

EIGENVALUE BOUNDS FOR MATRIX POLYNOMIALS IN GENERALIZED BASES

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ABSTRACT. We derive inclusion regions for the eigenvalues of matrix polynomials expressed in a general polynomial basis, which can lead to significantly better results than traditional bounds. We present several applications to engineering problems.

1. INTRODUCTION

A polynomial eigenvalue problem consists in computing a nonzero complex eigenvector v and a complex eigenvalue z such that $P(z)v = 0$, where P is a matrix polynomial of the form

$$A_n z^n + A_{n-1} z^{n-1} + \dots + A_1 z + A_0,$$

and A_j ($j = 0, 1, \dots, n$) are complex $m \times m$ matrices. If A_n is singular, then there are infinite eigenvalues, and if A_0 is singular, then zero is an eigenvalue. There are nm eigenvalues, including possibly infinite ones. The finite eigenvalues are the solutions of $\det P(z) = 0$. Throughout we will assume that the eigenvalue problem is *regular*, i.e., that $\det P(z)$ is not identically zero. We refer to [1] and [12] for an overview of engineering applications.

It is, in general, a computationally intensive task to solve these problems, although bounds on the eigenvalues are relatively easy to compute. Such bounds are useful, e.g., in eigenvalue computation by iterative methods ([11]) and when computing pseudospectra ([5], [13]). Most localization results for polynomial eigenvalues found in the literature apply to matrix polynomials that are expressed in the regular polynomial power basis $\{1, z, z^2, \dots\}$. However, using a different basis can lead to significantly better results for particular classes of problems. It is what we propose to do here.

Bounds for matrix polynomials are often based on bounds for scalar polynomials, many examples of which can be found in [6], and our approach will be similar. Specifically, we were inspired by Theorem 8.4.6 in [10], where a zero inclusion region is derived for scalar polynomials, expressed in a weakly interlacing basis, namely, a basis consisting of polynomials with real zeros such that the zeros of consecutive polynomials interlace, while some or all zeros may coincide. We found such bases to be too restrictive, and we will derive a matrix version of a generalization of this theorem to more general bases. For one of those bases, the Newton basis with complex nodes, our result, applied to scalar polynomials, is Theorem 8.6.3 of [10].

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We therefore unify and generalize to matrix polynomials inclusion regions not only for weakly interlacing and Newton bases, but for more general ones as well.

To make our exposition reasonably self-contained, we now state a few theorems and definitions that we will use later. The first is an extension to matrix-valued analytical functions of Rouché's theorem from [2] and [9].

Theorem 1.1. *Let $A, B : \Omega \rightarrow \mathbb{C}^{m \times m}$ be analytic matrix-valued functions, where Ω is an open connected subset of \mathbb{C} and assume that $A(z)$ is nonsingular for all z on the simple closed curve $\Gamma \subseteq \Omega$.*

If, for any matrix norm $\|\cdot\|$, $\|A(z)^{-1}B(z)\| < 1$ for all $z \in \Gamma$, then $\det(A + B)$ and $\det(A)$ have the same number of zeros inside Γ , counting multiplicities.

Theorem 1.1 is a convenient (although not the only) way to prove the following generalization to matrix polynomials of a result by Cauchy from 1829 ([3], [8, Theorem (27, 1), p. 122]). It can be found, with slight variations, in [2], [6], and [9].

Theorem 1.2. *The eigenvalues of the regular matrix polynomial $P(z) = A_n z^n + A_{n-1} z^{n-1} + \cdots + A_1 z + A_0$, with $A_j \in \mathbb{C}^{m \times m}$ and A_n nonsingular, are contained in the disk $|z| \leq \rho$, where ρ is the unique positive root of*

$$\|A_n^{-1}\|^{-1} x^n - \|A_{n-1}\| x^{n-1} - \cdots - \|A_1\| x - \|A_0\| = 0,$$

for any matrix norm $\|\cdot\|$.

We note that the smaller $\|A_n^{-1}\|$ and $\|A_j\|$ ($0 \leq j \leq n-1$) are, all else being equal, the smaller ρ will be.

Theorem 1.2 leads to the following definition.

Definition 1.1 (Cauchy radius). The quantity ρ in Theorem 1.2 is called the *Cauchy radius of the matrix polynomial P* . It depends on the matrix norm used in its statement.

The computational cost of solving the real equation in Theorem 1.2 is negligible compared to that of computing the eigenvalues of the matrix polynomial. Many standard rootfinders exist that easily solve such equations, such as Laguerre's method or even a simple accelerated Newton method, and we will not dwell on it here.

Theorem 1.2 will be the reference inclusion region to which we will compare our results, since it generally appears to be among the best bounds attainable for matrix polynomials expressed in the standard power basis, judging from the extensive results in [6], where a large number of such eigenvalue bounds were compared.

Throughout the paper, we will use I for the identity matrix without specifying its size, which is usually clear from the context. On those occasions where it is not, a $k \times k$ identity matrix will be denoted by I_k .

The paper is organized as follows. In Section 2 we derive an inclusion region for the eigenvalues of a matrix polynomial expressed in a generalized basis, which is then applied to several engineering problems in Section 3.

2. MAIN RESULT

The following theorem, our main result, is stated for a scalar polynomial basis $\{q_j\}_{j=0}^n$, where q_j is a polynomial of degree j , whose zeros are denoted by r_{ij} , $i = 1, \dots, j$. It relies on an inequality that must be satisfied by these basis polynomials.

To better explain it, consider the special case of a basis where for each j the roots r_{ij} are all distinct. Then from the partial fraction expansion of $q_{j-1}(z)/q_j(z)$ we obtain

$$(2.1) \quad \left| \frac{q_{j-1}(z)}{q_j(z)} \right| = \left| \sum_{i=1}^j \frac{\beta_i^{(j)}}{z - r_{ij}} \right| \leq \sum_{i=1}^j \frac{|\beta_i^{(j)}|}{|z - r_{ij}|},$$

for appropriate numbers $\beta_i^{(j)}$. For each j , these numbers will be assumed to satisfy $\sum_{i=1}^j |\beta_i^{(j)}| = \gamma_j$, with γ_j such that $\gamma_j \leq \gamma$ for all j , for some $\gamma_j, \gamma > 0$.

An inequality as in (2.1) can be obtained for many bases. As was shown in Lemma 8.4.5 in [10], it holds with $\gamma_j = 1$ when the q_j basis polynomials form a weakly interlacing system, i.e., when the zeros of the basis polynomials are real and when the zeros of q_{j-1} interlace those of q_j , while allowed to coincide with zeros of q_j . Weakly interlacing bases include all classical orthogonal bases: Hermite, Legendre, Chebyshev, etc.

Another important special case is obtained by choosing the Newton basis with complex nodes $\{a_j\}$, $j = 1, 2, \dots$, for which the basis polynomials are defined by $q_0(z) = 1$ and $q_j(z) = (z - a_j)q_{j-1}(z)$. This means that the zeros of q_j are a_1, a_2, \dots, a_j . Since

$$\frac{q_{j-1}(z)}{q_j(z)} = \frac{1}{z - a_j},$$

the inequality in (2.1) is once again satisfied with $\gamma_j = 1$.

We now state the theorem.

Theorem 2.1. *Let $\{q_j\}_{j=0}^n$ be a scalar polynomial basis, where q_j is a polynomial of degree j for $j = 0, 1, \dots, n$, and denote by r_{ij} the i th zero of q_j ($i \leq j, j \geq 1$). If for every $j \geq 1$ there exist nonnegative numbers $\alpha_1^{(j)}, \dots, \alpha_j^{(j)}$ so that $\sum_{i=1}^j \alpha_i^{(j)} \leq \gamma$, with $\gamma > 0$, and, for $z \neq r_{ij}$,*

$$(2.2) \quad \left| \frac{q_{j-1}(z)}{q_j(z)} \right| \leq \sum_{i=1}^j \frac{\alpha_i^{(j)}}{|z - r_{ij}|},$$

then the eigenvalues of the regular matrix polynomial

$$P(z) = A_n q_n(z) + A_{n-1} q_{n-1}(z) + \dots + A_1 q_1(z) + A_0 q_0(z),$$

with $A_j \in \mathbb{C}^{m \times m}$ and A_n nonsingular, are contained in the union of the at most $n(n+1)/2$ distinct disks

$$\mathcal{R} = \bigcup_{\substack{i,j=1 \\ i \leq j}}^n \{z \in \mathbb{C} : |z - r_{ij}| \leq \gamma \rho\},$$

where ρ is the Cauchy radius of $\sum_{j=0}^n A_j z^j$ for any matrix norm. Moreover, if the region \mathcal{R} is composed of disjoint components, then each component contains m times as many eigenvalues of P as it contains zeros of q_n .

Proof. The norm $\|\cdot\|$ used in the proof stands for any matrix norm. If z is an eigenvalue of P such that $z \neq r_{ij}$, then

$$\det(A_n q_n(z) + \dots + A_1 q_1(z) + A_0 q_0(z)) = 0$$

implies that

$$\det\left(I + (A_n q_n(z))^{-1}(A_{n-1}q_{n-1}(z) + \cdots + A_1q_1(z) + A_0q_0(z))\right) = 0,$$

which is only possible ([7, p. 351]) if

$$(2.3) \quad \left\| (A_n q_n(z))^{-1}(A_{n-1}q_{n-1}(z) + \cdots + A_1q_1(z) + A_0q_0(z)) \right\| \geq 1.$$

Since $\|A^{-1}\| \|B\| \geq \|A^{-1}B\|$, inequality (2.3) implies that

$$(2.4) \quad \|A_{n-1}q_{n-1}(z) + \cdots + A_1q_1(z) + A_0q_0(z)\| \geq \|A_n^{-1}\|^{-1} |q_n(z)|,$$

so that, with $\|A\| + \|B\| \geq \|A + B\|$, inequality (2.4) yields

$$(2.5) \quad \|A_{n-1}\| |q_{n-1}(z)| + \cdots + \|A_1\| |q_1(z)| + \|A_0\| |q_0(z)| \geq \|A_n^{-1}\|^{-1} |q_n(z)|.$$

To express the left-hand side of (2.5) in a more useful way, we define for each j ($j = 1, 2, \dots, n$),

$$d_j(z) = \min_{1 \leq i, k \leq j} |z - r_{ik}|,$$

namely, the distance of a point z to the set of all the zeros of q_1, \dots, q_j . Clearly, $d_1(z) \geq d_2(z) \geq \cdots \geq d_n(z)$. Inequality (2.2) then implies that

$$(2.6) \quad \left| \frac{q_{j-1}(z)}{q_j(z)} \right| \leq \sum_{i=1}^j \frac{\alpha_i^{(j)}}{|z - r_{ij}|} \leq \frac{\sum_{i=1}^j \alpha_i^{(j)}}{\min_{1 \leq i \leq j} |z - r_{ij}|} \leq \frac{\gamma}{d_j(z)} \leq \frac{\gamma}{d_n(z)} \quad (j = 1, \dots, n).$$

Repeated application of (2.6) yields

$$(2.7) \quad \left| \frac{q_j(z)}{q_n(z)} \right| = \left| \frac{q_j(z)}{q_{j+1}(z)} \right| \cdots \left| \frac{q_{n-1}(z)}{q_n(z)} \right| \leq \left(\frac{\gamma}{d_n(z)} \right)^{n-j}.$$

Dividing the left-hand side of (2.5) by $|q_n(z)|$ and majorizing it in terms of $d_n(z)$ using (2.7) yields

$$(2.8) \quad \begin{aligned} & \|A_{n-1}\| \left| \frac{q_{n-1}(z)}{q_n(z)} \right| + \cdots + \|A_1\| \left| \frac{q_1(z)}{q_n(z)} \right| + \|A_0\| \left| \frac{q_0(z)}{q_n(z)} \right| \\ & \leq \|A_{n-1}\| \left(\frac{\gamma}{d_n(z)} \right) + \cdots + \|A_1\| \left(\frac{\gamma}{d_n(z)} \right)^{n-1} + \|A_0\| \left(\frac{\gamma}{d_n(z)} \right)^n \\ & \leq \left(\|A_{n-1}\| \left(\frac{d_n(z)}{\gamma} \right)^{n-1} + \cdots + \|A_1\| \left(\frac{d_n(z)}{\gamma} \right) + \|A_0\| \right) \left(\frac{\gamma}{d_n(z)} \right)^n. \end{aligned}$$

Combining (2.8) with (2.5), we obtain that if z is an eigenvalue of P , then

$$\|A_{n-1}\| \left(\frac{d_n(z)}{\gamma} \right)^{n-1} + \cdots + \|A_1\| \left(\frac{d_n(z)}{\gamma} \right) + \|A_0\| \geq \|A_n^{-1}\|^{-1} \left(\frac{d_n(z)}{\gamma} \right)^n.$$

By the definition of ρ this means that $d_n(z)/\gamma \leq \rho$, where ρ is the Cauchy radius of $\sum_{j=0}^n A_j z^j$, or

$$\min_{1 \leq i, k \leq n} |z - r_{ik}| \leq \gamma \rho,$$

i.e., z must lie in the union \mathcal{R} of disks centered at the zeros of q_1, \dots, q_n with radius $\gamma\rho$. The number of distinct disks is at most $\sum_{j=1}^n j = n(n+1)/2$ (basis polynomials can have common zeros).

If \mathcal{R} is composed of disjoint subregions (each subregion necessarily a connected union of disks), then the boundary Γ of such a subregion is a simple closed curve on which $A_n q_n(z)$ is nonsingular and $d_n(z) = \gamma\rho$. Now consider the collection of points z for which $d_n(z) = \gamma\rho + \varepsilon$ for $\varepsilon > 0$. It is the boundary of the union \mathcal{R}_1 of disks with the same centers as those that determine \mathcal{R} , but with a larger radius. Clearly, this boundary does not contain any of the centers r_{ij} . If \mathcal{R} consists of disjoint subregions, then we can choose ε small enough so that \mathcal{R}_1 does as well. One of those will necessarily enclose Γ , and we define Γ_1 as its boundary. It is a simple closed curve on which $A_n q_n(z)$ is nonsingular and $d_n(z) = \gamma\rho + \varepsilon$.

Using the same arguments as in (2.4), (2.5), and (2.8), one sees that, for $z \neq r_{ij}$, the inequality

$$(2.9) \quad \left\| (A_n q_n(z))^{-1} (A_{n-1} q_{n-1}(z) + \dots + A_1 q_1(z) + A_0 q_0(z)) \right\| < 1$$

will certainly be satisfied when

$$(2.10) \quad \|A_{n-1}\| \left(\frac{d_n(z)}{\gamma} \right)^{n-1} + \dots + \|A_1\| \left(\frac{d_n(z)}{\gamma} \right) + \|A_0\| < \|A_n^{-1}\|^{-1} \left(\frac{d_n(z)}{\gamma} \right)^n .$$

Since for any $z \in \Gamma_1$ we have $d_n(z)/\gamma = \rho + \varepsilon/\gamma > \rho$, by the definition of ρ we get that (2.10) is satisfied on Γ_1 . This, in turn, implies that (2.9) is satisfied, from which we obtain with Theorem 1.1 that P and $A_n q_n$ have the same number of eigenvalues in the open region enclosed by Γ_1 . Since Γ_1 encloses Γ , we conclude, by letting $\varepsilon \rightarrow 0^+$, that the closed subregion of \mathcal{R} bounded by Γ contains a number of eigenvalues of P equal to the number of eigenvalues of $A_n q_n$ that it contains. Because $\det(A_n q_n(w)) = 0 \iff \det(A_n) q_n^m(w) = 0$, and $\det(A_n) \neq 0$, this number is m times the number of zeros of q_n in the closed region. \square

We remark that for the standard power basis $\{z_j\}_{j=0}^\infty$ (for which $r_{ij} = 0$ for all i, j , and $\gamma = 1$), Theorem 2.1 reduces to Theorem 1.2, i.e., it is an extension of Theorem 1.2 to more general bases.

In the special case where P is a scalar polynomial and the polynomials q_j form a weakly interlacing system, Theorem 2.1 essentially reduces to Theorem 8.4.6 in [10]. In this case, the region derived in Theorem 8.4.6 in [10] is the convex hull of the one in Theorem 2.1.

For the Newton basis with complex nodes $\{a_j\}$, $j = 1, 2, \dots$, condition (2.2) is satisfied with $\alpha_1^{(j)} = \alpha_2^{(j)} = \dots = \alpha_{j-1}^{(j)} = 0$, $\alpha_j^{(j)} = 1$, and $\gamma = 1$. For scalar polynomials, this is Theorem 8.6.3 in [10]. However, Theorem 2.1 allows for more general bases where the zeros of different basis polynomials do not need to be interlaced or satisfy a similar property, and we will see an example of this further on.

Obviously, changing the basis does not universally improve results for all problems. As with other localization results, some problems lend themselves better to certain bounds than others. In addition, when the degree is high and the polynomial is expressed in the standard power basis, it may be too cumbersome to express it in a general basis. On the other hand, if the polynomial is already given in a general basis, then Theorem 2.1 makes it unnecessary to first transform it to a power basis.

It should be mentioned that the degree of matrix polynomials appearing in engineering applications tends to be low; many of them are quadratic. They are easily

expressed in a different basis, while the computation of bounds, including the solution of the real scalar polynomial equation appearing in Theorem 1.2, is several orders of magnitude less onerous than the computation of the actual eigenvalues, so that not much is lost by trying a different basis.

To illustrate how Theorem 2.1 can improve classical bounds, we turn to the literature on quadratic eigenvalue problems with their many applications in engineering.

3. EXAMPLES

We establish a few preliminary results concerning monic quadratic matrix polynomials before applying them to numerical examples. Since it is easy to compute, we choose the 1-norm throughout this section.

3.1. Quadratic matrix polynomials. We consider the monic quadratic matrix polynomial $P(z) = Iz^2 + A_1z + A_0$, expressed in the standard power basis, and we define the Newton basis $\mathcal{N} = \{f_0, f_1, f_2\}$ and the more general basis $\mathcal{B} = \{q_0, q_1, q_2\}$, respectively, by

$$\begin{cases} f_0(z) = 1, \\ f_1(z) = z - a, \\ f_2(z) = (z - a)(z - b), \end{cases} \quad \text{and} \quad \begin{cases} q_0(z) = 1, \\ q_1(z) = z - a, \\ q_2(z) = (z - b)(z - c). \end{cases}$$

These bases were chosen for no particular reason, other than that they are very different from the power basis. The Newton basis is relatively commonly used for *scalar* polynomials, and a zero inclusion exists in this case (Theorem 8.6.3 in [10]). No such results exist for the basis \mathcal{B} , making it a natural basis candidate.

For the basis \mathcal{B} , either $a = b = c$ in which case it becomes a Newton basis, or $b \neq c$. When $b = c \neq a$, inequality (2.2) is not satisfied, which can be seen from the partial fraction expansion of $q_1(z)/q_2(z)$, whose denominator has a double zero that is different from the zero of the numerator. The choice of the nodes a, b , and c , which are generally different for different bases, is tailored to the particular matrix polynomial. The power basis is easily expressed in terms of \mathcal{N} and \mathcal{B} :

$$\begin{cases} 1 = f_0(z), \\ z = f_1(z) + af_0(z), \\ z^2 = f_2(z) + (a + b)f_1(z) + a^2f_0(z), \end{cases}$$

and

$$\begin{cases} 1 = q_0(z), \\ z = q_1(z) + aq_0(z), \\ z^2 = q_2(z) + (b + c)q_1(z) + (a(b + c) - bc)q_0(z). \end{cases}$$

The quadratic P in the bases \mathcal{N} and \mathcal{B} then becomes, respectively,

$$(3.1) \quad \begin{aligned} P(z) &= If_2(z) + (A_1 + (a + b)I)f_1(z) + (aA_1 + A_0 + a^2I)f_0(z), \\ P(z) &= Iq_2(z) + (A_1 + (b + c)I)q_1(z) + (aA_1 + A_0 + (a(b + c) - bc)I)q_0(z). \end{aligned}$$

Let us verify condition (2.2) in Theorem 2.1 for these bases. For \mathcal{N} , this was already done in the remarks following that theorem. For the basis \mathcal{B} with $b \neq c$, we obtain

$$\frac{q_0(z)}{q_1(z)} = \frac{1}{z - a} \quad \text{and} \quad \frac{q_1(z)}{q_2(z)} = \frac{(a - b)/(c - b)}{z - b} + \frac{(c - a)/(c - b)}{z - c},$$

so that (2.2) is satisfied with

$$\alpha_1^{(1)} = 1, \alpha_1^{(2)} = \left| \frac{a-b}{c-b} \right|, \alpha_2^{(2)} = \left| \frac{a-c}{c-b} \right|, \text{ and } \gamma = \left| \frac{a-b}{c-b} \right| + \left| \frac{a-c}{c-b} \right|.$$

We note that $\gamma \geq 1$.

The nodes determining the bases \mathcal{N} and \mathcal{B} should be chosen so as to make the norms of the coefficient matrices as small as possible, since this will make the radii of the disks in the inclusion region smaller, as was observed immediately after the statement of Theorem 1.2. To do this for the numerical examples below, we will use the observation that for real numbers $\{\beta_j\}_{j=1}^n$, ordered in increasing order, the solution of the minimization problem

$$(3.2) \quad \min_x \max_{1 \leq j \leq n} |\beta_j - x|$$

is obtained for $x^* = (\beta_1 + \beta_n)/2$. This implies that if the numbers β_j are the diagonal of a diagonal matrix M , then $\|M - x^*I\|_1 \leq \|M\|_1$. When the matrix M is not diagonal, but strongly diagonally dominant, then we expect this inequality to still be true in most cases. When the numbers β_j are complex and the minimization in x is to be carried out over the complex plane, then, to keep matters simple, we will carry out the minimization separately for the real and complex parts.

3.2. Numerical examples.

Example 1. We consider the connected damped mass-spring system in [12, p. 259]. Its vibration is described by a second-order differential equation of the form $A_2y''(t) + A_1y'(t) + A_0y(t) = f(t)$, where A_2, A_1 , and A_0 are $m \times m$ matrices and $y(t)$ is an m -vector. The solution of the differential equation can be expressed in terms of the eigenvalues and eigenvectors of the quadratic eigenvalue problem $(A_2z^2 + A_1z + A_0)v = 0$. Here, the mass matrix A_2 is diagonal, and the damping and stiffness matrices A_1 and A_0 , respectively, are symmetric tridiagonal. In [12], $A_2 = I, A_1 = \tau \text{tridiag}(-1, 3, -1), A_0 = \kappa \text{tridiag}(-1, 3, -1), \tau, \kappa \in \mathbb{R}$, and $m = 50$. We will compare the standard power basis $\{1, z, z^2\}$ with the bases \mathcal{N} and \mathcal{B} from Subsection 3.1.

We now determine the nodes a, b , and c defining those bases, and start with the Newton basis \mathcal{N} , where from (3.1) the coefficients of f_1 and f_0 are given, respectively, by

$$\begin{aligned} A_1 + (a+b)I &= \text{tridiag}(-\tau, 3\tau + a + b, -\tau), \\ aA_1 + A_0 + a^2I &= \text{tridiag}(-\tau a - \kappa, a^2 + 3\tau a + 3\kappa, -\tau a - \kappa). \end{aligned}$$

In choosing a and b , we aim to make the 1-norm of the coefficients as small as possible. Without attempting an elaborate optimization, we will choose a and b such as to make the diagonals of the matrix coefficients zero, i.e., $a^2 + 3\tau a + 3\kappa = 0$ and $a + b = -3\tau$. This means that a and b are the two zeros of $z^2 + 3\tau z + 3\kappa$, and we choose a as the zero for which $|a\tau + \kappa|$ is smaller.

For the basis \mathcal{B} , we have more flexibility since we now have three nodes a, b , and c . Here, from (3.1), the coefficients of q_1 and q_0 are given, respectively, by

$$\begin{aligned} A_1 + (b+c)I &= \text{tridiag}(-\tau, 3\tau + b + c, -\tau), \\ aA_1 + A_0 + (a(b+c) - bc)I &= \text{tridiag}(-\tau a - \kappa, a(b+c) - bc + 3\tau a + 3\kappa, -\tau a - \kappa). \end{aligned}$$

Arguing similarly as before, we choose the nodes such that $b + c = -3\tau$, $bc = 3\kappa$ and $a = -\kappa/\tau$. This makes the diagonal of the coefficient matrix of q_1 zero, while making the coefficient matrix of q_0 vanish. The nodes b and c are the same as the nodes a and b we found for the Newton basis since they are the zeros of the same quadratic polynomial. If $b = c$, then we set $a = b = c$, reverting to a Newton basis. Potentially better results could be obtained than for the Newton basis when $b \neq c$ although there is a price to pay in the form of a larger value for γ . It is therefore not a priori clear which basis is preferable. Fortunately, it is a simple matter to compute the 1-norm, so that both bases can easily be compared.

The following figures show the eigenvalue inclusion regions for a few representative values of τ and κ . All eigenvalues have negative real parts since the coefficient matrices are all strictly positive definite (see [12]). In each figure, the large circle centered at the origin is the circle obtained from Theorem 1.2, namely, Cauchy's theorem for matrix polynomials; its radius is the Cauchy radius of P and we will refer to it as the *Cauchy disk* of P . On the left, the smaller disks represent the inclusion region obtained from Theorem 2.1 for the Newton basis, while those on the right are for the basis \mathcal{B} . Figure 1 and Figure 2 show the eigenvalue inclusion regions for $\tau = 3, \kappa = 5$ and $\tau = 10, \kappa = 5$, respectively, which are the values used in [12]. The dots are the eigenvalues, which are added for reference. For Figures 2, 3, 4, and 5, the values for the pair (τ, κ) are $(1, 8)$, $(5, 20)$, $(5, 30)$, and $(5, 80)$, respectively. When τ is large relative to κ , the inclusion regions are almost identical for both bases, as for the $(10, 5)$ case. This can be seen from the roots

$$\left(-1 \pm \left(1 - \frac{4\kappa}{3\tau^2}\right)^{1/2}\right) \cdot \frac{3\tau}{2}$$

of the quadratic polynomial $z^2 + 3\tau z + 3\kappa$ (which are a and b in the basis \mathcal{N} and b and c in the basis \mathcal{B}): as κ/τ approaches zero, the root with the “+” sign approaches $-\kappa/\tau$, which is a in the basis \mathcal{B} . This follows (as $\kappa/\tau \rightarrow 0$) from

$$\left(-1 + \left(1 - \frac{4\kappa}{3\tau^2}\right)^{1/2}\right) \cdot \frac{3\tau}{2} \approx \left(-1 + \left(1 - \frac{2\kappa}{3\tau^2}\right)\right) \cdot \frac{3\tau}{2} = -\frac{\kappa}{\tau}.$$

As a result, two of the inclusion disks for the basis \mathcal{B} become almost identical with a disk for the basis \mathcal{N} when κ/τ is small.

These figures clearly show that using a more general basis can significantly reduce the eigenvalue inclusion regions, when compared to the disk obtained from Theorem 1.2, which is often the best one can obtain for the power basis. Sometimes the Newton basis is better, and sometimes it is the more general basis \mathcal{B} that produces the smaller inclusion region. Of special interest is Figure 3, where the eigenvalues are split among the top and bottom disks, 50 in each disk, as predicted by the theorem, since the middle disk does not contain any zeros of q_2 . We remark that it would not be possible to obtain such an inclusion region from any of the classical bounds.

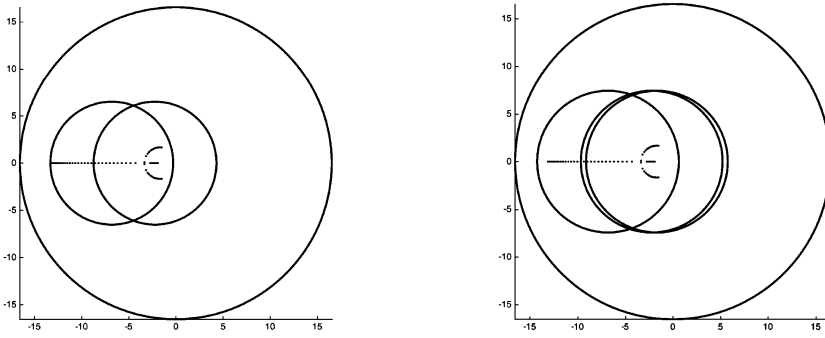


FIGURE 1. Inclusion regions for Example 1 with $\tau = 3$ and $\kappa = 5$.

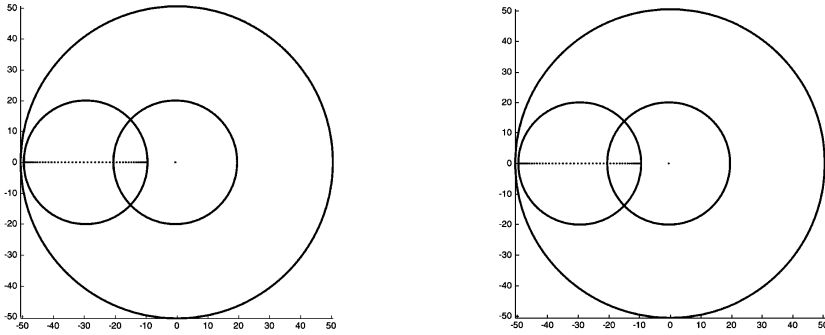


FIGURE 2. Inclusion regions for Example 1 with $\tau = 10$ and $\kappa = 5$.

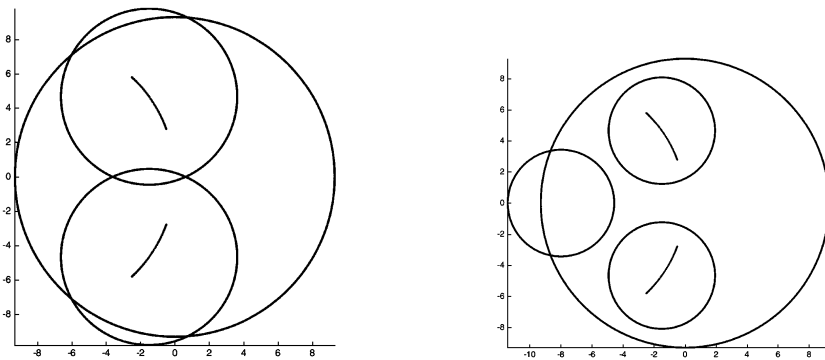


FIGURE 3. Inclusion regions for Example 1 with $\tau = 1$ and $\kappa = 8$.

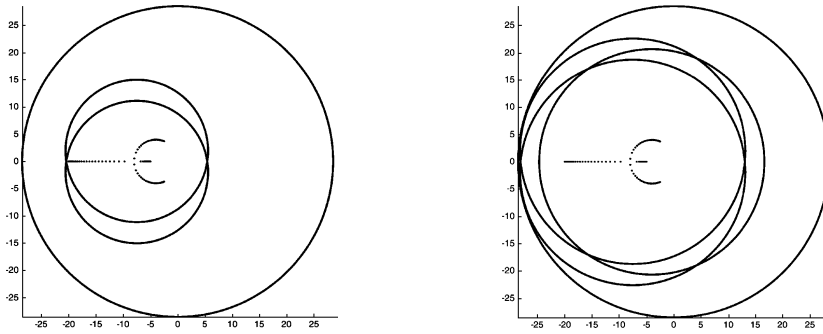


FIGURE 4. Inclusion regions for Example 1 with $\tau = 5$ and $\kappa = 20$.

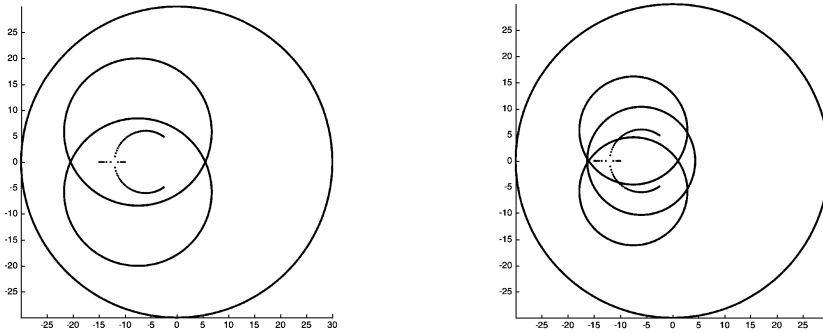


FIGURE 5. Inclusion regions for Example 1 with $\tau = 5$ and $\kappa = 30$.

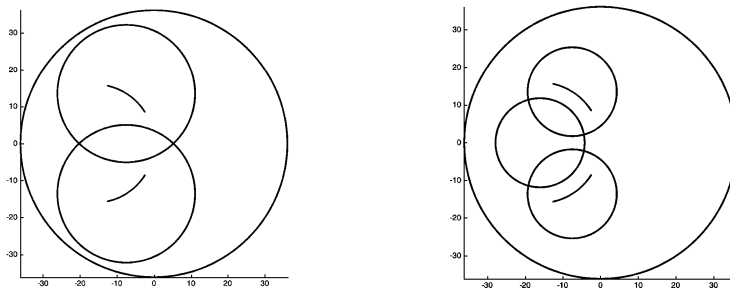


FIGURE 6. Inclusion regions for Example 1 with $\tau = 5$ and $\kappa = 80$.

Example 2. In this example from [1] and [4], we consider a quadratic polynomial produced by the finite-element discretization of a time-harmonic wave equation for the acoustic pressure on the unit square $[0, 1] \times [0, 1]$. The eigenvalues lie in the upper half of the complex plane. Here we have $m = \ell(\ell - 1)$, where $\ell = 1/h$ and h

is the mesh size. Defining the $\ell \times \ell$ matrix S_ℓ and the $(\ell - 1) \times (\ell - 1)$ matrix $T_{\ell-1}$ as

$$S_\ell = \begin{pmatrix} 4 & -1 & & & \\ -1 & \ddots & \ddots & & \\ & \ddots & 4 & -1 & \\ & & & -1 & 2 \end{pmatrix} \quad \text{and} \quad T_{\ell-1} = \begin{pmatrix} 0 & -1 & & & \\ -1 & \ddots & \ddots & & \\ & \ddots & \ddots & -1 & \\ & & & -1 & 0 \end{pmatrix},$$

the coefficients of the quadratic matrix polynomial $P(z) = A_2z^2 + A_1z + A_0$ are given by

$$A_0 = I_{\ell-1} \otimes S_\ell + T_{\ell-1} \otimes \left(-I_\ell + \frac{1}{2}e_\ell e_\ell^T\right), \quad A_1 = \frac{2\pi}{\ell\zeta} I_{\ell-1} \otimes e_\ell e_\ell^T,$$

$$A_2 = -\frac{4\pi^2}{\ell^2} I_{\ell-1} \otimes \left(I_\ell - \frac{1}{2}e_\ell e_\ell^T\right),$$

where the complex number ζ is the impedance, e_i is the i th standard unit vector, and the Kronecker product of two matrices $A \otimes B$ is the block matrix $(a_{ij}B)$. Since A_2 is nonsingular and diagonal, it is an easy matter to compute $A_2^{-1}P = Iz^2 + B_1z + B_0$, where $B_1 = A_2^{-1}A_1$ and $B_0 = A_2^{-1}A_0$. The matrix B_0 is diagonally dominant for most of its rows and columns. We can now conveniently use the results from Subsection 3.1 to express P in the basis \mathcal{N} . In this example, as in the next, we will only consider the Newton basis, since the disks in both bases are not significantly different in size.

The diagonals are not constant, and to minimize their 1-norm we use the observation about the minimization problem in (3.2). We will once again aim to choose nodes that minimize the 1-norms of the coefficients of P . We set $\text{diag}(B_j) = C_j + iD_j$ for $j = 0, 1$, and, in light of the above observation about (3.2), we define

$$\mu_j = \frac{1}{2}(\min(C_j) + \max(C_j)) + \frac{i}{2}(\min(D_j) + \max(D_j)) \quad (j = 0, 1).$$

From the expression in (3.1) for the matrix coefficients of f_1 , we see that a reasonable choice for the nodes is to choose them so that $a + b = -\mu_1$. Since from (3.1) the coefficient of f_0 can be written as

$$aB_1 + B_0 + a^2I = a^2I + \mu_1aI + \mu_0I + a(B_1 - \mu_1I) + (B_0 - \mu_0I),$$

we choose a as that solution of $a^2 + \mu_1a + \mu_0 = 0$ that minimizes $\|aB_1 + B_0 + a^2I\|_1$.

Figures 7 and 8 show the eigenvalue inclusion regions for $\zeta = 0.1 + 0.1i$ and $2 + 2i$, respectively, with $h = 0.05$, so that the matrix coefficients are of size 380×380 . The large circle centered at the origin is, as before, the Cauchy disk of P , while the black dots are the eigenvalues.

For very small or very large values of $|\zeta|$, the disks in the Newton basis are not significantly different from the ones in Figure 7 and Figure 8, respectively. From these results, it is clear that using a generalized basis here clearly allows a significant part of the Cauchy disk to be discarded as a possible location for the eigenvalues.

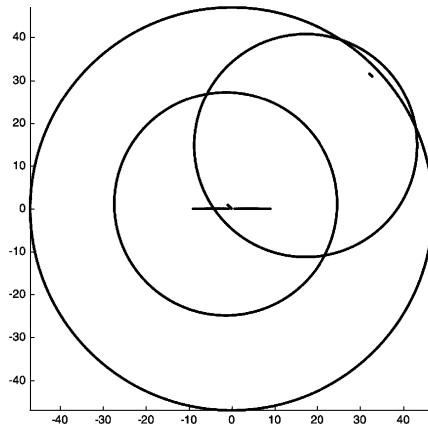


FIGURE 7. Inclusion regions for Example 2 with $\zeta = 0.1 + 0.1i$.

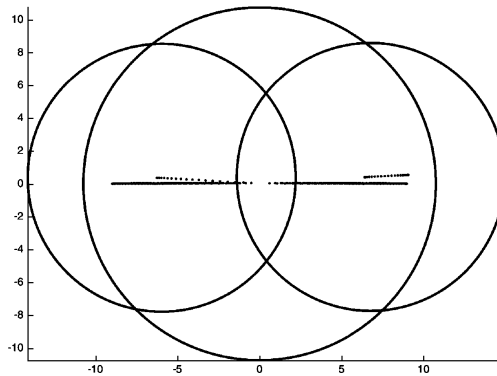


FIGURE 8. Inclusion regions for Example 2 with $\zeta = 2 + 2i$.

Example 3. This example is taken from [6]. Its quadratic matrix polynomial $Iz^2 + A_1z + A_0$ originates from a Galerkin method with n basis functions applied to a second-order partial differential equation describing the free vibration of a string, clamped at both ends in a spatially inhomogeneous environment. Here the matrix coefficients are given, for $\epsilon, \delta > 0$, by

$$A_0 = \pi \operatorname{diag} (j^2)_{1 \leq j \leq n} \quad , \quad (A_1)_{k\ell} = 2\epsilon \int_0^\pi (x^2(\pi - x)^2 - \delta) \sin(kx) \sin(\ell x) dx .$$

With $n = 50$, $\epsilon = 0.1$, and $\delta = 2.7$ as in [6], we proceed as in the previous example, using similar arguments for the choice of the nodes. As for the previous example, we have shown results only for the Newton basis as there is very little difference in the size of the disks between the \mathcal{N} and \mathcal{B} bases. Figure 9 shows the inclusion region for the Newton basis. The eigenvalues are concentrated along the imaginary axis, but they are not purely imaginary.

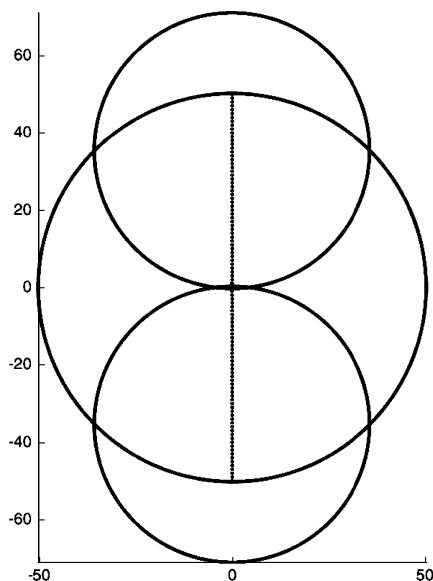


FIGURE 9. Inclusion regions for Example 3.

Summary. We have derived inclusion regions for the eigenvalues of matrix polynomials expressed in a general basis and have shown the advantages this can provide at the hand of several examples from the engineering literature. Not every problem benefits from a change of basis, but there is apparently no shortage of problems that do. We further remark that the relatively crude estimations we have used to determine the nodes of the bases \mathcal{N} and \mathcal{B} will generally be different for different problems and may be refined, depending on the properties of the coefficient matrices and the choice of matrix norm. Fortunately, the computational cost involved is negligible compared to the computation of the eigenvalues themselves, so that there is no reason not to try and use a more general basis, especially since the eigenvalues must lie in the intersection of all the inclusion regions obtained for different bases, further reducing the size of those regions. Finally, we mention that the reverse polynomial of a matrix polynomial with no zero eigenvalues can be used to generate additional information on the location of the eigenvalues. This polynomial is obtained by the transformation $z \rightarrow 1/z$, and it can be expressed in a general basis to which Theorem 2.1 can be applied. This leads to inclusion disks for the reciprocals of the eigenvalues. Because $1/z$ is a Möbius transformation, these disks become inclusion or exclusion disks or half-planes for the eigenvalues themselves.

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