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# UNIFORM ERROR ESTIMATES OF THE CONSERVATIVE FINITE DIFFERENCE METHOD FOR THE ZAKHAROV SYSTEM IN THE SUBSONIC LIMIT REGIME

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Abstract. We rigorously analyze the error estimates of the conservative finite difference method (CNFD) for the Zakharov system (ZS) with a dimensionless parameter $\varepsilon \in (0,1]$, which is inversely proportional to the ion acoustic speed. When $\varepsilon \to 0^+$, ZS collapses to the standard nonlinear Schrödinger equation (NLS). In the subsonic limit regime, i.e., $\varepsilon \to 0^+$, there exist highly oscillatory initial layers in the solution. The initial layers propagate with $O(\varepsilon)$ wavelength in time, $O(1)$ and $O(\varepsilon^2)$ amplitudes, for the ill-prepared initial data and well-prepared initial data, respectively. This oscillatory behavior brings significant difficulties in analyzing the errors of numerical methods for solving the Zakharov system. In this work, we show the CNFD possesses the error bounds $h^2/\varepsilon+\tau^2/\varepsilon^3$ in the energy norm for the ill-prepared initial data, where $h$ is mesh size and $\tau$ is time step. For the well-prepared initial data, CNFD is uniformly convergent for $\varepsilon \in (0,1]$, with second-order accuracy in space and $O(\tau^{4/3})$ accuracy in time. The main tools involved in the analysis include cut-off technique, energy methods, $\varepsilon$-dependent error estimates of the ZS, and $\varepsilon$-dependent error bounds between the numerical approximate solution of the ZS and the solution of the limit NLS. Our approach works in one, two and three dimensions, and can be easily extended to the generalized Zakharov system and nonconservative schemes. Numerical results suggest that the error bounds are sharp for the plasma densities and the error bounds of the CNFD for the electric fields are the same as those of the splitting methods.

## 1. Introduction

In this paper, we consider the Zakharov system (ZS) in $d$ ($d = 1, 2, 3$) dimensions describing the propagation of Langmuir waves in plasma [6, 18, 19, 23],

(1.1)
$$\begin{cases} i\partial_t E^\varepsilon(\mathbf{x},t) + \nabla^2 E^\varepsilon(\mathbf{x},t) - N^\varepsilon(\mathbf{x},t)E^\varepsilon(\mathbf{x},t) = 0, & \mathbf{x}\in\mathbb{R}^d,\ t>0,\\ \varepsilon^2\partial_{tt}N^\varepsilon(\mathbf{x},t) - \nabla^2 N^\varepsilon(\mathbf{x},t) - \nabla^2|E^\varepsilon(\mathbf{x},t)|^2 = 0, & \mathbf{x}\in\mathbb{R}^d,\ t>0,\\ E^\varepsilon(\mathbf{x},0)=E_0(\mathbf{x}),\quad N^\varepsilon(\mathbf{x},0)=N_0^\varepsilon(\mathbf{x}),\quad \partial_t N^\varepsilon(\mathbf{x},0)=N_1^\varepsilon(\mathbf{x}), & \mathbf{x}\in\mathbb{R}^d, \end{cases}$$

where $E^\varepsilon := E^\varepsilon(\mathbf{x},t)$ is a complex function describing the envelope of electric field and $N^\varepsilon := N^\varepsilon(\mathbf{x},t)$ is a real function representing the plasma ion density deviation

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from its equilibrium position, $\mathbf{x}$ is the spatial coordinate, $t$ is the time, $0 < \varepsilon \le 1$ is a dimensionless parameter inversely proportional to the ion acoustic speed, and $\Delta = \nabla^2$ is the Laplace operator in $d$-dimensional space.

In the subsonic limit, i.e., $\varepsilon \to 0^+$, as proven by [1, 14, 16, 18], the Zakharov system (1.1) converges to the standard cubic nonlinear Schrödinger equation (NLS),

$$\begin{cases} i\partial_t E(\mathbf{x},t) + \nabla^2 E(\mathbf{x},t) + |E(\mathbf{x},t)|^2 E(\mathbf{x},t) = 0, & \mathbf{x} \in \mathbb{R}^d,\ t > 0, \\ E(\mathbf{x},0) = E_0(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^d. \end{cases} \tag{1.2}$$

However, due to the incompatibility of the initial data $(E_0, N_0^\varepsilon, N_1^\varepsilon)$ of (1.1), convergence rates are different. Noticing that for the solution $E(\mathbf{x},t)$ of (1.2),

$$\partial_t |E(\mathbf{x},t)|^2\big|_{t=0} = -2\,\mathrm{Im}(\Delta E_0(\mathbf{x})\overline{E_0(\mathbf{x})}),$$

and the compatibility of the initial data between the ZS (1.1) and the NLS (1.2) can be characterized as

$$N_0^\varepsilon(\mathbf{x}) = -|E_0(\mathbf{x})|^2 + \varepsilon^\alpha w_0(\mathbf{x}), \quad N_1^\varepsilon(\mathbf{x}) = 2\,\mathrm{Im}(\Delta E_0(\mathbf{x})\overline{E_0(\mathbf{x})}) + \varepsilon^{\beta-1} w_1(\mathbf{x}), \tag{1.3}$$

where $\alpha, \beta \ge 0$ are nonnegative parameters, $\mathrm{Im}(f)$ and $\bar{f}$ denote the imaginary part and complex conjugate of function $f$, respectively.

As $\varepsilon \to 0^+$, the solution of ZS (1.1) oscillates with wavelength $O(\varepsilon)$ in time $t$ due to the wave operator term and/or the initial data $(E_0, N_0^\varepsilon, N_1^\varepsilon)$. Based on the theoretical results [14, 16, 18], we can find the asymptotic form of the solution $(E^\varepsilon(\mathbf{x},t), N^\varepsilon(\mathbf{x},t))$ for Zakharov system (1.1) as $\varepsilon \to 0^+$,

$$\begin{aligned} E^\varepsilon(\mathbf{x},t) =& E(\mathbf{x},t) + \varepsilon^2 E^{(1)}(\mathbf{x},t) + \varepsilon^3 E^{(2)}(\mathbf{x},t) + \cdots \\ &+ \varepsilon^{1+\min\{\alpha,\beta,2\}} U(\mathbf{x}, t/\varepsilon) + \varepsilon^{2+\min\{\alpha,\beta,1\}} U^{(1)}(\mathbf{x}, t/\varepsilon) + \cdots, \\ N^\varepsilon(\mathbf{x},t) =& -|E(\mathbf{x},t)|^2 + \varepsilon^2 N^{(1)}(\mathbf{x},t) + \varepsilon^3 N^{(2)}(\mathbf{x},t) + \cdots \\ &+ \varepsilon^\alpha \Lambda^{(1)}(\mathbf{x}, t/\varepsilon) + \varepsilon^\beta \Lambda^{(2)}(\mathbf{x}, t/\varepsilon) + \varepsilon^{1+\min\{\alpha,\beta,1\}} \Lambda^{(3)}(\mathbf{x}, t/\varepsilon) + O(\varepsilon^2), \end{aligned}$$

where $E(\mathbf{x},t)$ is the solution of the limit NLS (1.2), $\Lambda^{(1)}(\mathbf{x},s)$ and $\Lambda^{(2)}(\mathbf{x},s)$ are the first and the second initial layers, respectively, satisfying the following equations:

$$\begin{aligned} &\partial_{ss}\Lambda^{(1)}(\mathbf{x},s) - \nabla^2\Lambda^{(1)}(\mathbf{x},s) = 0, \quad \partial_{ss}\Lambda^{(2)}(\mathbf{x},s) - \nabla^2\Lambda^{(2)}(\mathbf{x},s) = 0,\ \mathbf{x} \in \mathbb{R}^d,\ s > 0, \\ &\Lambda^{(1)}(\mathbf{x},0) = w_0(\mathbf{x}), \quad \partial_s\Lambda^{(1)}(\mathbf{x},0) = 0, \quad \Lambda^{(2)}(\mathbf{x},0) = 0, \quad \partial_s\Lambda^{(2)}(\mathbf{x},0) = w_1(\mathbf{x}). \end{aligned} \tag{1.4}$$

From the results in literature [1, 14, 16] and the above expansion, it is clear that leading oscillations due to the initial layers are determined by $\alpha$ and $\beta$. Moreover, we can classify the initial data into well-prepared ($\alpha \ge 2$, $\beta \ge 2$), less-ill-prepared ($1 \le \alpha, \beta < 2$) and ill-prepared ($0 \le \alpha, \beta < 1$) cases. Indeed, when $\alpha \ge 2$ and $\beta \ge 2$, the leading oscillation in the density $N^\varepsilon$ comes from the $\varepsilon^2\partial_{tt}$ term in the equation, and $\partial_{tt}N^\varepsilon$ is uniformly bounded w.r.t. $\varepsilon$; when $1 \le \alpha, \beta < 2$, the leading oscillation term in the density $N^\varepsilon$ is of order $O(\varepsilon^{\min\{\alpha,\beta\}})$ which comes from the initial layers, and $\partial_t N^\varepsilon$ is uniformly bounded w.r.t. $\varepsilon$; when $0 \le \alpha, \beta < 1$, the leading oscillation comes from the initial layers, and $\partial_t N^\varepsilon = O(\varepsilon^{\min\{\alpha,\beta\}-1})$ is unbounded w.r.t. $\varepsilon$.

There have been extensive theoretical studies on the ZS (1.1), including the well-posedness of the Cauchy problem [1, 9, 13, 18], the subsonic limit [1, 14, 16, 18] and the blow-up [15], etc. For the numerical part, different kinds of numerical methods have been proposed for solving the Zakharov system and/or the nonlinear

Schrödinger equation [2–4, 6, 8, 10, 12, 16]. For the case $\varepsilon = O(1)$, Glassey [10] obtained the first-order convergence in both spatial and temporal discretizations for the conservative finite difference scheme solving the ZS (1.1). Later, Chang et al. [8] improved the estimates to the optimal second-order convergence. Despite the finite difference methods, exponential-wave-integrator spectral methods [6, 17], the Legendre-Galerkin method [11], the Jacobi-type method [7], the discontinuous-Galerkin method [22], the multi-symplectic method [21] and time-splitting spectral methods [5, 12] have been proposed for solving the ZS. Concerning the ZS (1.1) in the subsonic limit regime, i.e., $\varepsilon \ll 1$, the issue is rather complicated due to the initial layer phenomenon and the oscillatory behavior of the solution. The rapid oscillations bring significant difficulties in analyzing the numerical methods as $\varepsilon \ll 1$. There are very few works addressing the numerical issues as $\varepsilon \to 0^+$. Thus, the aim of this paper is to carry out rigorous error analysis for the conservative finite difference (CNFD) methods of Zakharov equations in the subsonic limit regime. We pay particular attention to how the errors depend on mesh size $h$, time step $\tau$ and the parameter $\varepsilon$. It is shown that the CNFD is uniformly convergent at $O(h^2+\tau^{4/3})$ and $O(h^2+\tau^{2/3})$ for the well-prepared and less-ill-prepared initial data, respectively; for the ill-prepared initial data, the error of CNFD is at the order $O(h^2/\varepsilon+\tau^2/\varepsilon^3)$.

The rest of the paper is organized as follows. In section 2, we introduced the conservative finite difference method and our main results. Section 3 is devoted to the error analysis. Numerical results are shown in section 4 to demonstrate the error behavior. Finally, some conclusions are drawn in section 5. Throughput the paper, we adopt the standard notations for Sobolev spaces and write $p \lesssim q$ to mean that there exists a constant $C > 0$ independent of $\varepsilon$, $h$ and $\tau$, such that $p \le Cq$.

## 2. Numerical method and main results

In practical computation, ZS (1.1) is usually truncated on a bounded interval $\Omega = (a, b)$ in 1D, or a rectangle $\Omega = (a, b)\times(c, d)$ in 2D or a box $\Omega = (a, b)\times(c, d)\times(e, f)$ in 3D [5, 6, 8, 10, 12, 17, 22], with zero Dirichlet boundary condition (or periodic boundary condition). By doing such a truncation, we assume that the solutions are well localized in the bounded domain and the error due to the truncation is negligible. For the simplicity of notation, we only deal with the case in 1D, i.e., $d = 1$ and $\Omega = (a, b)$. Extensions to 2D and 3D are straightforward, and the error estimates in $l^2$-norm and discrete semi-$H^1$-norm are the same in 2D and 3D. In 1D, ZS (1.1) is truncated on an interval $\Omega = (a, b)$ as

(2.1)
$$\begin{cases} i\partial_t E^\varepsilon(x,t) + \partial_{xx}E^\varepsilon(x,t) - N^\varepsilon(x,t)E^\varepsilon(x,t) = 0, & x\in\Omega,\ t>0,\\ \varepsilon^2\partial_{tt}N^\varepsilon(x,t) - \partial_{xx}N^\varepsilon(x,t) - \partial_{xx}|E^\varepsilon(x,t)|^2 = 0, & x\in\Omega,\ t>0,\\ E^\varepsilon(x,0) = E_0(x), \quad N^\varepsilon(x,0) = N_0^\varepsilon(x), \quad \partial_t N^\varepsilon(x,0) = N_1^\varepsilon(x), & x\in\Omega,\\ E^\varepsilon(x,t)|_{\partial\Omega} = 0, \quad N^\varepsilon(x,t)|_{\partial\Omega} = 0, & t>0. \end{cases}$$

To describe the important conserved quantity of ZS (2.1), we introduce the potential function $U^\varepsilon := U^\varepsilon(x,t) \in \mathbb{R}$ as the solution of the elliptic equation:

$$-\partial_{xx}U^\varepsilon = N_t^\varepsilon, \qquad U^\varepsilon|_{\partial\Omega} = 0. \tag{2.2}$$

It is well-known that the following quantities are conserved by ZS (2.1), i.e., the energy

(2.3)
$$H(t) := \int_\Omega \left( |\partial_x E^\varepsilon(x,t)|^2 + N^\varepsilon |E^\varepsilon|^2 + \frac{1}{2}(\varepsilon^2 |\partial_x U^\varepsilon|^2 + |N^\varepsilon(x,t)|^2) \right) dx \equiv H(0)$$

and the mass

$$\|E^\varepsilon(\cdot,t)\|_{L^2(\Omega)} = \|E^\varepsilon(\cdot,0)\|_{L^2(\Omega)}. \tag{2.4}$$

Formally, as $\varepsilon \to 0^+$, ZS (2.1) collapses to the standard NLS [1,14,16,18],

$$\begin{cases} i\partial_t E(x,t) + \partial_{xx} E(x,t) + |E(x,t)|^2 E(x,t) = 0, & x \in \Omega \subset \mathbb{R},\ t > 0, \\ E(x,0) = E_0(x), & x \in \overline{\Omega}, \\ E(x,t)\big|_{\partial\Omega} = 0, & t > 0. \end{cases} \tag{2.5}$$

We assume that the initial data $N_0^\varepsilon$ and $N_1^\varepsilon$ are given as

$$\begin{cases} N_0^\varepsilon(x) = -|E_0(x)|^2 + \varepsilon^\alpha w_0(x), & \alpha \ge 0, \quad x \in \Omega, \\ N_1^\varepsilon(x) = 2\,\mathrm{Im}(\Delta E_0(x)\overline{E_0(x)}) + \varepsilon^{\beta-1} w_1(x), & \beta \ge 0, \quad x \in \Omega, \end{cases} \tag{2.6}$$

where $\alpha, \beta \ge 0$ are parameters describing the consistency of the initial data of ZS (2.1) with respect to the initial value of NLS (2.5). In particular, we require that $\int_\Omega w_1(x)dx = 0$ in accordance with the whole space case where $(-\nabla^2)^{-1/2} N_1^\varepsilon \in L^2$ [1,14,16,18].

2.1. **Numerical methods.** Choose time step $\tau := \Delta t$ and denote time steps as $t_k := k\,\tau$ for $k = 0,1,2,\ldots$; choose mesh size $\Delta x := \frac{b-a}{M}$ with $M$ being a positive integer and denote $h := \Delta x$ and grid points as $x_j := a + j\,\Delta x$, $j = 0,1,\ldots,M$. Define the index sets

$$\mathcal{T}_M = \{j \mid j = 1,2,\ldots,M-1\}, \qquad \mathcal{T}_M^0 = \{j \mid j = 0,1,2,\ldots,M\}.$$

Let $E_j^{\varepsilon,k}$, $E_j^k$ and $N_j^{\varepsilon,k}$ be the numerical approximations of $E^\varepsilon(x_j,t_k)$, $E(x_j,t_k)$ and $N^\varepsilon(x_j,t_k)$, respectively, for $j \in \mathcal{T}_M^0$ and $k \ge 0$, and denote $E^{\varepsilon,k}$, $E^k \in \mathbb{C}^{M+1}$ and $N^{\varepsilon,k} \in \mathbb{R}^{M+1}$ to be the numerical solution vectors at time $t = t_k$. We introduce the following finite difference operators:

$$\delta_x^+ E_j^k = \frac{1}{h}(E_{j+1}^k - E_j^k), \quad \delta_x^- E_j^k = \frac{1}{h}(E_j^k - E_{j-1}^k), \quad \delta_x E_j^k = \frac{E_{j+1}^k - E_{j-1}^k}{2\,h},$$
$$\delta_t^+ E_j^k = \frac{1}{\tau}(E_j^{k+1} - E_j^k), \quad \delta_t^- E_j^k = \frac{1}{\tau}(E_j^k - E_j^{k-1}), \quad \delta_t E_j^k = \frac{E_j^{k+1} - E_j^{k-1}}{2\tau},$$
$$\delta_t^2 u_j^k = \frac{E_j^{k+1} - 2E_j^k + E_j^{k-1}}{\tau^2}, \quad \delta_x^2 E_j^k = \frac{E_{j+1}^k - 2E_j^k + E_{j-1}^k}{h^2}.$$

The classical *conservative Crank-Nicolson finite difference* (CNFD) discretization of ZS (2.1) reads as [8,10],

$$\begin{aligned} i\delta_t^- E_j^{\varepsilon,k} &= \left[-\delta_x^2 + \frac{N_j^{\varepsilon,k-1} + N_j^{\varepsilon,k}}{2}\right] \frac{E_j^{\varepsilon,k-1} + E_j^{\varepsilon,k}}{2}, \quad j \in \mathcal{T}_M, \quad k \ge 1, \\ \varepsilon^2 \delta_t^2 N_j^{\varepsilon,k} &= \frac{1}{2}\delta_x^2 (N_j^{\varepsilon,k+1} + N_j^{\varepsilon,k-1}) + \delta_x^2 |E_j^{\varepsilon,k}|^2, \quad j \in \mathcal{T}_M, \quad k \ge 1, \end{aligned} \tag{2.7}$$

with boundary and initial conditions for $j \in \mathcal{T}_M$ as

(2.8)

$$E_0^{\varepsilon,k} = E_M^{\varepsilon,k} = 0, \quad N_0^{\varepsilon,k} = N_M^{\varepsilon,k} = 0, \quad E_j^{\varepsilon,0} = E_0(x_j),$$
$$N_j^{\varepsilon,0} = -|E_0(x_j)|^2 + \varepsilon^\alpha w_0(x_j), \qquad N_{1,j}^\varepsilon = 2\,\mathrm{Im}(\delta_x^2 E_0(x_j)\overline{E_0(x_j)}) + \varepsilon^{\beta-1} w_1(x_j).$$

To complete the scheme, we need to assign the value $N^{\varepsilon,k}$ at the first step $k = 1$.

*Choice of the first step value and the initial layer.* Under the hypothesis of suitable regularity of $N^\varepsilon(x,t)$, one may use Taylor expansion to have the second-order accurate initial step as

(2.9)

$$N_j^{\varepsilon,1} \approx N_0^\varepsilon(x_j) + \tau\partial_t N^\varepsilon(x_j,0) + \frac{\tau^2}{2}\partial_{tt}N^\varepsilon(x_j,0), \quad \partial_t N^\varepsilon(x_j,0) = N_1^\varepsilon(x_j), \quad j \in \mathcal{T}_M,$$
$$\partial_{tt}N^\varepsilon(x_j,0) = \frac{1}{\varepsilon^2}\partial_{xx}\left(N^\varepsilon(x_j,0) + |E^\varepsilon(x_j,0)|^2\right) \approx \varepsilon^{\alpha-2}\delta_x^2 w_0(x_j), \quad j \in \mathcal{T}_M.$$

Using (2.8), we then obtain

(2.10)

$$N_j^{\varepsilon,1} \approx N_0^\varepsilon(x_j) + \tau\left(2\,\mathrm{Im}(\delta_x^2 E_0(x_j)\overline{E_0(x_j)}) + \varepsilon^{\beta-1}w_1(x_j)\right) + \frac{\tau^2}{2}\varepsilon^{\alpha-2}\delta_x^2 w_0(x_j).$$

When $0 \le \alpha < 2, 0 \le \beta < 1$, the above approximation (2.9) is not appropriate if $\varepsilon \ll 1$, since the time step has to be very small to make the first step $N^{\varepsilon,1}$ bounded, while we know that the exact $N^\varepsilon$ is uniformly bounded. On the other hand, due to the rapid oscillation, small time step $\tau$ is required to maintain the accuracy of approximation (2.10) in view of Taylor expansion, where the local truncation error induced by the discretization (2.7) shares the same property. Based on these observations and the current finite difference discretization, we are going to derive a better approximation of $N^\varepsilon(x_j,\tau)$ which is uniformly bounded w.r.t. $\varepsilon$ and has the same accuracy as the local truncation error of (2.10) (cf. (3.3)).

The idea is to absorb the unbounded terms when $\varepsilon \to 0^+$ (the terms involving $\varepsilon^{\beta-1}$ and $\varepsilon^{\alpha-2}$) in (2.10) using trigonometric functions $\sin(\tau/\varepsilon)$, resulting in the following initial step:

(2.11)

$$N_j^{\varepsilon,1} \approx N_0^\varepsilon(x_j) + 2\tau\,\mathrm{Im}(\delta_x^2 E_0(x_j)\overline{E_0(x_j)}) + \varepsilon^\beta \sin(\frac{\tau}{\varepsilon})w_1(x_j) + 2\varepsilon^\alpha \sin^2(\frac{\tau}{2\varepsilon})\delta_x^2 w_0(x_j).$$

It is easy to verify that (2.11) maintains the same temporal accuracy as (2.10) when $\tau$ is small enough and (2.11) ensures $N_j^{\varepsilon,1}$ is uniformly bounded for all $\tau$ and $\varepsilon$. Meanwhile, (2.11) will preserve the Schrödinger limit of the ZS (1.1) as $\varepsilon \to 0^+$ if $\alpha, \beta > 0$.

Now, (2.7), (2.8) and (2.11) complete the CNFD scheme for the Zakharov system (2.1). We notice that another type initialization (cf. (4.6)) can be used instead of (2.11) while the results obtained will remain the same.

2.2. **Main results.** Before stating our main results on the error estimates for CNFD (2.7)–(2.11), we introduce some notation. Denote

$$X_M = \left\{ v = (v_j)_{j \in \mathcal{T}_M^0} \mid v_0 = v_M = 0 \right\} \subset \mathbb{C}^{M+1},$$

and define the norms and inner product over $X_M$ as

$$\|v\|^2 = h\sum_{j=0}^{M-1}|v_j|^2,\ \|\delta_x^+ v\|^2 = h\sum_{j=0}^{M-1}\left|\delta_x^+ v_j\right|^2,$$
$$\|\delta_x^2 v\|^2 = h\sum_{j=1}^{M-1}\left|\delta_x^2 v_j\right|^2,\ \|v\|_\infty = \sup_{j\in\mathcal{T}_M^0}|v_j|,$$
$$(u,v) = h\sum_{j=0}^{M-1}u_j\bar{v}_j, \quad \langle u,v\rangle = h\sum_{j=1}^{M-1}u_j\bar{v}_j \quad \forall u,v\in X_M. \tag{2.12}$$

For simplicity of notation, we denote

$$\alpha^* = \min\{1,\alpha,\beta\}, \qquad \alpha^\dagger = \min\{\alpha,\beta,2\}. \tag{2.13}$$

Let $T^*$ be the maximum common existence time for the solution ($E^\varepsilon = E^\varepsilon(x,t)$, $N^\varepsilon = N^\varepsilon(x,t)$) to the ZS (2.1) and the solution $E = E(x,t)$ to the NLS (2.5). For $0 < T < T^*$, according to the known results in [1, 14, 16, 18] and the asymptotic expansion in section 1, we assume the exact solution $(E^\varepsilon(x,t), N^\varepsilon(x,t))$ of ZS (2.1) and the exact solution $E(x,t)$ of NLS (2.5) are smooth enough satisfying the homogeneous boundary conditions. More precisely, we assume $E^\varepsilon, N^\varepsilon, E \in L^\infty([0,T];H_0^1(\Omega))$ and

$$\text{(A)}\quad \begin{aligned}
&\|E^\varepsilon\|_{L^\infty([0,T];W^{5,\infty}(\Omega))} + \|E^\varepsilon\|_{W^{1,\infty}([0,T];W^{1,\infty}(\Omega))}\\
&\quad + \varepsilon^{1-\alpha^*}\|E^\varepsilon\|_{W^{2,\infty}([0,T];W^{1,\infty}(\Omega))}\\
&\quad + \varepsilon^{2-\alpha^\dagger}\|E^\varepsilon\|_{W^{3,\infty}([0,T];W^{1,\infty}(\Omega))} + \|E\|_{W^{3,\infty}([0,T];W^{1,\infty}(\Omega))} \lesssim 1,\\
&\|N^\varepsilon\|_{L^\infty([0,T];W^{5,\infty}(\Omega))} + \varepsilon^{\alpha^{1-\alpha^*}}\|N^\varepsilon\|_{W^{1,\infty}([0,T];L^\infty(\Omega))}\\
&\quad + \varepsilon^{2-\alpha^\dagger}\|N^\varepsilon\|_{W^{2,\infty}([0,T];W^{3,\infty}(\Omega))}\\
&\quad + \varepsilon^{3-\alpha^\dagger}\|N^\varepsilon\|_{W^{3,\infty}([0,T];L^\infty(\Omega))} + \varepsilon^{4-\alpha^\dagger}\|N^\varepsilon\|_{W^{4,\infty}([0,T];W^{1,\infty}(\Omega))} \lesssim 1,
\end{aligned}$$

with the convergence

$$\text{(B)}\quad \|E^\varepsilon - E\|_{L^\infty([0,T];H^1(\Omega))} \lesssim \varepsilon^{1+\alpha^*}, \qquad \|N^\varepsilon - |E|^2\|_{L^\infty([0,T];L^2(\Omega))} \lesssim \varepsilon^{\alpha^\dagger},$$

as well as the initial data

$$\text{(C)}\quad \|E_0\|_{H^4(\Omega)} + \|w_0\|_{H^2(\Omega)} + \|w_1\|_{H^2(\Omega)} \lesssim 1.$$

Define the 'error' functions $e^{\varepsilon,k}, n^{\varepsilon,k} \in X_M$ for $k \geq 0$ as

$$e_j^{\varepsilon,k} = E^\varepsilon(x_j,t_k) - E_j^{\varepsilon,k}, \quad n_j^{\varepsilon,k} = N^\varepsilon(x_j,t_k) - N_j^{\varepsilon,k}, \quad j \in \mathcal{T}_M, \tag{2.14}$$

and we have the following estimates on the errors.

**Theorem 2.1** (Well- and less-ill-prepared initial data)**.** *For well- and less-ill-prepared initial data, i.e., $\alpha,\beta \geq 1$, under the assumptions* (A), (B) *and* (C)*, there exist $h_0 > 0$ and $\tau_0 > 0$ sufficiently small, when $0 < h \leq h_0$ and $0 < \tau \leq \tau_0$, CNFD method* (2.7) *with* (2.8) *and* (2.11) *satisfy the following optimal error estimates:*

$$\|e^{\varepsilon,k}\| + \|\delta_x^+ e^{\varepsilon,k}\| + \|n^{\varepsilon,k}\| \lesssim h^2 + \frac{\tau^2}{\varepsilon^{3-\alpha^\dagger}}, \quad 0 \leq k \leq \frac{T}{\tau}, \tag{2.15}$$

$$\|e^{\varepsilon,k}\| + \|\delta_x^+ e^{\varepsilon,k}\| + \|n^{\varepsilon,k}\| \leq h^2 + \tau^2 + \varepsilon^{\alpha^\dagger}, \quad 0 \leq k \leq \frac{T}{\tau}. \tag{2.16}$$

*Thus, by taking the minimum, we have the $\varepsilon$-independent convergence rate for the errors as*

$$\|e^{\varepsilon,k}\| + \|\delta_x^+ e^{\varepsilon,k}\| + \|n^{\varepsilon,k}\| \lesssim h^2 + \tau^{2\alpha^\dagger/3}, \quad 0 \le k \le \frac{T}{\tau}. \tag{2.17}$$

**Theorem 2.2** (Ill-prepared initial data)**.** *For ill-prepared initial data, i.e., $\alpha, \beta \in [0, 1)$, under the assumptions* (A)*,* (B) *and* (C)*, there exist $h_0 > 0$ and $\tau_0 > 0$ sufficiently small, when $0 < h \le h_0$ and $0 < \tau \le \tau_0$, CNFD method* (2.7) *with* (2.8) *and* (2.11) *satisfy the following optimal error estimates:*

$$\|e^{\varepsilon,k}\| + \|\delta_x^+ e^{\varepsilon,k}\| + \|n^{\varepsilon,k}\| \lesssim \frac{h^2}{\varepsilon^{1-\alpha^*}} + \frac{\tau^2}{\varepsilon^{3-\alpha^*}}, \quad 0 \le k \le \frac{T}{\tau}. \tag{2.18}$$

*Remark* 2.1. For the 1D cases, Theorems 2.1 and 2.2 do not assume any compatible conditions among mesh size $h$, time step $\tau$ and parameter $\varepsilon$. For extensions to the higher dimensions case, i.e., 2D and 3D, some additional technical conditions on $h$, $\tau$ and $\varepsilon$ are necessary by the same proof of the paper (cf. Remark 3.1).

Define $U_j^{\varepsilon,k+\frac{1}{2}}$ ($j \in \mathcal{T}_M^0$, $k \ge 0$) as the solution of

$$-\delta_x^2 U_j^{\varepsilon,k+1/2} = \delta_t^+ N_j^{\varepsilon,k}, \quad j \in \mathcal{T}_M, \tag{2.19}$$

with boundary condition $U_0^{\varepsilon,k+1/2} = U_M^{\varepsilon,k+1/2} = 0$. For simplicity of notation, we write

$$U^{\varepsilon,k+1/2} = (-\delta_x^2)^{-1}\delta_t^+ N^{\varepsilon,k}, \tag{2.20}$$

i.e., $(-\delta_x^2)^{-1}$ is the inverse of discrete Laplacian with homogeneous Dirichlet boundary conditions, and the discrete Sobolev inequality implies that

$$\|U^{\varepsilon,k+1/2}\| \lesssim \|\delta_x^+ U^{\varepsilon,k+1/2}\| \lesssim \|\delta_t^+ N^{\varepsilon,k}\|. \tag{2.21}$$

Then, the CNFD method (2.7), (2.8) and (2.11) satisfy the following conservation laws at the discrete level.

**Lemma 2.1.** *For CNFD* (2.7)*,* (2.8) *and* (2.11)*, the following quantities are conserved:*

$$\|E^{\varepsilon,k}\| \equiv \|E^{\varepsilon,0}\|, \quad k \ge 0, \tag{2.22}$$

*and*

(2.23)

$$\begin{aligned} H^k =& \frac{1}{2}(\|\delta_x^+ E^{\varepsilon,k+1}\|^2 + \|\delta_x^+ E^{\varepsilon,k}\|^2) + \frac{\varepsilon^2}{2}\|\delta_x^+ U^{\varepsilon,k+1/2}\|^2 + \frac{1}{4}(\|N^{\varepsilon,k+1}\|^2 + \|N^{\varepsilon,k}\|^2) \\ &+ \frac{h}{4}\sum_{j=0}^{M-1}(N_j^{\varepsilon,k+1} + N_j^{\varepsilon,k})(|E_j^{\varepsilon,k+1}|^2 + |E_j^{\varepsilon,k}|^2) \equiv H^0, \quad k \ge 0. \end{aligned}$$

*In particular, we have the a priori bounds in 1D under assumption* (C) *for $0 < \tau \le 1$,*

$$\|E^{\varepsilon,k}\|_\infty \le \sqrt{2\|E^{\varepsilon,k}\|\,\|\delta_x^+ E^{\varepsilon,k}\|} \le C_a, \quad k \ge 0. \tag{2.24}$$

*Proof.* The proof is standard [8,10] and we only show $H^k$ is conserved here. Multiplying the first equation in (2.7) by $\overline{E_j^{\varepsilon,k} - E_j^{\varepsilon,k-1}}$, summing all together for $j \in \mathcal{T}_M$ and taking real parts, we get

$$\frac{1}{2}\|\delta_x^+ E^{\varepsilon,k}\|^2 - \frac{1}{2}\|\delta_x^+ E^{\varepsilon,k-1}\|^2 + \frac{1}{4}\left(N^{\varepsilon,k} + N^{\varepsilon,k-1}, |E^{\varepsilon,k}|^2 - |E^{\varepsilon,k-1}|^2\right) = 0 \tag{2.25}$$

which implies for $k \geq 1$,

$$\begin{aligned}&\frac{1}{2}\|\delta_x^+ E^{\varepsilon,k+1}\|^2 - \frac{1}{2}\|\delta_x^+ E^{\varepsilon,k-1}\|^2 + \frac{1}{4}\left(N^{\varepsilon,k+1} + N^{\varepsilon,k}, |E^{\varepsilon,k+1}|^2 - |E^{\varepsilon,k}|^2\right)\\ &+ \frac{1}{4}\left(N^{\varepsilon,k} + N^{\varepsilon,k-1}, |E^{\varepsilon,k}|^2 - |E^{\varepsilon,k-1}|^2\right) = 0.\end{aligned} \tag{2.26}$$

Multiplying the second equation in (2.7) by $\tau(U_j^{\varepsilon,k-\frac{1}{2}} + U_j^{\varepsilon,k+\frac{1}{2}})$, summing all together for $j \in \mathcal{T}_M$ and making use of (2.19), we have for $k \geq 1$,

$$\begin{aligned}&\varepsilon^2\|\delta_x^+ U^{\varepsilon,k+\frac{1}{2}}\|^2 - \varepsilon^2\|\delta_x^+ U^{\varepsilon,k-\frac{1}{2}}\|^2 + \frac{1}{2}\|N^{\varepsilon,k+1}\|^2 - \frac{1}{2}\|N^{\varepsilon,k-1}\|^2\\ &+(|E^{\varepsilon,k}|^2, N^{\varepsilon,k+1} - N^{\varepsilon,k-1}) = 0.\end{aligned} \tag{2.27}$$

Furthermore, for (2.26) + $\frac{1}{2}\cdot$ (2.27), we can compute that for $k \geq 1$,

$$\begin{aligned}&\frac{1}{2}\|\delta_x^+ E^{\varepsilon,k+1}\|^2 + \frac{\varepsilon^2}{2}\|\delta_x^+ U^{\varepsilon,k+\frac{1}{2}}\|^2 + \frac{1}{4}\|N^{\varepsilon,k+1}\|^2\\ &\qquad + \frac{1}{4}\left(N^{\varepsilon,k+1} + N^{\varepsilon,k}, |E^{\varepsilon,k+1}|^2 + |E^{\varepsilon,k}|^2\right)\\ &= \frac{1}{2}\|\delta_x^+ E^{\varepsilon,k-1}\|^2 + \frac{\varepsilon^2}{2}\|\delta_x^+ U^{\varepsilon,k-\frac{1}{2}}\|^2 + \frac{1}{4}\|N^{\varepsilon,k-1}\|^2\\ &\qquad + \frac{1}{4}\left(N^{\varepsilon,k} + N^{\varepsilon,k-1}, |E^{\varepsilon,k}|^2 + |E^{\varepsilon,k-1}|^2\right).\end{aligned}$$

Summing the above equation together from 1 to $k$, we can draw the conclusion.

From the initial data (2.11), using Taylor expansion and assumption (C), we have

$$\varepsilon\|\delta_x^+ U^{\varepsilon,1/2}\| \lesssim \varepsilon\left\|\delta_t^- N^{\varepsilon,1}\right\| \lesssim \varepsilon\|E_0\delta_x^2 E_0\| + \varepsilon^\beta\|(-\delta_x^2)^{1/2}w_1\| + \varepsilon^\alpha\|(-\delta_x^2)^{1/2}w_0\| \lesssim 1$$

and

$$\|N^{\varepsilon,0}\| + \|N^{\varepsilon,1}\| \lesssim \||E_0|^2\| + \tau\|E_0\delta_x^2 E_0\| + \varepsilon^\beta\|w_1\| + \varepsilon^\alpha\|w_0\| \lesssim 1.$$

From (2.25), we have

$$\begin{aligned}&\frac{1}{2}\|\delta_x^+ E^{\varepsilon,1}\|^2 + \frac{1}{4}\left(N^{\varepsilon,1} + N^{\varepsilon,0}, |E^{\varepsilon,1}|^2\right)\\ &\quad = \frac{1}{2}\|\delta_x^+ E^{\varepsilon,0}\|^2 + \frac{1}{4}\left(N^{\varepsilon,1} + N^{\varepsilon,0}, |E^{\varepsilon,0}|^2\right) \lesssim 1.\end{aligned}$$

Thus $H^0$ is uniformly bounded w.r.t. $\varepsilon$ and the *a priori* bounds for $E^{\varepsilon,k}$ hold by a standard argument [8]. ☐

## 3. Error analysis

We start with the following lemmas.

**Lemma 3.1** (Summation by parts formula)**.** *For the sequences $u^j$ and $v^j$ $(j=0, 1,2,\ldots)$, we have*

$$\sum_{j=1}^{J} u^j\delta_t^- v^j = -\sum_{j=2}^{J}\delta_t^- u^j v^{j-1} - \frac{1}{\tau}u^1v^0 + \frac{1}{\tau}u^J v^J. \tag{3.1}$$

*The following holds for $\tilde u, \tilde v \in X_M$:*

$$\langle -\delta_x^2\tilde u, \tilde v\rangle = (\delta_x^+\tilde u, \delta_x^+\tilde v), \quad \langle (-\delta_x^2)^{-1}\tilde u, \tilde v\rangle = \langle \tilde u, (-\delta_x^2)^{-1}\tilde v\rangle. \tag{3.2}$$

**Lemma 3.2** (Solvability of CNFD)**.** *For any given initial data $E^{\varepsilon,0}, N^{\varepsilon,0}, N^{\varepsilon,1}\in X_M$, there exists a unique solution $(E^{\varepsilon,k}, N^{\varepsilon,k})$ to CNFD* (2.7) *for $k>1$.*

*Proof.* The proof is standard [10]. The uniqueness is a direct consequence of the fact that the solutions in (2.7) are updated as $N^{\varepsilon,1}\to E^{\varepsilon,1}\to N^{\varepsilon,2}\to E^{\varepsilon,2}\ldots$, where a linear system is solved at each step. ☐

Define the local truncation error $\eta^{\varepsilon,k}, \zeta^{\varepsilon,k}\in X_M$ of CNFD (2.7) with (2.8) and (2.11) for $k\ge 1$ and $j\in\mathcal{T}_M$ as

(3.3)

$$\begin{aligned}
\eta_j^{\varepsilon,k} :=& i\delta_t^- E^\varepsilon(x_j,t_k) + \frac{1}{2}(\delta_x^2 E^\varepsilon(x_j,t_{k-1}) + \delta_x^2 E^\varepsilon(x_j,t_k))\\
&- \frac{1}{4}(N^\varepsilon(x_j,t_{k-1}) + N^\varepsilon(x_j,t_k))(E^\varepsilon(x_j,t_{k-1}) + E^\varepsilon(x_j,t_k)),\\
\zeta_j^{\varepsilon,k} :=& \varepsilon^2\delta_t^2 N^\varepsilon(x_j,t_k) - \frac{1}{2}(\delta_x^2 N^\varepsilon(x_j,t_{k+1}) + \delta_x^2 N^\varepsilon(x_j,t_{k-1})) - \delta_x^2|E^\varepsilon(x_j,t_k)|^2.
\end{aligned}$$

**Lemma 3.3** (Local truncation error)**.** *Under assumption* (A)*, we have*

$$\begin{aligned}
&\|\eta^{\varepsilon,k}\| + \|\zeta^{\varepsilon,k}\| + \|\delta_x^+\eta^{\varepsilon,k}\| \lesssim h^2 + \frac{\tau^2}{\varepsilon^{2-\alpha^\dagger}}, \quad 1\le k\le \frac{T}{\tau},\\
&\|\delta_t\zeta^{\varepsilon,k}\| \lesssim \frac{h^2}{\varepsilon^{1-\alpha^*}} + \frac{\tau^2}{\varepsilon^{3-\alpha^\dagger}}, \quad 2\le k\le\frac{T}{\tau}-1.
\end{aligned} \tag{3.4}$$

*Proof.* Using Taylor expansion, we have, for $j\in\mathcal{T}_M$,

$$\begin{aligned}
\eta_j^{\varepsilon,k} =& \frac{i\tau^2}{8}\int_0^1\int_0^\theta\int_{-s}^{s} E_{ttt}^\varepsilon(x_j, \frac{\sigma\tau}{2} + t_{k-\frac12})\,d\sigma ds d\theta\\
&+ \frac{\tau^2}{8}\int_0^1\int_{-\theta}^{\theta} E_{xxtt}^\varepsilon(x_j, \frac{s\tau}{2} + t_{k-\frac12}) ds d\theta\\
&+ \frac{h^2}{2}\int_0^1\int_0^\theta\int_0^s\int_{-\sigma}^{\sigma}\sum_{m=0,1} E_{xxxx}^\varepsilon(x_j + s_1 h, t_{k-1} + m\tau) ds_1 d\sigma ds d\theta\\
&- \frac{\tau^2}{16}(N^\varepsilon(x_j,t_{k-1}) + N^\varepsilon(x_j,t_k))\int_0^1\int_{-\theta}^{\theta} E_{tt}^\varepsilon(x_j, s\tau/2 + t_{k-1/2}) ds d\theta\\
&- \frac{\tau^2}{8}E^\varepsilon(x_j,t_{k-1/2})\int_0^1\int_{-\theta}^{\theta} N_{tt}^\varepsilon(x_j, s\tau/2 + t_{k-1/2}) ds d\theta
\end{aligned}$$

and

$$\begin{aligned}\zeta_j^{\varepsilon,k} &= \varepsilon^2\tau^2\int_0^1\int_0^\theta\int_0^s\int_{-\sigma}^{\sigma} N_{tttt}^\varepsilon(x_j,t_k+z\tau)dzd\sigma dsd\theta \\ &\quad -\frac{\tau^2}{2}\int_0^1\int_{-\theta}^{\theta} N_{xxtt}^\varepsilon(x_j,s\tau+t_k)dsd\theta \\ &\quad -\frac{h^2}{2}\int_0^1\int_0^\theta\int_0^s\int_{-\sigma}^{\sigma}\sum_{m=\pm1} N_{xxxx}^\varepsilon(x_j+s_1h,t_k+m\tau)ds_1d\sigma dsd\theta \\ &\quad - h^2\int_0^1\int_0^\theta\int_0^s\int_{-\sigma}^{\sigma}\partial_{xxxx}(|E^\varepsilon|^2)(x_j+s_1h,t_k)ds_1d\sigma dsd\theta.\end{aligned}$$

Using assumption (A), we conclude that

$$\begin{aligned}|\eta_j^{\varepsilon,k}| &\lesssim h^2\|N_{xxxx}^\varepsilon\|_{L^\infty} \\ &\qquad + \tau^2\left(\|E_{ttt}^\varepsilon\|_{L^\infty}+\|E_{xxtt}^\varepsilon\|_{L^\infty}+\|N^\varepsilon\|_{L^\infty}\|E_{tt}^\varepsilon\|_{L^\infty}+\|E^\varepsilon\|_{L^\infty}\|N_{tt}^\varepsilon\|_{L^\infty}\right) \\ &\lesssim h^2+\frac{\tau^2}{\varepsilon^{2-\alpha^\dagger}}, \\ |\zeta_j^{\varepsilon,k}| &\lesssim h^2(\|N_{xxxx}^\varepsilon\|_{L^\infty}+\|\partial_{xxxx}|E^\varepsilon|^2\|_{L^\infty})+\tau^2\left(\varepsilon^2\|N_{tttt}^\varepsilon\|_{L^\infty}+\|N_{xxtt}^\varepsilon\|_{L^\infty}\right) \\ &\lesssim h^2+\frac{\tau^2}{\varepsilon^{2-\alpha^\dagger}},\end{aligned}$$

and the first inequality in (3.4) holds. For $j=1,\ldots,M-2$, applying $\delta_x^+$ to $\eta_j^{\varepsilon,k}$, we get

$$\begin{aligned}|\delta_x^+\eta_j^{\varepsilon,k}| &\lesssim h^2\|N_{xxxxx}^\varepsilon\|_{L^\infty}+\tau^2\Big(\|E_{tttx}^\varepsilon\|_{L^\infty}+\|E_{xxxtt}^\varepsilon\|_{L^\infty}+\|N^\varepsilon\|_{L^\infty}\|E_{ttx}^\varepsilon\|_{L^\infty} \\ &\qquad +\|E^\varepsilon\|_{L^\infty}\|N_{ttx}^\varepsilon\|_{L^\infty}+\|N_x^\varepsilon\|_{L^\infty}\|E_{tt}^\varepsilon\|_{L^\infty}+\|E_x^\varepsilon\|_{L^\infty}\|N_{tt}^\varepsilon\|_{L^\infty}\Big) \\ &\lesssim h^2+\frac{\tau^2}{\varepsilon^{2-\alpha^\dagger}};\end{aligned}$$

for $j=0,M-1$, using the boundary conditions which implies that $E_{xxxx}^\varepsilon|_{\partial\Omega}=E_{xx}^\varepsilon|_{\partial\Omega}=0$, we can find the same bound for $\delta_x^+\eta_j^{\varepsilon,k}$ when $j=0,M-1$. Thus, we obtain the estimates for $\|\delta_x^+\eta^{\varepsilon,k}\|$.

It remains to show the estimates of $\delta_t\zeta^{\varepsilon,k}$ for $2\le k\le \frac{T}{\tau}-1$. Applying $\delta_t$ to $\zeta^{\varepsilon,k}$ and making use of assumption (A), we derive

$$\begin{aligned}|\delta_t\zeta_j^{\varepsilon,k}| &\lesssim h^2(\|N_{xxxxt}^\varepsilon\|_{L^\infty}+\|\partial_{xxxxt}|E^\varepsilon|^2\|_{L^\infty})+\tau^2\left(\varepsilon^2\|N_{ttttt}^\varepsilon\|_{L^\infty}+\|N_{xxttt}^\varepsilon\|_{L^\infty}\right) \\ &\lesssim \frac{h^2}{\varepsilon^{1-\alpha^*}}+\frac{\tau^2}{\varepsilon^{3-\alpha^\dagger}},\end{aligned}$$

and the estimates for $\delta_t\zeta^{\varepsilon,k}$ follow. □

For the initial step, we have the following.

**Lemma 3.4.** *Under the assumptions* (A) *and* (C), *the first step errors of CNFD* (2.7)–(2.11) *satisfy*

$$\|n_j^{\varepsilon,1}\|\lesssim\frac{\tau h^2}{\varepsilon^{1-\alpha^*}}+\frac{\tau^3}{\varepsilon^{3-\alpha^\dagger}},\quad \|\delta_t^- n_j^{\varepsilon,1}\|\lesssim\frac{h^2}{\varepsilon^{1-\alpha^*}}+\frac{\tau^2}{\varepsilon^{3-\alpha^\dagger}}. \tag{3.5}$$

*Proof.* Using Taylor expansion, (2.8) and (2.11), we get for $j = 1, 2, \ldots, M-1$,

$$n_j^{\varepsilon,1} = -\frac{\tau h^2}{6}\mathrm{Im}\left(E_0(x_j)\partial_{xxxx}\overline{E_0(x_\theta)}\right) + \frac{\tau^3}{6\varepsilon^3}\varepsilon^\beta\cos(\frac{s_1}{\varepsilon})w_1(x_j) + \frac{\tau^3}{6}\partial_{ttt}N^\varepsilon(x_j, s_2) + \frac{\tau^3}{6\varepsilon^3}\sin(\frac{s_3}{\varepsilon})\varepsilon^\alpha\partial_{xx}w_0(x_j) - \frac{\tau h^2}{\varepsilon}\varepsilon^\alpha\sin(\frac{s_4}{\varepsilon})\partial_{xxxx}w_0(x'_\theta),$$

where $s_j \in (0, \tau)$ ($j = 1, 2, 3, 4$) and $x_\theta, x'_\theta \in (x_{j-1}, x_{j+1})$. Then (3.5) follows from the assumptions (A) and (B). □

Lemma 2.1 implies $\|E^{\varepsilon,k}\|_\infty$ is uniformly bounded and our error analysis can be done in a similar way as that in [8]. However, we want to use a unified approach that works in all dimensions ($d = 1, 2, 3$). One main difficulty in deriving error bounds for CNFD in high dimensions (2D and 3D) is the $l^\infty$ bounds for the finite difference solutions (cf. Lemma 2.1 in 1D). For nonlinear Schrödinger equations [2, 3, 20], this difficulty was overcome by truncating the nonlinearity to a global Lipschitz function with compact support in $d$-dimensions ($d = 1, 2, 3$). This is guaranteed if the continuous solution is bounded and the numerical solution is close to the continuous solution. Here, we apply the same idea. Choose $M'_0 > 0$ and a smooth function $\rho(s) \in C^\infty(\mathbb{R}^1)$ such that

$$M'_0 = \max\left\{\|E(x,t)\|_{L^\infty(\Omega_T)},\ \sup_{\varepsilon\in(0,1]}\|E^\varepsilon(x,t)\|_{L^\infty(\Omega_T)}\right\}, \tag{3.6}$$

$$\rho(s) = \begin{cases} 1, & |s| \le 1\,, \\ \in [0,1], & |s| \le 2\,, \\ 0, & |s| \ge 2\,, \end{cases}$$

where $\Omega_T = \Omega \times (0, T)$. By assumption (A), $M'_0$ is well-defined and let us choose $M_0 = \max\{M'_0, C_a\}$ in $d = 1$ dimension ($C_a$ given in (2.24)) and $M_0 = M'_0$ in $d = 2, 3$ dimensions.

For $s \in \mathbb{R}$, define

$$f_B(s) = s\rho(s/B), \quad B = (M_0 + 1)^2. \tag{3.7}$$

Choose $\hat{E}^{\varepsilon,0} = E^{\varepsilon,0}$, $\hat{N}^{\varepsilon,0} = N^{\varepsilon,0}$, $\hat{N}^{\varepsilon,1} = N^{\varepsilon,1}$ and define $\hat{E}^{\varepsilon,k}, \hat{N}^{\varepsilon,k+1} \in X_M$ ($k \ge 1$) for $j \in \mathcal{T}_M$ as

$$\begin{aligned} i\delta_t^-\hat{E}_j^{\varepsilon,k} &= \left[-\delta_x^2 + \frac{\hat{N}_j^{\varepsilon,k-1} + \hat{N}_j^{\varepsilon,k}}{2}g(\hat{E}_j^{\varepsilon,k}, \hat{E}_j^{\varepsilon,k-1})\right]\frac{\hat{E}_j^{\varepsilon,k-1} + \hat{E}_j^{\varepsilon,k}}{2}, \\ \varepsilon^2\delta_t^2\hat{N}_j^{\varepsilon,k} &= \frac{1}{2}\delta_x^2(\hat{N}_j^{\varepsilon,k+1} + \hat{N}_j^{\varepsilon,k-1}) + \delta_x^2 f_B(|\hat{E}_j^{\varepsilon,k}|^2), \end{aligned} \tag{3.8}$$

with

$$g(\hat{E}_j^{\varepsilon,k}, \hat{E}_j^{\varepsilon,k-1}) = \int_0^1 f'_B(\theta|\hat{E}_j^{\varepsilon,k}|^2 + (1-\theta)|\hat{E}_j^{\varepsilon,k-1}|^2)\,d\theta. \tag{3.9}$$

In fact, $(\hat{E}^{\varepsilon,k}, \hat{N}^{\varepsilon,k})$ can be viewed as another approximation of $(E^\varepsilon(x, t_k), N^\varepsilon(x, t_k))$. Using the nice properties of $f_B(\cdot)$, it is easy to see that (3.8) is uniquely solvable for sufficiently small time step $\tau$.

In the subsequent discussion, we will prove different type estimates in Theorems 2.1 and 2.2 for $(\hat{E}^{\varepsilon,k}, \hat{N}^{\varepsilon,k})$, separately.

3.1. (2.15) **type estimates.** Define the 'error' functions $\hat{e}^{\varepsilon,k}, \hat{n}^{\varepsilon,k} \in X_M$ $(k \geq 0)$ as

$$\hat{e}_j^{\varepsilon,k} = E^\varepsilon(x_j,t_k) - \hat{E}_j^{\varepsilon,k}, \quad \hat{n}_j^{\varepsilon,k} = N^\varepsilon(x_j,t_k) - \hat{N}_j^{\varepsilon,k}, \quad j \in \mathcal{T}_M, \tag{3.10}$$

and the local truncation error $\hat{\eta}^{\varepsilon,k}, \hat{\zeta}^{\varepsilon,k} \in X_M$ for $k \geq 1$ and $j \in \mathcal{T}_M$ as

$$\begin{aligned}\hat{\eta}_j^{\varepsilon,k} :=& i\delta_t^- E^\varepsilon(x_j,t_k) + \Big[\frac{1}{2}\delta_x^2 - \frac{1}{4} g\left(E^\varepsilon(x_j,t_k), E^\varepsilon(x_j,t_{k-1})\right) \\ &\times (N^\varepsilon(x_j,t_{k-1}) + N^\varepsilon(x_j,t_k))\Big](E^\varepsilon(x_j,t_{k-1}) + E^\varepsilon(x_j,t_k)), \qquad k \geq 1, \\ \hat{\zeta}_j^{\varepsilon,k} :=& \varepsilon^2 \delta_t^2 N^\varepsilon(x_j,t_k) - \frac{1}{2}(\delta_x^2 N^\varepsilon(x_j,t_{k+1}) + \delta_x^2 N^\varepsilon(x_j,t_{k-1})) \\ &- \delta_x^2 f_B\left(|E^\varepsilon(x_j,t_k)|^2\right), \quad k \geq 1.\end{aligned} \tag{3.11}$$

As in Lemma 3.3, we have the bounds for $\hat{\eta}^{\varepsilon,k}, \hat{\zeta}^{\varepsilon,k} \in X_M$ $(k \geq 1)$,

$$\|\hat{\eta}^{\varepsilon,k}\| + \|\delta_x^+ \hat{\eta}^{\varepsilon,k}\| + \|\hat{\zeta}^{\varepsilon,k}\| \lesssim h^2 + \frac{\tau^2}{\varepsilon^{2-\alpha^\dagger}}. \quad \|\delta_t \hat{\zeta}^{\varepsilon,k}\| \lesssim \frac{h^2}{\varepsilon^{1-\alpha^*}} + \frac{\tau^2}{\varepsilon^{3-\alpha^\dagger}}. \tag{3.12}$$

For analyzing the error of the scheme (2.7), it is natural to work with the energy space of ZS (2.1). To this aim, we introduce $\hat{u}^{\varepsilon,k-\frac{1}{2}} = (-\delta_x^2)^{-1}(\delta_t^- \hat{n}^{\varepsilon,k})$ $(k \geq 1)$, i.e., $\hat{u}^{\varepsilon,k-\frac{1}{2}}$ is the solution to the equation

$$-\delta_x^2 \hat{u}_j^{\varepsilon,k-\frac{1}{2}} = \delta_t^- \hat{n}_j^{\varepsilon,k}, \quad j \in \mathcal{T}_M, \tag{3.13}$$

with $\hat{u}_0^{\varepsilon,k-\frac{1}{2}} = \hat{u}_M^{\varepsilon,k-\frac{1}{2}} = 0$.

Similarly to Lemma 3.4, the initial errors satisfy $\hat{e}^{\varepsilon,0} = \mathbf{0}$, $\hat{n}^{\varepsilon,0} = \mathbf{0}$,

$$\|\hat{n}_j^{\varepsilon,1}\| \lesssim \frac{\tau h^2}{\varepsilon^{1-\alpha^*}} + \frac{\tau^3}{\varepsilon^{3-\alpha^\dagger}}, \quad \|\hat{u}^{\frac{1}{2}}\| + \|\delta_x^+ \hat{u}^{\frac{1}{2}}\| \lesssim \|\delta_t^- \hat{n}^{\varepsilon,1}\| \lesssim \frac{h^2}{\varepsilon^{1-\alpha^*}} + \frac{\tau^2}{\varepsilon^{3-\alpha^\dagger}}. \tag{3.14}$$

Subtracting (3.8) from (3.11), we get the error equations for $j \in \mathcal{T}_M$ and $k \geq 1$

$$i\delta_t^- \hat{e}_j^{\varepsilon,k} = -\frac{1}{2}\delta_x^2(\hat{e}_j^{\varepsilon,k-1} + \hat{e}_j^{\varepsilon,k}) + R_j^k + \hat{\eta}_j^{\varepsilon,k}, \tag{3.15}$$

$$\varepsilon^2 \delta_t^2 \hat{n}_j^{\varepsilon,k} = \frac{1}{2}\delta_x^2(\hat{n}_j^{\varepsilon,k-1} + \hat{n}_j^{\varepsilon,k+1}) + \delta_x^2 P_j^k + \hat{\zeta}_j^{\varepsilon,k}, \tag{3.16}$$

where $R^k, P^k \in X_M$ and for $j \in \mathcal{T}_M$,

$$\begin{aligned}R_j^k =& \frac{1}{4}(N^\varepsilon(x_j,t_{k-1}) + N^\varepsilon(x_j,t_k)) \\ &\times g(E^\varepsilon(x_j,t_k), E^\varepsilon(x_j,t_{k-1}))(E^\varepsilon(x_j,t_{k-1}) + E^\varepsilon(x_j,t_k)) \\ &- \frac{1}{4}(N_j^{\varepsilon,k-1} + N_j^{\varepsilon,k}) g(\hat{E}_j^{\varepsilon,k}, \hat{E}_j^{\varepsilon,k-1})(\hat{E}_j^{\varepsilon,k-1} + \hat{E}_j^{\varepsilon,k}), \quad k \geq 1,\end{aligned} \tag{3.17}$$

$$P_j^k = f_B(|E^\varepsilon(x_j,t_k)|^2) - f_B(|\hat{E}_j^{\varepsilon,k}|^2), \quad k \geq 0. \tag{3.18}$$

It is convenient to write $R_j^k$ as

$$\begin{aligned}R_j^k =&\frac{1}{4}(\hat{n}_j^{\varepsilon,k}+\hat{n}_j^{\varepsilon,k-1})g(\hat{E}_j^{\varepsilon,k},\hat{E}_j^{\varepsilon,k-1})(\hat{E}_j^{\varepsilon,k-1}+\hat{E}_j^{\varepsilon,k})\\ &+\frac{1}{4}(N^\varepsilon(x_j,t_k)+N^\varepsilon(x_j,t_{k-1}))g(\hat{E}_j^{\varepsilon,k},\hat{E}_j^{\varepsilon,k-1})(\hat{e}_j^{\varepsilon,k-1}+\hat{e}_j^{\varepsilon,k})\\ &+\Big(g(E^\varepsilon(x_j,t_k),E^\varepsilon(x_j,t_{k-1}))-g(\hat{E}_j^{\varepsilon,k},\hat{E}_j^{\varepsilon,k-1})\Big)\\ &\times\frac{1}{4}(N^\varepsilon(x_j,t_k)+N^\varepsilon(x_j,t_{k-1}))(E^\varepsilon(x_j,t_k)+E^\varepsilon(x_j,t_{k-1})).\end{aligned} \tag{3.19}$$

Using the property of $f_B(\cdot)$, we find that there exists $C_B>0$ depending on $M_0$ and $\rho(\cdot)$ (see (3.6)), such that

$$|P_j^k|\le\sqrt{C_B}|\hat{e}_j^{\varepsilon,k}|,\qquad k\ge 0,\quad j\in\mathcal{T}_M. \tag{3.20}$$

Denoting

$$g_e(z_1,z_2)=g(z_1,z_2)(z_1+z_2),\quad z_1,z_2\in\mathbb{C}, \tag{3.21}$$

and using the properties of $f_B(\cdot)$, it is easy to check that the following holds.

**Lemma 3.5.** *For $k\ge1$ and $j\in\mathcal{T}_M$, we have* $\left|g_e(\hat{E}_j^{\varepsilon,k},\hat{E}_j^{\varepsilon,k-1})\right|\lesssim 1$ *and*

$$\left|g(E^\varepsilon(x_j,t_k),E^\varepsilon(x_j,t_{k-1}))-g(\hat{E}_j^{\varepsilon,k},\hat{E}_j^{\varepsilon,k-1})\right|\lesssim|\hat{e}_j^k|+|\hat{e}_j^{k-1}|, \tag{3.22}$$

$$\left|g_e(E^\varepsilon(x_j,t_k),E^\varepsilon(x_j,t_{k-1}))-g_e(\hat{E}_j^{\varepsilon,k},\hat{E}_j^{\varepsilon,k-1})\right|\lesssim|\hat{e}_j^k|+|\hat{e}_j^{k-1}|, \tag{3.23}$$

$$\begin{aligned}&|\delta_x^+(g_e(E^\varepsilon(x_j,t_k),E^\varepsilon(x_j,t_{k-1}))-g_e(\hat{E}_j^{\varepsilon,k},\hat{E}_j^{\varepsilon,k-1}))|\\ &\quad\lesssim\sum_{l=k-1}^{k}(|\hat{e}_j^l|+|\hat{e}_{j+1}^l|+|\delta_x^+\hat{e}_j^l|).\end{aligned} \tag{3.24}$$

*Proof.* The proof of the claims in the lemma is essentially done in [4]. Here, we prove (3.22) and the other conclusions can be proved analogously (see [4]). Recalling the definition of $g(\cdot,\cdot)$ (3.9), we have

$$g(E^\varepsilon(x_j,t_k),E^\varepsilon(x_j,t_{k-1}))-g(\hat{E}_j^{\varepsilon,k},\hat{E}_j^{\varepsilon,k-1})=\int_0^1(f_B'(\rho_\theta)-f_B'(\hat{\rho}_\theta))\,d\theta,$$

where $\rho_\theta=\theta|E^\varepsilon(x_j,t_k)|^2+(1-\theta)|E^\varepsilon(x_j,t_{k-1})|^2$, $\hat{\rho}_\theta=\theta|\hat{E}_j^{\varepsilon,k}|^2+(1-\theta)|\hat{E}_j^{\varepsilon,k-1}|^2$ ($\theta\in[0,1]$). Since

$$\sqrt{\rho_\theta}+\sqrt{\hat{\rho}_\theta}\ge\max\{\sqrt{\theta}(|E^\varepsilon(x_j,t_k)|+|\hat{E}_j^{\varepsilon,k}|),\sqrt{1-\theta}(|E^\varepsilon(x_j,t_{k-1})|+|\hat{E}_j^{\varepsilon,k-1}|)\},$$

we know that

$$\begin{aligned}|\sqrt{\rho_\theta}-\sqrt{\hat{\rho}_\theta}|=&\frac{\left|\theta(|E^\varepsilon(x_j,t_k)|^2-|\hat{E}_j^{\varepsilon,k}|^2)+(1-\theta)(|E^\varepsilon(x_j,t_{k-1})|^2-|\hat{E}_j^{\varepsilon,k-1}|^2)\right|}{\sqrt{\rho_\theta}+\sqrt{\hat{\rho}_\theta}}\\ \le&\sqrt{\theta}\left||E^\varepsilon(x_j,t_k)|-|\hat{E}_j^{\varepsilon,k}|\right|+\sqrt{1-\theta}\left||E^\varepsilon(x_j,t_{k-1})|-|\hat{E}_j^{\varepsilon,k-1}|\right|\\ \le&|\hat{e}_j^{\varepsilon,k}|+|\hat{e}_j^{\varepsilon,k-1}|.\end{aligned}$$

In view of the properties of $f_B(\cdot)$ (compactly supported), $f_B'(s)$ ($s\ge0$) is Lipschitz in $\sqrt{s}$ and (3.22) holds by combining the above results together. □

Under assumption (A), using Lemma 3.5, it is easy to check that for $k \geq 1$,

$$|R_j^k| \lesssim |\hat{n}_j^{\varepsilon,k}| + |\hat{n}_j^{\varepsilon,k-1}| + |\hat{e}_j^{\varepsilon,k}| + |\hat{e}_j^{\varepsilon,k-1}|, \quad j \in \mathcal{T}_M. \tag{3.25}$$

To control the growth of $\|\hat{e}^{\varepsilon,k}\|$, multiplying both sides of equality (3.15) by $\tau \cdot \overline{\left(\hat{e}_j^{\varepsilon,k} + \hat{e}_j^{\varepsilon,k-1}\right)}$, summing together for $j \in \mathcal{T}_M$ and taking imaginary parts, we have for $k \geq 1$,

$$\|\hat{e}^{\varepsilon,k}\|^2 - \|\hat{e}^{\varepsilon,k-1}\|^2 = \tau \operatorname{Im}(R^k, \hat{e}^{\varepsilon,k} + \hat{e}^{\varepsilon,k-1}) + \tau \operatorname{Im}(\hat{\eta}^{\varepsilon,k}, \hat{e}^{\varepsilon,k} + \hat{e}^{\varepsilon,k-1}). \tag{3.26}$$

To control the growth of $\|\delta_x^+ \hat{e}^{\varepsilon,k}\|$, multiplying both sides of equality (3.15) by $\overline{\hat{e}_j^{\varepsilon,k} - \hat{e}_j^{\varepsilon,k-1}}$, summing together for $j \in \mathcal{T}_M$ and taking real parts, we have for $k \geq 1$,

$$\frac{1}{2}(\|\delta_x^+ \hat{e}^{\varepsilon,k}\|^2 - \|\delta_x^+ \hat{e}^{\varepsilon,k-1}\|^2) + \operatorname{Re}(R^k, \hat{e}^{\varepsilon,k} - \hat{e}^{\varepsilon,k-1}) = -\operatorname{Re}(\hat{\eta}^{\varepsilon,k}, \hat{e}^{\varepsilon,k} - \hat{e}^{\varepsilon,k-1}). \tag{3.27}$$

To control the growth of $\|\hat{n}^{\varepsilon,k}\|$, multiplying both sides of equality (3.16) by $\tau(\hat{u}_j^{\varepsilon,k+\frac{1}{2}} + \hat{u}_j^{\varepsilon,k-\frac{1}{2}})$, summing together for $j \in \mathcal{T}_M$, making use of (3.13) and the summation by parts formula, we have for $k \geq 1$,

$$\begin{aligned} &\varepsilon^2 \|\delta_x^+ \hat{u}^{\varepsilon,k+\frac{1}{2}}\|^2 - \varepsilon^2 \|\delta_x^+ \hat{u}^{\varepsilon,k-\frac{1}{2}}\|^2 \\ &\quad + \frac{1}{2}(\|\hat{n}^{\varepsilon,k+1}\|^2 - \|\hat{n}^{\varepsilon,k-1}\|^2) + (P^k, \hat{n}^{\varepsilon,k+1} - \hat{n}^{\varepsilon,k-1}) \\ &= -\tau(\hat{\zeta}^{\varepsilon,k}, \hat{u}^{\varepsilon,k+\frac{1}{2}} + \hat{u}^{\varepsilon,k-\frac{1}{2}}). \end{aligned} \tag{3.28}$$

Combing $5C_B \cdot (3.26) + 4 \cdot (3.27) + (3.28)$, we get for $k \geq 1$

$$\begin{aligned} &5C_B \|\hat{e}^{\varepsilon,k}\|^2 + 2\|\delta_x^+ \hat{e}^{\varepsilon,k}\|^2 + \varepsilon^2 \|\delta_x^+ \hat{u}^{\varepsilon,k+\frac{1}{2}}\|^2 \\ &\quad + \frac{1}{2}\|\hat{n}^{\varepsilon,k+1}\|^2 + 4\operatorname{Re}(R^k, \hat{e}^{\varepsilon,k} - \hat{e}^{\varepsilon,k-1}) \\ &\quad + (P^k, \hat{n}^{\varepsilon,k+1} - \hat{n}^{\varepsilon,k-1}) - 5C_B \|\hat{e}^{\varepsilon,k-1}\|^2 - 2\|\delta_x^+ \hat{e}^{\varepsilon,k-1}\|^2 \\ &\quad - \varepsilon^2 \|\delta_x^+ \hat{u}^{\varepsilon,k-\frac{1}{2}}\|^2 - \frac{1}{2}\|\hat{n}^{\varepsilon,k-1}\|^2 \\ &= 5\tau C_B \operatorname{Im}(R^k, \hat{e}^{\varepsilon,k} + \hat{e}^{\varepsilon,k-1}) + 5\tau C_B \operatorname{Im}(\hat{\eta}^{\varepsilon,k}, \hat{e}^{\varepsilon,k} + \hat{e}^{\varepsilon,k-1}) - 4\operatorname{Re}(\hat{\eta}^{\varepsilon,k}, \hat{e}^{\varepsilon,k} - \hat{e}^{\varepsilon,k-1}) \\ &\quad - \tau(\hat{\zeta}^{\varepsilon,k}, \hat{u}^{\varepsilon,k+\frac{1}{2}} + \hat{u}^{\varepsilon,k-\frac{1}{2}}). \end{aligned} \tag{3.29}$$

To control the terms appearing in (3.29), we need the following lemmas. For the terms coming from (3.26), we have the following estimates.

**Lemma 3.6.** *Under assumption* (A), *for* $1 \leq k \leq \frac{T}{\tau}$, *we have*

$$\left|\operatorname{Im}(R^k, \hat{e}^{\varepsilon,k} + \hat{e}^{\varepsilon,k-1})\right| \lesssim \sum_{l=k,k-1} (\|\hat{n}^{\varepsilon,l}\|^2 + \|\hat{e}^{\varepsilon,l}\|^2), \tag{3.30}$$

$$\left|\operatorname{Im}(\hat{\eta}^{\varepsilon,k}, \hat{e}^{\varepsilon,k} + \hat{e}^{\varepsilon,k-1})\right| \leq 2\|\hat{\eta}^{\varepsilon,k}\|^2 + \frac{1}{4}\left(\|\hat{e}^{\varepsilon,k}\|^2 + \|\hat{e}^{\varepsilon,k-1}\|^2\right). \tag{3.31}$$

*Proof.* In view of (3.19), we find

$$\begin{aligned}&\left|\mathrm{Im}(R^k, \hat{e}^{\varepsilon,k} + \hat{e}^{\varepsilon,k-1})\right| \\ &\lesssim \left|\mathrm{Im}\left((\hat{n}_j^{\varepsilon,k} + \hat{n}_j^{\varepsilon,k-1})g(\hat{E}_j^{\varepsilon,k}, \hat{E}_j^{\varepsilon,k-1})(\hat{E}_j^{\varepsilon,k} + \hat{E}_j^{\varepsilon,k-1}), \hat{e}_j^{\varepsilon,k} + \hat{e}_j^{\varepsilon,k-1}\right)\right| \\ &\quad + \|\hat{e}^{\varepsilon,k}\|^2 + \|\hat{e}^{\varepsilon,k-1}\|^2 \lesssim \sum_{l=k,k-1} (\|\hat{n}^{\varepsilon,l}\|^2 + \|\hat{e}^{\varepsilon,l}\|^2),\end{aligned} \tag{3.32}$$

where (3.30) is proved. (3.31) is a direct consequence of the Cauchy inequality. ☐

**Lemma 3.7.** *Under assumption* (A), *for* $1 \le k \le \frac{T}{\tau}$, *we have*

$$\begin{aligned}|\mathrm{Re}(\hat{\eta}^{\varepsilon,k}, \hat{e}^{\varepsilon,k} - \hat{e}^{\varepsilon,k-1})| \lesssim & \tau \sum_{l=k-1}^{k} (\|\hat{e}^{\varepsilon,l}\|^2 + \|\delta_x^+ \hat{e}^{\varepsilon,l}\|^2 + \|\hat{n}^{\varepsilon,l}\|^2) \\ &+ \tau(\|\delta_x^+ \hat{\eta}^{\varepsilon,k}\|^2 + \|\hat{\eta}^{\varepsilon,k}\|^2)\end{aligned} \tag{3.33}$$

*and*

$$\begin{aligned}&4\mathrm{Re}(R^k, \hat{e}^{\varepsilon,k} - \hat{e}^{\varepsilon,k-1}) + (P^k, \hat{n}^{\varepsilon,k+1} - \hat{n}^{\varepsilon,k-1}) \\ &= \left(\hat{n}^{\varepsilon,k+1} + \hat{n}^{\varepsilon,k}, P^k\right) - \left(\hat{n}^{\varepsilon,k} + \hat{n}^{\varepsilon,k-1}, P^{k-1}\right) + Q^k,\end{aligned} \tag{3.34}$$

*where* $Q^k$ (3.38) *is bounded as*

$$|Q^k| \lesssim \tau \left(\|\hat{\eta}^{\varepsilon,l}\|^2 + \sum_{l=k-1}^{k} \left(\|\hat{n}^{\varepsilon,l}\|^2 + \|\hat{e}^{\varepsilon,l}\|^2 + \|\delta_x^+ \hat{e}^{\varepsilon,l}\|^2\right)\right). \tag{3.35}$$

*In addition, we know that*

$$\begin{aligned}&-\tau \sum_{l=1}^{k} (\hat{\zeta}^{\varepsilon,l}, \hat{u}^{\varepsilon,l+\frac{1}{2}} + \hat{u}^{\varepsilon,l-\frac{1}{2}}) \\ &\le \frac{\tau}{2} \sum_{l=2}^{k-1} \left(C\|\delta_t \hat{\zeta}^{\varepsilon,l}\|^2 + \|\hat{n}^{\varepsilon,l}\|^2\right) \\ &\quad + \sum_{l=0}^{1} (C\|\hat{\zeta}^{\varepsilon,l+1}\|^2 + \frac{1}{4}\|\hat{n}^{\varepsilon,l}\|^2) + \sum_{l=k}^{k+1} (C\|\hat{\zeta}^{\varepsilon,l-1}\|^2 + \frac{1}{4}\|\hat{n}^{\varepsilon,l}\|^2),\end{aligned} \tag{3.36}$$

*where* $C$ *is a constant independent of* $h$, $\tau$ *and* $\varepsilon$.

*Proof.* In view of Lemma 3.3, (3.15) and (3.19), using the Cauchy inequality, we find

$$\begin{aligned}\left|\mathrm{Re}(\hat{\eta}^{\varepsilon,k}, \hat{e}^{\varepsilon,k} - \hat{e}^{\varepsilon,k-1})\right| &= \tau \left|\mathrm{Im}\langle \hat{\eta}^{\varepsilon,k}, -\frac{1}{2}(\delta_x^2 \hat{e}^{\varepsilon,k} + \delta_x^2 \hat{e}^{\varepsilon,k-1}) + R^k + \hat{\eta}^{\varepsilon,k}\rangle\right| \\ &= \tau \left|\mathrm{Im}\langle \hat{\eta}^{\varepsilon,k}, -\frac{1}{2}(\delta_x^2 \hat{e}^{\varepsilon,k} + \delta_x^2 \hat{e}^{\varepsilon,k-1}) + R^k\rangle\right| \\ &\lesssim \tau(\|\hat{\eta}^{\varepsilon,k}\|^2 + \|\hat{n}^{\varepsilon,k-1}\|^2 + \|\hat{n}^{\varepsilon,k}\|^2 + \|\hat{e}^{\varepsilon,k}\|^2 + \|\hat{e}^{\varepsilon,k-1}\|^2) \\ &\quad + \tau(\|\delta_x^+ \hat{\eta}^{\varepsilon,k}\|^2 + \|\delta_x^+ \hat{e}^{\varepsilon,k}\|^2 + \|\delta_x^+ \hat{e}^{\varepsilon,k-1}\|^2),\end{aligned}$$

where (3.33) follows.

Next, we prove (3.34). Recalling (3.19) and by a similar computation in [8], we get

$$\operatorname{Re}(R^k, \hat{e}^{\varepsilon,k} - \hat{e}^{\varepsilon,k-1}) = \frac{1}{4}\left(\hat{n}^{\varepsilon,k} + \hat{n}^{\varepsilon,k-1}, P^k - P^{k-1}\right) + Q^k, \tag{3.37}$$

with

$$\begin{aligned} Q^k =& \frac{1}{4}\operatorname{Re}\Big((N^\varepsilon(\cdot,t_{k-1}) + N^\varepsilon(\cdot,t_k))\big(g_e(E^\varepsilon(\cdot,t_k),E^\varepsilon(\cdot,t_{k-1})) - g_e(\hat{E}^{\varepsilon,k},\hat{E}^{\varepsilon,k-1})\big), \\ & \hat{e}^{\varepsilon,k} - \hat{e}^{\varepsilon,k-1}\Big) \\ & - \frac{1}{4}\operatorname{Re}\Big((\hat{n}^{\varepsilon,k} + \hat{n}^{\varepsilon,k-1})\big(g_e(E^\varepsilon(\cdot,t_k),E^\varepsilon(\cdot,t_{k-1})) - g_e(\hat{E}^{\varepsilon,k},\hat{E}^{\varepsilon,k-1})\big), \\ & E^\varepsilon(\cdot,t_k) - E^\varepsilon(\cdot,t_{k-1})\Big), \end{aligned} \tag{3.38}$$

where $N^\varepsilon(\cdot,t_k)$ and $E^\varepsilon(\cdot,t_k)$ represent the exact solution vectors on the grid points. Thus, (3.34) is a direct consequence of (3.37).

To control $Q^k$, we estimate the individual terms as follows. Using assumption (A) and Lemma 3.5, we find

$$\begin{aligned} &|\operatorname{Re}\big((\hat{n}^{\varepsilon,k} + \hat{n}^{\varepsilon,k-1})\,(g_e(E^\varepsilon(\cdot,t_k),E^\varepsilon(\cdot,t_{k-1})) - g_e(\hat{E}^{\varepsilon,k},\hat{E}^{\varepsilon,k-1})), \\ &\qquad E^\varepsilon(\cdot,t_k) - E^\varepsilon(\cdot,t_{k-1})\big)| \\ &\lesssim \tau\|\partial_t E^\varepsilon(\cdot,t)\|_{L^\infty}\left(\|\hat{n}^{\varepsilon,k}\|^2 + \|\hat{n}^{\varepsilon,k-1}\|^2 + \|\hat{e}^{\varepsilon,k}\|^2 + \|\hat{e}^{\varepsilon,k-1}\|^2\right) \\ &\lesssim \tau\left(\|\hat{n}^{\varepsilon,k}\|^2 + \|\hat{n}^{\varepsilon,k-1}\|^2 + \|\hat{e}^{\varepsilon,k}\|^2 + \|\hat{e}^{\varepsilon,k-1}\|^2\right). \end{aligned}$$

Using error equation (3.15), Lemma 3.5, (3.25), assumption (A) and summation by parts, we obtain

$$\begin{aligned} &|\operatorname{Re}\big((N^\varepsilon(\cdot,t_{k-1}) + N^\varepsilon(\cdot,t_k))\,(g_e(E^\varepsilon(\cdot,t_k),E^\varepsilon(\cdot,t_{k-1})) \\ &\qquad - g_e(\hat{E}^{\varepsilon,k},\hat{E}^{\varepsilon,k-1})), \hat{e}^{\varepsilon,k} - \hat{e}^{\varepsilon,k-1}\big)| \\ &= \tau\Big|\operatorname{Im}\Big\langle (N^\varepsilon(\cdot,t_{k-1}) + N^\varepsilon(\cdot,t_k)) \times \big(g_e(E^\varepsilon(\cdot,t_k),E^\varepsilon(\cdot,t_{k-1})) - g_e(\hat{E}^{\varepsilon,k},\hat{E}^{\varepsilon,k-1})\big), \\ &\qquad -\frac{1}{2}(\delta_x^2\hat{e}^{\varepsilon,k} + \delta_x^2\hat{e}^{\varepsilon,k-1}) + R^k + \hat{\eta}^{\varepsilon,k}\Big\rangle\Big| \\ &\lesssim \tau\left(\|\hat{\eta}^{\varepsilon,k}\|^2 + \sum_{l=k-1}^{k}\left(\|\hat{n}^{\varepsilon,l}\|^2 + \|\hat{e}^{\varepsilon,l}\|^2 + \|\delta_x^+\hat{e}^{\varepsilon,l}\|^2\right)\right). \end{aligned}$$

Then we immediately get (3.35).

Using the summation by parts formula, the Cauchy inequality and (3.13), we derive

$$
\begin{aligned}
&-\tau\sum_{l=1}^{k}(\hat{\zeta}^{\varepsilon,l},\hat{u}^{\varepsilon,l+\frac{1}{2}}+\hat{u}^{\varepsilon,l-\frac{1}{2}})=-\sum_{l=1}^{k}\Big((-\delta_x^2)^{-1}\hat{\zeta}^{\varepsilon,l},\hat{n}^{\varepsilon,l+1}-\hat{n}^{\varepsilon,l-1}\Big)\\
&\quad=\tau\sum_{l=2}^{k-1}\Big(\delta_t(-\delta_x^2)^{-1}\hat{\zeta}^{\varepsilon,l},\hat{n}^{\varepsilon,l}\Big)+\sum_{l=0}^{1}\Big((-\delta_x^2)^{-1}\hat{\zeta}^{\varepsilon,l+1},\hat{n}^{\varepsilon,l}\Big)\\
&\qquad-\sum_{l=k}^{k+1}\Big((-\delta_x^2)^{-1}\hat{\zeta}^{\varepsilon,l-1},\hat{n}^{\varepsilon,l}\Big)\\
&\quad\le\frac{\tau}{2}\sum_{l=2}^{k-1}\Big(C_1\|\delta_t\hat{\zeta}^{\varepsilon,l}\|^2+\|\hat{n}^{\varepsilon,l}\|^2\Big)+\sum_{l=0}^{1}\left(C_1\|\hat{\zeta}^{\varepsilon,l+1}\|^2+\frac{1}{4}\|\hat{n}^{\varepsilon,l}\|^2\right)\\
&\qquad+\sum_{l=k}^{k+1}\left(C_1\|\hat{\zeta}^{\varepsilon,l-1}\|^2+\frac{1}{4}\|\hat{n}^{\varepsilon,l}\|^2\right),
\end{aligned}
$$

where $C_1$ is a constant independent of $\varepsilon$, $h$ and $\tau$. This completes the proof of Lemma 3.7. □

Now, we are ready to prove the error estimates of the modified scheme (3.8). Summing (3.29) together for time step $1,2,\ldots,k<\frac{T}{\tau}$, using Lemma 3.7, we have for some constant $C>0$ independent of $h$, $\tau$ and $\varepsilon$,

$$
\begin{aligned}
&5C_B\|\hat{e}^{\varepsilon,k}\|^2+2\|\delta_x^+\hat{e}^{\varepsilon,k}\|^2+\varepsilon^2\|\delta_x^+\hat{u}^{\varepsilon,k+\frac{1}{2}}\|^2+\frac{1}{2}\|\hat{n}^{\varepsilon,k+1}\|^2\\
&\qquad+\frac{1}{2}\|\hat{n}^{\varepsilon,k}\|^2+(P^k,\hat{n}^{\varepsilon,k+1}+\hat{n}^{\varepsilon,k})\\
&\quad\le 5C_B\|\hat{e}^{\varepsilon,0}\|^2+2\|\delta_x^+\hat{e}^{\varepsilon,0}\|^2+\varepsilon^2\|\delta_x^+\hat{u}^{\varepsilon,\frac{1}{2}}\|^2+\frac{1}{2}\|\hat{n}^{\varepsilon,1}\|^2\\
&\qquad+\frac{1}{2}\|\hat{n}^{\varepsilon,0}\|^2+(P^0,\hat{n}^{\varepsilon,1}+\hat{n}^{\varepsilon,0})\\
&\qquad+\frac{1}{4}\sum_{l=k}^{k+1}\|\hat{n}^{\varepsilon,l}\|^2+\frac{1}{4}\sum_{l=0}^{1}\|\hat{n}^{\varepsilon,l}\|^2+C\sum_{l=1}^{2}\|\hat{\zeta}^{\varepsilon,l}\|^2+C\sum_{l=k-1}^{k}\|\hat{\zeta}^{\varepsilon,l}\|^2\\
&\qquad+C\tau\sum_{l=1}^{k}\Big(\|\hat{\eta}^{\varepsilon,l}\|^2+\|\delta_t\hat{\zeta}^{\varepsilon,l}\|^2+\|\delta_x^+\hat{\eta}^{\varepsilon,l}\|^2+\|\hat{n}^{\varepsilon,l}\|^2+\|\hat{e}^{\varepsilon,l}\|^2+\|\delta_x^+\hat{e}^{\varepsilon,l}\|^2\Big).
\end{aligned}
\tag{3.39}
$$

Denote

$$
\begin{aligned}
\hat{\mathcal{E}}^k=&5C_B\|\hat{e}^{\varepsilon,k}\|^2+2\|\delta_x^+\hat{e}^{\varepsilon,k}\|^2+\varepsilon^2\|\delta_x^+\hat{u}^{\varepsilon,k+\frac{1}{2}}\|^2+\frac{1}{4}\|\hat{n}^{\varepsilon,k+1}\|^2\\
&+\frac{1}{4}\|\hat{n}^{\varepsilon,k}\|^2+(P^k,\hat{n}^{\varepsilon,k+1}+\hat{n}^{\varepsilon,k}),
\end{aligned}
$$

where (3.20) and the Cauchy inequality imply

$$
\hat{\mathcal{E}}^k\ge C_B\|\hat{e}^{\varepsilon,k}\|^2+2\|\delta_x^+\hat{e}^{\varepsilon,k}\|^2+\varepsilon^2\|\delta_x^+\hat{u}^{\varepsilon,k+\frac{1}{2}}\|^2+\frac{1}{8}\|\hat{n}^{\varepsilon,k+1}\|^2+\frac{1}{8}\|\hat{n}^{\varepsilon,k}\|^2,\quad k\ge 1.
\tag{3.40}
$$

Recalling assumption (A), (3.12), (3.14) and (3.39), we have for $1 \le k \le \frac{T}{\tau}$

$$\hat{\mathcal{E}}^k \lesssim \left(\frac{h^2}{\varepsilon^{1-\alpha^*}} + \frac{\tau^2}{\varepsilon^{3-\alpha^\dagger}}\right)^2 + \tau \sum_{l=1}^{k} \hat{\mathcal{E}}^l, \quad 1 \le k \le \frac{T}{\tau}. \tag{3.41}$$

Using the discrete Gronwall inequality, for sufficiently small $\tau$, we have

$$\hat{\mathcal{E}}^k \lesssim \left(\frac{h^2}{\varepsilon^{1-\alpha^*}} + \frac{\tau^2}{\varepsilon^{3-\alpha^\dagger}}\right)^2, \quad 1 \le k \le \frac{T}{\tau}, \tag{3.42}$$

and the following estimates hold in view of (3.40):

$$\|\hat{e}^{\varepsilon,k}\| + \|\delta_x^+ \hat{e}^{\varepsilon,k}\| + \varepsilon\|\delta_x^+ \hat{u}^{\varepsilon,k+\frac{1}{2}}\| + \|\hat{n}^{\varepsilon,k}\| \lesssim \frac{h^2}{\varepsilon^{1-\alpha^*}} + \frac{\tau^2}{\varepsilon^{3-\alpha^\dagger}}, \quad 0 \le k \le \frac{T}{\tau}. \tag{3.43}$$

3.2. (2.16) **type estimates.** Here, we prove the second type error estimates for the numerical approximation $(\hat{E}^{\varepsilon,k}, \hat{N}^{\varepsilon,k})$ in the modified CNFD scheme (3.8). Define the 'biased' error function $\tilde{e}^{\varepsilon,k}, \tilde{n}^{\varepsilon,k} \in X_M$ for $k \ge 0$ as

$$\tilde{e}_j^{\varepsilon,k} = E(x_j, t_k) - \hat{E}_j^{\varepsilon,k}, \quad \tilde{n}_j^{\varepsilon,k} = N(x_j, t_k) - \hat{N}_j^{\varepsilon,k}, \quad j \in \mathcal{T}_M, \tag{3.44}$$

where $E(x,t)$ is the solution of the limit NLS (2.5) and we denote the density $N(x,t)$ as

$$N(x,t) = -|E(x,t)|^2, \quad x \in \Omega,\ t \ge 0. \tag{3.45}$$

We introduce $\tilde{u}^{\varepsilon,k-\frac{1}{2}} = (-\delta_x^2)^{-1}(\delta_t^- \tilde{n}^{\varepsilon,k})$ $(k \ge 1)$, i.e., $\tilde{u}^{\varepsilon,k-\frac{1}{2}}$ is the solution of the equation

$$-\delta_x^2 \tilde{u}_j^{\varepsilon,k-\frac{1}{2}} = \delta_t^- \tilde{n}_j^{\varepsilon,k}, \quad j \in \mathcal{T}_M, \tag{3.46}$$

with $\tilde{u}_0^{\varepsilon,k-\frac{1}{2}} = \tilde{u}_M^{\varepsilon,k-\frac{1}{2}} = 0$.

Then we define the local truncation error $\tilde{\eta}^{\varepsilon,k}, \tilde{\zeta}^{\varepsilon,k} \in X_M$ for $k \ge 1$ and $j \in \mathcal{T}_M$ as

(3.47)

$$\begin{aligned}\tilde{\eta}_j^{\varepsilon,k} :=& i\delta_t^- E(x_j,t_k) + \left[\frac{1}{2}\delta_x^2 - \frac{1}{4}g(E(x_j,t_k), E(x_j,t_{k-1}))\,(N(x_j,t_{k-1}) + N(x_j,t_k))\right] \\ &\times (E(x_j,t_{k-1}) + E(x_j,t_k)), \\ \tilde{\zeta}_j^{\varepsilon,k} :=& \varepsilon^2 \delta_t^2 N(x_j,t_k) - \frac{1}{2}(\delta_x^2 N(x_j,t_{k+1}) + \delta_x^2 N(x_j,t_{k-1})) - \delta_x^2 f_B\left(|E(x_j,t_k)|^2\right).\end{aligned}$$

As in the proof of the first type estimates (2.15), under the assumption (A), we could obtain the following bounds on the local truncation error,

$$\|\tilde{\eta}^{\varepsilon,k}\| + \|\delta_x^+ \tilde{\eta}^{\varepsilon,k}\| \lesssim h^2 + \tau^2, \quad \|\tilde{\zeta}^{\varepsilon,k}\| + \|\delta_t \tilde{\zeta}^{\varepsilon,k}\| \lesssim h^2 + \tau^2 + \varepsilon^2, \quad k \ge 1. \tag{3.48}$$

The initial errors satisfy $\tilde{e}^{\varepsilon,0} = \mathbf{0}$ and

(3.49)

$$\|\tilde{n}^{\varepsilon,0}\| \lesssim \varepsilon^\alpha,\ \|\tilde{n}^{\varepsilon,1}\| \lesssim \tau h^2 + \tau^2 + \varepsilon^\beta + \varepsilon^\alpha,\ \|\tilde{u}^{\varepsilon,1/2}\| \lesssim \|\delta_t^- \tilde{n}^{\varepsilon,1}\| \lesssim h^2 + \tau + \varepsilon^{\beta-1} + \varepsilon^{\alpha-1}.$$

Subtracting (3.8) from (3.47), we get the error equations for $j \in \mathcal{T}_M$ and $k \ge 1$

$$i\delta_t^- \tilde{e}_j^{\varepsilon,k} = -\frac{1}{2}\delta_x^2(\tilde{e}_j^{\varepsilon,k-1} + \tilde{e}_j^{\varepsilon,k}) + \tilde{R}_j^k + \tilde{\eta}_j^{\varepsilon,k}, \tag{3.50}$$

$$\varepsilon^2 \delta_t^2 \tilde{n}_j^{\varepsilon,k} = \frac{1}{2}\delta_x^2(\tilde{n}_j^{\varepsilon,k-1} + \tilde{n}_j^{\varepsilon,k+1}) + \delta_x^2 \tilde{P}_j^k + \tilde{\zeta}_j^{\varepsilon,k}, \tag{3.51}$$

where $\tilde{R}^k, \tilde{P}^k \in X_M$ and for $j \in \mathcal{T}_M$

$$\begin{aligned}\tilde{R}_j^k =& \frac{1}{4}(N(x_j,t_{k-1}) + N^\varepsilon(x_j,t_k)) \\ & \times g(E(x_j,t_k),E(x_j,t_{k-1}))(E(x_j,t_{k-1}) + E(x_j,t_k)) \\ & - \frac{1}{4}(N_j^{\varepsilon,k-1} + N_j^{\varepsilon,k})g(\hat{E}_j^{\varepsilon,k},\hat{E}_j^{\varepsilon,k-1})(\hat{E}_j^{\varepsilon,k-1} + \hat{E}_j^{\varepsilon,k}), \quad k \geq 1, \end{aligned} \tag{3.52}$$

$$\tilde{P}_j^k = f_B(|E(x_j,t_k)|^2) - f_B(|\hat{E}_j^{\varepsilon,k}|^2), \quad k \geq 0. \tag{3.53}$$

Using the property of $f_B(\cdot)$, we find that for the same constant $C_B > 0$ in (3.20), the following holds:

$$|\tilde{P}_j^k| \leq \sqrt{C_B}|\tilde{e}_j^{\varepsilon,k}|, \qquad k \geq 0, \quad j \in \mathcal{T}_M. \tag{3.54}$$

Similarly to Lemma 3.5, we have the results below.

**Lemma 3.8.** *Under assumption* (A), *for* $k \geq 1$ *and* $j \in \mathcal{T}_M$, *we have*

$$\left|g(E(x_j,t_k),E(x_j,t_{k-1})) - g(\hat{E}_j^{\varepsilon,k},\hat{E}_j^{\varepsilon,k-1})\right| \lesssim |\tilde{e}_j^k| + |\tilde{e}_j^{k-1}|, \tag{3.55}$$

$$\left|g_e(E(x_j,t_k),E(x_j,t_{k-1})) - g_e(\hat{E}_j^{\varepsilon,k},\hat{E}_j^{\varepsilon,k-1})\right| \lesssim |\tilde{e}_j^k| + |\tilde{e}_j^{k-1}|, \tag{3.56}$$

$$\begin{aligned}&\left|\delta_x^+(g_e(E(x_j,t_k),E(x_j,t_{k-1})) - g_e(\hat{E}_j^{\varepsilon,k},\hat{E}_j^{\varepsilon,k-1}))\right| \\ &\quad \lesssim \sum_{l=k-1}^{k}(|\tilde{e}_j^l| + |\tilde{e}_{j+1}^l| + |\delta_x^+\tilde{e}_j^l|).\end{aligned} \tag{3.57}$$

Following the proof of the estimates (3.43) for $(\hat{e}^{\varepsilon,k},\hat{n}^{\varepsilon,k})$, we proceed as follows.

First, multiplying both sides of (3.50) by $\tau \cdot \overline{\tilde{e}_j^{\varepsilon,k} + \tilde{e}_j^{\varepsilon,k-1}}$, summing together for $j \in \mathcal{T}_M$ and taking imaginary parts, we have for $k \geq 1$,

$$\|\tilde{e}^{\varepsilon,k}\|^2 - \|\tilde{e}^{\varepsilon,k-1}\|^2 = \tau \mathrm{Im}(\tilde{R}^k, \tilde{e}^{\varepsilon,k} + \tilde{e}^{\varepsilon,k-1}) + \tau \mathrm{Im}(\tilde{\eta}^{\varepsilon,k}, \tilde{e}^{\varepsilon,k} + \tilde{e}^{\varepsilon,k-1}). \tag{3.58}$$

Second, multiplying both sides of (3.50) by $\overline{\tilde{e}_j^{\varepsilon,k} - \tilde{e}_j^{\varepsilon,k-1}}$, summing together for $j \in \mathcal{T}_M$ and taking real parts, we have for $k \geq 1$,

$$\frac{1}{2}(\|\delta_x^+\tilde{e}^{\varepsilon,k}\|^2 - \|\delta_x^+\tilde{e}^{\varepsilon,k-1}\|^2) + \mathrm{Re}(\tilde{R}^k, \tilde{e}^{\varepsilon,k} - \tilde{e}^{\varepsilon,k-1}) = -\mathrm{Re}(\tilde{\eta}^{\varepsilon,k}, \tilde{e}^{\varepsilon,k} - \tilde{e}^{\varepsilon,k-1}). \tag{3.59}$$

Third, multiplying both sides of (3.51) by $\tau(\tilde{u}_j^{\varepsilon,k+\frac{1}{2}} + \tilde{u}_j^{\varepsilon,k-\frac{1}{2}})$, summing together for $j \in \mathcal{T}_M$, making use of (3.46) and summation by parts formula, we have for $k \geq 1$,

$$\begin{aligned}&\varepsilon^2\|\delta_x^+\tilde{u}^{\varepsilon,k+\frac{1}{2}}\|^2 - \varepsilon^2\|\delta_x^+\tilde{u}^{\varepsilon,k-\frac{1}{2}}\|^2 + \frac{1}{2}(\|\tilde{n}^{\varepsilon,k+1}\|^2 - \|\tilde{n}^{\varepsilon,k-1}\|^2) \\ &\qquad + (\tilde{P}^k, \tilde{n}^{\varepsilon,k+1} - \tilde{n}^{\varepsilon,k-1}) \\ &\quad = -\tau(\tilde{\zeta}^{\varepsilon,k}, \tilde{u}^{\varepsilon,k+\frac{1}{2}} + \tilde{u}^{\varepsilon,k-\frac{1}{2}}).\end{aligned} \tag{3.60}$$

Finally, combining $5C_B \cdot (3.58) + 4\cdot(3.59) + (3.60)$, we get for $k \ge 1$

$$\begin{aligned}
&5C_B\|\tilde{e}^{\varepsilon,k}\|^2 + 2\|\delta_x^+\tilde{e}^{\varepsilon,k}\|^2 + \varepsilon^2\|\delta_x^+\tilde{u}^{\varepsilon,k+\frac12}\|^2 + \frac12\|\tilde{n}^{\varepsilon,k+1}\|^2\\
&\quad + 4\mathrm{Re}(\tilde{R}^k, \tilde{e}^{\varepsilon,k} - \tilde{e}^{\varepsilon,k-1}) + (\tilde{P}^k, \tilde{n}^{\varepsilon,k+1} - \tilde{n}^{\varepsilon,k-1})\\
&\quad - 5C_B\|\tilde{e}^{\varepsilon,k-1}\|^2 - 2\|\delta_x^+\tilde{e}^{\varepsilon,k-1}\|^2 - \varepsilon^2\|\delta_x^+\tilde{u}^{\varepsilon,k-\frac12}\|^2 - \frac12\|\tilde{n}^{\varepsilon,k-1}\|^2\\
&= 5\tau C_B \mathrm{Im}(\tilde{R}^k, \tilde{e}^{\varepsilon,k} + \tilde{e}^{\varepsilon,k-1}) + 5\tau C_B\mathrm{Im}(\tilde{\eta}^{\varepsilon,k}, \tilde{e}^{\varepsilon,k} + \tilde{e}^{\varepsilon,k-1})\\
&\quad - 4\mathrm{Re}(\tilde{\eta}^{\varepsilon,k}, \tilde{e}^{\varepsilon,k} - \tilde{e}^{\varepsilon,k-1}) - \tau(\tilde{\zeta}^{\varepsilon,k}, \tilde{u}^{\varepsilon,k+\frac12} + \tilde{u}^{\varepsilon,k-\frac12}).
\end{aligned} \tag{3.61}$$

Following the proofs of Lemma 3.6 and 3.7, we can prove similar estimates below and omit the details here.

**Lemma 3.9.** *Under the assumption* (A), *for $k \ge 1$, we have*

$$\left|\mathrm{Im}(\tilde{R}^k, \tilde{e}^{\varepsilon,k} + \tilde{e}^{\varepsilon,k-1})\right| \lesssim \|\tilde{n}^{\varepsilon,k-1}\|^2 + \|\tilde{e}^{\varepsilon,k-1}\|^2 + \|\tilde{n}^{\varepsilon,k}\|^2 + \|\tilde{e}^{\varepsilon,k}\|^2, \tag{3.62}$$

$$\left|\mathrm{Im}(\tilde{\eta}^{\varepsilon,k}, \tilde{e}^{\varepsilon,k} + \tilde{e}^{\varepsilon,k-1})\right| \le 2\|\tilde{\eta}^{\varepsilon,k}\|^2 + \frac14\left(\|\tilde{e}^{\varepsilon,k}\|^2 + \|\tilde{e}^{\varepsilon,k-1}\|^2\right), \tag{3.63}$$

$$\begin{aligned}
\left|\mathrm{Re}(\tilde{\eta}^{\varepsilon,k}, \tilde{e}^{\varepsilon,k} - \tilde{e}^{\varepsilon,k-1})\right| &\lesssim \tau\sum_{l=k-1}^{k}(\|\tilde{e}^{\varepsilon,l}\|^2 + \|\delta_x^+\tilde{e}^{\varepsilon,l}\|^2 + \|\tilde{n}^{\varepsilon,l}\|^2)\\
&\quad + \tau(\|\delta_x^+\tilde{\eta}^{\varepsilon,k}\|^2 + \|\tilde{\eta}^{\varepsilon,k}\|^2),
\end{aligned} \tag{3.64}$$

*and*

$$\begin{aligned}
&4\mathrm{Re}(\tilde{R}^k, \tilde{e}^{\varepsilon,k} - \tilde{e}^{\varepsilon,k-1}) + (\tilde{P}^k, \tilde{n}^{\varepsilon,k+1} - \tilde{n}^{\varepsilon,k-1})\\
&=(\tilde{n}^{\varepsilon,k+1} + \tilde{n}^{\varepsilon,k}, \tilde{P}^k) - (\tilde{n}^{\varepsilon,k} + \tilde{n}^{\varepsilon,k-1}, \tilde{P}^{k-1}) + \tilde{Q}^k,
\end{aligned} \tag{3.65}$$

*where $\tilde{Q}^k$ is defined similarly to* (3.38) *and*

$$|\tilde{Q}^k| \lesssim \tau\left(\|\tilde{\eta}^{\varepsilon,l}\|^2 + \sum_{l=k-1}^{k}\left(\|\tilde{n}^{\varepsilon,l}\|^2 + \|\tilde{e}^{\varepsilon,l}\|^2 + \|\delta_x^+\tilde{e}^{\varepsilon,l}\|^2\right)\right). \tag{3.66}$$

*In addition, we have*

$$\begin{aligned}
-\tau\sum_{l=1}^{k}(\tilde{\zeta}^{\varepsilon,l}, \tilde{u}^{\varepsilon,l+\frac12} + \tilde{u}^{\varepsilon,l-\frac12}) &\le \frac{\tau}{2}\sum_{l=2}^{k-1}\left(C\|\delta_t\tilde{\zeta}^{\varepsilon,l}\|^2 + \|\tilde{n}^{\varepsilon,l}\|^2\right)\\
&\quad + \sum_{l=0}^{1}(C\|\tilde{\zeta}^{\varepsilon,l+1}\|^2 + \frac14\|\tilde{n}^{\varepsilon,l}\|^2) + \sum_{l=k}^{k+1}(C\|\tilde{\zeta}^{\varepsilon,l-1}\|^2 + \frac14\|\tilde{n}^{\varepsilon,l}\|^2),
\end{aligned} \tag{3.67}$$

*where $C$ is a constant independent of $\tau$, $h$ and $\varepsilon$.*

Now, we are ready to prove (2.16) type estimates. Summing (3.61) together for time step $1, 2, \ldots, k < \frac{T}{\tau}$, using Lemma 3.9, we have for some constant $C > 0$

independent of $h$, $\tau$ and $\varepsilon$,

$$\begin{aligned}
&5C_B\|\tilde{e}^{\varepsilon,k}\|^2 + 2\|\delta_x^+\tilde{e}^{\varepsilon,k}\|^2 + \varepsilon^2\|\delta_x^+\tilde{u}^{\varepsilon,k+\frac{1}{2}}\|^2 + \frac{1}{2}\|\tilde{n}^{\varepsilon,k+1}\|^2 \\
&\quad + \frac{1}{2}\|\tilde{n}^{\varepsilon,k}\|^2 + (\tilde{P}^k, \tilde{n}^{\varepsilon,k+1} + \tilde{n}^{\varepsilon,k}) \\
&\quad \le 5C_B\|\tilde{e}^{\varepsilon,0}\|^2 + 2\|\delta_x^+\tilde{e}^{\varepsilon,0}\|^2 + \varepsilon^2\|\delta_x^+\tilde{u}^{\varepsilon,\frac{1}{2}}\|^2 + \frac{1}{2}\|\tilde{n}^{\varepsilon,1}\|^2 + \frac{1}{2}\|\tilde{n}^{\varepsilon,0}\|^2 + (\tilde{P}^0, \tilde{n}^{\varepsilon,1} + \tilde{n}^{\varepsilon,0}) \\
&\quad + \frac{1}{4}\sum_{l=k}^{k+1}\|\tilde{n}^{\varepsilon,l}\|^2 + \frac{1}{4}\sum_{l=0}^{1}\|\tilde{n}^{\varepsilon,l}\|^2 + C\sum_{l=1}^{2}\|\tilde{\zeta}^{\varepsilon,l}\|^2 + C\sum_{l=k-1}^{k}\|\tilde{\zeta}^{\varepsilon,l}\|^2 \\
&\quad + C\tau\sum_{l=1}^{k}\left(\|\tilde{\eta}^{\varepsilon,l}\|^2 + \|\delta_t\tilde{\zeta}^{\varepsilon,l}\|^2 + \|\delta_x^+\tilde{\eta}^{\varepsilon,l}\|^2 + \|\tilde{n}^{\varepsilon,l}\|^2 + \|\tilde{e}^{\varepsilon,l}\|^2 + \|\delta_x^+\tilde{e}^{\varepsilon,l}\|^2\right).
\end{aligned} \tag{3.68}$$

Denote

$$\begin{aligned}
\tilde{\mathcal{E}}^k &= 5C_B\|\tilde{e}^{\varepsilon,k}\|^2 + 2\|\delta_x^+\tilde{e}^{\varepsilon,k}\|^2 + \varepsilon^2\|\delta_x^+\tilde{u}^{\varepsilon,k+\frac{1}{2}}\|^2 + \frac{1}{4}\|\tilde{n}^{\varepsilon,k+1}\|^2 \\
&\quad + \frac{1}{4}\|\tilde{n}^{\varepsilon,k}\|^2 + (\tilde{P}^k, \tilde{n}^{\varepsilon,k+1} + \tilde{n}^{\varepsilon,k}),
\end{aligned}$$

where (3.54) and the Cauchy inequality imply

$$\tilde{\mathcal{E}}^k \ge C_B\|\tilde{e}^{\varepsilon,k}\|^2 + 2\|\delta_x^+\tilde{e}^{\varepsilon,k}\|^2 + \varepsilon^2\|\delta_x^+\tilde{u}^{\varepsilon,k+\frac{1}{2}}\|^2 + \frac{1}{8}\|\tilde{n}^{\varepsilon,k+1}\|^2 + \frac{1}{8}\|\tilde{n}^{\varepsilon,k}\|^2, \quad k \ge 1. \tag{3.69}$$

Recalling assumptions (A) and (B), (3.48), (3.49) and (3.68), we have for $1 \le k \le \frac{T}{\tau}$,

$$\tilde{\mathcal{E}}^k \lesssim \left(h^2 + \tau^2 + \varepsilon^\beta + \varepsilon^\alpha + \varepsilon^2\right)^2 + \tau\sum_{l=1}^{k}\tilde{\mathcal{E}}^l, \quad 1 \le k \le \frac{T}{\tau}. \tag{3.70}$$

Using the discrete Gronwall inequality, for sufficiently small $\tau$, we have

$$\tilde{\mathcal{E}}^k \lesssim \left(h^2 + \tau^2 + \varepsilon^\alpha + \varepsilon^\beta + \varepsilon^2\right)^2, \quad 1 \le k \le \frac{T}{\tau}, \tag{3.71}$$

and the following estimates hold in view of (3.69):

$$\|\tilde{e}^{\varepsilon,k}\| + \|\delta_x^+\tilde{e}^{\varepsilon,k}\| + \varepsilon\|\delta_x^+\tilde{u}^{\varepsilon,k+\frac{1}{2}}\| + \|\tilde{n}^{\varepsilon,k}\| \lesssim h^2 + \tau^2 + \varepsilon^\beta + \varepsilon^\alpha + \varepsilon^2, \quad 0 \le k \le \frac{T}{\tau}. \tag{3.72}$$

Using assumption (B), we obtain

$$\begin{aligned}
\|\hat{e}^{\varepsilon,k}\| + \|\delta_x^+\hat{e}^{\varepsilon,k}\| &\le \|\tilde{e}^{\varepsilon,k}\| + \|\delta_x^+\tilde{e}^{\varepsilon,k}\| + \|E^\varepsilon(\cdot,t_k) - E(\cdot,t_k)\|_{H^1} \\
&\lesssim h^2 + \tau^2 + \varepsilon^{\alpha^\dagger} + \varepsilon^{1+\alpha^*},
\end{aligned} \tag{3.73}$$

$$\|\hat{n}^{\varepsilon,k}\| \le \|\tilde{n}^{\varepsilon,k}\| + \|N^\varepsilon(\cdot,t_k) - N(\cdot,t_k)\|_{L^2} \lesssim h^2 + \tau^2 + \varepsilon^{\alpha^\dagger}. \tag{3.74}$$

The (2.16) type estimate is established.

3.3. **Proof of Theorems 2.1 and 2.2.** We have proved the estimates (3.43), (3.73) and (3.74) for $(\hat{E}^{\varepsilon,k}, \hat{N}^{\varepsilon,k})$ $(k \ge 0)$, the solution of modified CNFD (3.8) with (2.8) and (2.9). In order to get the error bounds (2.15) and (2.16) for $(E^{\varepsilon,k}, N^{\varepsilon,k})$ in Theorem 2.1, recalling (3.7), we only need show that

$$\|\hat{E}^{\varepsilon,k}\|_\infty \le M_0 + 1, \quad k \ge 0, \tag{3.75}$$

which implies the modified CNFD (3.8) collapses to (2.7) and $(\hat{E}^{\varepsilon,k}, \hat{N}^{\varepsilon,k})$ $(k \geq 0)$ are identical to $(E^{\varepsilon,k}, N^{\varepsilon,k})$ $(k \geq 0)$. For the 1D case, similarly to Lemma 2.1, we can derive the *a priori* bound of $(\hat{E}^{\varepsilon,k}, \hat{N}^{\varepsilon,k})$. Following the proof of Lemma 2.1 (see also [8]), it is easy to show that (3.8) conserves the mass $\|\hat{E}^{\varepsilon,k}\|$ and the energy

(3.76)

$$\hat{H}^k = \frac{1}{2}(\|\delta_x^+ \hat{E}^{\varepsilon,k+1}\|^2 + \|\delta_x^+ \hat{E}^{\varepsilon,k}\|^2) + \frac{\varepsilon^2}{2}\|\delta_x^+ \hat{U}^{\varepsilon,k+1/2}\|^2 + \frac{1}{4}(\|\hat{N}^{\varepsilon,k+1}\|^2 + \|\hat{N}^{\varepsilon,k}\|^2) + \frac{h}{4}\sum_{j=0}^{M-1}(\hat{N}_j^{\varepsilon,k+1} + \hat{N}_j^{\varepsilon,k})\left(f_B\left(|\hat{E}_j^{\varepsilon,k+1}|^2\right) + f_B\left(|\hat{E}_j^{\varepsilon,k}|^2\right)\right) \equiv \hat{H}^0, \quad k \geq 0,$$

where $\hat{U}^{\varepsilon,k+1/2} = (-\delta_x^2)^{-1}\delta_t^+ \hat{N}^{\varepsilon,k}$.

Noticing $f_B(s) \leq s$ $(s \geq 0)$, using the same arguments in Lemma 2.1, we could obtain the same bounds for $\hat{E}^{\varepsilon,k}$ $(k \geq 0)$ as

$$\|\hat{E}^{\varepsilon,k}\|_\infty \leq C_a, \quad k \geq 0. \tag{3.77}$$

By the choice of $M_0$ in the 1D case, we have

$$\|\hat{E}^{\varepsilon,k}\|_\infty \leq C_a \leq M_0 + 1, \quad k \geq 0. \tag{3.78}$$

This implies $(\hat{E}^{\varepsilon,k}, \hat{N}^{\varepsilon,k})$ is identical to $(E^{\varepsilon,k}, N^{\varepsilon,k})$, and conclusions (2.15), (2.16) and (2.18) follow. ☐

*Remark* 3.1. In the 2D $(d = 2)$ and 3D $(d = 3)$ cases, the above approach needs modification. Instead of bounding $\|\hat{E}^{\varepsilon,k}\|_\infty$ by conserved quantities, we control $\|\hat{E}^{\varepsilon,k}\|_\infty$ by the error function $\hat{e}^{\varepsilon,k}$, i.e.,

$$\|\hat{E}^{\varepsilon,k}\|_\infty \leq \|E^\varepsilon(x, t_k)\|_{L^\infty} + \|\hat{e}^{\varepsilon,k}\|_\infty \leq M_0 + \frac{1}{C_d(h)}\|\hat{e}^{\varepsilon,k}\|_{H^1}, \tag{3.79}$$

and we use discrete Sobolev inequalities [20] in 2D and 3D to control the $l^\infty$-norm as

$$\|\psi_h\|_\infty \lesssim \frac{1}{C_d(h)}\|\psi_h\|_{H^1}, \quad d = 2, 3, \tag{3.80}$$

where $C_d(h) = 1/|\ln h|$ when $d = 2$ and $C_d(h) = h^{1/2}$ when $d = 3$, and $\psi_h$ are 2D/3D mesh functions with zero at the boundary. Thus, by assuming the additional conditions with $h = o(\varepsilon^{2/3})$, $\tau = o(\varepsilon^{3/2}C_d(h))$ for the ill-prepared initial data, $\tau = o(C_d(h)^{3/2})$ for less-ill-prepared initial data, and $\tau = o(C_d(h)^{2/3})$ for the well-prepared initial data, we could obtain the same error estimates as those in Theorems 2.1 and 2.2.

## 4. Numerical results

In this section, we present numerical results for ZS (2.1) by CNFD (2.7) with (2.8) and (2.11). The initial data of ZS (2.1) is chosen as those given in (2.1) and (2.6) with

$$E_0(x) = e^{-x^2/2}, \quad \omega_0(x) = e^{-x^2/4}, \quad \omega_1(x) = xe^{-x^2/4}. \tag{4.1}$$

and the parameters $\alpha$ and $\beta$ in (2.6) are set as:

*Case* I. Well-prepared initial data, i.e., $\alpha = 2$ and $\beta = 2$.

*Case* II. Less-ill-prepared initial data, i.e., $\alpha = 1$ and $\beta = 1$.

TABLE 4.1. Spatial error analysis of CNFD at time $t = 1$ for the well-prepared initial data Case I, i.e., $\alpha = 2$ and $\beta = 2$.

| $e^\varepsilon(t=1)$ | $h = 0.1$ | $h = \frac{0.1}{2}$ | $h = \frac{0.1}{2^2}$ | $h = \frac{0.1}{2^3}$ | $h = \frac{0.1}{2^4}$ | $h = \frac{0.1}{2^5}$ |
|---|---|---|---|---|---|---|
| $\varepsilon_0 = 1$ | 7.31E-3 | 1.83E-3 | 4.59E-4 | 1.15E-4 | 2.87E-5 | 7.17E-6 |
| order | - | 1.99 | 1.99 | 2.00 | 2.00 | 2.00 |
| $\varepsilon = 1/2$ | 6.01E-3 | 1.52E-3 | 3.76E-4 | 9.40E-5 | 2.35E-5 | 5.90E-6 |
| order | - | 1.98 | 2.02 | 2.00 | 2.00 | 1.99 |
| $\varepsilon = 1/2^2$ | 6.22E-3 | 1.61E-3 | 3.90E-4 | 9.76E-5 | 2.44E-5 | 6.11E-6 |
| order | - | 1.95 | 2.04 | 2.00 | 2.00 | 1.99 |
| $\varepsilon = 1/2^3$ | 6.81E-3 | 1.71E-3 | 4.24E-4 | 1.06E-4 | 2.66E-5 | 6.66E-6 |
| order | - | 1.99 | 2.00 | 2.01 | 1.99 | 2.00 |
| $\varepsilon = 1/2^4$ | 6.83E-3 | 1.71E-3 | 4.27E-4 | 1.07E-4 | 2.68E-5 | 6.30E-6 |
| order | - | 2.00 | 2.00 | 2.00 | 2.00 | 2.08 |
| $\varepsilon = 1/2^5$ | 6.84E-3 | 1.71E-3 | 4.27E-4 | 1.07E-4 | 2.68E-5 | 6.74E-6 |
| order | - | 2.00 | 2.00 | 2.00 | 2.00 | 1.99 |
| $\varepsilon = 1/2^6$ | 6.84E-3 | 1.71E-3 | 4.28E-4 | 1.07E-4 | 2.68E-5 | 6.74E-6 |
| order | - | 2.00 | 2.00 | 2.00 | 2.00 | 1.99 |

| $n^\varepsilon(t=1)$ | $h = 0.1$ | $h = \frac{0.1}{2}$ | $h = \frac{0.1}{2^2}$ | $h = \frac{0.1}{2^3}$ | $h = \frac{0.1}{2^4}$ | $h = \frac{0.1}{2^5}$ |
|---|---|---|---|---|---|---|
| $\varepsilon = 1$ | 1.72E-3 | 4.31E-4 | 1.08E-4 | 2.69E-5 | 6.70E-6 | 1.68E-6 |
| order | - | 2.00 | 2.00 | 2.01 | 2.01 | 2.00 |
| $\varepsilon = 1/2$ | 2.52E-3 | 6.12E-4 | 1.53E-4 | 3.82E-5 | 9.52E-6 | 2.37E-6 |
| order | - | 2.04 | 2.00 | 2.00 | 2.00 | 2.00 |
| $\varepsilon = 1/2^2$ | 2.02E-3 | 5.00E-4 | 1.25E-4 | 3.13E-5 | 7.83E-6 | 1.99E-6 |
| order | - | 2.01 | 2.00 | 2.00 | 2.00 | 1.98 |
| $\varepsilon = 1/2^3$ | 1.22E-3 | 3.04E-4 | 7.60E-5 | 1.90E-5 | 4.77E-6 | 1.22E-6 |
| order | - | 2.00 | 2.00 | 2.00 | 1.99 | 1.97 |
| $\varepsilon = 1/2^4$ | 1.13E-3 | 2.83E-4 | 7.14E-5 | 1.77E-5 | 4.43E-6 | 1.24E-6 |
| order | - | 2.00 | 2.00 | 2.01 | 2.00 | 1.84 |
| $\varepsilon = 1/2^5$ | 1.12E-3 | 2.79E-4 | 6.97E-5 | 1.74E-5 | 4.38E-6 | 1.15E-6 |
| order | - | 2.00 | 2.00 | 2.00 | 1.99 | 1.93 |
| $\varepsilon = 1/2^6$ | 1.12E-3 | 2.78E-4 | 6.94E-5 | 1.74E-5 | 4.39E-6 | 1.25E-6 |
| order | - | 2.00 | 2.00 | 2.00 | 1.99 | 1.81 |

*Case* III. Ill-prepared initial data, i.e., $\alpha = 0$ and $\beta = 0$.

The problem is truncated onto $\Omega = (-200, 200)$, such that the error due to the truncation with homogeneous Dirichlet boundary condition is negligible. The 'exact' solution $(E^\varepsilon(x,t), N^\varepsilon(x,t))$ is computed by the time splitting spectral method [5,6] with very fine mesh size $h = 1/32$ and time step $\tau = 10^{-7}$. In order to measure the error of the numerical solution of ZS (2.1) obtained by CNFD at time $t_k$ as $E^{\varepsilon,k}, N^{\varepsilon,k} \in X_M$, we introduce the $H^1$-norm of error for $E^{\varepsilon,k}$ and $l^2$-norm of the error for $N^{\varepsilon,k}$ as

$$e^\varepsilon(t_k) := \|e^{\varepsilon,k}\| + \|\delta_x^+ e^{\varepsilon,k}\|, \quad n^\varepsilon(t_k) := \|n^{\varepsilon,k}\|, \tag{4.2}$$

with $e^{\varepsilon,k} = E^\varepsilon(\cdot, t_k) - E^{\varepsilon,k}$ and $n^{\varepsilon,k} = N^\varepsilon(\cdot, t_k) - N^{\varepsilon,k}$.

TABLE 4.2. Temporal error analysis of CNFD at time $t = 1$ for the well-prepared initial data Case I, i.e., $\alpha = 2$ and $\beta = 2$.

| $e^\varepsilon(t=1)$ | $\tau_0 = 0.05$ | $\frac{\tau_0}{2}$ | $\frac{\tau_0}{2^2}$ | $\frac{\tau_0}{2^3}$ | $\frac{\tau_0}{2^4}$ | $\frac{\tau_0}{2^5}$ | $\frac{\tau_0}{2^6}$ | $\frac{\tau_0}{2^7}$ |
|---|---|---|---|---|---|---|---|---|
| $\varepsilon = 1$ | 4.38E-2 | 1.63E-2 | 4.82E-3 | 1.21E-3 | 3.12E-4 | 7.83E-5 | 1.96E-5 | 4.94E-6 |
| order | - | 1.43 | 1.76 | 1.99 | 1.96 | 1.99 | 2.00 | 1.99 |
| $\varepsilon = 1/2$ | 3.39E-2 | 1.39E-2 | 4.12E-3 | 1.11E-3 | 2.68E-4 | 6.71E-5 | 1.68E-5 | 4.22E-6 |
| order | - | 1.29 | 1.75 | 1.89 | 2.05 | 2.00 | 2.00 | 1.99 |
| $\varepsilon = 1/2^2$ | 3.57E-2 | 1.36E-2 | 4.03E-3 | 1.01E-3 | 2.58E-4 | 6.45E-5 | 1.61E-5 | 4.05E-6 |
| order | - | 1.39 | 1.75 | 2.00 | 1.97 | 2.00 | 2.00 | 1.99 |
| $\varepsilon = 1/2^3$ | 4.07E-2 | 1.44E-2 | 4.02E-3 | 1.01E-3 | 2.62E-4 | 6.56E-5 | 1.65E-5 | 4.22E-6 |
| order | - | 1.50 | 1.84 | 1.99 | 1.95 | 2.00 | 1.99 | 1.97 |
| $\varepsilon = 1/2^4$ | 3.97E-2 | 1.43E-2 | 4.12E-3 | 1.02E-3 | 2.63E-4 | 6.57E-5 | 1.66E-5 | 5.12E-6 |
| order | - | 1.47 | 1.80 | 2.01 | 1.96 | 2.00 | 1.98 | 1.70 |
| $\varepsilon = 1/2^5$ | 3.97E-2 | 1.43E-2 | 4.12E-3 | 1.02E-3 | 2.64E-4 | 6.60E-5 | 1.66E-5 | 4.28E-6 |
| order | - | 1.47 | 1.80 | 2.01 | 1.95 | 2.00 | 1.99 | 1.96 |
| $\varepsilon = 1/2^6$ | 3.96E-2 | 1.43E-2 | 4.11E-3 | 1.02E-3 | 2.64E-4 | 6.61E-5 | 1.66E-5 | 4.23E-6 |
| order | - | 1.47 | 1.80 | 2.01 | 1.95 | 2.00 | 1.99 | 1.97 |
| $\varepsilon = 1/2^7$ | 3.96E-2 | 1.43E-2 | 4.12E-3 | 1.02E-3 | 2.64E-4 | 6.62E-5 | 1.66E-5 | 4.26E-6 |
| order | - | 1.47 | 1.80 | 2.01 | 1.95 | 2.00 | 2.00 | 1.96 |
| $\eta^\varepsilon(t=1)$ | $\tau_0 = 0.05$ | $\frac{\tau_0}{2}$ | $\frac{\tau_0}{2^2}$ | $\frac{\tau_0}{2^3}$ | $\frac{\tau_0}{2^4}$ | $\frac{\tau_0}{2^5}$ | $\frac{\tau_0}{2^6}$ | $\frac{\tau_0}{2^7}$ |
| $\varepsilon = 1$ | 6.41E-3 | 1.62E-3 | 3.97E-4 | 9.91E-5 | 2.47E-5 | 6.17E-6 | 1.55E-6 | 3.90E-7 |
| order | - | 1.98 | 2.03 | 2.00 | 2.00 | 2.00 | 1.99 | 1.99 |
| $\varepsilon = 1/2$ | 1.68E-2 | 4.61E-3 | 1.22E-3 | 2.95E-4 | 7.39E-5 | 1.85E-5 | 4.61E-6 | 1.16E-6 |
| order | - | 1.87 | 1.92 | 2.05 | 2.00 | 2.00 | 2.00 | 1.99 |
| $\varepsilon = 1/2^2$ | 3.53E-2 | 1.29E-2 | 3.91E-3 | 9.96E-4 | 2.51E-4 | 6.27E-5 | 1.57E-5 | 3.93E-6 |
| order | - | 1.45 | 1.72 | 1.97 | 1.99 | 2.00 | 2.00 | 2.00 |
| $\varepsilon = 1/2^3$ | 2.56E-2 | 9.71E-3 | 3.51E-3 | 1.42E-3 | 3.77E-4 | 9.52E-5 | 2.38E-5 | 5.96E-6 |
| order | - | 1.40 | 1.47 | 1.31 | 1.91 | 1.99 | 2.00 | 1.99 |
| $\varepsilon = 1/2^4$ | 1.79E-2 | 8.61E-3 | 4.01E-3 | 1.32E-3 | 4.83E-4 | 1.53E-4 | 4.11E-5 | 1.58E-5 |
| order | - | 1.06 | 1.10 | 1.60 | 1.45 | 1.66 | 1.90 | 1.38 |
| $\varepsilon = 1/2^5$ | 8.91E-3 | 4.01E-3 | 2.62E-3 | 1.51E-3 | 5.87E-4 | 1.84E-4 | 6.90E-5 | 1.99E-5 |
| order | - | 1.15 | 0.61 | 0.80 | 1.36 | 1.67 | 1.42 | 1.79 |
| $\varepsilon = 1/2^6$ | 8.81E-3 | 2.82E-3 | 1.01E-3 | 7.48E-4 | 4.99E-4 | 2.43E-4 | 8.08E-5 | 2.94E-5 |
| order | - | 1.64 | 1.48 | 0.43 | 0.58 | 1.04 | 1.59 | 1.46 |
| $\varepsilon = 1/2^7$ | 8.91E-3 | 3.22E-3 | 5.29E-4 | 2.71E-4 | 2.05E-4 | 1.55E-4 | 9.08E-5 | 3.66E-5 |
| order | - | 1.47 | 2.60 | 0.97 | 0.40 | 0.40 | 0.77 | 1.31 |

For spatial error analysis, we take time step $\tau$ small enough, e.g., $\tau = 10^{-4}$, such that the temporal error can be ignored; for temporal error analysis, we take mesh size $h$ small enough, e.g., $h = 2.5 \times 10^{-4}$, such that the spatial error can be neglected. Tables 4.1, 4.3, 4.5 display the spatial errors for Cases I, II, III, respectively, while Tables 4.2, 4.4, 4.6 depict the temporal errors for Cases I, II, III, respectively. From Tables 4.1–4.6, Tables 4.7–4.8 and additional numerical results

TABLE 4.3. Spatial error analysis of CNFD at time $t = 1$ for the less-ill-prepared initial data Case II, i.e., $\alpha = 1$ and $\beta = 1$.

| $e^\varepsilon(t=1)$ | $h = 0.1$ | $h = \frac{0.1}{2}$ | $h = \frac{0.1}{2^2}$ | $h = \frac{0.1}{2^3}$ | $h = \frac{0.1}{2^4}$ | $h = \frac{0.1}{2^5}$ |
|---|---|---|---|---|---|---|
| $\varepsilon = 1$ | 7.31E-3 | 1.83E-3 | 4.59E-4 | 1.15E-4 | 2.87E-5 | 7.17E-6 |
| order | - | 1.99 | 1.99 | 2.00 | 2.00 | 2.00 |
| $\varepsilon = 1/2$ | 6.32E-3 | 1.62E-3 | 3.95E-4 | 9.88E-5 | 2.47E-5 | 6.22E-6 |
| order | - | 1.96 | 2.04 | 2.00 | 2.00 | 1.99 |
| $\varepsilon = 1/2^2$ | 6.31E-3 | 1.61E-3 | 3.96E-4 | 9.90E-5 | 2.48E-5 | 6.21E-6 |
| order | - | 1.97 | 2.02 | 2.00 | 2.00 | 2.00 |
| $\varepsilon = 1/2^3$ | 6.81E-3 | 1.71E-3 | 4.24E-4 | 1.06E-4 | 2.66E-5 | 6.67E-6 |
| order | - | 1.99 | 2.01 | 2.00 | 1.99 | 2.00 |
| $\varepsilon = 1/2^4$ | 6.82E-3 | 1.71E-3 | 4.27E-4 | 1.07E-4 | 2.67E-5 | 6.72E-6 |
| order | - | 2.00 | 2.00 | 2.00 | 2.00 | 1.99 |
| $\varepsilon = 1/2^5$ | 6.83E-3 | 1.71E-3 | 4.28E-4 | 1.07E-4 | 2.66E-5 | 6.74E-6 |
| order | - | 2.00 | 2.00 | 2.00 | 2.00 | 1.98 |
| $\varepsilon = 1/2^6$ | 6.83E-3 | 1.71E-3 | 4.28E-4 | 1.07E-4 | 2.68E-5 | 6.75E-6 |
| order | - | 2.00 | 2.00 | 2.00 | 2.00 | 1.99 |
| $n^\varepsilon(t=1)$ | $h = 0.1$ | $h = \frac{0.1}{2}$ | $h = \frac{0.1}{2^2}$ | $h = \frac{0.1}{2^3}$ | $h = \frac{0.1}{2^4}$ | $h = \frac{0.1}{2^5}$ |
| $\varepsilon = 1$ | 1.72E-3 | 4.31E-4 | 1.08E-4 | 2.69E-5 | 6.70E-6 | 1.68E-6 |
| order | - | 2.00 | 2.00 | 2.01 | 2.01 | 2.00 |
| $\varepsilon = 1/2$ | 2.51E-3 | 6.21E-4 | 1.55E-4 | 3.88E-5 | 9.67E-6 | 2.40E-6 |
| order | - | 2.02 | 2.0 | 2.00 | 2.00 | 2.01 |
| $\varepsilon = 1/2^2$ | 2.13E-3 | 5.32E-4 | 1.33E-4 | 3.32E-5 | 8.33E-6 | 2.11E-6 |
| order | - | 2.00 | 2.00 | 2.00 | 1.99 | 1.98 |
| $\varepsilon = 1/2^3$ | 1.30E-3 | 3.33E-4 | 8.32E-5 | 2.08E-5 | 5.28E-6 | 1.34E-6 |
| order | - | 1.96 | 2.00 | 2.00 | 1.98 | 1.98 |
| $\varepsilon = 1/2^4$ | 1.21E-3 | 3.12E-4 | 7.81E-5 | 1.97E-5 | 4.93E-6 | 1.25E-6 |
| order | - | 1.96 | 2.00 | 1.99 | 2.00 | 1.98 |
| $\varepsilon = 1/2^5$ | 1.21E-3 | 3.08E-4 | 7.79E-5 | 1.96E-5 | 4.92E-6 | 1.24E-6 |
| order | - | 1.97 | 1.98 | 1.99 | 1.99 | 1.99 |
| $\varepsilon = 1/2^6$ | 1.22E-3 | 3.06E-4 | 7.63E-5 | 1.91E-5 | 4.77E-6 | 1.21E-6 |
| order | - | 2.00 | 2.00 | 2.00 | 2.00 | 1.98 |

not shown here for brevity, we can draw the following conclusions for the CNFD method in the subsonic limit regime:

(i) For the spatial discretization, CNFD is uniformly second order accurate w.r.t. $\varepsilon \in (0, 1]$ for Case I, II (cf. Tables 4.1, 4.3), i.e., the well-prepared and less-ill-prepared initial data. For Case III, i.e., the ill-prepared initial data, the error bounds of CNFD depend on $\varepsilon$ as $O(h^2/\varepsilon)$ (cf. Table 4.5, each column for $n^\varepsilon$).

(ii) For the temporal discretization, CNFD is uniformly convergent for Case I, II (cf. Tables 4.2, 4.4), i.e., the well-prepared and less-ill-prepared initial data. For well-prepared initial data in Table 4.2, i.e., $\alpha = 2$, $\beta = 2$, upper triangle part of the table shows the order of the errors at $O(\tau^2/\varepsilon)$ and $O(\tau^2)$, for $n^\varepsilon$ and $e^\varepsilon$ (cf. each column), respectively, and the lower triangle part of the table depicts the error

TABLE 4.4. Temporal error analysis of CNFD at time $t = 1$ for the less-ill-prepared initial data Case II, i.e., $\alpha = 1$ and $\beta = 1$.

| $e^\varepsilon(t=1)$ | $\tau_0 = 0.1$ | $\frac{\tau_0}{2}$ | $\frac{\tau_0}{2^2}$ | $\frac{\tau_0}{2^3}$ | $\frac{\tau_0}{2^4}$ | $\frac{\tau_0}{2^5}$ | $\frac{\tau_0}{2^6}$ | $\frac{\tau_0}{2^7}$ |
|---|---|---|---|---|---|---|---|---|
| $\varepsilon = 1$ | 4.38E-2 | 1.63E-2 | 4.81E-3 | 1.21E-3 | 3.13E-4 | 7.84E-5 | 1.97E-5 | 5.05E-6 |
| order | - | 1.48 | 1.78 | 1.95 | 1.99 | 1.99 | 2.00 | 2.00 |
| $\varepsilon = 1/2$ | 3.60E-2 | 1.44E-2 | 4.34E-3 | 1.13E-3 | 2.80E-4 | 7.03E-5 | 1.77E-5 | 4.51E-6 |
| order | - | 1.32 | 1.73 | 1.94 | 2.01 | 1.99 | 1.99 | 1.97 |
| $\varepsilon = 1/2^2$ | 3.84E-2 | 1.42E-2 | 4.12E-3 | 1.13E-3 | 2.69E-4 | 6.75E-5 | 1.69E-5 | 4.32E-6 |
| order | - | 1.44 | 1.79 | 1.87 | 2.07 | 1.99 | 2.00 | 1.97 |
| $\varepsilon = 1/2^3$ | 4.20E-2 | 1.46E-2 | 4.14E-3 | 1.13E-3 | 2.67E-4 | 6.67E-5 | 1.73E-5 | 4.63E-6 |
| order | - | 1.52 | 1.82 | 1.87 | 2.08 | 2.00 | 1.95 | 1.90 |
| $\varepsilon = 1/2^4$ | 4.06E-2 | 1.44E-2 | 4.12E-3 | 1.03E-3 | 2.63E-4 | 6.58E-5 | 1.66E-5 | 4.28E-6 |
| order | - | 1.50 | 1.81 | 2.00 | 1.97 | 2.00 | 1.99 | 1.96 |
| $\varepsilon = 1/2^5$ | 4.05E-2 | 1.45E-2 | 4.13E-3 | 1.13E-3 | 2.64E-4 | 6.62E-5 | 1.66E-5 | 4.24E-6 |
| order | - | 1.48 | 1.81 | 1.87 | 2.09 | 2.00 | 2.00 | 1.97 |
| $\varepsilon = 1/2^6$ | 3.94E-2 | 1.44E-2 | 4.13E-3 | 1.12E-3 | 2.64E-4 | 6.62E-5 | 1.66E-5 | 4.24E-6 |
| order | - | 1.45 | 1.80 | 1.87 | 2.08 | 2.00 | 2.00 | 1.97 |
| $\varepsilon = 1/2^7$ | 3.94E-2 | 1.43E-2 | 4.12E-3 | 1.11E-3 | 2.65E-4 | 6.63E-5 | 1.67E-5 | 4.26E-6 |
| order | - | 1.45 | 1.80 | 1.89 | 2.07 | 2.00 | 1.99 | 1.97 |
| $n^\varepsilon(t=1)$ | $\tau_0 = 0.1$ | $\frac{\tau_0}{2}$ | $\frac{\tau_0}{2^2}$ | $\frac{\tau_0}{2^3}$ | $\frac{\tau_0}{2^4}$ | $\frac{\tau_0}{2^5}$ | $\frac{\tau_0}{2^6}$ | $\frac{\tau_0}{2^7}$ |
| $\varepsilon = 1$ | 6.41E-3 | 1.62E-3 | 3.97E-4 | 9.91E-5 | 2.47E-5 | 6.17E-6 | 1.55E-6 | 3.90E-7 |
| order | - | 1.98 | 2.03 | 2.00 | 2.00 | 2.00 | 1.99 | 1.99 |
| $\varepsilon = 1/2$ | 2.02E-2 | 5.41E-3 | 1.42E-3 | 3.46E-4 | 8.66E-5 | 2.17E-5 | 5.44E-6 | 1.39E-6 |
| order | - | 1.90 | 1.93 | 2.04 | 2.00 | 2.00 | 2.00 | 1.97 |
| $\varepsilon = 1/2^2$ | 5.35E-2 | 1.67E-2 | 4.72E-3 | 1.24E-3 | 3.02E-4 | 7.55E-5 | 1.89E-5 | 4.76E-6 |
| order | - | 1.68 | 1.82 | 1.93 | 2.04 | 2.00 | 2.00 | 1.99 |
| $\varepsilon = 1/2^3$ | 1.23E-1 | 4.08E-2 | 1.11E-2 | 3.02E-3 | 7.73E-4 | 1.94E-4 | 5.00E-5 | 1.36E-5 |
| order | - | 1.59 | 1.88 | 1.88 | 1.97 | 1.99 | 1.96 | 1.88 |
| $\varepsilon = 1/2^4$ | 1.45E-1 | 9.18E-2 | 3.65E-2 | 1.03E-2 | 2.62E-3 | 6.69E-4 | 1.68E-4 | 4.31E-5 |
| order | - | 0.66 | 1.33 | 1.83 | 1.98 | 1.97 | 1.99 | 1.96 |
| $\varepsilon = 1/2^5$ | 8.64E-2 | 7.94E-2 | 6.03E-2 | 3.08E-2 | 9.81E-3 | 2.62E-3 | 6.45E-4 | 1.62E-4 |
| order | - | 0.12 | 0.40 | 0.97 | 1.65 | 1.90 | 2.02 | 1.99 |
| $\varepsilon = 1/2^6$ | 4.39E-2 | 4.31E-2 | 4.22E-2 | 3.59E-2 | 2.29E-2 | 9.13E-3 | 2.53E-3 | 6.36E-4 |
| order | - | 0.03 | 0.03 | 0.23 | 0.65 | 1.33 | 1.85 | 1.99 |
| $\varepsilon = 1/2^7$ | 2.42E-2 | 2.23E-2 | 2.14E-2 | 2.20E-2 | 2.01E-2 | 1.50E-2 | 7.71E-3 | 2.51E-3 |
| order | - | 0.12 | 0.06 | -0.04 | 0.13 | 0.42 | 0.96 | 1.62 |

bounds at the order $O(\tau^2 + \varepsilon^2)$. For less-ill-prepared initial data in Table 4.4 , i.e., $\alpha = 1$, $\beta = 1$, upper triangle part of the table implies the convergence order at $\tau^2/\varepsilon^2$, $\tau^2/\varepsilon$ for $n^\varepsilon$ and $e^\varepsilon$ (cf. each column), respectively, and the lower triangle part of the table demonstrates the convergence order at $O(\tau^2+\varepsilon)$. Noticing that the uniform convergence order is attained as the two types estimates are comparable, the uniform rates can be confirmed by the degeneracy of the error estimates listed in Tables 4.7 and 4.8, for $n^\varepsilon$ as $\tau^2 \sim \varepsilon^3$ and $e^\varepsilon$ as $\tau \sim \varepsilon$ (for Case II only).

TABLE 4.5. Spatial error analysis of CNFD at time $t = 1$ for the ill-prepared initial data Case III, i.e., $\alpha = 0$ and $\beta = 0$.

| $e^\varepsilon(t=1)$ | $h = 0.1$ | $h = \frac{0.1}{2}$ | $h = \frac{0.1}{2^2}$ | $h = \frac{0.1}{2^3}$ | $h = \frac{0.1}{2^4}$ | $h = \frac{0.1}{2^5}$ |
|---|---|---|---|---|---|---|
| $\varepsilon = 1$ | 7.31E-3 | 1.83E-3 | 4.59E-4 | 1.15E-4 | 2.87E-5 | 7.17E-6 |
| order | - | 1.99 | 1.99 | 2.00 | 2.00 | 2.00 |
| $\varepsilon = 1/2$ | 7.62E-3 | 1.91E-3 | 4.77E-4 | 1.20E-4 | 3.02E-5 | 7.54E-6 |
| order | - | 2.00 | 2.00 | 1.99 | 1.99 | 2.00 |
| $\varepsilon = 1/2^2$ | 8.81E-3 | 2.21E-3 | 5.53E-4 | 1.38E-4 | 3.46E-5 | 8.66E-6 |
| order | - | 2.00 | 2.00 | 2.00 | 2.00 | 2.00 |
| $\varepsilon = 1/2^3$ | 7.81E-3 | 2.01E-3 | 4.92E-4 | 1.24E-4 | 3.12E-5 | 7.81E-6 |
| order | - | 1.96 | 2.03 | 1.99 | 1.99 | 2.00 |
| $\varepsilon = 1/2^4$ | 7.01E-3 | 1.71E-3 | 4.35E-4 | 1.08E-4 | 2.72E-5 | 6.58E-6 |
| order | - | 2.04 | 1.97 | 2.01 | 1.99 | 2.05 |
| $\varepsilon = 1/2^5$ | 6.92E-3 | 1.72E-3 | 4.27E-4 | 1.07E-4 | 2.67E-5 | 6.68E-6 |
| order | - | 2.00 | 2.01 | 2.00 | 2.00 | 2.00 |
| $\varepsilon = 1/2^6$ | 6.92E-3 | 1.72E-3 | 4.29E-4 | 1.07E-4 | 2.68E-5 | 6.72E-6 |
| order | - | 2.00 | 2.00 | 2.00 | 2.00 | 2.00 |
| $n^\varepsilon(t=1)$ | $h = 0.1$ | $h = \frac{0.1}{2}$ | $h = \frac{0.1}{2^2}$ | $h = \frac{0.1}{2^3}$ | $h = \frac{0.1}{2^4}$ | $h = \frac{0.1}{2^5}$ |
| $\varepsilon = 1$ | 1.72E-3 | 4.31E-4 | 1.08E-4 | 2.69E-5 | 6.70E-6 | 1.68E-6 |
| order | - | 2.00 | 2.00 | 2.01 | 2.01 | 2.00 |
| $\varepsilon = 1/2$ | 2.71E-3 | 6.81E-4 | 1.70E-4 | 4.26E-5 | 1.07E-5 | 2.68E-6 |
| order | - | 1.99 | 2.00 | 2.00 | 1.99 | 2.00 |
| $\varepsilon = 1/2^2$ | 3.22E-3 | 7.89E-4 | 1.97E-4 | 4.93E-5 | 1.23E-5 | 3.10E-6 |
| order | - | 2.03 | 2.00 | 2.00 | 2.00 | 1.99 |
| $\varepsilon = 1/2^3$ | 4.52E-3 | 1.11E-3 | 2.78E-4 | 6.95E-5 | 1.74E-5 | 4.36E-6 |
| order | - | 2.03 | 2.00 | 2.00 | 2.00 | 2.00 |
| $\varepsilon = 1/2^4$ | 8.22E-3 | 2.11E-3 | 5.23E-4 | 1.29E-4 | 3.21E-5 | 8.05E-6 |
| order | - | 1.96 | 2.01 | 2.02 | 2.00 | 2.00 |
| $\varepsilon = 1/2^5$ | 1.62E-2 | 4.10E-3 | 1.05E-3 | 2.63E-4 | 6.58E-5 | 1.65E-5 |
| order | - | 1.98 | 1.97 | 2.00 | 2.00 | 2.00 |
| $\varepsilon = 1/2^6$ | 3.30E-2 | 8.73E-3 | 2.04E-3 | 5.06E-4 | 1.26E-4 | 3.13E-5 |
| order | - | 1.92 | 2.09 | 2.01 | 2.00 | 2.00 |

(iii) For the ill-prepared initial data in Table 4.6, i.e., $\alpha = 0$, $\beta = 0$, the error is at the order $O(\tau^2/\varepsilon^3)$ and $O(\tau^2/\varepsilon)$, for $n^\varepsilon$ and $e^\varepsilon$ (cf. each column), respectively. In addition, the lower triangle part shows $e^\varepsilon$ is uniform in $\varepsilon$ with the order $O(\tau^2 + \varepsilon)$, where the degeneracy of the error bounds for $e^\varepsilon$ is clearly shown in Table 4.8 as $\tau \sim \varepsilon$.

It is observed that the errors for the electric field $E^{\varepsilon,k}$ are much better than the predicted estimates in Theorems 2.1 and 2.2. Indeed, the numerical results suggest the following estimates on $e^{\varepsilon,k}$,

$$\|e^{\varepsilon,k}\| + \|\delta_x^+ e^{\varepsilon,k}\| \lesssim h^2 + \frac{\tau^2}{\varepsilon^{1-\alpha^*}}, \quad \|e^{\varepsilon,k}\| + \|\delta_x^+ e^{\varepsilon,k}\| \lesssim h^2 + \tau^2 + \varepsilon^{1+\alpha^*}. \tag{4.3}$$

Table 4.6. Temporal error analysis of CNFD at time $t = 1$ for the ill-prepared initial data Case III, i.e., $\alpha = 0$ and $\beta = 0$.

| $e^\varepsilon(t=1)$ | $\tau_0 = 0.1$ | $\frac{\tau_0}{2}$ | $\frac{\tau_0}{2^2}$ | $\frac{\tau_0}{2^3}$ | $\frac{\tau_0}{2^4}$ | $\frac{\tau_0}{2^5}$ | $\frac{\tau_0}{2^6}$ | $\frac{\tau_0}{2^7}$ |
|---|---|---|---|---|---|---|---|---|
| $\varepsilon = 1$ | 4.38E-2 | 1.63E-2 | 4.81E-3 | 1.22E-3 | 3.13E-4 | 7.84E-5 | 1.97E-5 | 5.05E-6 |
| order | - | 1.48 | 1.78 | 1.95 | 1.99 | 1.99 | 2.00 | 2.00 |
| $\varepsilon = 1/2$ | 4.85E-2 | 1.71E-2 | 5.03E-3 | 1.30E-3 | 3.25E-4 | 8.15E-5 | 2.04E-5 | 5.24E-6 |
| order | - | 1.50 | 1.77 | 1.95 | 2.00 | 2.00 | 2.00 | 1.96 |
| $\varepsilon = 1/2^2$ | 7.89E-2 | 2.41E-2 | 6.62E-3 | 1.71E-3 | 4.23E-4 | 1.06E-4 | 2.67E-5 | 6.79E-6 |
| order | - | 1.71 | 1.86 | 1.95 | 2.02 | 2.00 | 1.99 | 1.98 |
| $\varepsilon = 1/2^3$ | 5.56E-2 | 1.89E-2 | 5.91E-3 | 1.62E-3 | 3.97E-4 | 9.95E-5 | 2.50E-5 | 6.72E-6 |
| order | - | 1.56 | 1.68 | 1.87 | 2.03 | 2.00 | 1.99 | 1.90 |
| $\varepsilon = 1/2^4$ | 1.28E-1 | 1.85E-2 | 5.00E-3 | 1.31E-3 | 3.23E-4 | 8.08E-5 | 2.01E-5 | 5.09E-6 |
| order | - | 2.76 | 1.89 | 1.93 | 2.02 | 2.00 | 2.00 | 1.98 |
| $\varepsilon = 1/2^5$ | 1.68E-1 | 2.93E-2 | 6.92E-3 | 1.73E-3 | 4.38E-4 | 1.09E-4 | 2.74E-5 | 6.85E-6 |
| order | - | 2.52 | 2.08 | 2.00 | 1.98 | 2.00 | 1.99 | 2.00 |
| $\varepsilon = 1/2^6$ | 1.54E-1 | 8.03E-2 | 1.24E-2 | 2.82E-3 | 7.23E-4 | 1.82E-4 | 4.57E-5 | 1.16E-5 |
| order | - | 0.94 | 2.70 | 2.14 | 1.96 | 1.99 | 1.99 | 1.98 |
| $\varepsilon = 1/2^7$ | 1.71E-1 | 6.20E-2 | 3.99E-2 | 5.84E-3 | 1.34E-3 | 3.37E-4 | 8.52E-5 | 2.14E-5 |
| order | - | 1.46 | 0.64 | 2.77 | 2.12 | 1.99 | 1.98 | 1.99 |

| $n^\varepsilon(t=1)$ | $\tau_0 = 0.1$ | $\frac{\tau_0}{2}$ | $\frac{\tau_0}{2^2}$ | $\frac{\tau_0}{2^3}$ | $\frac{\tau_0}{2^4}$ | $\frac{\tau_0}{2^5}$ | $\frac{\tau_0}{2^6}$ | $\frac{\tau_0}{2^7}$ |
|---|---|---|---|---|---|---|---|---|
| $\varepsilon = 1$ | 6.41E-3 | 1.62E-3 | 3.97E-4 | 9.91E-5 | 2.47E-5 | 6.17E-6 | 1.55E-6 | 3.90E-7 |
| order | - | 1.98 | 2.03 | 2.00 | 2.00 | 2.00 | 1.99 | 1.99 |
| $\varepsilon = 1/2$ | 2.99E-2 | 7.81E-3 | 2.02E-3 | 4.94E-4 | 1.24E-4 | 3.09E-5 | 7.74E-6 | 1.97E-6 |
| order | - | 1.94 | 1.95 | 2.03 | 1.99 | 2.00 | 2.00 | 1.97 |
| $\varepsilon = 1/2^2$ | 1.59E-1 | 4.32E-2 | 1.12E-2 | 2.82E-3 | 7.05E-4 | 1.77E-4 | 4.42E-5 | 1.11E-5 |
| order | - | 1.88 | 1.95 | 1.99 | 2.00 | 1.99 | 2.00 | 1.99 |
| $\varepsilon = 1/2^3$ | 9.35E-1 | 3.06E-1 | 8.19E-2 | 2.08E-2 | 5.21E-3 | 1.32E-3 | 3.26E-4 | 8.41E-5 |
| order | - | 1.61 | 1.90 | 1.98 | 2.00 | 1.98 | 2.02 | 1.95 |
| $\varepsilon = 1/2^4$ | 1.03 | 1.44 | 5.68E-1 | 1.60E-1 | 4.06E-2 | 1.02E-2 | 2.51E-3 | 6.36E-4 |
| order | - | -0.48 | 1.34 | 1.83 | 1.98 | 1.99 | 2.02 | 1.98 |
| $\varepsilon = 1/2^5$ | 2.72 | 2.52 | 1.92 | 9.75E-1 | 3.10E-1 | 8.05E-2 | 2.02E-2 | 5.10E-3 |
| order | - | 0.11 | 0.39 | 0.98 | 1.65 | 1.95 | 1.99 | 1.99 |
| $\varepsilon = 1/2^6$ | 2.73 | 2.72 | 2.69 | 2.28 | 1.45 | 5.82E-1 | 1.60E-1 | 4.04E-2 |
| order | - | 0.01 | 0.02 | 0.24 | 0.65 | 1.32 | 1.86 | 1.99 |
| $\varepsilon = 1/2^7$ | 2.73 | 2.79 | 2.73 | 2.81 | 2.56 | 1.91 | 9.85E-1 | 3.13E-1 |
| order | - | -0.03 | 0.03 | -0.04 | 0.13 | 0.42 | 0.96 | 1.65 |

The reason can be formally argued as the following. The ZS (1.1) in 1D can be reformulated as

$$\begin{cases} i\partial_t E^\varepsilon(x,t) + \partial_{xx}E^\varepsilon - \widetilde{N}^\varepsilon E^\varepsilon - (\varepsilon^\alpha \Lambda^{(1)}(x,t/\varepsilon) + \varepsilon^\beta \Lambda^{(2)}(x,t/\varepsilon))E^\varepsilon = 0, \\ \varepsilon^2 \partial_{tt}\widetilde{N}^\varepsilon(x,t) - \partial_{xx}\widetilde{N}^\varepsilon(x,t) - \partial_{xx}|E^\varepsilon|^2 = 0, \\ E^\varepsilon(x,0) = E_0(x), \quad \widetilde{N}^\varepsilon(x,0) = |E_0|^2, \quad \partial_t\widetilde{N}(x,0) = 2\mathrm{Im}(E_0\partial_{xx}\overline{E}_0), \end{cases}$$

TABLE 4.7. Degeneracy of temporal error of CNFD at time $t = 1$ for plasma densities. The convergence orders are calculated with respect to time step $\tau$.

| $\alpha = 2$, $\beta = 2$ | $\tau_0 = 0.2, \varepsilon_0 = \frac{1}{2}$ | $\tau_0/2^3, \varepsilon_0/2^2$ | $\tau_0/2^6, \varepsilon_0/2^4$ | $\tau_0/2^9, \varepsilon_0/2^6$ |
|---|---|---|---|---|
| $n^\varepsilon(t = 1)$ | 5.94E-2 | 9.71E-3 | 5.87E-4 | 3.66E-5 |
| order | - | 2.61/3 | 4.05/3 | 4.00/3 |
| $\alpha = 1$, $\beta = 1$ | $\tau_0 = 0.2, \varepsilon_0 = \frac{1}{2}$ | $\tau_0/2^3, \varepsilon_0/2^2$ | $\tau_0/2^6, \varepsilon_0/2^4$ | $\tau_0/2^9, \varepsilon_0/2^6$ |
| $n^\varepsilon(t = 1)$ | 6.62E-2 | 1.11E-2 | 2.62E-3 | 6.32E-4 |
| order | - | 2.58/3 | 2.08/3 | 2.05/3 |

TABLE 4.8. Degeneracy of temporal error of CNFD at time $t = 1$ for electric fields. The convergence orders are calculated with respect to time step $\tau$.

| $\alpha = 1$, $\beta = 1$ | $\varepsilon_0 = 1/2^2$ $\tau_0 = 0.1$ | $\tau_0/2, \varepsilon_0/2$ | $\tau_0/2^2, \varepsilon_0/2^2$ | $\tau_0/2^3, \varepsilon_0/2^3$ | $\tau_0/2^4, \varepsilon_0/2^4$ | $\tau_0/2^5, \varepsilon_0/2^5$ |
|---|---|---|---|---|---|---|
| $e^\varepsilon(t = 1)$ | 3.84E-2 | 1.46E-2 | 4.12E-3 | 1.13E-3 | 2.64E-4 | 6.63E-5 |
| order | - | 1.40 | 1.83 | 1.87 | 2.10 | 1.99 |
| $\alpha = 0$, $\beta = 0$ | $\varepsilon_0 = 1/2^2$ $\tau_0 = 0.1$ | $\tau_0/2, \varepsilon_0/2$ | $\tau_0/2^2, \varepsilon_0/2^2$ | $\tau_0/2^3, \varepsilon_0/2^3$ | $\tau_0/2^4, \varepsilon_0/2^4$ | $\tau_0/2^5, \varepsilon_0/2^5$ |
| $e^\varepsilon(t = 1)$ | 7.89E-2 | 1.89E-2 | 5.00E-3 | 1.73E-3 | 7.23E-4 | 3.37E-4 |
| order | - | 2.06 | 1.92 | 1.53 | 1.26 | 1.10 |

where $N^\varepsilon = \widetilde{N}^\varepsilon + \varepsilon^\alpha \Lambda^{(1)}(x, t/\varepsilon) + \varepsilon^\beta \Lambda^{(2)}(x, t/\varepsilon)$, $\Lambda^{(1)}$ and $\Lambda^{(2)}$ are the initial layers given in the Introduction. D'alembert's formula gives that
(4.4)

$$\Lambda^{(1)}(x, t/\varepsilon) = \frac{1}{2}(w_0(x + t/\varepsilon) + w_0(x - t/\varepsilon)), \quad \Lambda^{(2)}(x, t/\varepsilon) = \frac{1}{2}\int_{x-t/\varepsilon}^{x+t/\varepsilon} w_1(\xi)d\xi,$$

which implies that for well localized initial data $E_0$, $w_0$ and $\int_{\mathbb{R}} w_1(x)dx = 0$, the first and second initial layers' impact on the electric field will be gone instantly as $\varepsilon \to 0^+$. This phenomenon is due to the fact that sound wave travels much faster than the electric group velocity, and is the key idea in the analysis presented in [16]. The above observations imply that the electric field $E^\varepsilon$ almost satisfies the equation

$$i\partial_t E^\varepsilon(x, t) + \partial_{xx} E^\varepsilon - \widetilde{N}^\varepsilon E^\varepsilon \approx 0, \tag{4.5}$$

which suggests that the behavior of $E^\varepsilon$ is roughly independent of the initial layers. By the asymptotic expansion, it is then expected that $\partial_{ttt} E^\varepsilon = O(\partial_{tt} \tilde{N}^\varepsilon E^\varepsilon) = O(\varepsilon^{\min\{\alpha,\beta,1\}-1})$ and the local truncation error of the electric field equation for $E^\varepsilon$ is at the order $O(h^2 + \tau^2)$ for Cases I, II and $O(h^2 + \tau^2/\varepsilon)$ for Case III. Thus, the temporal accuracy of $e^{\varepsilon,k}$ is uniformly $O(\tau^2)$ for Cases I, II, and $O(\tau^2/\varepsilon)$ for Case III, respectively, which agrees with the numerical results.

On the other hand, the above equation (4.5) differs from the NLS (1.2) with an $O(\varepsilon^{1+\min\{\alpha,\beta,1\}}+\varepsilon^2)$ term from the asymptotic expansion, suggesting that the error $e^{\varepsilon,k}$ has the order of $O(h^2+\tau^2+\varepsilon)$ for Case III. Together with the error bounds $O(h^2+\tau^2/\varepsilon)$, we see $e^{\varepsilon,k}=O(\tau)$ uniformly w.r.t. $\varepsilon\in(0,1]$ for Case III. However, the analysis of such observations are beyond the scope of the current study.

Finally, we numerically compare the time splitting methods introduced in [5, 12] with CNFD (2.7), (2.8), (2.11) for ZS in the subsonic limit regime. Let $E_j^{\varepsilon,n}=\sum_{l=1}^{M-1}\widetilde{E}_l^{\varepsilon,n}\sin(\mu_l(x_j-a))$ ($j\in\mathcal{T}_M$) and $N_j^{\varepsilon,n}=\sum_{l=1}^{M-1}\widetilde{N}_l^{\varepsilon,n}\sin(\mu_l(x_j-a))$ be the numerical approximations of $E^\varepsilon(x_j,t_n)$ and $N^\varepsilon(x_j,t_n)$, respectively, where $\mu_l=\frac{l\pi}{b-a}$ ($l\geq 1$). $\widetilde{E}_l^{\varepsilon,n}$ is the discrete sine transform coefficients of grid vector functions $E^{\varepsilon,n}$ as

$$\widetilde{E}_l^{\varepsilon,n}=\frac{2}{M-1}\sum_{j=1}^{M-1}E_j^{\varepsilon,n}\sin(\mu_l(x_j-a)),\quad l=1,2,\ldots,M-1,$$

and $\tilde{N}_l^{\varepsilon,n}$ is the discrete sine transform coefficients of grid vector functions $N^{\varepsilon,n}$. We consider the time splitting spectral method (TSSP, also referred to as PSAS-TSSP in [5], where $E_j^{\varepsilon,n+1}$ and $N_j^{\varepsilon,n+1}=\sum_{l=1}^{M-1}\widetilde{N}_l^{\varepsilon,n+1}\sin(\mu_l(x_j-a))$ ($n\geq 1$) are updated as

$$\widetilde{N}_l^{\varepsilon,n+1}=2\cos(\mu_l\tau/\varepsilon)\widetilde{N}_l^{\varepsilon,n}-\widetilde{N}_l^{\varepsilon,n-1}-2(1-\cos(\mu_l\tau/\varepsilon))(\widetilde{|E^{\varepsilon,n}|^2})_l,\quad n\geq 1,$$

and

$$\begin{aligned}
E_j^{*}&=\sum_{l=1}^{M-1}e^{-i\mu_l^2\tau/2}\widetilde{E}_l^{\varepsilon,n}\sin(\mu_l(x_j-a)),\\
E_j^{**}&=e^{-i\tau(N_j^{\varepsilon,n}+N_j^{\varepsilon,n+1})/2}E_j^{*},\\
E_j^{\varepsilon,n+1}&=\sum_{l=1}^{M-1}e^{-i\mu_l^2\tau/2}\widetilde{E}_l^{**}\sin(\mu_l(x_j-a)),\quad n\geq 0.
\end{aligned}$$

The initial values are set as $E_j^{\varepsilon,0}=E_0(x_j)$, $N_j^{\varepsilon,0}=N^{\varepsilon,0}(x_j)$ and

$$\widetilde{N}_l^{\varepsilon,1}=\cos(\mu_l\tau/\varepsilon)\widetilde{N}_l^{\varepsilon,0}+\frac{\varepsilon}{\mu_l}\sin(\mu_l\tau/\varepsilon)\widetilde{(N_1^\varepsilon)}_l-(1-\cos(\mu_l\tau/\varepsilon))(\widetilde{|E^{\varepsilon,0}|^2})_l, \tag{4.6}$$

with $N_{1,j}^\varepsilon=N_1^\varepsilon(x_j)$. We remark that (4.6) can also be used as an initialization for the CNFD (2.7), but the errors of the CNFD scheme remain the same since the truncation error of the finite difference discretization does not change.

Because of the spectral discretization, we do not observe the spatial error dependence on $\varepsilon$. Thus, we only include the temporal error results of the above TSSP for brevity. From Tables 4.9–4.11, we have the following observations.

(1) The temporal error of TSSP on the electric field $E^\varepsilon$ behaves similarly to that of CNFD, where the error of TSSP on $E^\varepsilon$ is uniformly second-order $O(\tau^2)$ accurate for Cases I, II (Tables 4.9–4.10), and $O(\tau^2/\varepsilon)$ for Case III (each column of Table 4.11).

(2) The temporal error of TSSP on the density $N^\varepsilon$ behaves much better than the CNFD case, where the errors on $N^\varepsilon$ are also uniformly second-order accurate for Cases I, II (Tables 4.9–4.10), and $O(\tau^2/\varepsilon)$ for Case III (each column of Table 4.11).

(3) TSSP can improve the $\varepsilon$ resolution on the density $N^\varepsilon$.

From Tables 4.9–4.11 and numerical results not shown here for brevity, we find that spectral discretization in space and splitting technique would enhance the accuracy of the algorithm to solve ZS (1.1). However, a rigorous analysis of the splitting methods [5, 12] becomes extremely challenging because of the loss-of-derivative [1, 9, 13], and investigations of such methods will be carried out in our future work.

TABLE 4.9. Temporal error analysis of TSSP at time $t = 1$ for the well-prepared initial data Case I, i.e., $\alpha = 2$ and $\beta = 2$.

| $e^\varepsilon(t=1)$ | $\tau_0 = 0.1$ | $\frac{\tau_0}{2}$ | $\frac{\tau_0}{2^2}$ | $\frac{\tau_0}{2^3}$ | $\frac{\tau_0}{2^4}$ | $\frac{\tau_0}{2^5}$ | $\frac{\tau_0}{2^6}$ | $\frac{\tau_0}{2^7}$ |
|---|---|---|---|---|---|---|---|---|
| $\varepsilon = 1$ | 3.53E-3 | 8.35E-4 | 2.07E-4 | 5.18E-5 | 1.29E-5 | 3.23E-6 | 8.06E-7 | 2.07E-7 |
| order | - | 1.48 | 1.78 | 1.95 | 1.99 | 1.99 | 2.00 | 2.00 |
| $\varepsilon = 1/2$ | 3.83E-3 | 8.79E-4 | 2.19E-4 | 5.45E-5 | 1.36E-5 | 3.40E-6 | 8.46E-7 | 2.08E-7 |
| order | - | 1.50 | 1.77 | 1.95 | 2.00 | 2.00 | 2.00 | 1.96 |
| $\varepsilon = 1/2^2$ | 3.73E-3 | 8.47E-4 | 2.10E-4 | 5.25E-5 | 1.31E-5 | 3.27E-6 | 8.14E-7 | 2.96E-7 |
| order | - | 1.71 | 1.86 | 1.95 | 2.02 | 2.00 | 1.99 | 1.98 |
| $\varepsilon = 1/2^3$ | 4.12E-3 | 9.13E-4 | 2.26E-4 | 5.63E-5 | 1.41E-5 | 3.52E-6 | 8.78E-7 | 2.18E-7 |
| order | - | 1.56 | 1.68 | 1.87 | 2.03 | 2.00 | 1.99 | 1.90 |
| $\varepsilon = 1/2^4$ | 6.61E-3 | 9.39E-4 | 2.27E-4 | 5.65E-5 | 1.41E-5 | 3.53E-6 | 8.82E-7 | 2.20E-7 |
| order | - | 2.76 | 1.89 | 1.93 | 2.02 | 2.00 | 2.00 | 1.98 |
| $\varepsilon = 1/2^5$ | 3.64E-2 | 2.73E-3 | 2.34E-4 | 5.68E-5 | 1.41E-5 | 3.53E-6 | 8.84E-7 | 2.21E-7 |
| order | - | 2.52 | 2.08 | 2.00 | 1.98 | 2.00 | 1.99 | 2.00 |
| $\varepsilon = 1/2^6$ | 3.82E-2 | 1.80E-2 | 1.32E-3 | 5.84E-5 | 1.42E-5 | 3.54E-6 | 8.84E-7 | 2.21E-7 |
| order | - | 0.94 | 2.70 | 2.14 | 1.96 | 1.99 | 1.99 | 1.98 |
| $\varepsilon = 1/2^7$ | 3.82E-2 | 1.90E-2 | 9.02E-3 | 6.84E-4 | 1.46E-5 | 3.55E-6 | 8.85E-7 | 2.21E-7 |
| $n^\varepsilon(t=1)$ | $\tau_0 = 0.1$ | $\frac{\tau_0}{2}$ | $\frac{\tau_0}{2^2}$ | $\frac{\tau_0}{2^3}$ | $\frac{\tau_0}{2^4}$ | $\frac{\tau_0}{2^5}$ | $\frac{\tau_0}{2^6}$ | $\frac{\tau_0}{2^7}$ |
| $\varepsilon = 1$ | 1.92E-3 | 4.65E-4 | 1.16E-4 | 2.88E-5 | 7.19E-6 | 1.83E-6 | 5.61E-7 | 3.68E-7 |
| order | - | 1.98 | 2.03 | 2.00 | 2.00 | 2.00 | 1.99 | 1.99 |
| $\varepsilon = 1/2$ | 2.92E-3 | 6.83E-4 | 1.69E-4 | 4.21E-5 | 1.05E-5 | 2.62E-6 | 6.74E-7 | 2.69E-7 |
| order | - | 1.94 | 1.95 | 2.03 | 1.99 | 2.00 | 2.00 | 1.97 |
| $\varepsilon = 1/2^2$ | 4.12E-3 | 7.92E-4 | 1.91E-4 | 4.73E-5 | 1.18E-5 | 2.95E-6 | 7.65E-7 | 2.96E-7 |
| order | - | 1.88 | 1.95 | 1.99 | 2.00 | 1.99 | 2.00 | 1.99 |
| $\varepsilon = 1/2^3$ | 5.21E-3 | 4.59E-4 | 9.96E-5 | 2.42E-5 | 6.00E-6 | 1.50E-6 | 3.94E-7 | 1.55E-7 |
| order | - | 1.61 | 1.90 | 1.98 | 2.00 | 1.98 | 2.02 | 1.95 |
| $\varepsilon = 1/2^4$ | 2.09E-2 | 8.25E-4 | 8.58E-5 | 1.99E-5 | 4.89E-6 | 1.22E-6 | 3.15E-7 | 1.11E-7 |
| order | - | -0.48 | 1.34 | 1.83 | 1.98 | 1.99 | 2.02 | 1.98 |
| $\varepsilon = 1/2^5$ | 1.07E-1 | 1.33E-2 | 2.04E-4 | 2.06E-5 | 4.84E-6 | 1.19E-6 | 3.02E-7 | 9.39E-8 |
| order | - | 0.11 | 0.39 | 0.98 | 1.65 | 1.95 | 1.99 | 1.99 |
| $\varepsilon = 1/2^6$ | 1.14E-1 | 7.56E-2 | 9.41E-3 | 7.39E-5 | 5.12E-6 | 1.20E-6 | 2.99E-7 | 8.35E-8 |
| order | - | 0.01 | 0.02 | 0.24 | 0.65 | 1.32 | 1.86 | 1.99 |
| $\varepsilon = 1/2^7$ | 1.13E-1 | 8.04E-2 | 5.36E-2 | 6.61E-3 | 4.02E-5 | 1.28E-6 | 3.01E-7 | 7.95E-8 |
| order | - | -0.03 | 0.03 | -0.04 | 0.13 | 0.42 | 0.96 | 1.65 |

TABLE 4.10. Temporal error analysis of TSSP at time $t = 1$ for the less-ill-prepared initial data Case II, i.e., $\alpha = 1$ and $\beta = 1$.

| $e^\varepsilon(t=1)$ | $\tau_0 = 0.1$ | $\frac{\tau_0}{2}$ | $\frac{\tau_0}{2^2}$ | $\frac{\tau_0}{2^3}$ | $\frac{\tau_0}{2^4}$ | $\frac{\tau_0}{2^5}$ | $\frac{\tau_0}{2^6}$ | $\frac{\tau_0}{2^7}$ |
|---|---|---|---|---|---|---|---|---|
| $\varepsilon = 1$ | 3.53E-3 | 8.35E-4 | 2.07E-4 | 5.18E-5 | 1.29E-5 | 3.23E-6 | 8.06E-7 | 2.07E-7 |
| order | - | 1.48 | 1.78 | 1.95 | 1.99 | 1.99 | 2.00 | 2.00 |
| $\varepsilon = 1/2$ | 4.02E-3 | 9.32E-4 | 2.32E-4 | 5.78E-5 | 1.45E-5 | 3.61E-6 | 8.99E-7 | 2.23E-7 |
| order | - | 1.50 | 1.77 | 1.95 | 2.00 | 2.00 | 2.00 | 1.96 |
| $\varepsilon = 1/2^2$ | 4.11E-3 | 9.53E-4 | 2.37E-4 | 5.92E-5 | 1.48E-5 | 3.69E-6 | 9.22E-7 | 2.29E-7 |
| order | - | 1.71 | 1.86 | 1.95 | 2.02 | 2.00 | 1.99 | 1.98 |
| $\varepsilon = 1/2^3$ | 4.22E-3 | 9.36E-4 | 2.32E-4 | 5.78E-5 | 1.44E-5 | 3.61E-6 | 9.01E-7 | 2.25E-7 |
| order | - | 1.56 | 1.68 | 1.87 | 2.03 | 2.00 | 1.99 | 1.90 |
| $\varepsilon = 1/2^4$ | 6.71E-3 | 9.95E-4 | 2.41E-4 | 6.00E-5 | 1.49E-5 | 3.75E-6 | 9.37E-7 | 2.34E-7 |
| order | - | 2.76 | 1.89 | 1.93 | 2.02 | 2.00 | 2.00 | 1.98 |
| $\varepsilon = 1/2^5$ | 3.64E-2 | 2.81E-3 | 2.52E-4 | 6.14E-5 | 1.53E-5 | 3.82E-6 | 9.56E-7 | 2.39E-7 |
| order | - | 2.52 | 2.08 | 2.00 | 1.98 | 2.00 | 1.99 | 2.00 |
| $\varepsilon = 1/2^6$ | 3.77E-2 | 1.80E-2 | 1.32E-3 | 6.35E-5 | 1.55E-5 | 3.86E-6 | 9.65E-7 | 2.41E-7 |
| order | - | 0.94 | 2.70 | 2.14 | 1.96 | 1.99 | 1.99 | 1.98 |
| $\varepsilon = 1/2^7$ | 3.78E-2 | 1.89E-2 | 9.02E-3 | 6.84E-4 | 1.59E-5 | 3.89E-6 | 9.70E-7 | 2.42E-7 |
| order | - | 1.46 | 0.64 | 2.77 | 2.12 | 1.99 | 1.98 | 1.99 |
| $n^\varepsilon(t=1)$ | $\tau_0 = 0.1$ | $\frac{\tau_0}{2}$ | $\frac{\tau_0}{2^2}$ | $\frac{\tau_0}{2^3}$ | $\frac{\tau_0}{2^4}$ | $\frac{\tau_0}{2^5}$ | $\frac{\tau_0}{2^6}$ | $\frac{\tau_0}{2^7}$ |
| $\varepsilon = 1$ | 1.92E-3 | 4.65E-4 | 1.16E-4 | 2.88E-5 | 7.19E-6 | 1.83E-6 | 5.61E-7 | 3.68E-7 |
| order | - | 1.98 | 2.03 | 2.00 | 2.00 | 2.00 | 1.99 | 1.99 |
| $\varepsilon = 1/2$ | 3.04E-3 | 7.06E-4 | 1.74E-4 | 4.35E-5 | 1.09E-5 | 2.70E-6 | 6.79E-7 | 2.17E-7 |
| order | - | 1.94 | 1.95 | 2.03 | 1.99 | 2.00 | 2.00 | 1.97 |
| $\varepsilon = 1/2^2$ | 4.12E-3 | 7.92E-4 | 1.91E-4 | 4.73E-5 | 1.18E-5 | 2.95E-6 | 7.49E-7 | 2.50E-7 |
| order | - | 1.88 | 1.95 | 1.99 | 2.00 | 1.99 | 2.00 | 1.99 |
| $\varepsilon = 1/2^3$ | 5.21E-3 | 4.76E-4 | 1.05E-4 | 2.55E-5 | 6.33E-6 | 1.59E-6 | 4.11E-7 | 1.52E-7 |
| order | - | 1.61 | 1.90 | 1.98 | 2.00 | 1.98 | 2.02 | 1.95 |
| $\varepsilon = 1/2^4$ | 2.08E-2 | 8.34E-4 | 9.24E-5 | 2.17E-5 | 5.34E-6 | 1.33E-6 | 3.45E-7 | 1.27E-7 |
| order | - | -0.48 | 1.34 | 1.83 | 1.98 | 1.99 | 2.02 | 1.98 |
| $\varepsilon = 1/2^5$ | 1.07E-1 | 1.33E-2 | 2.07E-4 | 2.24E-5 | 5.29E-6 | 1.31E-6 | 3.31E-7 | 9.83E-8 |
| order | - | 0.11 | 0.39 | 0.98 | 1.65 | 1.95 | 1.99 | 1.99 |
| $\varepsilon = 1/2^6$ | 1.14E-1 | 7.56E-2 | 9.41E-3 | 7.44E-5 | 5.56E-6 | 1.32E-6 | 3.29E-7 | 9.57E-7 |
| order | - | 0.01 | 0.02 | 0.24 | 0.65 | 1.32 | 1.86 | 1.99 |
| $\varepsilon = 1/2^7$ | 1.12E-1 | 8.04E-2 | 5.36E-2 | 6.61E-3 | 4.03E-5 | 1.39E-6 | 3.30E-7 | 8.60E-8 |
| order | - | -0.03 | 0.03 | -0.04 | 0.13 | 0.42 | 0.96 | 1.65 |

TABLE 4.11. Temporal error analysis of TSSP at time $t = 1$ for the ill-prepared initial data Case III, i.e., $\alpha = 0$ and $\beta = 0$.

| $e^\varepsilon(t=1)$ | $\tau_0 = 0.1$ | $\frac{\tau_0}{2}$ | $\frac{\tau_0}{2^2}$ | $\frac{\tau_0}{2^3}$ | $\frac{\tau_0}{2^4}$ | $\frac{\tau_0}{2^5}$ | $\frac{\tau_0}{2^6}$ | $\frac{\tau_0}{2^7}$ |
|---|---|---|---|---|---|---|---|---|
| $\varepsilon = 1$ | 3.53E-3 | 8.35E-4 | 2.07E-4 | 5.18E-5 | 1.29E-5 | 3.23E-6 | 8.06E-7 | 2.07E-7 |
| order | - | 1.48 | 1.78 | 1.95 | 1.99 | 1.99 | 2.00 | 2.00 |
| $\varepsilon = 1/2$ | 4.82E-3 | 1.21E-3 | 2.88E-4 | 7.19E-5 | 1.79E-5 | 4.49E-6 | 1.12E-6 | 2.88E-7 |
| order | - | 1.50 | 1.77 | 1.95 | 2.00 | 2.00 | 2.00 | 1.96 |
| $\varepsilon = 1/2^2$ | 9.62E-3 | 2.41E-3 | 5.95E-4 | 1.49E-4 | 3.72E-5 | 9.31E-6 | 2.33E-6 | 5.88E-7 |
| order | - | 1.71 | 1.86 | 1.95 | 2.02 | 2.00 | 1.99 | 1.98 |
| $\varepsilon = 1/2^3$ | 1.38E-2 | 3.43E-3 | 8.42E-4 | 2.10E-4 | 5.26E-5 | 1.32E-5 | 3.29E-6 | 8.28E-7 |
| order | - | 1.56 | 1.68 | 1.87 | 2.03 | 2.00 | 1.99 | 1.90 |
| $\varepsilon = 1/2^4$ | 2.81E-2 | 6.52E-3 | 1.63E-3 | 3.97E-4 | 9.92E-5 | 2.48E-5 | 6.20E-6 | 1.55E-6 |
| order | - | 2.76 | 1.89 | 1.93 | 2.02 | 2.00 | 2.00 | 1.98 |
| $\varepsilon = 1/2^5$ | 8.18E-2 | 1.40E-2 | 3.22E-3 | 7.99E-4 | 1.99E-4 | 4.98E-5 | 1.25E-5 | 3.11E-6 |
| order | - | 2.52 | 2.08 | 2.00 | 1.98 | 2.00 | 1.99 | 2.00 |
| $\varepsilon = 1/2^6$ | 8.28E-2 | 4.06E-2 | 7.06E-3 | 1.64E-3 | 4.01E-4 | 9.99E-5 | 2.50E-5 | 6.24E-6 |
| order | - | 0.94 | 2.70 | 2.14 | 1.96 | 1.99 | 1.99 | 1.98 |
| $\varepsilon = 1/2^7$ | 7.53E-2 | 4.16E-2 | 2.02E-2 | 3.53E-3 | 8.11E-4 | 2.00E-4 | 4.99E-5 | 1.25E-5 |
| order | - | 1.46 | 0.64 | 2.77 | 2.12 | 1.99 | 1.98 | 1.99 |

| $n^\varepsilon(t=1)$ | $\tau_0 = 0.1$ | $\frac{\tau_0}{2}$ | $\frac{\tau_0}{2^2}$ | $\frac{\tau_0}{2^3}$ | $\frac{\tau_0}{2^4}$ | $\frac{\tau_0}{2^5}$ | $\frac{\tau_0}{2^6}$ | $\frac{\tau_0}{2^7}$ |
|---|---|---|---|---|---|---|---|---|
| $\varepsilon = 1$ | 1.92E-3 | 4.65E-4 | 1.16E-4 | 2.88E-5 | 7.19E-6 | 1.83E-6 | 5.61E-7 | 3.68E-7 |
| order | - | 1.98 | 2.03 | 2.00 | 2.00 | 2.00 | 1.99 | 1.99 |
| $\varepsilon = 1/2$ | 3.41E-3 | 8.07E-4 | 1.99E-4 | 4.98E-5 | 1.24E-5 | 3.11E-6 | 8.06E-7 | 3.11E-7 |
| order | - | 1.94 | 1.95 | 2.03 | 1.99 | 2.00 | 2.00 | 1.97 |
| $\varepsilon = 1/2^2$ | 4.12E-3 | 8.32E-4 | 2.02E-4 | 5.02E-5 | 1.25E-5 | 3.14E-6 | 8.29E-7 | 3.48E-7 |
| order | - | 1.88 | 1.95 | 1.99 | 2.00 | 1.99 | 2.00 | 1.99 |
| $\varepsilon = 1/2^3$ | 6.51E-3 | 1.22E-3 | 2.84E-4 | 7.07E-5 | 1.77E-5 | 4.43E-6 | 1.15E-6 | 4.15E-7 |
| order | - | 1.61 | 1.90 | 1.98 | 2.00 | 1.98 | 2.02 | 1.95 |
| $\varepsilon = 1/2^4$ | 2.31E-2 | 2.42E-3 | 5.57E-4 | 1.39E-4 | 3.46E-5 | 8.66E-6 | 2.18E-6 | 5.94E-7 |
| order | - | -0.48 | 1.34 | 1.83 | 1.98 | 1.99 | 2.02 | 1.98 |
| $\varepsilon = 1/2^5$ | 1.09E-1 | 1.42E-2 | 1.11E-3 | 2.79E-4 | 6.94E-5 | 1.73E-5 | 4.34E-6 | 1.09E-6 |
| order | - | 0.11 | 0.39 | 0.98 | 1.65 | 1.95 | 1.99 | 1.99 |
| $\varepsilon = 1/2^6$ | 1.16E-1 | 7.65E-2 | 9.73E-3 | 5.67E-4 | 1.39E-4 | 3.47E-5 | 8.68E-6 | 2.17E-6 |
| order | - | 0.01 | 0.02 | 0.24 | 0.65 | 1.32 | 1.86 | 1.99 |
| $\varepsilon = 1/2^7$ | 1.12E-1 | 8.09E-2 | 5.38E-2 | 6.71E-3 | 2.84E-4 | 6.96E-5 | 1.74E-5 | 4.34E-6 |
| order | - | -0.03 | 0.03 | -0.04 | 0.13 | 0.42 | 0.96 | 1.65 |

## 5. Conclusion

We have rigorously analyzed the error estimates of the conservative finite difference method (CNFD) for the Zakharov system in $d$ ($d = 1, 2, 3$) dimensions, involving a dimensionless parameter $\varepsilon \in (0, 1]$, which is inversely proportional to the ion acoustic speed. Particular attention has been paid to the dependence of the error on the parameter $\varepsilon$, mesh size $h$ and time step $\tau$. In the subsonic limit regime, i.e., $0 < \varepsilon \ll 1$, there exist highly oscillatory initial layers in the plasma density $N^\varepsilon$, which propagate with $O(\varepsilon)$ wavelength in time and $O(1)$ wavelength in space, and $O(\varepsilon^2)$, $O(\varepsilon)$ and $O(1)$ amplitudes for the well-prepared, less-ill-prepared and ill-prepared initial data, respectively. This rapid oscillation in time brought significant difficulties in the error analysis. We established the uniform convergence rate at $O(h^2 + \tau^{4/3})$ and $O(h^2 + \tau^{2/3})$, for the well-prepared initial data and less-ill-prepared initial data, respectively. For the ill-prepared initial data, we obtained the error bounds at $O(h^2/\varepsilon + \tau^2/\varepsilon^3)$. Numerical results confirmed our results and showed that the time splitting approach [5, 12] could perform better than the CNFD. The analysis presented here will provide important insights to the numerical studies of the Zakharov system in the subsonic limit regime, including the splitting methods.

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