

where the sum is over all divisors of n, and $\mu(n)$ is the Möbius function (see Graham et al. [17, p. 136]), defined as

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^k & \text{if } n \text{ is the product of } k \text{ distinct primes}, \\ 0 & \text{if } n \text{ has one or more repeated prime factors}. \end{cases}$$

Equation (1.8) also counts the number of aperiodic cyclic equivalence classes of q-ary strings of length m [16], [27]. An aperiodic string of length m returns to its original configuration after exactly m cyclic shifts, and fewer than m cyclic shifts results in a different string. The lexicographically smallest of these cyclic shifts is referred to as a Lyndon word and by convention is chosen as the string representing the equivalence class [25].

From (1.8), we establish the prime polynomial (Lyndon word) counting function as

$$\pi_q(m) = \sum_{n=1}^{m} N_q(n)\,. \tag{1.9}$$

When enumerating Lyndon words such that the zero-length word is allowed, the sum (1.9) must be increased by one. Sharp upper and lower bounds on $N_q(n)$ are provided in [24, pp. 123–124]. These bounds imply the simple estimate derived in [32, Theorem 2.2]

$$N_q(n) = \frac{q^n}{n} + O\left(\frac{q^{\frac{n}{2}}}{n}\right). \tag{1.10}$$

Substituting (1.10) into (1.9) yields

$$\pi_q(m) = \sum_{n=1}^{m}\frac{q^n}{n} + O\left(\frac{q^{\frac{m}{2}}}{m}\right), \tag{1.11}$$

with the error term following from [31, Lemma 3].

Several approximations to (1.9) have been developed. To the best of our knowledge, the first correct result is due to Kruse et al., who provided a first order approximation in 1990 [21]. More recently, Wang et al. extended this result to a second order approximation in [39]. Pollack was the first to explore a finite field analog similar to the series expansion (1.6) in 2010 [31]. His approach is slightly different in that he considers the number of irreducible polynomials less than integers that encode univariate polynomials over a finite field in a bijective mapping. However, as in [21], [39], the asymptotic expansion provided in [31, Theorem 2] rests on the approximation of the sum in (1.11). An estimate is given in the form of a series [31, Lemma 6] that depends on coefficients that involve the evaluation of infinite series. An asymptotic result for these coefficients is provided in [31, Lemma 7]. However, while the resulting asymptotic expansion resembles that of (1.6), it yields inferior numerical results when compared with those of [21], [39].

Theorem 1.1 provides a Poincaré-type expansion for (1.9) that is based on (1.11) and analogous to (1.6). It is one of the main results of this paper and is a significant improvement on the results in [21], [39], [31].

Theorem 1.1. *Let $\mathbb{F}_q[T]$ denote the ring of polynomials over the finite field with q elements. Then for $m \in \mathbb{N}^+$, $m \to \infty$, the number of monic irreducible polynomials in $\mathbb{F}_q[T]$ of degree less than or equal to m is given by*

$$\pi_{q,N}(m) \sim \frac{q}{q-1}\frac{X}{\log_q X}\left[\sum_{n=0}^{N-1}\frac{(q-1)^{-n}\mathcal{A}_n(q)}{(\log_q X)^n} + R_{q,N}(X)\right], \tag{1.12}$$

where $X = q^m$ with

$$R_{q,N}(X) \leqslant C_N \frac{q-1}{q\,(\log q)^{N+1}}\frac{N!}{(\log_q X)^N}, \tag{1.13}$$

where $N \in \mathbb{N}^+$ is the truncation order of the expansion, C_N is a constant independent of X, and $\mathcal{A}_n(z)$ denotes the nth Eulerian polynomial as in Definition 2.3. The remainder term becomes exponentially small when the series is optimally truncated as per Definition 2.2.

The proof of Theorem 1.1 relies on asymptotic expansions of the Eulerian polynomials and the truncated polylogarithm function, which are discussed in detail in the next section.

2. Asymptotic expansion of the truncated polylogarithm function

In this section, an accurate asymptotic expansion of the truncated polylogarithm function is presented. While the results given here are required for the proof of Theorem 1.1 in Section 3, they find applications in many areas of combinatorics other than the enumeration of irreducible polynomials.

Definition 2.1. The truncated polylogarithm function is given by the finite series

$$\mathcal{L}(z,s,m) = \sum_{n=1}^{m}\frac{z^n}{n^s} \qquad (z \in \mathbb{C};\quad s \in \mathbb{C};\quad m \in \mathbb{N}^+). \tag{2.1}$$

Definition 2.1 is the mth partial sum resulting from truncating the infinite series representation of the polylogarithm. The polylogarithm, also known as Jonquière's function (see Jonquière [19] and Truesdell [37]), is defined as

$$\begin{gathered}\operatorname{Li}_s(z)=\sum_{n=1}^{\infty}\frac{z^n}{n^s}=z\,\Phi(z,s,1)\\ (z\in\mathbb{C};\quad s\in\mathbb{C}\text{ when }|z|<1;\quad \Re(s)>1\text{ when }|z|=1),\end{gathered}\tag{2.2}$$

where $\Phi(z,s,1)$ denotes the Lerch transcendent (see Srivastava et al. [33, p. 121], which is given by the power series

$$\begin{gathered}\Phi(z,s,a)=\sum_{n=0}^{\infty}\frac{z^n}{(a+n)^s}\\ (z\in\mathbb{C};\quad s\in\mathbb{C}\text{ when }|z|<1;\quad \Re(s)>1\text{ when }|z|=1;\quad a\in\mathbb{C}\setminus\mathbb{Z}_0^-).\end{gathered}\tag{2.3}$$

The Lerch transcendent is analytically continued via the following integral representation valid for the cut z-plane with $z\in\mathbb{C}\setminus[1,\infty)$ (see Erdélyi et al. [11, p. 27])

$$\begin{gathered}\Phi(z,s,a)=\frac{1}{\Gamma(s)}\int_0^{\infty}\frac{t^{s-1}\,e^{-(a-1)t}}{e^t-z}\,dt\\ (\Re(s)>0\text{ when }|z|\leqslant 1,\ z\neq 1;\quad \Re(s)>1\text{ when }z=1;\quad \Re(a)>0),\end{gathered}\tag{2.4}$$

where $\Gamma(s)$ denotes the gamma function, and the integrant has simple poles located at

$$t_k=\log z+2k\pi i\qquad (k=0,\,\pm1,\,\pm2,\,\dots)\,.\tag{2.5}$$

The Lerch transcendent plays an important role in many applications in applied and pure mathematics. A thorough discussion of its properties is provided in Chaudhry et al. [9, pp. 316–318], Ferreira et al. [12], and more recently Lagarias et al. [22]. These results predominately focus on the analytic continuation and approximation of the Lerch transcendent for the domain $z\in\mathbb{C}\setminus[1,\infty)$, as then the above integrant (or an expansion of this integrant), can be integrated along a suitable Hankel contour that avoids the poles t_k.

The truncated polylogarithm function can be expressed in terms of the Lerch transcendent as

$$\mathcal{L}(z,s,m)=z\Phi(z,s,1)-z^{m+1}\Phi(z,s,m+1)\,.\tag{2.6}$$

However, excluding $z\in[1,\infty)$ from the domain precludes the use of the truncated polylogarithm function for many practical applications, among them the enumeration of irreducible polynomials over finite fields. Hence, in the subsequent discussion we develop a Poincaré-type expansion that allows us to evaluate (2.6) for $|z|>1$ with remarkable accuracy. For this we consider a combination of two divergent series expansions of the Lerch transcendent. Despite divergence, these series expansions are extremely accurate when "optimally truncated" as per the following definition due to Bender and Orszag [2, Ch. 3].

Definition 2.2 (The optimal truncation rule)**.** Consider a function $f(t)$ and let $\{f_n(t)\}$ be an asymptotic sequence for $t \to t_0$ such that

$$f(t) \sim \sum_{n=0}^{N-1} a_n\, f_n(t),$$

is an asymptotic series expansion of $f(t)$ as $t \to t_0$. For a divergent series expansion, assume the magnitude of successive series terms initially decreases until a minimum is reached and thereafter increases without bound due to the divergent nature of the series. Optimal truncation is then defined as the partial sum up to but not including the least series term [2, Ch. 3]. The truncation order of such an expansion is denoted by $N_\star$, with the least term being an estimate for the approximation error

$$\left| f(t) - \sum_{n=0}^{N_\star - 1} a_n\, f_n(t) \right| = O(\, f_{N_\star}(t)\,),$$

that thereby is minimized.

The optimal truncation rule given by Definition 2.2 is by no means strictly valid for all divergent series and is justified more often by empirical evidence rather than by rigorous proof. The resulting asymptotic expansion is also referred to as superasymptotic and typically exhibits an exponentially small error term [4].

The proof of Theorem 1.1 requires Lemma 2.4 and Theorem 2.7. Lemma 2.4 provides an approximation for Eulerian polynomials not previously found in the literature. Eulerian polynomials (not to be confused with the Euler polynomials [11, pp. 40–43]), were introduced by Euler in the 18th century and have since found numerous applications in enumerative, algebraic, and geometric combinatorics. A general introduction to these polynomials can be found in [8], [10], [30]. The definitions associated with Eulerian polynomials in the literature are not consistent and we largely draw on [13] for our definitions and notation.

Definition 2.3. The nth Eulerian polynomial is given by

$$\mathcal{A}_n(z) = \sum_{k=0}^{n} A(n,k)\, z^k, \qquad z \in \mathbb{C}; \quad n \in \mathbb{N}_0\,. \tag{2.7}$$

The coefficients $A(n,k)$ are positive integers, commonly referred to as Eulerian numbers, and are generated by the recurrence relation

$$A(n,k) = (k+1)\, A(n-1,k) + (n-k)\, A(n-1,k-1), \qquad 1 \leqslant k \leqslant n-1, \tag{2.8}$$

subject to the boundary conditions

$$\begin{aligned} A(n,0) &= 1, &\quad n &\geqslant 0, \quad \text{and} \\ A(n,k) &= 0, & k &\geqslant n\,. \end{aligned}$$

Eulerian numbers are perhaps best known for their combinatorial interpretation as the number of permutations in the symmetric group S_n having exactly k ascents (see Graham et al. [17, pp. 253–255] and Carlitz et al. [7]). While the asymptotic properties of Eulerian numbers have been well studied (see for example [6], [15], [36]), those of the Eulerian polynomials have not received an equally rigorous treatment. In what follows we take a generating function approach to derive a simple yet accurate approximation formula for these polynomials.

Lemma 2.4. *For fixed* $z \in \mathbb{C}\backslash\{0, 1\}$*, with* $|\arg(z)| < \pi$*,* $|\log z| < 2K\pi$*, and* $n \in \mathbb{N}^+$*, the nth Eulerian polynomial* $\mathcal{A}_n(z)$ *is given by*

$$\mathcal{A}_{n,K}(z) = \frac{(z-1)^{n+1}}{z}\left[\frac{1}{(\log z)^{n+1}} + T_K(z, n+1)\right] n!, \tag{2.9}$$

where $K \in \mathbb{N}^+$ *is the truncation order of the expansion and*

$$T_K(z,n) = 2\sum_{k=1}^{K-1}\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor}\binom{n}{2j}\frac{(-1)^j\,(2\pi k)^{2j}(\log z)^{n-2j}}{(4\pi^2k^2 + (\log z)^2)^n} + R_K(z,n),$$

with

$$|R_K(z,n)| \leqslant \frac{C_K\,|z|}{|\log z + 2K\pi\,|^{n-1}},$$

and C_K *is a finite quantity independent of* z*.*

Proof. Euler's bivariate exponential generating function enumerating the Eulerian polynomials is provided in Foata [13, (3.1)] as

$$f(z,u) = \frac{z-1}{z - e^{(z-1)\,u}} = \sum_{n=0}^{\infty}\mathcal{A}_n(z)\frac{u^n}{n!}\,. \tag{2.10}$$

Substituting $u = t/(z-1)$ and multiplying by $1/(1-z)$ yields

$$g(z,t) = \frac{1}{e^t - z} = -\sum_{n=0}^{\infty} a_n(z)\,t^n, \qquad a_n(z) = \frac{\mathcal{A}_n(z)}{(z-1)^{n+1}\,n!}\,. \tag{2.11}$$

The generating function $g(z,t)$ is meromorphic on $\mathbb{C}$ and has simple poles located at

$$t_k = \log z + 2k\pi i, \qquad k = 0, \pm 1, \pm 2, \ldots.$$

Hence, the power series of $g(z,t)$ is convergent in the disk about the origin of radius $R_0 < |\log z|$. Consider now the Laurent series of $g(z,t)$ about each of the poles t_k. Their principal part is given by

$$\mathrm{PP}(g,t_k) = \frac{\mathrm{Res}(g,t_k)}{t - t_k} = -\sum_{n=0}^{\infty} b_{n,k}(z)\,t^n, \qquad b_{n,k}(z) = \frac{\mathrm{Res}(g,t_k)}{t_k^{n+1}}, \tag{2.12}$$

where $\mathrm{Res}(g,t_k)$ denotes the residue of $g(z,t)$ at t_k which is easily obtained using L'Hôpital's rule as

$$\mathrm{Res}(g,t_k) = \lim_{t\to t_k}\frac{t - t_k}{e^t - z} \overset{\mathrm{H}}{=} \frac{1}{z}\,. \tag{2.13}$$

Following Wilf [41, pp. 142–146], we find that for any fixed integer K the function

$$h_K(z,t) = g(z,t) - \sum_{-K<k<K}\mathrm{PP}(g,t_k) \tag{2.14}$$

$$= -\sum_{n=0}^{\infty} a_n(z)\,t^n + \sum_{n=0}^{\infty}\left[\sum_{-K<k<K} b_{n,k}(z)\right]t^n = \sum_{n=0}^{\infty} c_n(z)\,t^n, \tag{2.15}$$

is analytic at t_k, $k = 0, \pm 1, \dots, \pm[K-1]$, and its power series expansion about the origin converges in the disk of radius $R_K < |t_K|$. By the Cauchy–Hadamard Theorem [14, p. 142], we may bound the growth of the coefficients $c_n(z)$ as $n \to \infty$. In particular, by Theorem 2.4.3 in [41, p. 49], for any given $\epsilon > 0$ there exists an integer N such that for all $n > N$,

$$|c_n(z)| < \left(\frac{1}{R_K} + \epsilon\right)^n = r_K(z)^n \,. \tag{2.16}$$

Comparing the absolute value of the coefficients in (2.15) as n approaches infinity, we see that $|a_n(z)|$ is much larger than $|c_n(z)|$ when $n > N$. More generally, by Theorem 5.2.1 in [41, p. 174] the coefficients $a_n(z)$ can be approximated by

$$a_{n,K}(z) = \sum_{-K<k<K} b_{n,k}(z) + O(r_K(z)^n), \tag{2.17}$$

which yields

$$a_{n,K}(z) = \frac{\mathcal{A}_{n,K}(z)}{(z-1)^{n+1}\, n!} = \sum_{-K<k<K} \operatorname{Res}(g, t_k)/(\log z + 2\pi k i)^{n+1} + O(r_K(z)^n)\,. \tag{2.18}$$

The partial sum in (2.18) is a special case of the series studied by Lindelöf and Wirtinger [42]. Expanding the terms of the sum in binomial series and extracting the term due to the pole closest to the origin, we obtain the Kth order asymptotic formula

$$\begin{gathered}\mathcal{A}_{n,K}(z) = \frac{(z-1)^{n+1}}{z}\left[\frac{1}{(\log z)^{n+1}} + T_K(z, n+1)\right] n! \\ (z \in \mathbb{C}\backslash\{0,\,1\},\ |\arg(z)| < \pi,\ |\log z| < 2K\pi; \quad n \in \mathbb{N}^+; \quad K \in \mathbb{N}^+),\end{gathered} \tag{2.19}$$

where $T_K(z, n)$ is given by

$$T_K(z,n) = 2\sum_{k=1}^{K-1} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j} \frac{(-1)^j\,(2\pi k)^{2j} (\log z)^{n-2j}}{(4\pi^2 k^2 + (\log z)^2)^n} + z\, O(r_K(z)^{n-1})\,. \tag{2.20}$$

For fixed n and $|\log z| < 2K\pi$, $r_K(z)$ is of order $O(|\log z + 2K\pi|^{-1})$ so that (2.19) is accurate up to an arbitrary small error that depends only on K. □

Remark 2.5. $T_1(z, n)$ can be expressed in an alternative form involving Bernoulli polynomials as follows. The generating function for Bernoulli polynomials (see Apostol [1, p. 264]), is given by

$$b(t,x) = \frac{t\, e^{xt}}{e^t - 1} = \sum_{j=0}^{\infty} \mathcal{B}_j(x) \frac{t^n}{j!} \qquad (|t| < 2\pi)\,, \tag{2.21}$$

from which $g(t, z)$ in (2.11) is obtained in terms of $b(t, x)$ as

$$g(t,z) = \frac{\operatorname{Res}(g, t_0)}{t - t_0}\, b(t - t_0, 0) = \frac{1}{e^t - z} \tag{2.22}$$

$$= \frac{1}{z(t - \log z)} + \frac{1}{z}\sum_{j=0}^{\infty} \frac{B_{j+1}}{(j+1)!}(t - \log z)^j, \tag{2.23}$$

where $\mathcal{B}_n(0) = B_n$ denotes the nth Bernoulli number with $B_0 = 1$. Substituting the power series expansion from (2.11) for $g(z,t)$ and expanding the right-hand side of (2.23) as a binomial series gives after simplification

$$\mathcal{A}_{n,1}(z) = \frac{(z-1)^{n+1}}{z}\left[\frac{1}{(\log z)^{n+1}} - \sum_{j=0}^{\infty}\frac{(-1)^j\,B_{n+j+1}}{(n+j+1)\,n!\,j!}(\log z)^j\right]n! \tag{2.24}$$

$$(z \in \mathbb{C}\backslash\{0,\,1\},\ |\arg(z)| < \pi,\ |\log z| < 2\pi;\quad n \in \mathbb{N}^+),$$

from which it can be deduced that

$$T_1(z,n) = \sum_{j=0}^{\infty}\frac{(-1)^{j+1}\,B_{n+j}}{(n+j)\,(n-1)!\,j!}(\log z)^j. \tag{2.25}$$

Remark 2.6. An asymptotic expression for the polylogarithm of negative integer order follows directly from Lemma 2.4. From the Dirichlet series for the polylogarithm provided by (2.2), for negative integer orders we obtain

$$\mathrm{Li}_{-s}(z) = \sum_{n=1}^{\infty} n^s z^n = z\,\Phi(z,-s,1) \tag{2.26}$$

$$(z \in \mathbb{C},\ |z| < 1;\quad s \in \mathbb{N}_0).$$

We further have that $\mathrm{Li}_{-s}(z)$ is related to the Eulerian polynomials [13, (3.2)]) by

$$\mathrm{Li}_{-s}(z) = \frac{z\,\mathcal{A}_s(z)}{(1-z)^{s+1}} \qquad (s \geqslant 0). \tag{2.27}$$

From (2.27), it is observed that $\mathrm{Li}_{-s}(z)$ extends to a meromorphic function on $\mathbb{C}$ with a pole of multiplicity $s+1$ located at $z_0 = 1$. Thus, we may analytically continue (2.27), and after applying Lemma 2.4 we obtain the following approximation for the polylogarithms of negative integer order which is valid beyond the unit disk in which its power series converges

$$\mathrm{Li}_{-s,K}(z) = (-1)^{s+1}\left[\frac{1}{(\log z)^{s+1}} + T_K(z,s+1)\right]s! \tag{2.28}$$

$$(z \in \mathbb{C}\backslash\{1\},\ |\arg(z)| < \pi,\ |\log z| < 2K\pi;\quad n \in \mathbb{N}^+;\quad K \in \mathbb{N}^+).$$

Using the definition for $T_1(z,n)$ in (2.25), an expression for $\mathrm{Li}_{-s,1}(z)$ valid for $|\log z| < 2\pi$ in terms of Bernoulli polynomials can be obtained which is a well-known result in special functions theory (see for example Erdélyi et al. [11, p. 30] and Srivastava et al. [34, p. 198]).

The following theorem provides the asymptotic expansion of the truncated polylogarithm function for $|z| > 1$ which is a key result of this paper.

Theorem 2.7. *For fixed $z \in \mathbb{C}$, $|z| > 1$, $|\arg(z)| < \pi$, fixed $s \in \mathbb{N}^+$, and $m \in \mathbb{N}^+$ with $m \to \infty$, the truncated polylogarithm function is given by the asymptotic expansion*

$$\mathcal{L}_N(z,s,m) = \frac{z}{z-1}\,\frac{z^m}{m}\left[\sum_{n=0}^{N-1}\binom{n+s-1}{n}\frac{\mathcal{A}_n(z)}{(z-1)^n\,m^{n+s-1}} + R_N(z,s,m)\right]. \tag{2.29}$$

The error term is asymptotically bounded by

$$|R_N(z,s,m)| \leqslant C_N \frac{|z-1|\,(N+s-1)!}{|z\,(\log z)^{N+1}|\,m^{N+s-1}}, \tag{2.30}$$

where C_N is a finite quantity independent of m, and $\mathcal{A}_n(z)$ denotes the nth Eulerian polynomial. The error term becomes exponentially small when the series is optimally truncated.

Proof. Consider the definition of the truncated polylogarithm function in terms of the Lerch transcendent given in (2.6) with the integral representation of the Lerch transcendent given in (2.4). We may rewrite (2.4) as a Laplace type integral as

$$\Phi(z,s,a) = \frac{1}{\Gamma(s)} \int_0^\infty \left[t^{s-1} g(z,t) \right] e^{-(a-1)t}\, dt, \qquad g(z,t) = \frac{1}{e^t - z}. \tag{2.31}$$

In the following we restrict z such that $|z| > 1$ and further limit s and a to positive integers to simplify the notation; however, the results are readily extended to $|z| < 1$ and suitable sets of complex values for s and a (see [12] for a discussion).

We continue by noting that the Maclaurin series of $g(z,t)$ is convergent for $0 < t < |\log z|$, so we have

$$g(z,t) = -\sum_{n=0}^{N-1} a_n(z)\, t^n + r_N(z), \qquad a_n(z) = \frac{\mathcal{A}_n(z)}{(z-1)^{n+1}\, n!}, \tag{2.32}$$

which is a series we are already familiar with from Lemma 2.4. However, the interval of convergence for this series is only a small portion of the entire integration region of the integral (2.31). Nevertheless, with the integral being of Laplace type, we obtain an approximation for $\Phi(z,s,a)$ using Watson's Lemma [40] (see also Olver [28, pp. 71–72] for a contemporary version of the proof). As required by Watson's Lemma, we have that for fixed s and z and positive c,

$$|t^{s-1} g(z,t)| = O(e^{ct}),$$

as $t \to \infty$. This allows for the use of a truncated Maclaurin series in the integrant of (2.31) in place of $g(z,t)$. A reversal of summation and integration yields the asymptotic series expansion

$$\Phi_N(z,s,a) = \frac{-1}{(s-1)!} \sum_{n=0}^{N-1} \left[a_n(z) \int_0^\infty t^{n+s-1}\, e^{-(a-1)\,t}\, dt \right] + R_N(z,s,a), \tag{2.33}$$

where

$$R_N(z,s,a) = \frac{-1}{(s-1)!} \int_0^\infty \left[t^{s-1}\, r_N(z) \right] e^{-(a-1)\,t}\, dt, \tag{2.34}$$

and the integral in (2.33) is a special case of the gamma function that using [9, (7.47), p. 294] evaluates to

$$\int_0^\infty t^{n+s-1}\, e^{-(a-1)t}\, dt = \frac{(n+s-1)!}{(a-1)^{n+s}}. \tag{2.35}$$

After simplification and reordering of terms, (2.33) can be written as

$$\Phi_N(z,s,a) = -\sum_{n=0}^{N-1} \binom{n+s-1}{n} \frac{(z-1)^{-n-1}\,\mathcal{A}_n(z)}{(a-1)^{n+s}} + R_N(z,s,a)\,. \tag{2.36}$$

We note that (2.36) is not defined for $a = 1$, and being the result of the termwise integration of a divergent series does not converge. However, as will be shown later via (2.48) and (2.49), optimal truncation given by Definition 2.2 holds. Examining (2.18) to (2.20), from Lemma 2.4 we find that the dominant contribution to the magnitude of $a_n(z)$ in (2.32) is due to the principal part of the Laurent series about the pole nearest the origin, while the contributions of the remaining poles are negligible as n approaches infinity. Hence, an asymptotic bound on the error in (2.36) can be obtained using a first order approximation to $a_N(z)$ based on (2.18), giving

$$|R_N(z,s,a)| \leqslant C_N \frac{(N+s-1)!}{|z\,(\log z)^{N+1}|\,(a-1)^{N+s}}, \tag{2.37}$$

where C_N is a constant.

Now consider an alternative choice of a Laplace type integral given by

$$\Phi(z,s,a) = \frac{1}{\Gamma(s)} \int_0^\infty \left[\, t^{s-1}\tilde{g}(z,t)\,\right] e^{-at}\,dt, \qquad \tilde{g}(z,t) = \frac{1}{1-ze^{-t}}\,. \tag{2.38}$$

The Maclaurin series of $\tilde{g}(z,t)$ resembles that of $g(z,t)$, a fact easily deduced from

$$\left(\frac{\partial}{\partial t}\right)^n \frac{1}{e^t - z}\bigg|_{t=0} = \left(\frac{\partial}{\partial t}\right)^n \frac{1}{1-ze^t}\bigg|_{t=0} = \left(-z\frac{\partial}{\partial z}\right)^n \frac{1}{1-z}, \tag{2.39}$$

and we have

$$\tilde{g}(z,t) = -\sum_{n=0}^{N-1} a_n(z)\,t^n + \tilde{r}_N(z), \qquad a_n(z) = \frac{\mathcal{A}_n(z)}{(z-1)^{n+1}\,n!}, \tag{2.40}$$

which converges for $0 < t < |\log z|$.

Applying Watson's Lemma to (2.38), a slightly different expansion formula is obtained

$$\Phi_N(z,s,a) = -\sum_{n=0}^{N-1} \binom{n+s-1}{n} \frac{(z-1)^{-n-1}\,\mathcal{A}_n(z)}{a^{n+s}} + R_N(z,s,a), \tag{2.41}$$

which, contrary to (2.36), is well defined for all $a > 0$. In general, the error terms in (2.36) and (2.41) are distinct. In particular, an asymptotic error bound for (2.41) is obtained from Lemma 2.4 analogous to (2.37) by replacing $\operatorname{Res}(g,t_k)$ with $\operatorname{Res}(\tilde{g},t_k) = 1$ in (2.18), giving

$$|R_N(z,s,a)| \leqslant C_N \frac{(N+s-1)!}{|(\log z)^{N+1}|\,a^{N+s}}, \tag{2.42}$$

where C_N is a constant.

Combining the expansions (2.36) and (2.41) gives

$$\Phi_N(z,s,a) = -\sum_{n=0}^{N-1} \binom{n+s-1}{n} \frac{(z-1)^{-n-1}\,\mathcal{A}_n(z)}{(a-\lambda_a(z))^{n+s}} + R_N(z,s,a) \tag{2.43}$$
$$(z \in \mathbb{C}\backslash\{1\},\ |\arg(z)| < \pi; \quad s \in \mathbb{N}^+; \quad a \in \mathbb{N}^+,\ a \to \infty; \quad N \in \mathbb{N}^+),$$

where $\lambda_a(z)$ is defined as

$$\lambda_a(z) = \begin{cases} 1 & \text{if } a > 1 \text{ and } |z| > 1, \\ 0 & \text{otherwise,} \end{cases} \qquad z \in \mathbb{C}, \quad a \in \mathbb{N}^+, \tag{2.44}$$

and the asymptotic error term for (2.43) is bounded by

$$|R_N(z,s,a)| \leqslant C_N \frac{(N+s-1)!}{|z^{\lambda_a(z)}(\log z)^{N+1}|\,(a-\lambda_a(z))^{N+s}}, \tag{2.45}$$

where C_N is a constant. The purpose of $\lambda_a(z)$ is to employ (2.36) when $|z| > 1$ and $a > 1$, and (2.41) otherwise. This has the advantage that the magnitude of the asymptotic error term in (2.45) is reduced by a factor proportional to $|z|$ and also ensures that (2.43) is defined for all $a > 0$.

A related expansion formula for $\Phi(z,s,a)$ was defined for $z \notin [1,\infty)$ by Ferreira et al. [12, Theorem 1]. In this case, a Maclaurin expansion of $\tilde{g}(z,t)$ is employed for all $a > 0$, and for $|z| > 1$ the error in the approximation is worse compared with (2.43). Nonetheless, the results of [12] can be used to prove that (2.43) remains valid when $|z| < 1$.

For fixed z and s, the error term in (2.45) is of order $O(a^{-N-s})$ and so is negligible as $a \to \infty$. This is consistent with Poincaré's definition for asymptotic series expansions. Moreover, when (2.43) is optimally truncated according to Definition 2.2, the error term in the expansion is exponentially small. In order to show this, rewrite (2.43) as

$$\Phi_N(z,s,a) = -\sum_{n=0}^{N-1} w_n(z,s,a) + R_N(z,s,a),$$

where

$$w_n(z,s,a) = \binom{n+s-1}{n} \frac{(z-1)^{-n-1}\,\mathcal{A}_n(z)}{(a-\lambda_a(z))^{n+s}}.$$

Then an estimate for the truncation order of the optimally truncated series expansion (or the index of the least term of the series), denoted by $\hat{N}_\star(z,s,a)$, is obtained by bounding the ratio of successive series terms as follows:

$$1 \geqslant \left| \frac{w_{n+1}(z,s,a)}{w_n(z,s,a)} \right| \tag{2.46}$$

$$\geqslant \frac{|\,\mathcal{A}_{n+1}(z)\,|}{|\,\mathcal{A}_n(z)\,|} \frac{n+s}{(n+1)(z-1)(a-\lambda_a(z))} \sim \frac{n+s}{(a-\lambda_a(z))\,|\log z|}, \tag{2.47}$$

where the first order approximation for Eulerian polynomials provided by Lemma 2.4 has been used. Solving (2.47) for n and noting that $N_\star - 1 = n$, we have

$$N_\star \sim \hat{N}_\star(z,s,a) \leqslant \big\lfloor (a-\lambda_a(z))\,|\log z|\,\big\rfloor - s + 1\,. \tag{2.48}$$

From Ursell's strong form of Watson's Lemma (see Ursell [38] and Paris [29, p. 76]), we have that when (2.43) is truncated such that the order of the expansion is given by

$$\hat{N}_\star(z,a) = \big\lfloor r\,(a-\lambda_a(z))\,\big\rfloor + 1 \qquad (\,0 < r < |\log z|\,), \tag{2.49}$$

the error term satisfies $R_{\hat{N}_\star}(z,a) = O(e^{-r\,(a-\lambda_a(z))})$ as $a \to \infty$. Noting that for fixed s (2.48) and (2.49) are asymptotically equivalent, we conclude that the approximation error of (2.43) is exponentially small when the series is optimally truncated.

For $|z| > 1$ the truncated polylogarithm as defined in (2.6) may be expressed in terms of the optimally truncated series expansion of the Lerch transcendent given in (2.43), and we have

$$\mathcal{L}_{\widetilde{N}_\star,N_\star}(z,s,m) = z\,\Phi_{\widetilde{N}_\star}(z,s,1) - z^{m+1}\Phi_{N_\star}(z,s,m+1) \qquad (|z|>1). \tag{2.50}$$

From (2.48) we observe that for fixed s and z, the optimal truncation order of (2.43) depends only on the value of a. Contrary to the second term on the right-hand side of (2.50), the value of a in the first term is not a function of m, but is fixed. This implies that $\widetilde{N}_\star$ is fixed and so the contribution of the first term is a finite quantity, which when omitted results in a constant error that is negligible as $m \to \infty$. Therefore, for fixed s and z, $|z| > 1$, we have

$$\mathcal{L}_N(z,s,m) = -z^{m+1}\Phi_N(z,s,m+1) + O(1) \qquad (m \to \infty), \tag{2.51}$$

from which it follows that

$$\begin{gathered}\mathcal{L}_N(z,s,m) = \frac{z}{z-1}\,\frac{z^m}{m}\left[\sum_{n=0}^{N-1}\binom{n+s-1}{n}\frac{\mathcal{A}_n(z)}{(z-1)^n\,m^{n+s-1}} + R_N(z,s,m)\right] \\ (z\in\mathbb{C},\ |z|>1,\ |\arg(z)|<\pi;\quad s\in\mathbb{N}^+;\quad m\in\mathbb{N}^+,\ m\to\infty;\quad N\in\mathbb{N}^+),\end{gathered} \tag{2.52}$$

where

$$|R_N(z,s,m)| \leqslant C_N\,\frac{|z-1|\,(N+s-1)!}{|z\,(\log z)^{N+1}|\,m^{N+s-1}}, \tag{2.53}$$

and C_N is a finite quantity independent of m that absorbs the constant error term observed in (2.51). □

3. A prime number theorem analog for finite fields

In this section we provide a proof of the finite field analog of the prime counting function given in Theorem 1.1. The truncated polylogarithm function in the previous section is central to this proof.

Proof of Theorem 1.1. From (1.11) and Definition 2.1 it directly follows that

$$\pi_q(m) = \mathcal{L}(q,1,m) + O\left(\frac{q^{\frac{m}{2}}}{m}\right). \tag{3.1}$$

Replacing the truncated polylogarithm function in (3.1) with its asymptotic series expansion from (2.29) gives

$$\begin{gathered}\pi_{q,N}(m) \sim \frac{q}{q-1}\frac{q^m}{m}\left[\sum_{n=0}^{N-1}\frac{\mathcal{A}_n(q)}{(q-1)^n\, m^n} + R_{q,N}(m)\right]\\ (q\in\mathbb{R},\ q>1;\quad m\in\mathbb{N}^+,\ m\to\infty;\quad N\in\mathbb{N}^+),\end{gathered} \tag{3.2}$$

where

$$R_{q,N}(m) \leqslant C_N\frac{q-1}{q\,(\log q)^{N+1}}\frac{N!}{m^N}, \tag{3.3}$$

and C_N is a finite quantity independent of m. Substituting $X = q^m$ yields Theorem 1.1, which completes the proof. □

Remark 3.1. Aside from the structural resemblance of the asymptotic expansion formulas for the prime and prime polynomial counting functions, there is an interesting and less obvious connection between the two series expansions. If $q^m = x$ and $q = 1+\varepsilon$ with $\varepsilon \to 0^+$, then

$$\lim_{\varepsilon\to 0^+}\pi_{q,N}(m) = \pi_N(x). \tag{3.4}$$

This result is easily obtained from (1.6), (1.7), (1.12), (1.13) and noting that

$$\lim_{\varepsilon\to 0^+}\mathcal{A}_n(1+\varepsilon) = n! \qquad \text{and} \qquad \lim_{\varepsilon\to 0^+}\frac{\log(1+\varepsilon)}{\varepsilon} = 1.$$

4. Computational results

In this section, numerical examples are presented to illustrate the accuracy of the asymptotic expansions presented in Lemma 2.4 and Theorems 1.1 and 2.7 (Tables 1–10 and Figures 1 and 2). In all tables, ϵ_{abs} and ϵ_{rel} indicate absolute and relative approximation errors, respectively. Round parenthesis that follow numeric entries indicate the power of 10 multiplying the entry. Optimal truncation is employed according to Definition 2.2. The index of the least expansion term (or the truncation order of the optimally truncated expansion) is indicated by $N_\star$. For comparison purposes, an expansion truncated at the term $\underline{N}$ which provides the smallest absolute approximation error is given. This term was determined empirically and thus is referred to as empirical truncation.

TABLE 1. Asymptotic approximations to $\mathcal{A}_{n,K}(z)$ for $z = 0.2$ and three values of n using the series expansion in (2.9). As K and n grow the precision of the approximation increases.

	n		
K	3	6	12
1	1.83140457	26.99061729332598	270,839.39932067465073432075399
2	1.83887199	26.98754353174761	270,839.39845645427252513115984
4	1.83988563	26.98752012507686	270,839.39845767155530943918198
8	1.83998774	26.98752000071216	270,839.39845767167998984994921
16	1.83999859	26.98752000000460	270,839.39845767167999999908535
32	1.83999983	26.98752000000003	270,839.39845767167999999999991
	Exact value of $\mathcal{A}_n(z)$ for $z = 0.2$		
	1.84	26.98752	270,839.39845767168
	Relative approximation error at $K = 32$		
	9.1312e (−8)	1.2011 (−15)	3.4493 (−28)

TABLE 2. Asymptotic approximations to $\mathcal{A}_{n,K}(z)$ for $z = 2$ and three values of n using the series expansion in (2.9).

	n		
K	3	6	12
1	12.99629051	4,683.00124726225744	28,091,567,594.98157244071518917609953
2	12.99969112	4,683.00000565855005	28,091,567,594.99999841159980097513895
4	12.99997138	4,683.00000002665499	28,091,567,594.99999999989018454893781
8	12.99999699	4,683.00000000014752	28,091,567,594.99999999999999182402913
16	12.99999966	4,683.00000000000095	28,091,567,594.99999999999999999927866
32	12.99999996	4,683.00000000000001	28,091,567,594.99999999999999999999993
	Exact value of $\mathcal{A}_n(z)$ for $z = 2$		
	13	4,683	28,091,567,595
	Relative approximation error at $K = 32$		
	3.1563 (−9)	1.4222 (−18)	2.6095 (−33)

Table 3. Asymptotic approximations to $\mathcal{A}_{n,K}(z)$ for $z = -7 + 11i$ and three values of n using the series expansion in (2.9).

	n		
K	3	6	12
1	$-93.11893 -$ $85.45339i$	$-531{,}598.89055533 +$ $874{,}262.38289886i$	$475{,}150{,}327{,}070{,}173.35238548 -$ $3{,}061{,}637{,}862{,}706{,}965.99701066i$
2	$-100.51924 -$ $109.67358i$	$-528{,}605.03041391 +$ $945{,}617.73392225i$	$466{,}334{,}909{,}934{,}807.37188130 -$ $3{,}040{,}705{,}750{,}998{,}494.88615944i$
4	$-99.14327 -$ $110.03634i$	$-528{,}881.39280995 +$ $945{,}440.53805549i$	$466{,}335{,}674{,}969{,}793.31438924 -$ $3{,}040{,}705{,}929{,}911{,}602.98592405i$
8	$-99.01480 -$ $110.00537i$	$-528{,}881.99788632 +$ $945{,}439.00909223i$	$466{,}335{,}674{,}976{,}418.96557659 -$ $3{,}040{,}705{,}929{,}869{,}754.08981405i$
16	$-99.00168 -$ $110.00065i$	$-528{,}881.99998840 +$ $945{,}439.00005903i$	$466{,}335{,}674{,}976{,}418.00011617 -$ $3{,}040{,}705{,}929{,}869{,}751.00026510i$
32	$-99.00020 -$ $110.00008i$	$-528{,}881.99999992 +$ $945{,}439.00000042i$	$466{,}335{,}674{,}976{,}418.00000001 -$ $3{,}040{,}705{,}929{,}869{,}751.00000003i$
	Exact value of $\mathcal{A}_n(z)$ for $z = -7 + 11i$		
	$-99 - 110i$	$-528{,}882 + 945{,}439i$	$466{,}335{,}674{,}976{,}418 -$ $3{,}040{,}705{,}929{,}869{,}751i$
	Relative approximation error at $K = 32$		
	1.4553 (−6)	3.9132 (−13)	9.5960 (−24)

Table 4. The Eulerian number triangle generated using Definition 2.3. Due to the symmetry of these numbers, only about half of the entries in each row have to be computed. This triangle permits the efficient evaluation of the asymptotic expansion of the truncated polylogarithm function as no derivatives need to be calculated for the Maclaurin series on which (2.29) is based.

	k						
n	0	1	2	3	4	5	$\cdots$
0	1						
1	1	0					
2	1	1	0				
3	1	4	1	0			
4	1	11	11	1	0		
5	1	26	66	26	1	0	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

TABLE 5. Asymptotic approximations to $\mathcal{L}_N(z, s, m)$ for $z = 3$, $s = 1$ and six values of m using the series expansion from Theorem 2.7 for optimal and empirical truncation. Empirical truncation provides the smallest absolute approximation error. These results confirm that optimal truncation gives very accurate results with a rapid degradation in approximation accuracy as N exceeds $N_\star$. The optimal and empirical truncation orders are in close agreement (see also Figure 1 for a comparison of optimal versus empirical truncation, and Figure 2 for the absolute approximation error under optimal truncation).

	m					
N	1	2	4	8	10	12
1	$\underline{4.5000}$	$\underline{6.7500}$	30.3750	1,230.1875	8,857.3500	66,430.1250
2	6.7500	8.4375	34.1719	1,307.0742	9,300.2175	69,198.0469
3	11.2500	10.1250	36.0703	1,326.2959	9,388.7910	69,659.3672
4	23.6250	12.4453	$\underline{37.3755}$	1,332.9034	9,413.1487	69,765.0864
5	68.6250	16.6641	38.5620	1,335.9067	9,422.0061	69,797.1226
6	273.3750	26.2617	39.9117	1,337.6149	9,426.0362	69,809.2696
7	1,391.6250	52.4707	41.7545	1,338.7811	9,428.2372	69,814.7980
8	8,516.8125	135.9690	44.6900	$\underline{1,339.7099}$	9,429.6397	69,817.7335
9	60,401.8125	439.9827	50.0340	1,340.5553	9,430.6609	69,819.5149
10	485,451.5625	1,685.2456	60.9787	1,341.4211	$\underline{9,431.4975}$	69,820.7309
11	4,354,421.0625	7,352.6814	85.8844	1,342.4061	9,432.2591	69,821.6534
12	43,092,987.1875	35,725.6546	148.2274	1,343.6390	9,433.0216	$\underline{69,822.4230}$
13	466,229,337.1875	190,682.8141	318.4684	1,345.3222	9,433.8544	69,823.1236
14	5,473,248,288.1875	1,107,495.3661	822.0886	1,347.8121	9,434.8399	69,823.8145
15	69,279,439,093.3125	6,949,126.6044	2,426.5405	1,351.7782	9,436.0958	69,824.5481
Exact value of $\mathcal{L}(z, s, m)$ for $z = 3$ and $s = 1$						
	3	7.5	36.75	1,339.4036	9,431.3036	69,822.3263
Absolute and relative approximation error for empirical ($\underline{N}$) and optimal ($N_\star$) truncation						
$\underline{N}$	1	1	4	8	10	12
$\epsilon_{abs}\left[\mathcal{L}_{\underline{N}}\right]$	1.5	0.75	0.6255	0.3063	0.1940	0.0967
$\%\,\epsilon_{rel}\left[\mathcal{L}_{\underline{N}}\right]$	50	10	1.7	0.023	0.0021	0.00014
$N_\star$	1	2	4	8	10	13
$\epsilon_{abs}\left[\mathcal{L}_{N_\star}\right]$	1.5	0.9375	0.6255	0.3063	0.1940	0.7973
$\%\,\epsilon_{rel}\left[\mathcal{L}_{N_\star}\right]$	50	13	1.7	0.023	0.0021	0.0011

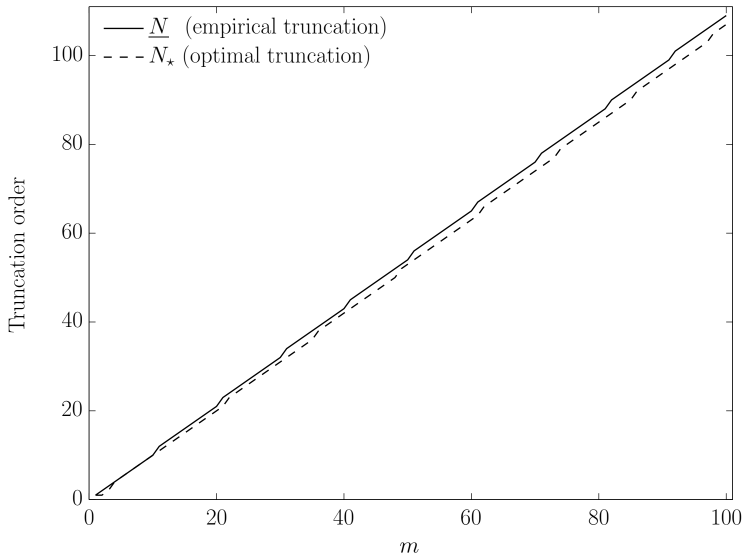

FIGURE 1. Comparison of empirical and optimal truncation of $\mathcal{L}_N(z,s,m)$ for $z=3$, $s=1$.

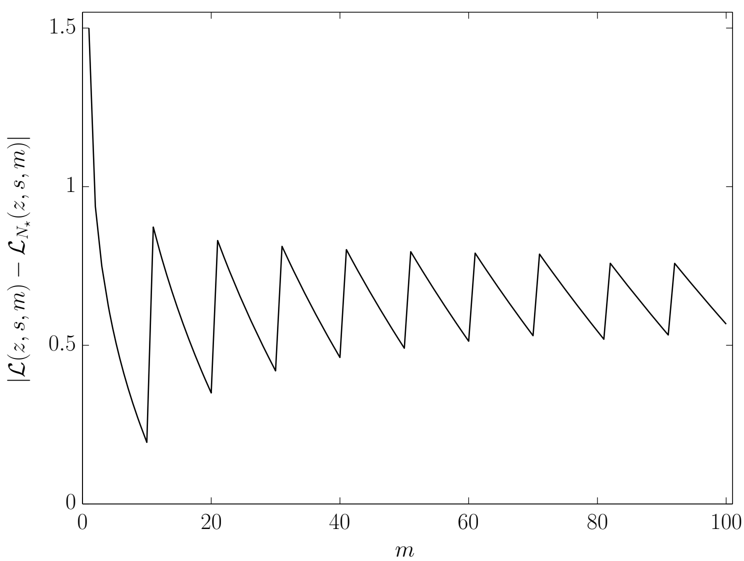

FIGURE 2. Absolute approximation error of $\mathcal{L}_N(z,s,m)$ for $z=3$, $s=1$ under optimal truncation.

TABLE 6. Relative approximation error of $\mathcal{L}_N(z, s, m)$ for $z = 3$, $s = 1$ under optimal truncation with the maximum truncation order limited to $N = 500$. As m grows the optimal number of terms increases allowing for an ever more accurate approximation. For $m = 100$, the relative error is less than 1.04×10^{-46}, which is remarkable.

		m			
	N	10	100	500	1000
$\mathcal{L}_N$	1	8.85735 (3)	7.73066 (45)	1.09081 (236)	1.98311 (474)
$\epsilon_{rel}[\mathcal{L}_N]$	1	6.08562 (−2)	5.07695 (−3)	1.00302 (−3)	5.00752 (−4)
	5	9.85814 (−4)	4.79114 (−9)	1.47063 (−12)	4.57272 (−14)
	10	2.05661 (−5)	9.51681 (−15)	8.97529 (−22)	8.68040 (−25)
	25	—	1.17310 (−26)	3.15510 (−44)	9.17894 (−52)
	50	—	3.15964 (−38)	2.07677 (−73)	1.75535 (−88)
		Exact value of $\mathcal{L}(z, s, m)$ for $z = 3$ and $s = 1$			
		9.43130 (3)	7.77011 (45)	1.09190 (236)	1.98410 (474)
		(Optimal) truncation at $N_\star^{500} = \min(500, N_\star)$			
$N_\star^{500}$		10	109	500	500
$\mathcal{L}_{N_\star^{500}}$		9.43150 (3)	7.77011 (45)	1.09190 (236)	1.98410 (474)
$\epsilon_{rel}[\mathcal{L}_{N_\star^{500}}]$		1.01311 (−4)	1.03734 (−46)	1.25639 (−235)	5.15309 (−387)

TABLE 7. Relative approximation error of $\mathcal{L}_N(z, s, m)$ for $z = 1.25$ and $s = 2$ under optimal truncation with the maximum truncation order limited to $N = 500$.

		m			
	N	10	100	500	1000
$\mathcal{L}_N$	1	4.65661 (−1)	2.45455 (6)	5.70212 (43)	4.06427 (91)
$\epsilon_{rel}[\mathcal{L}_N]$	1	8.21847 (−1)	8.53779 (−2)	1.61821 (−2)	8.04474 (−3)
	5	—	1.62688 (−4)	3.92053 (−8)	1.19469 (−9)
	10	—	3.53141 (−6)	1.32108 (−13)	1.22567 (−16)
	25	—	—	3.04892 (−25)	7.87530 (−33)
	50	—	—	1.09064 (−36)	6.70461 (−52)
		Exact value of $\mathcal{L}(z, s, m)$ for $z = 1.25$ and $s = 2$			
		2.61382	2.68367 (6)	5.79591 (43)	4.09724 (91)
		(Optimal) truncation at $N_\star^{500} = \min(500, N_\star)$			
$N_\star^{500}$		1	21	110	222
$\mathcal{L}_{N_\star^{500}}$		4.65661 (−1)	2.68367 (6)	5.79591 (43)	4.09724 (91)
$\epsilon_{rel}[\mathcal{L}_{N_\star^{500}}]$		8.21847 (−1)	8.15237 (−7)	3.80060 (−44)	5.32815 (−92)

TABLE 8. Relative approximation error of $\mathcal{L}_N(z, s, m)$ for $z = -9 + 2.5i$ and $s = 5$ under optimal truncation with the maximum truncation order limited to $N = 500$.

		m			
	N	10	100	500	1000
$\mathcal{L}_N$	1	$[-4.20914 - 1.81022i]\,(4)$	$[-4.00940 - 9.04816i]\,(86)$	$[-4.13030 + 1.79962i]\,(471)$	$[1.53050 - 1.56284i]\,(955)$
$\epsilon_{rel}[\mathcal{L}_N]$	1	6.56708 (−2)	4.99458 (−3)	9.75804 (−4)	4.86484 (−4)
	5	2.70157 (−4)	9.54729 (−10)	2.97703 (−13)	9.27309 (−15)
	10	1.82678 (−4)	2.47836 (−17)	2.51721 (−24)	2.45558 (−27)
	25	1.82932 (−4)	1.16817 (−35)	3.74139 (−53)	1.10843 (−60)
	50	—	2.90336 (−59)	3.03589 (−94)	2.67054 (−109)
		Exact values of $\mathcal{L}(z, s, m)$ for $z = -9 + 2.5i$ and $s = 5$			
		$[-3.98121 - 1.64277i]\,(4)$	$[-4.00090 - 8.99971i]\,(86)$	$[-4.12597 + 1.79890i]\,(471)$	$[1.52960 - 1.56228i]\,(955)$
		(Optimal) truncation at $N_\star^{500} = \min(500, N_\star)$			
$N_\star^{500}$		31	359	500	500
$\mathcal{L}_{N_\star^{500}}$		$[-3.98045 - 1.64295i]\,(4)$	$[-4.00090 - 8.99971i]\,(86)$	$[-4.12597 + 1.79890i]\,(471)$	$[1.52960 - 1.56228i]\,(955)$
$\epsilon_{rel}[\mathcal{L}_{N_\star^{500}}]$		1.82931 (−4)	7.99936 (−87)	1.75037 (−471)	4.10294 (−638)

TABLE 9. Absolute approximation error for the prime polynomial counting function $\mathcal{L}(q, 1, m)$ and $\pi_{q,N}(m)$, $q = 2$ under optimal truncation with the maximum truncation order limited to $N = 1000$.

m	$\pi_q(m),\ q = 2$	$\epsilon_{abs}[\mathcal{L}(q, 1, m)]$	$\epsilon_{abs}[\pi_{q,1}(m)]$	$N_\star^{1000}$	$\epsilon_{abs}[\pi_{q,N_\star^{1000}}(m)]$
2^1	3.000000	1.000000	1.000000	1	1.000000
2^2	8.000000	2.666667	0	2	2.000000
2^3	7.100000 (1)	7.019048	7.000000	5	6.796875
2^4	8.800000 (3)	4.579598 (1)	6.080000 (2)	11	4.597922 (1)
2^5	2.777378 (8)	4.505926 (3)	9.302341 (6)	22	4.506007 (3)
2^6	5.859217 (17)	1.389416 (8)	9.460913 (15)	44	1.389416 (8)
2^7	5.359458 (36)	2.929609 (17)	4.254646 (34)	88	2.929609 (17)
2^8	9.082015 (74)	2.679729 (36)	3.575822 (72)	177	2.679729 (36)
2^9	5.247715 (151)	4.541008 (74)	1.028980 (149)	354	4.541008 (74)
2^{10}	3.514558 (305)	2.623857 (151)	3.438916 (302)	709	2.623857 (151)
2^{11}	3.157501 (613)	1.757279 (305)	1.543257 (610)	1000	1.757279 (305)
2^{12}	5.100801 (1229)	1.578750 (613)	1.245921 (1226)	1000	1.578750 (613)
2^{13}	2.663285 (2462)	2.550401 (1229)	3.251874 (2458)	1000	5.714684 (1275)
2^{14}	1.452398 (4928)	1.331642 (2462)	8.865814 (4923)	1000	2.627392 (3440)
2^{15}	8.639552 (9859)	7.261988 (4927)	2.636743 (9855)	1000	1.391382 (8071)
2^{16}	6.114381 (19723)	4.319776 (9859)	9.330090 (19718)	1000	8.982976 (17633)
2^{17}	6.125126 (39451)	3.057191 (19723)	4.673172 (39446)	1000	8.304716 (37060)
2^{18}	1.229349 (78908)	3.062563 (39451)	4.689629 (78902)	1000	1.546957 (76216)
2^{19}	9.904387 (157820)	6.146743 (78907)	1.889119 (157815)	1000	1.159938 (154828)
2^{20}	1.285772 (315647)	4.952193 (157820)	1.226210 (315641)	1000	1.403384 (312353)

Table 10. Relative approximation error for the prime polynomial counting function estimates $\mathcal{L}(q,1,m)$ and $\pi_{q,N}(m)$, $q=2$ under optimal truncation with the maximum truncation order limited to $N=1000$. Using optimal truncation significantly improves the approximation accuracy when compared to a first order approximation.

m	$\pi_q(m),\ q=2$	$\epsilon_{rel}\left[\mathcal{L}(q,1,m)\right]$	$\epsilon_{rel}\left[\pi_{q,1}(m)\right]$	$N_\star^{1000}$	$\epsilon_{rel}\left[\pi_{q,N_\star^{1000}}(m)\right]$
2^1	3.000000	3.333333 (−1)	3.333333 (−1)	1	3.333333 (−1)
2^2	8.000000	3.333333 (−1)	0	2	2.500000 (−1)
2^3	7.100000 (1)	9.885983 (−2)	9.859155 (−2)	5	9.573063 (−2)
2^4	8.800000 (3)	5.204089 (−3)	6.909091 (−2)	11	5.224911 (−3)
2^5	2.777378 (8)	1.622367 (−5)	3.349325 (−2)	22	1.622396 (−5)
2^6	5.859217 (17)	2.371334 (−10)	1.614706 (−2)	44	2.371334 (−10)
2^7	5.359458 (36)	5.466241 (−20)	7.938575 (−3)	88	5.466241 (−20)
2^8	9.082015 (74)	2.950589 (−39)	3.937256 (−3)	177	2.950589 (−39)
2^9	5.247715 (151)	8.653305 (−78)	1.960815 (−3)	354	8.653305 (−78)
2^{10}	3.514558 (305)	7.465682 (−155)	9.784773 (−4)	709	7.465682 (−155)
2^{11}	3.157501 (613)	5.565411 (−309)	4.887590 (−4)	1000	5.565411 (−309)
2^{12}	5.100801 (1229)	3.095103 (−617)	2.442600 (−4)	1000	3.095103 (−617)
2^{13}	2.663285 (2462)	9.576147 (−1234)	1.221001 (−4)	1000	2.145728 (−1187)
2^{14}	1.452398 (4928)	9.168579 (−2467)	6.104261 (−5)	1000	1.809003 (−1488)
2^{15}	8.639552 (9859)	8.405514 (−4933)	3.051944 (−5)	1000	1.610479 (−1789)
2^{16}	6.114381 (19723)	7.064944 (−9865)	1.525925 (−5)	1000	1.469155 (−2090)
2^{17}	6.125126 (39451)	4.991229 (−19729)	7.629511 (−6)	1000	1.355844 (−2391)
2^{18}	1.229349 (78908)	2.491208 (−39457)	3.814726 (−6)	1000	1.258355 (−2692)
2^{19}	9.904387 (157820)	6.206082 (−78914)	1.907356 (−6)	1000	1.171136 (−2993)
2^{20}	1.285772 (315647)	3.851534 (−157827)	9.536761 (−7)	1000	1.091472 (−3294)

5. Conclusion

An asymptotic approximation formula was derived for Eulerian polynomials and the polylogarithm of negative integer orders with an arbitrary small error. An accurate approximation of the Lerch transcendent $\Phi(z,s,a)$ was presented such that, when optimally truncated, provides an exponentially accurate Poincaré-type expansion formula for $|z|>1$. Finally, the asymptotic expansion of the Lerch transcendent was used to compute the truncated polylogarithm function $\mathcal{L}(z,s,m)$ which in turn provides an approximation of the prime polynomial counting function that is analogous to the well-known asymptotic expansion of the prime number theorem. This approximation allows for efficient computation and provides significantly better numerical accuracy than the results presented in [21], [31], [39]. The expansion formulas for the Eulerian polynomials and truncated polylogarithm function are general and find applications in many areas other than the enumeration of irreducible polynomials. The accuracy of the expressions presented was verified using extensive numerical evaluation.

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