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# A CONTINUED FRACTION OF ORDER TWELVE AS A MODULAR FUNCTION

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Abstract. We study a continued fraction $U(\tau)$ of order twelve using the modular function theory. We obtain the modular equations of $U(\tau)$ by computing the affine models of modular curves $X(\Gamma)$ with $\Gamma = \Gamma_1(12) \cap \Gamma_0(12n)$ for any positive integer $n$; this is a complete extension of the previous result of Mahadeva Naika et al. and Dharmendra et al. to every positive integer $n$. We point out that we provide an explicit construction method for finding the modular equations of $U(\tau)$. We also prove that these modular equations satisfy the Kronecker congruence relations. Furthermore, we show that we can construct the ray class field modulo 12 over imaginary quadratic fields by using $U(\tau)$ and the value $U(\tau)$ at an imaginary quadratic argument is a unit. In addition, if $U(\tau)$ is expressed in terms of radicals, then we can express $U(r\tau)$ in terms of radicals for a positive rational number $r$.

## 1. Introduction

The Rogers-Ramanujan continued fraction $r(\tau)$ is a holomorphic function on the complex upper half-plane $\mathfrak{H}$ defined by

$$r(\tau) = \cfrac{q^{\frac{1}{5}}}{1 + \cfrac{q}{1 + \cfrac{q^2}{1 + \cfrac{q^3}{1 + \cdots}}}} = q^{\frac{1}{5}} \prod_{n=1}^{\infty} (1 - q^n)^{\left(\frac{n}{5}\right)},$$

where $q = e^{2\pi i \tau}$ and $\left(\frac{\cdot}{5}\right)$ is the Legendre symbol. Ramanujan stated that the exact value $r(\sqrt{-n}/2)$ can be found whenever $n$ is any positive rational quantity. Gee and Honsbeek proved that $r(\tau)$ is a generator of the modular function field of level 5. They also showed that any singular value of $r(\tau)$ at imaginary quadratic argument is contained in some ray class field over an imaginary quadratic field [5]; hence its minimal polynomial is solvable by radicals.

Another important subject is the modular equation. For each positive integer $n$, there is a certain polynomial giving the relation between $r(\tau)$ and $r(n\tau)$ because the modular function field of level 5 is of genus 0. This polynomial is called *the*

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*modular equation of level $n$*. By the aid of the theory of arithmetic models of modular curves, Cais and Conrad studied the modular equations of the Rogers-Ramanujan continued fraction [1]. They found the Kronecker congruence relations for the modular equations which are similar to the modular equations of the elliptic modular function $j(\tau)$.

Since then there have been active research activities on the study of holomorphic functions on $\mathfrak{H}$ similar to $r(\tau)$. Recently, the authors studied a Ramanujan-Selberg continued fraction $S(\tau)$ as a modular function of level 8, and they also obtained the modular equations for every level and the ray class field modulo 4 over an imaginary quadratic field by the value $S(\tau)$ [9].

A *continued fraction $U(\tau)$ of order twelve* is defined by

$$U(\tau) := \cfrac{q(1-q)}{1-q^3+\cfrac{q^3(1-q^2)(1-q^4)}{(1-q^3)(1+q^6)+\cfrac{q^3(1-q^8)(1-q^{10})}{(1-q^3)(1+q^{12})+\cdots}}}.$$

Mahadeva Naika et al. [10] expressed $U(\tau)$ as the following infinite product form:

$$U(\tau) = q\prod_{n=1}^{\infty}\frac{(1-q^{12n-1})(1-q^{12n-11})}{(1-q^{12n-5})(1-q^{12n-7})}.$$

They found modular equations of $U(\tau)$ of levels 3 and 5 using the theory of hypergeometric series. They also evaluated $U(\tau)$ at $\tau = i\sqrt{m}$ for several rational numbers of $m$ [10]. Afterwards, Dharmendra et al. [3] extended their result on modular equations to other levels $n = 2, 3, 5, 7, 9$ in a similar way as [10]. Moreover, Mahadeva Naika et al. obtained modular equations of $U(\tau)$ of degrees $4, 6, 8, 10, 12, 3/2, 5/2$ and $7/2$ [11, 12].

In this paper we study $U(\tau)$ using the modular function theory. We first find the generators of modular function fields on $\Gamma_1(12)$ and $\Gamma_0(12)$ by using the modularity of $U(\tau)$ (Theorem 1.1). We then obtain the modular equations of $U(\tau)$ by computing the affine models of modular curves $X(\Gamma)$ with $\Gamma = \Gamma_1(12)\cap\Gamma_0(12n)$ for any positive integer $n$ (Theorem 1.2); Mahadeva Naika et al. [10] and Dharmendra et al. got their result for some positive integers, and our work is a complete extension of their result to every positive integer $n$. We point out that we provide an explicit construction method for finding the modular equations of $U(\tau)$ (Algorithm 3.4). We also prove that these modular equations satisfy the Kronecker congruence relations (Theorem 1.3). Furthermore, we show that we can construct the ray class field modulo 12 over imaginary quadratic fields by using $U(\tau)$ (Theorem 1.4). We show that the value $U(\tau)$ at an imaginary quadratic argument is a unit (Theorem 1.6). In addition, if $U(\tau)$ is expressed in terms of radicals, then we can express $U(r\tau)$ in terms of radicals for a positive rational number $r$.

We state our main results as follows.

**Theorem 1.1.**

(1) *The field of modular functions on $\Gamma_1(12)$ is generated by $U(\tau)$.*
(2) *The field of modular functions on $\Gamma_0(12)$ is generated by $U(\tau)+1/U(\tau)$.*
(3) *We can write $U(\tau)$ in terms of Weierstrass $\wp$-functions:*

$$U(\tau) = \frac{\wp_{12,(2,0)}(12\tau)-\wp_{12,(5,0)}(12\tau)}{\wp_{12,(1,0)}(12\tau)-\wp_{12,(2,0)}(12\tau)},$$

*where*

$$\wp_{N,(a_1,a_2)}(\tau) = \wp\left(\frac{a_1\tau + a_2}{N}, \mathbb{Z}\tau + \mathbb{Z}\right).$$

(4) *We can express* $U(\tau) + 1/U(\tau)$ *in terms of Dedekind eta functions:*

$$U(\tau) + \frac{1}{U(\tau)} = \frac{\eta(3\tau)^3\eta(4\tau)}{\eta(\tau)\eta(12\tau)^3}.$$

**Theorem 1.2.** *For any positive integer* $n$*, one can find a modular equation* $F_n(X,Y)$ *of* $U(\tau)$ *of level* $n$ *by an explicit construction method in Algorithm* 3.4.

**Theorem 1.3** (Kronecker congruence relation)**.** *Let* $F_n(X,Y)$ *be the modular equation of* $U(\tau)$ *of level* $n$*. Then for any prime* $p \geq 5$ *we have*

$$F_p(X,Y) \equiv \begin{cases} (X^p - Y)(X - Y^p) \pmod p & \text{if } p \equiv \pm 1 \pmod{12}, \\ (X^p - Y)(XY^p - 1) \pmod p & \text{if } p \equiv \pm 5 \pmod{12}. \end{cases}$$

**Theorem 1.4.** *Let* $K$ *be an imaginary quadratic field with discriminant* $d_K$ *and let* $\tau \in K \cap \mathfrak{H}$ *be a root of the primitive equation* $ax^2+bx+c = 0$ *such that* $b^2-4ac = d_K$ *and* $(a,6) = 1$*, where* $a,b,c \in \mathbb{Z}$*. Then* $K(U(\tau))$ *is the ray class field modulo* 12 *over* $K$*.*

**Corollary 1.5.** *Let* $K$ *be an imaginary quadratic field. If* $\mathbb{Z}[\tau]$ *is the integral closure of* $\mathbb{Z}$ *in* $K$*, then* $K(U(\tau))$ *is the ray class field modulo* 12 *over* $K$*.*

**Theorem 1.6.** *Let* $K$ *be an imaginary quadratic field. Then* $U(\tau)$ *is an algebraic unit for every* $\tau \in K - \mathbb{Q}$*.*

**Theorem 1.7.** *If* $U(\tau)$ *is expressed in terms of radicals, then we can express* $U(r\tau)$ *in terms of radicals for a positive rational number* $r$*.*

This paper is organized as follows. In Section 2, we present some basic and necessary notions about modular functions and Klein forms, and we give several lemmas about the cusps of a congruence subgroup, which will be used in Section 3. Then we prove Theorems 1.1, 1.2 and 1.3 and give the properties of modular equations in Section 3. In Section 4, we prove Theorems 1.4, 1.6 and 1.7. We use the MAPLE program to find some examples.

## 2. Preliminaries

We introduce some definitions and properties in the theory of modular functions. Let $\mathfrak{H} = \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$ be the complex upper half plane, $\mathfrak{H}^* := \mathfrak{H} \cup \mathbb{Q} \cup \{\infty\}$ and $\Gamma(1) := \mathrm{SL}_2(\mathbb{Z})$. For any positive integer $N$, we have congruent subgroups $\Gamma(N), \Gamma_1(N), \Gamma_0(N)$, $\Gamma^1(N)$ and $\Gamma^0(N)$ of $\Gamma(1)$ consisting of matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ congruent modulo $N$ to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & 0 \\ * & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} * & 0 \\ * & * \end{smallmatrix}\right)$, respectively.

A congruence subgroup $\Gamma$ acts on $\mathfrak{H}^*$ by linear fractional transformations as $\gamma(\tau) = (a\tau + b)/(c\tau + d)$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, and its quotient space $\Gamma\backslash\mathfrak{H}^*$ is a compact Riemann surface with an appropriate complex structure. We identify $\gamma$ with its action on $\mathfrak{H}^*$. An element $s$ of $\mathbb{Q} \cup \{\infty\}$ is called a *cusp*, and two cusps $s_1, s_2$ are equivalent under $\Gamma$ if there exists $\gamma \in \Gamma$ such that $\gamma(s_1) = s_2$. The equivalence class of such $s$ is also called a *cusp*. Indeed, there exist at most finitely many inequivalent cusps of $\Gamma$. Let $s$ be any cusp of $\Gamma$, and let $\rho$ be an element of $\mathrm{SL}_2(\mathbb{Z})$ such that $\rho(s) = \infty$. We define the *width* of the cusp $s$ in $\Gamma\backslash\mathfrak{H}^*$ by the smallest positive integer $h$ satisfying $\rho^{-1}\left(\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right)\rho \in \{\pm 1\} \cdot \Gamma$. Then the width of the cusp $s$ depends only on the equivalence class of $s$ under $\Gamma$, and it is independent of the choice of $\rho$.

The modular function $f(\tau)$ on a congruence subgroup $\Gamma$ is a $\mathbb{C}$-valued function $f(\tau)$ defined on $\mathfrak{H}$ satisfying the following three conditions:

(1) $f(\tau)$ is meromorphic on $\mathfrak{H}$.
(2) $f(\tau)$ is invariant under $\Gamma$, i.e., $f \circ \gamma = f$ for all $\gamma \in \Gamma$.
(3) $f(\tau)$ is meromorphic at all cusps of $\Gamma$.

The meaning of the last condition is as follows. For a cusp $s$ for $\Gamma$, let $h$ be the width for $s$ and let $\rho$ be an element of $\mathrm{SL}_2(\mathbb{Z})$ such that $\rho(s) = \infty$. Since

$$(f \circ \rho^{-1})(\tau + h) = \left(f \circ \rho^{-1} \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \rho\right)(\rho^{-1}\tau) = (f \circ \rho^{-1})(\tau),$$

$f \circ \rho^{-1}$ has a Laurent series expansion in $q_h = e^{2\pi i \tau/h}$, namely for some integer $n_0$, $(f \circ \rho^{-1})(\tau) = \sum_{n \geq n_0} a_n q_h^n$ with $a_{n_0} \neq 0$. We call this integer $n_0$ the *order* of $f(\tau)$ at the cusp $s$ and denote $n_0$ by $\mathrm{ord}_s f(\tau)$. When $\mathrm{ord}_s f(\tau)$ is positive (respectively, negative), we say that $f(\tau)$ has a zero (respectively, a pole) at $s$. If a modular function $f(\tau)$ is holomorphic on $\mathfrak{H}$ and $\mathrm{ord}_s f(\tau)$ is nonnegative for all cusps $s$, then we say that $f(\tau)$ is holomorphic on $\mathfrak{H}^*$. Since we may identify a modular function on $\Gamma$ with a meromorphic function on the compact Riemann surface $\Gamma\backslash\mathfrak{H}^*$, any holomorphic modular function on $\Gamma$ is a constant function.

Let $A_0(\Gamma)$ be the field of all modular functions on $\Gamma$, and let $A_0(\Gamma)_\mathbb{Q}$ be the subfield of $A_0(\Gamma)$ which consists of all modular functions $f(\tau)$ whose Fourier coefficients are in $\mathbb{Q}$. We may identify $A_0(\Gamma)$ with the field $\mathbb{C}(\Gamma\backslash\mathfrak{H}^*)$ of all meromorphic functions on the compact Riemann surface $\Gamma\backslash\mathfrak{H}^*$, and if $f(\tau) \in A_0(\Gamma)$ is nonconstant, then the field extension degree $[A_0(\Gamma) : \mathbb{C}(f(\tau))]$ is finite and is equal to the total degree of poles of $f(\tau)$.

To recall the Klein form, consider the *Weierstrass $\sigma$-function* by

$$\sigma(z; L) := z \prod_{\omega \in L - \{0\}} \left(1 - \frac{z}{\omega}\right) e^{\frac{z}{\omega} + \frac{1}{2}(\frac{z}{\omega})^2},$$

where $L$ is any lattice in $\mathbb{C}$ and $z \in \mathbb{C}$. This is holomorphic with only simple zeros at all points $z \in L$. The *Weierstrass $\zeta$-function* is defined by

$$\zeta(z; L) := \frac{\sigma'(z; L)}{\sigma(z; L)} = \frac{1}{z} + \sum_{\omega \in L - \{0\}} \left(\frac{1}{z - \omega} + \frac{1}{\omega} + \frac{z}{\omega^2}\right)$$

by the logarithmic derivative of $\sigma(z; L)$. This is meromorphic with only simple poles at all points $z \in L$. We can see that $\sigma(\lambda z; \lambda L) = \lambda \sigma(z; L)$ and $\zeta(\lambda z; \lambda L) = \lambda^{-1}\zeta(z; L)$ for any $\lambda \in \mathbb{C}^\times$. In fact, $\zeta'(z; L)$ is $-\wp(z; L)$ with *Weierstrass $\wp$-function* defined by

$$\wp(z; L) := \frac{1}{z^2} + \sum_{\omega \in L - \{0\}} \left(\frac{1}{(z - \omega)^2} - \frac{1}{\omega^2}\right).$$

For any $\omega \in L$, $\wp(z + \omega; L) = \wp(z; L)$ and $\frac{d}{dz}[\zeta(z + \omega; L) - \zeta(z; L)] = 0$. In other words, $\zeta(z+\omega; L) - \zeta(z; L)$ depends only on a lattice point $\omega \in L$ and not on $z \in \mathbb{C}$, so we may let $\eta(\omega; L)$ be $\zeta(z + \omega; L) - \zeta(z; L)$ for all $\omega \in L$. When we fix the basis $\omega_1, \omega_2$ of $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, for $z = a_1\omega_1 + a_2\omega_2 \in \mathbb{C}$ with $a_1, a_2 \in \mathbb{R}$, we define the *Weierstrass $\eta$-function* by

$$\eta(z; L) := a_1\eta(\omega_1; L) + a_2\eta(\omega_2; L).$$

Since $\eta(z;L)$ does not depend on the choice of the basis $\{\omega_1,\omega_2\}$ of $L$, it is well-defined. Moreover, $\eta(z;L)$ is $\mathbb{R}$-linear so that $\eta(rz;L)=r\eta(z;L)$ for any $r\in\mathbb{R}$.

We define the *Klein form* by

$$K(z;L)=e^{-\eta(z;L)z/2}\sigma(z;L).$$

For $\mathbf{a}=(a_1,a_2)\in\mathbb{R}^2$ and $\tau\in\mathfrak{H}$, we define

$$K_{\mathbf{a}}(\tau)=K(a_1\tau+a_2;\mathbb{Z}\tau+\mathbb{Z})$$

as the *Klein form* by abuse of terminology.

We observe that $K_{\mathbf{a}}(\tau)$ is homogeneous of degree 1, i.e., $K(\lambda z;\lambda L)=\lambda K(z;L)$ and $K_{\mathbf{a}}(\tau)$ is holomorphic and nonvanishing on $\mathfrak{H}$ for $\mathbf{a}\in\mathbb{R}^2-\mathbb{Z}^2$.

Then the Klein form satisfies the following properties. Let $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})$ and $\mathbf{a}=(a_1,a_2)\in\mathbb{R}^2$.

(K0) $K_{-\mathbf{a}}(\tau)=-K_{\mathbf{a}}(\tau)$.

(K1) $K_{\mathbf{a}}(\gamma\tau)=(c\tau+d)^{-1}K_{\mathbf{a}\gamma}(\tau)$.

(K2) For $\mathbf{b}=(b_1,b_2)\in\mathbb{Z}^2$, we have that

$$K_{\mathbf{a}+\mathbf{b}}(\tau)=\varepsilon(\mathbf{a},\mathbf{b})K_{\mathbf{a}}(\tau),$$

where $\varepsilon(\mathbf{a},\mathbf{b})=(-1)^{b_1b_2+b_1+b_2}e^{\pi i(b_2a_1-b_1a_2)}$.

(K3) For $\mathbf{a}=(r/N,s/N)\in(1/N)\mathbb{Z}^2-\mathbb{Z}^2$ and $\gamma\in\Gamma(N)$ with an integer $N>1$, we obtain that

$$K_{\mathbf{a}}(\gamma\tau)=\varepsilon_{\mathbf{a}}(\gamma)(c\tau+d)^{-1}K_{\mathbf{a}}(\tau),$$

where $\varepsilon_{\mathbf{a}}(\gamma)=-(-1)^{((a-1)r+cs+N)(br+(d-1)s+N)/N^2}e^{\pi i(br^2+(d-a)rs-cs^2)/N^2}$.

(K4) Let $\tau\in\mathfrak{H}$ and $z=a_1\tau+a_2$ with $\mathbf{a}=(a_1,a_2)\in\mathbb{Q}^2-\mathbb{Z}^2$. For $q=e^{2\pi i\tau}$ and $q_z=e^{2\pi iz}=e^{2\pi ia_2}e^{2\pi ia_1\tau}$, we get

$$K_{\mathbf{a}}(\tau)=-\frac{1}{2\pi i}e^{\pi ia_2(a_1-1)}q^{a_1(a_1-1)/2}(1-q_z)\prod_{n=1}^{\infty}\frac{(1-q^nq_z)(1-q^nq_z^{-1})}{(1-q^n)^2},$$

and $\mathrm{ord}_qK_{\mathbf{a}}(\tau)=\langle a_1\rangle(\langle a_1\rangle-1)/2$, where $\langle a_1\rangle$ denotes the rational number such that $0\le\langle a_1\rangle<1$ and $a_1-\langle a_1\rangle\in\mathbb{Z}$.

(K5) Let $f(\tau)=\prod_{\mathbf{a}}K_{\mathbf{a}}^{m(\mathbf{a})}(\tau)$ be a finite product of Klein forms with $m(\mathbf{a})\in\mathbb{Z}$ and $\mathbf{a}=(r/N,s/N)=(1/N)\mathbb{Z}^2-\mathbb{Z}^2$ for an integer $N>1$, and let $k=-\sum_{\mathbf{a}}m(\mathbf{a})$. Then $f(\tau)$ is a modular form of weight $k$ on $\Gamma(N)$ if and only if

$$\begin{cases}\sum_{\mathbf{a}}m(\mathbf{a})r^2\equiv\sum_{\mathbf{a}}m(\mathbf{a})s^2\equiv\sum_{\mathbf{a}}m(\mathbf{a})rs\equiv0\pmod N & \text{if } N \text{ is odd},\\ \sum_{\mathbf{a}}m(\mathbf{a})r^2\equiv\sum_{\mathbf{a}}m(\mathbf{a})s^2\equiv0\pmod{2N}, & \\ \qquad\sum_{\mathbf{a}}m(\mathbf{a})rs\equiv0\pmod N & \text{if } N \text{ is even}.\end{cases}$$

For more details on Klein forms, we refer to [8].

The following lemmas are useful to get our results. Let $N$ and $m$ be positive integers and let $\Gamma=\Gamma_1(N)\cap\Gamma_0(mN)$. If we let $\Gamma\backslash\Gamma(1)/\Gamma(1)_\infty=\{\Gamma\gamma_1\Gamma(1)_\infty,\ldots,\Gamma\gamma_g\Gamma(1)_\infty\}$, then $\{\gamma_1(\infty),\ldots,\gamma_g(\infty)\}$ is a set of all inequivalent cusps of $\Gamma$ such that $\gamma_i(\infty)$ and $\gamma_j(\infty)$ are not equivalent under $\Gamma$ for any $i\neq j$. Let

$$M:=\left\{(\overline{c},\overline{d})\in(\mathbb{Z}/mN\mathbb{Z})^2:(\overline{c},\overline{d})=\overline{1},\ \text{i.e.,}\ (c,d,mN)=1\right\}$$

and let $\Delta$ be a subgroup of $(\mathbb{Z}/mN\mathbb{Z})^\times$ defined as

$$\Delta:=\left\{\overline{\pm(1+Nk)}\in(\mathbb{Z}/mN\mathbb{Z})^\times:k=0,\ldots,m-1\right\}.$$

For $(\overline{c_1}, \overline{d_1})$ and $(\overline{c_2}, \overline{d_2})$, we define an equivalence relation $\sim$ on $M$ by $(\overline{c_1}, \overline{d_1}) \sim (\overline{c_2}, \overline{d_2})$ if there exist $\overline{s} \in \Delta$ and $\overline{n} \in \mathbb{Z}/mN\mathbb{Z}$ such that $\overline{c_2} = \overline{s} \cdot \overline{c_1}$ and $\overline{d_2} = \overline{s} \cdot \overline{d_1} + \overline{n} \cdot \overline{c_1}$. Furthermore, we define a map $\phi : \Gamma \backslash \Gamma(1)/\Gamma(1)_\infty \to M/\sim$ by $\phi(\Gamma \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \Gamma(1)_\infty) = [(\overline{c}, \overline{d})]$. Then we see without difficulty that the map $\phi$ is well-defined and bijective. Throughout the paper, we regard $\pm 1/0$ as $\infty$. Therefore we get the following lemmas.

**Lemma 2.1.** *Suppose that $a, c, a', c' \in \mathbb{Z}$ with $(a, c) = (a', c') = 1$. Then with the notation $\Delta$ as above, $a/c$ and $a'/c'$ are equivalent under $\Gamma_1(N) \cap \Gamma_0(mN)$ if and only if there exist $\overline{s} \in \Delta \subset (\mathbb{Z}/mN\mathbb{Z})^\times$ and $n \in \mathbb{Z}$ such that $\left(\begin{smallmatrix} a' \\ c' \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} \overline{s}^{-1} a + nc \\ \overline{s} c \end{smallmatrix}\right)$ (mod $mN$).*

For a positive divisor $x$ of $mN$, let $\pi_x : (\mathbb{Z}/mN\mathbb{Z})^\times \to (\mathbb{Z}/x\mathbb{Z})^\times$ be the natural homomorphism which is surjective. For a positive divisor $c$ of $mN$, let $\overline{s'_{c,1}}, \ldots, \overline{s'_{c,n_c}} \in (\mathbb{Z}/(mN/c)\mathbb{Z})^\times$ be all the distinct coset representatives of $\pi_{mN/c}(\Delta)$ in $(\mathbb{Z}/(mN/c)\mathbb{Z})^\times$, where

$$n_c = \frac{\phi(mN/c)}{|\pi_{mN/c}(\Delta)|}$$

and $\phi$ is the Euler's $\phi$-function. Then for any $\overline{s'_{c,i}}$ with $i = 1, \ldots, n_c$, we choose $\overline{s_{c,i}} \in (\mathbb{Z}/mN\mathbb{Z})^\times$ such that $\pi_{mN/c}(\overline{s_{c,i}}) = \overline{s'_{c,i}}$. We further let

$$S_c := \left\{ \overline{s_{c,1}}, \ldots, \overline{s_{c,n_c}} \in (\mathbb{Z}/mN\mathbb{Z})^\times \right\}.$$

For a positive divisor $c$ of $mN$, let $\overline{a'_{c,1}}, \ldots, \overline{a'_{c,m_c}} \in (\mathbb{Z}/c\mathbb{Z})^\times$ be all the distinct coset representatives of $\pi_c\left(\Delta \cap \ker\left(\pi_{mN/c}\right)\right)$ in $(\mathbb{Z}/c\mathbb{Z})^\times$, where

$$m_c = \frac{\phi(c)}{|\pi_c(\Delta \cap \ker(\pi_{mN/c}))|} = \frac{\phi(c) \cdot |\pi_{mN/c}(\Delta)|}{\left|\pi_{mN/(c,mN/c)}(\Delta)\right|}.$$

For any $\overline{a'_{c,j}}$ with $j = 1, \ldots, m_c$, we take $\overline{a_{c,j}} \in (\mathbb{Z}/mN\mathbb{Z})^\times$ such that $\pi_c(\overline{a_{c,j}}) = \overline{a'_{c,j}}$. We can choose a representative $a_{c,j}$ of $\overline{a_{c,j}}$ so that $0 < a_{c,1}, \ldots, a_{c,m_c} < mN$, $(a_{c,j}, mN) = 1$ and the set $A_c := \{a_{c,1}, \ldots, a_{c,m_c}\}$.

**Lemma 2.2.** *With the notation as above, let*

$$S := \left\{ (\overline{c} \cdot \overline{s_{c,i}}, \overline{a_{c,j}}) \in (\mathbb{Z}/mN\mathbb{Z})^2 : c > 0, c \mid mN, \overline{s_{c,i}} \in S_c, a_{c,j} \in A_c \right\}.$$

*For given $(\overline{c} \cdot \overline{s_{c,i}}, \overline{a_{c,j}}) \in S$, we can take $x, y \in \mathbb{Z}$ such that $(x, y) = 1$, $\overline{x} = \overline{c} \cdot \overline{s_{c,i}}$ and $\overline{y} = \overline{a_{c,j}}$ because $(c \cdot s_{c,i}, a_{c,j}, mN) = 1$. Then the set of $y/x$ with such $x$ and $y$ is a set of all the inequivalent cusps of $\Gamma_1(N) \cap \Gamma_0(mN)$ and the number of such cusps is*

$$|S| = \sum_{\substack{c>0 \\ c \mid mN}} n_c \cdot m_c = \sum_{\substack{c>0 \\ c \mid mN}} \frac{\phi(c)\phi\left(mN/c\right)}{\left|\pi_{mN/(c,mN/c)}(\Delta)\right|}.$$

The following lemma is for finding the width of cusps.

**Lemma 2.3.** *Let $a/c$ be a cusp of $\Gamma = \Gamma_1(N) \cap \Gamma_0(mN)$ with $a, c \in \mathbb{Z}$ and $(a, c) = 1$. Then the width $h$ of a cusp $a/c$ in $\Gamma \backslash \mathfrak{H}^*$ is given by*

$$h = \begin{cases} \frac{m}{(c^2/4, m)} & \text{if } N = 4, (m, 2) = 1 \text{ and } (c, 4) = 2, \\ \frac{mN}{(c,N) \cdot (m, c^2/(c,N))} & \text{otherwise.} \end{cases}$$

The proofs of Lemmas 2.1, 2.2 and 2.3 are given in [4].

## 3. A continued fraction $U(\tau)$ of order 12

We note that

$$U(\tau) = q\prod_{n=1}^{\infty}\frac{(1-q^{12n-1})(1-q^{12n-11})}{(1-q^{12n-5})(1-q^{12n-7})} = \zeta_{12}^{-1}\prod_{j=0}^{11}\frac{K_{(1/12,j/12)}(\tau)}{K_{(5/12,j/12)}(\tau)}$$

by (K4), where $\zeta_N = e^{2\pi i/N}$.

*Proof of Theorem* 1.1.

(1) By (K5), $U(\tau)$ is a modular function on $\Gamma(12)$. Since $U(\tau+1) = U(\tau)$, $U(\tau)$ is invariant under $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$. Hence $U(\tau)\in A_0(\Gamma_1(12))$ because $\Gamma_1(12) = \langle\Gamma(12), \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\rangle$. Note that the genus of $A_0(\Gamma_1(12))$ is zero. Consider the subfield $\mathbb{C}(U(\tau))$ of $A_0(\Gamma_1(12))$ generated by $U(\tau)$ over $\mathbb{C}$. By Lemmas 2.1, 2.2 and 2.3, all the inequivalent cusps of $\Gamma_1(12)$ are $\infty, \frac{5}{12}, 0, \frac{1}{5}, \frac{1}{2}, \frac{1}{3}, \frac{1}{9}, \frac{1}{4}, \frac{1}{8}$ and $\frac{1}{6}$ with widths $1, 1, 12, 12, 6, 4, 4, 3, 3$ and $2$, respectively. Since $U(\tau)$ has a simple zero only at $\infty$ and a simple pole only at $5/12$, the total degree of poles is 1. Hence $[A_0(\Gamma_1(12)) : \mathbb{C}(U(\tau))] = 1$.

(2) Note that $\Gamma_0(12) = \langle\Gamma_1(12), \left(\begin{smallmatrix}5&12\\12&29\end{smallmatrix}\right)\rangle$. We can choose $\infty, 0, 1/2, 1/3, 1/4$ and $1/6$ as the inequivalnet cusps of $\Gamma_0(12)$. By using that

$$U\circ\begin{pmatrix}5&12\\12&29\end{pmatrix}(\tau) = \frac{1}{U(\tau)},$$

we have

$$U(\tau) + \frac{1}{U(\tau)} \in A_0(\Gamma_0(12)).$$

This function has poles only at cusps because $U(\tau)$ allows zeros and poles only at cusps. Since $U(\infty) = 0$ and the width of $\infty$ is 1, $U(\tau) + 1/U(\tau)$ has only a simple pole at $\infty$. Hence

$$A_0(\Gamma_0(12)) = \mathbb{C}\left(U(\tau) + \frac{1}{U(\tau)}\right).$$

(3) For a modular function, we call $f$ *normalized* if its $q$-series is

$$q^{-1} + 0 + a_1 q + a_2 q^2 + \cdots .$$

By (1) the normalized generator of $A_0(\Gamma_1(12))$ is $1/U(\tau) - 1$ because

$$\frac{1}{U(\tau)} = q^{-1} + 1 + q + q^2 + \cdots .$$

In [7, Theorem 3.7], they found the normalized generator

$$\mathcal{N}(j_{1,12})(\tau) = \frac{-1}{j_{1,12}(\tau) - 1} - 2,$$

where

$$j_{1,12}(\tau) = \frac{\wp_{12,(1,0)}(12\tau) - \wp_{12,(2,0)}(12\tau)}{\wp_{12,(1,0)}(12\tau) - \wp_{12,(5,0)}(12\tau)}.$$

From

$$\frac{1}{U(\tau)} = \mathcal{N}(j_{1,12})(\tau) + 1 = \frac{-1}{j_{1,12}(\tau) - 1} - 1 = \frac{j_{1,12}(\tau)}{1 - j_{1,12}(\tau)},$$

we get

$$U(\tau) = \frac{\wp_{12,(2,0)}(12\tau) - \wp_{12,(5,0)}(12\tau)}{\wp_{12,(1,0)}(12\tau) - \wp_{12,(2,0)}(12\tau)}.$$

(4) By [7, Table 10], one can get the values $\mathcal{N}(j_{1,12})(s)$ and $U(s)$ at cusps $s$:

| cusp $s$ | $\infty$ | $0$ | $1/2$ | $1/3$ | $1/4$ | $1/6$ |
|---|---|---|---|---|---|---|
| $\mathcal{N}(j_{1,12})(s)$ | $\infty$ | $1+\sqrt{3}$ | $-2$ | $-1-i$ | $(-1-\sqrt{3}i)/2$ | $0$ |
| $U(s)$ | $0$ | $2-\sqrt{3}$ | $-1$ | $i$ | $(1+\sqrt{3}i)/2$ | $1$ |

Consider the modular function

$$g(\tau) := \frac{\eta(3\tau)^3\eta(4\tau)}{\eta(\tau)\eta(12\tau)^3}$$

on $\Gamma_0(12)$ by [13, Proposition 1.64].

Moreover, $g(\tau)$ has a simple pole only at $\infty$ and simple zero only at $1/3$. Thus $g(\tau)$ is also a generator of $A_0(\Gamma_0(12)) = \mathbb{C}(U(\tau) + 1/U(\tau))$ and there are constants $a, b, c, d$ such that

$$U(\tau) + \frac{1}{U(\tau)} = \frac{a \cdot g(\tau) + b}{c \cdot g(\tau) + d}.$$

Note that

$$\lim_{\tau\to 0} g(\tau) = \lim_{\tau\to\infty} g| \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (\tau) = \lim_{\tau\to\infty} g\left(-\frac{1}{\tau}\right) = 4$$

because $\eta(-1/\tau) = \sqrt{-i\tau}\eta(\tau)$. By substituting the values $U(\tau)$ and $g(\tau)$ at $\infty, 0$ and $1/3$, we get $a = d$ and $b = c = 0$. Therefore, $U(\tau)+1/U(\tau) = g(\tau)$. □

**Proposition 3.1.** *We have*

$$\mathbb{Q}(U(\tau), U(n\tau)) = A_0(\Gamma_1(12) \cap \Gamma_0(12n))_{\mathbb{Q}}$$

*for a positive integer $n$.*

*Proof.* From $\mathbb{Q}(U(\tau)) = A_0(\Gamma_1(12))_{\mathbb{Q}}$, we see that for any $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$, $U(\alpha\tau) = U(\tau)$ if and only if $\alpha \in \mathbb{Q}^\times \cdot \Gamma_1(12)$. For $\beta = \left(\begin{smallmatrix} n & 0 \\ 0 & 1 \end{smallmatrix}\right)$, note that

$$\Gamma_1(12) \cap \beta^{-1}\Gamma_1(12)\beta = \Gamma_1(12) \cap \Gamma_0(12).$$

Hence we get $U(\tau), U(n\tau) \in A_0(\Gamma_1(12) \cap \Gamma_0(12n))_{\mathbb{Q}}$. It is sufficient to show that $\mathbb{Q}(U(\tau), U(n\tau))$ contains $A_0(\Gamma_1(12) \cap \Gamma_0(12n))_{\mathbb{Q}}$. Taking $M_i \in \Gamma_1(12)$, we write

$$\Gamma_1(12) = \bigcup_i \Gamma \cdot M_i$$

as a disjoint union where $\Gamma := \Gamma_1(12) \cap \Gamma_0(12n)$.

Let $f(\tau) = U(n\tau) = (U \circ \beta)(\tau)$. Assume that for distinct indices $i$ and $j$,

$$f \circ M_i = f \circ M_j.$$

Then $U\circ\beta\circ M_i = U\circ\beta\circ M_j$ and $U\circ\beta M_i M_j^{-1}\beta^{-1} = U$. It means that $\beta M_i M_j^{-1}\beta^{-1} \in \mathbb{Q}^\times \cdot \Gamma_1(12)$; thus $M_i M_j^{-1} \in \beta^{-1}\Gamma_1(12)\beta$. Since $M_i M_j^{-1} \in \Gamma_1(12)$, $M_i M_j^{-1} \in \Gamma_1(12) \cap \beta^{-1}\Gamma_1(12)\beta = \Gamma$. So we get a contradiction. We showed that all functions $f \circ M_i$ are distinct and $\mathbb{C}(U(\tau), U(n\tau)) = A_0(\Gamma_1(12) \cap \Gamma_0(12n))_{\mathbb{Q}}$. □

From the definition of $U(\tau)$, the $q$-expansion of $U(\tau)$ is $U(\tau) = q + O(q^2)$, and so $U(\tau)$ has a simple zero at $\infty$. We also check that $U(\tau)$ has a simple pole at $5/12$ by computing $U|\left(\begin{smallmatrix} 5 & 12 \\ 12 & 29 \end{smallmatrix}\right)(\tau)$. We can figure out the behavior of $U(\tau)$ at each $s \in \mathbb{Q} \cup \{\infty\}$ by checking the equivalence class in $\mathbb{Q} \cup \{\infty\}$.

**Lemma 3.2.** *Let $a, c, a', c' \in \mathbb{Z}$ and $U(\tau)$ as before. Then we obtain the following assertions:*

(1) *$U(\tau)$ has a pole at $a/c \in \mathbb{Q} \cup \{\infty\}$ with $(a, c) = 1$ if and only if $(a, c) = 1, c \equiv 0 \pmod{12}, a \equiv \pm 5 \pmod{12}$.*
(2) *$U(n\tau)$ has a pole at $a'/c' \in \mathbb{Q} \cup \{\infty\}$ with $(a, c) = 1$ if and only if there exist $a, c \in \mathbb{Z}$ such that $a/c = na'/c'$, $(a, c) = 1, c \equiv 0 \pmod{12}, a \equiv \pm 5 \pmod{12}$.*
(3) *$U(\tau)$ has a zero at $a/c \in \mathbb{Q} \cup \{\infty\}$ with $(a, c) = 1$ if and only if $(a, c) = 1, c \equiv 0 \pmod{12}, a \equiv \pm 1 \pmod{12}$.*
(4) *$U(n\tau)$ has a zero at $a/c \in \mathbb{Q} \cup \{\infty\}$ with $(a, c) = 1$ if and only if there exist $a, c \in \mathbb{Z}$ such that $a/c = na'/c'$, $(a, c) = 1, c \equiv 0 \pmod{12}, a \equiv \pm 1 \pmod{12}$.*

*Proof.* (1) By Lemma 2.1, $U(\tau)$ has a simple pole at $a/c$ if and only if

$$\begin{pmatrix} a \\ c \end{pmatrix} \equiv \pm \begin{pmatrix} 5 \\ 0 \end{pmatrix} \pmod{12}$$

because the subgroup in Lemma 2.1 is $\Delta = \{\pm 1\}$. Hence the all $a/c \in \mathbb{Q} \cup \{\infty\}$ are only $a, c \in \mathbb{Z}^2$ such that $(a, c) = 1, c \equiv 0 \pmod{12}$ and $a \equiv \pm 5 \pmod{12}$.

(3) Similarly, $U(\tau)$ has a simple zero at $a/c$ if and only if

$$\begin{pmatrix} a \\ c \end{pmatrix} \equiv \pm \begin{pmatrix} 1 \\ 0 \end{pmatrix} \pmod{12}$$

for $a, c \in \mathbb{Z}$ such that $(a, c) = 1$.

(2) and (4) are proved by (1) and (3), respectively. □

Now we study the modular equation of $U(\tau)$. Ishida and Ishii [6] showed the following lemma by means of the standard theory of algebraic functions; this lemma will be useful for checking which coefficients of modular functions are zero or not.

**Lemma 3.3.** *For any congruence subgroup $\Gamma$, let $f_1(\tau)$ and $f_2(\tau)$ be nonconstants such that $\mathbb{C}(f_1(\tau), f_2(\tau)) = A_0(\Gamma)$ with the total degree $D_j$ of poles of $f_j(\tau)$ for $j = 1, 2$, and let*

$$F(X, Y) = \sum_{\substack{0 \le i \le D_2 \\ 0 \le j \le D_1}} C_{i,j} X^i Y^j \in \mathbb{C}[X, Y]$$

*be such that $F(f_1(\tau), f_2(\tau)) = 0$. Let $S_\Gamma$ be a set of all the inequivalent cusps of $\Gamma$, and let*

$$S_{j,0} = \{s \in S_\Gamma : f_j(\tau) \text{ has zeros at } s\}$$

*and*

$$S_{j,\infty} = \{s \in S_\Gamma : f_j(\tau) \text{ has poles at } s\}$$

*for $j = 1, 2$. Let*

$$a = -\sum_{s \in S_{1,\infty} \cap S_{2,0}} \mathrm{ord}_s f_1(\tau), \ \ b = \sum_{s \in S_{1,0} \cap S_{2,0}} \mathrm{ord}_s f_1(\tau).$$

*Here we assume that* $a$ *(respectively,* $b$*) is* $0$ *if* $S_{1,\infty} \cap S_{2,0}$ *(respectively,* $S_{1,0} \cap S_{2,0}$*) is empty. Then we obtain the following assertions:*

(1) $C_{D_2,a} \neq 0$. *In addition, if* $S_{1,\infty} \subset S_{2,\infty} \cup S_{2,0}$, *then* $C_{D_2,j} = 0$ *for any* $j \neq a$.
(2) $C_{0,b} \neq 0$. *In addition, if* $S_{1,0} \subset S_{2,\infty} \cup S_{2,0}$, *then* $C_{0,j} = 0$ *for any* $j \neq b$.
(3) $C_{i,D_1} = 0$ *for* $0 \leq i < |S_{1,0} \cap S_{2,\infty}|, D_2 - |S_{1,\infty} \cap S_{2,\infty}| < i \leq D_2$.
(4) $C_{i,0} = 0$ *for* $0 \leq i < |S_{1,0} \cap S_{2,0}|, D_2 - |S_{1,\infty} \cap S_{2,0}| < i \leq D_2$.

*If we interchange the roles of* $f_1(\tau)$ *and* $f_2(\tau)$, *then we may have more properties similar to* (1)–(4). *Suppose that there exist* $r \in \mathbb{R}$ *and* $N, n_1, n_2 \in \mathbb{Z}$ *with* $N > 0$ *such that*

$$f_j(\tau + r) = \zeta_N^{n_j} f_j(\tau)$$

*for* $j = 1, 2$, *where* $\zeta_N = e^{2\pi i/N}$. *Then we get the following assertion:*

(5) *If* $n_1 i + n_2 j \not\equiv n_1 D_2 + n_2 a \pmod N$, *then* $C_{i,j} = 0$. *Here note that* $n_2 b \equiv n_1 D_2 + n_2 a \pmod N$.

*Proof.* See [6, Lemmas 3 and 6]. □

We provide an explicit construction method for finding the modular equations $U(\tau)$ in the following algorithm.

**Algorithm 3.4** (Finding the modular equation of $U(\tau)$)**.**
**Input:** *The functions* $U(\tau)$ *and* $U(n\tau)$ *with a positive integer* $n$.
**Output:** *The explicit modular equation* $F_n(X, Y)$ *of* $U(\tau)$ *of level* $n$.
**Steps:**

(a) *Let* $f_1(\tau) := U(\tau), f_2(\tau) := U(n\tau)$ *and* $S_{\Gamma_1(12)\cap\Gamma_0(12n)}$ *be the set of inequivalent cusps on* $\Gamma_1(12) \cap \Gamma_0(12n)$.
(b) *Find the subsets* $S_{1,0}, S_{1,\infty}, S_{2,0}$ *and* $S_{2,\infty}$ *of* $S_{\Gamma_1(12)\cap\Gamma_0(12n)}$ *using Lemmas* 2.1–2.3.
(c) *Calculate the total degrees* $d_1$ *and* $d_2$ *of poles of* $f_1(\tau)$ *and* $f_2(\tau)$, *respectively, i.e.,*
$$d_j = -\sum_{s \in S_{j,\infty}} \mathrm{ord}_s f_j(\tau)$$
*for* $j = 1, 2$.
(d) *We set*
$$F_n(X, Y) = \sum_{\substack{0 \leq i \leq d_2 \\ 0 \leq j \leq d_1}} C_{i,j} X^i Y^j$$
*with* $C_{i,j} \in \mathbb{Q}$ *because all coefficients of* $f_1(\tau)$ *and* $f_2(\tau)$ *are rational;* $C_{i,j}$ *will be determined in Step* (e).
(e) *Let*
$$a = \begin{cases} 0 & \text{if } S_{1,\infty} \cap S_{2,0} = \phi, \\ -\sum_{s \in S_{1,\infty} \cap S_{2,0}} \mathrm{ord}_s f_1(\tau) & \text{otherwise,} \end{cases}$$
*and let* $C_{d_2,a} = 1$. *Then by substituting* $q$*-expansions of* $U(\tau)$ *and* $U(n\tau)$ *to* $F_n(X, Y)$, *we get* $C_{i,j}$ *explicitly.*

*Proof of Theorem* 1.2. For nonconstants $f_1(\tau)$ and $f_2(\tau)$, assume $\mathbb{C}(f_1(\tau), f_2(\tau))$ is the field of all modular functions on some congruence subgroup. Then

$$[\mathbb{C}(f_1(\tau), f_2(\tau)) : \mathbb{C}(f_j(\tau))] = d_j,$$

where $d_j$ is the total degree of poles of $f_j(\tau)$ for $j = 1, 2$. So we can take a polynomial $\Phi(X,Y) \in \mathbb{C}[X,Y]$ such that $\Phi(f_1(\tau), Y)$ and $\Phi(X, f_2(\tau))$ are minimal polynomials of $f_2(\tau)$ and $f_1(\tau)$ over $\mathbb{C}(f_1(\tau))$ and $\mathbb{C}(f_2(\tau))$ with degrees $d_1$ and $d_2$, respectively. It means that

$$\Phi(X,Y) = \sum_{\substack{0\le i\le d_2 \\ 0\le j\le d_1}} C_{i,j} X^i Y^j \tag{3.1}$$

satisfies $\Phi(f_1(\tau), f_2(\tau)) = 0$ for some $C_{i,j} \in \mathbb{C}$.

For a positive integer $n$, let $f_1(\tau)$ and $f_2(\tau)$ be $U(\tau)$ and $U(n\tau)$, respectively. By Proposition 3.1, $\mathbb{C}(f_1(\tau), f_2(\tau))$ is the field of modular functions on $\Gamma_1(12)\cap\Gamma_0(12n)$. Since $f_j(\tau)$ has zeros and poles only at cusps, the sets $S_{j,0}$ and $S_{j,\infty}$ are easily obtained by (K1), (K4) and Lemma 2.2, where $S_{\Gamma_1(12)\cap\Gamma_0(12n)}$ is a set of equivalence classes under $\sim$:

$$s_1, s_2 \in \mathbb{Q}\cup\{\infty\},\ s_1 \sim s_2 \Leftrightarrow \gamma s_1 = s_2 \text{ for some } \gamma \in \Gamma_1(12)\cap\Gamma_0(12n),$$

$$S_{j,0} = \left\{s \in S_{\Gamma_1(12)\cap\Gamma_0(12n)} : f_j(\tau) \text{ has zeros at } s\right\},$$

and

$$S_{j,\infty} = \left\{s \in S_{\Gamma_1(12)\cap\Gamma_0(12n)} : f_j(\tau) \text{ has poles at } s\right\}$$

for $j = 1, 2$. Moreover, we can get the total degree $d_j = -\sum_{s\in S_{j,\infty}} \mathrm{ord}_s f_j(\tau)$ of poles of $f_j(\tau)$ for $j = 1, 2$. Hence the equation $\Phi(X,Y)$ in (3.1) is the modular equation $F_n(X,Y)$. By using Lemma 3.3, we can choose $C_{d_2,a} = 1$ without loss of generality, where

$$a = \begin{cases} 0 & \text{if } S_{1,\infty}\cap S_{2,0} = \phi, \\ -\sum_{s\in S_{1,\infty}\cap S_{2,0}} \mathrm{ord}_s f_1(\tau) & \text{otherwise.} \end{cases}$$

Hence we get the explicit form of modular equation $F_n(X,Y)$ by substituting $q$-expansions of $U(\tau)$ and $U(n\tau)$. $\square$

*Remark* 3.5. In the case that $n$ is a prime $p \ge 5$, then Step (e) can be much more simplified by using Theorem 3.7.

We apply Algorithm 3.4 to get the modular equations of levels 2 and 3.

**Theorem 3.6.** *The modular equations of $U(\tau)$ of levels 2 and 3 are the following:*

(1) ***(Modular equation of level** 2)*

$$\left(U^2(\tau) - U(2\tau)\right)(1 - U(2\tau)) + 2U(\tau)U(2\tau) = 0.$$

(2) ***(Modular equation of level** 3)*

$$\left(U^3(\tau) - U(3\tau)\right)\left(1 - U(3\tau) + U^2(3\tau)\right) + 3U(\tau)U(3\tau)(1 - U(\tau)U(3\tau)) = 0.$$

*Proof.* Following Algorithm 3.4, we present the explicit computation as below, and we use the same notation for the steps.

(1) (a) Let $f_1(\tau) = U(\tau),\ f_2(\tau) = U(2\tau)$.
(b)

$$S_{1,0} = \left\{\infty, \frac{1}{12}\right\}, S_{1,\infty} = \left\{\frac{5}{12}, \frac{5}{24}\right\}, S_{2,0} = \{\infty\},\ \text{and } S_{2,\infty} = \left\{\frac{5}{24}\right\}.$$

(c) Since

$$\mathrm{ord}_\infty f_1(\tau) = \mathrm{ord}_{1/12} f_1(\tau) = 1,$$
$$\mathrm{ord}_{5/12} f_1(\tau) = \mathrm{ord}_{5/24} f_1(\tau) = -1,$$
$$\mathrm{ord}_\infty f_2(\tau) = 2 \text{ and } \mathrm{ord}_{5/24} f_2(\tau) = -2,$$

we have

$$d_1 = d_2 = 2.$$

(d) Hence we get

$$F_2(X,Y) = \sum_{0\le i,j\le 2} C_{i,j} X^i Y^j.$$

(e) Since $S_{1,\infty} \cap S_{2,0} = \phi$, we let $C_{2,0} = 1$. From $U(\tau) = q + q^2 + q^6 - q^7 + q^8 - q^9 + O(q^{10})$, we determine that

$$F_2(X,Y) = X^2 - X^2Y - Y + Y^2 + 2XY$$

and

$$\left(U^2(\tau) - U(2\tau)\right)(1 - U(2\tau)) + 2U(\tau)U(2\tau) = 0.$$

(2) (a) Let $f_1(\tau) = U(\tau), f_2(\tau) = U(3\tau)$.

(b) We have the subsets

$$S_{1,0} = \left\{\infty, \frac{1}{12}, \frac{11}{12}\right\}, S_{1,\infty} = \left\{\frac{5}{12}, \frac{7}{12}, \frac{5}{36}\right\}, S_{2,0} = \{\infty\}, \text{ and } S_{2,\infty} = \left\{\frac{5}{36}\right\}.$$

(c) From

$$\mathrm{ord}_\infty f_1(\tau) = \mathrm{ord}_{1/12} f_1(\tau) = \mathrm{ord}_{11/12} f_1(\tau) = 1,$$
$$\mathrm{ord}_{5/12} f_1(\tau) = \mathrm{ord}_{7/12} f_1(\tau) = \mathrm{ord}_{5/36} f_1(\tau) = -1,$$
$$\mathrm{ord}_\infty f_2(\tau) = 3 \text{ and } \mathrm{ord}_{5/36} f_2(\tau) = -3,$$

we get

$$d_1 = d_2 = 3.$$

(d) Hence we may assume that

$$F_3(X,Y) = \sum_{0\le i,j\le 3} C_{i,j} X^i Y^j.$$

(e) We may assume that $C_{3,0} = 1$ because $S_{1,\infty} \cap S_{2,0} = \phi$. Then by substituting $q$-expansions of $U(\tau)$ and $U(3\tau)$ to $F_3(X,Y)$ we get

$$F_3(X,Y) = X^3 - X^3Y + X^3Y^2 - Y + Y^2 - Y^3 + 3XY - 3X^2Y^2;$$

equivalently,

$$\left(U^3(\tau) - U(3\tau)\right)\left(1 - U(3\tau) + U^2(3\tau)\right) + 3U(\tau)U(3\tau)\left(1 - U(\tau)U(3\tau)\right) = 0.$$

$\square$

**Theorem 3.7.** *With the notations as above, let $p$ be a prime $\ge 5$. Then $F_p(X,Y) = \sum_{0\le i,j\le p+1} C_{i,j}X^iY^j \in \mathbb{Q}[X,Y]$ satisfies the following conditions:*

(1) *if $p \equiv \pm 1 \pmod{12}$, then*

$$C_{p+1,0} \ne 0 \text{ and } C_{p+1,1} = C_{p+1,2} = \cdots = C_{p+1,p+1} = 0,$$

*and*

$$C_{0,p+1} \ne 0 \text{ and } C_{0,0} = C_{0,1} = \cdots = C_{0,p} = 0;$$

(2) *if* $p \equiv \pm 5 \pmod{12}$, *then*

$$C_{p+1,p} \neq 0 \text{ and } C_{p+1,0} = \cdots = C_{p+1,p-1} = C_{p+1,p+1} = 0,$$

*and*

$$C_{0,1} \neq 0 \text{ and } C_{0,0} = C_{0,2} = \cdots = C_{0,p+1} = 0.$$

*Proof.* The congruence subgroup which we should consider is $\Gamma = \Gamma_1(12) \cap \Gamma_0(12p)$ and hence

$$\Delta = \left\{ \overline{\pm(1+12k)} \in (\mathbb{Z}/12p\mathbb{Z})^\times : k = 0, \ldots, p-1 \right\},$$

where $\Delta$ is the subgroup as in Section 2. Choose a unique $x \in \{0, \ldots, p-1\}$ such that $12x \equiv -1 \pmod{p}$. Among $k = 0, \ldots, p-1$, this value $x$ is the only one of them which does not satisfy the condition $\overline{\pm(1+12k)} \in (\mathbb{Z}/12p\mathbb{Z})^\times$. By Lemmas 2.2 and 3.2 we have to consider $S_{12}, A_{12}, S_{12p}, A_{12p}$. We observe that

$$\left\{(1+12k) \in (\mathbb{Z}/p\mathbb{Z})^\times : k = 0, \ldots, p-1 \text{ such that } 1+12k \not\equiv 0 \pmod{p}\right\}$$

is equal to the whole set $(\mathbb{Z}/p\mathbb{Z})^\times$. Thus all the inequivalent cusps under consideration are $1/12, 5/12, 1/12p$ and $5/12p$ ( respectively, $1/12, 7/12, 1/60$ and $7/60$) if $p \neq 5$ (respectively, $p = 5$). Although we consider only the case $p \neq 5$, for convenience, all the statements below still hold by replacing with appropriate cusps. Hence we concentrate on the cusps $1/12, 5/12, 1/12p$ and $5/12p$ at which the widths are $p, p, 1$ and $1$, respectively, by Lemma 2.3. Note that $1/12p$ is equivalent to $\infty$ by Lemma 2.1. If we let $f_1(\tau) = U(\tau)$ and $f_2(\tau) = U(p\tau)$ in Lemma 3.3, then by Lemma 3.2 we know that

$$S_{1,\infty} = \left\{ \frac{5}{12}, \frac{5}{12p} \right\}, S_{1,0} = \left\{ \frac{1}{12}, \frac{1}{12p} \right\}.$$

Further we obtain that

$$S_{2,\infty} = \left\{ \frac{5}{12}, \frac{5}{12p} \right\}, S_{2,0} = \left\{ \frac{1}{12}, \frac{1}{12p} \right\} \text{ if } p \equiv \pm 1 \pmod{12},$$

$$S_{2,\infty} = \left\{ \frac{1}{12}, \frac{5}{12p} \right\}, S_{2,0} = \left\{ \frac{5}{12}, \frac{1}{12p} \right\} \text{ if } p \equiv \pm 5 \pmod{12}.$$

Note that

$$\mathrm{ord}_{5/12} f_1(\tau) = -p \text{ and } \mathrm{ord}_{5/12p} f_1(\tau) = -1.$$

Consider the orders of $f_2(\tau)$ at $1/12, 5/12$ and $5/12p$. For $c \in \{1, \ldots, 11\}$ such that $cp \equiv 1 \pmod{12}$:

$$\begin{aligned}
f_2 | \begin{pmatrix} 1 & 0 \\ 12 & 1 \end{pmatrix} (\tau) &= f_2\left(\frac{\tau}{12\tau+1}\right) = U\left(\frac{p\tau}{12\tau+1}\right) \\
&= U \circ \begin{pmatrix} p & (cp-1)/12 \\ 12 & c \end{pmatrix} \begin{pmatrix} 1 & (1-cp)/12 \\ 0 & p \end{pmatrix} (\tau) \\
&= U \circ \begin{pmatrix} p & (cp-1)/12 \\ 12 & c \end{pmatrix} \left(\frac{\tau + (1-cp)/12}{p}\right) \\
&= \begin{cases} q^{1/p} + \cdots & \text{if } p \equiv \pm 1 \pmod{12}, \\ q^{-1/p} + \cdots & \text{if } p \equiv \pm 5 \pmod{12}. \end{cases}
\end{aligned}$$

Similarly, for $c' = \{1, \ldots, 11\}$ such that $cp \equiv 5 \pmod{12}$,

$$\begin{aligned} f_2| \begin{pmatrix} 5 & 12 \\ 12 & 29 \end{pmatrix} (\tau) &= f_2 \left( \frac{5\tau + 12}{12\tau + 29} \right) = U \left( \frac{5p\tau + 12p}{12\tau + 29} \right) \\ &= U \circ \begin{pmatrix} 5p & (5cp-1)/12 \\ 12 & c \end{pmatrix} \begin{pmatrix} 1 & (29-cp)/12 \\ 0 & p \end{pmatrix} (\tau) \\ &= U \circ \begin{pmatrix} 5p & (5cp-1)/12 \\ 12 & c \end{pmatrix} \left( \frac{\tau + (29-cp)/12}{p} \right) \\ &= \begin{cases} q^{-1/p} + \cdots & \text{if } p \equiv \pm 1 \pmod{12}, \\ q^{1/p} + \cdots & \text{if } p \equiv \pm 5 \pmod{12}. \end{cases} \end{aligned}$$

Hence it turns out that $\mathrm{ord}_x f_2(\tau) = -1$ for $x = 5/12$ (respectively, $1/12$ ) if $p \equiv \pm 1 \pmod{12}$ (respectively, $\pm 5 \pmod{12}$). At $5/12p$, take $b$ and $d \in \mathbb{Z}$ such that $5d - 12bp = 1$. As

$$f_2| \begin{pmatrix} 5 & b \\ 12p & d \end{pmatrix} (\tau) = U \circ \begin{pmatrix} 5 & bp \\ 12 & d \end{pmatrix} (p\tau) = q^{-p} + \cdots ,$$

we get $\mathrm{ord}_{5/12p} f_2(\tau) = -p$. So the total degrees of poles of $f_1(\tau)$ and $f_2(\tau)$ are both $p+1$ and we may let the modular equation $F_p(X, Y)$ be $\sum_{0 \le i,j \le p+1} C_{i,j} X^i Y^j$.

Moreover, by using $S_{1,0} \cup S_{1,\infty} = S_{2,0} \cup S_{2,\infty}$ and

$$S_{1,\infty} \cap S_{2,0} = \begin{cases} \phi & \text{if } p \equiv \pm 1 \pmod{12}, \\ \{5/12\} & \text{if } p \equiv \pm 5 \pmod{12}, \end{cases}$$

we get

$$\begin{cases} C_{p+1,0} \neq 0, C_{p+1,1} = \cdots = C_{p+1,p} = C_{p+1,p+1} = 0 & \text{if } p \equiv \pm 1 \pmod{12}, \\ C_{p+1,p} \neq 0, C_{p+1,0} = \cdots = C_{p+1,p-1} = C_{p+1,p+1} = 0 & \text{if } p \equiv \pm 5 \pmod{12}. \end{cases}$$

On the other hand,

$$S_{1,0} \cap S_{2,0} = \begin{cases} \{1/12, 1/12p\} & \text{if } p \equiv \pm 1 \pmod{12}, \\ \{1/12p\} & \text{if } p \equiv \pm 5 \pmod{12} \end{cases}$$

and

$$\begin{cases} \mathrm{ord}_{1/12} f_1(\tau) + \mathrm{ord}_{1/12p} f_1(\tau) = p+1 & \text{if } p \equiv \pm 1 \pmod{12}, \\ \mathrm{ord}_{1/12p} f_1(\tau) = 1 & \text{if } p \equiv \pm 5 \pmod{12}. \end{cases}$$

Hence we get

$$\begin{cases} C_{0,p+1} \neq 0, C_{0,0} = C_{0,1} = \cdots = C_{0,p} = 0 & \text{if } p \equiv \pm 1 \pmod{12}, \\ C_{0,1} \neq 0, C_{0,0} = C_{0,2} = \cdots = C_{0,p+1} = 0 & \text{if } p \equiv \pm 5 \pmod{12}. \end{cases}$$

□

We can completely determine all the coefficients $C_{i,j}$ of the modular equation $F_p(X, Y)$, which is presented in Table 1; this is done by using Theorem 3.7 and substituting the Fourier expansions of $U(\tau)$ and $U(p\tau)$ into $F_p(X, Y) = 0$ with $X = U(\tau)$ and $Y = U(p\tau)$. Table 1 includes the results of Dharmendra, Rajesh, Kanna, and Jagadeesh [3] and Mahadeva Naika, Dharmendra, and Shivashankar [10]. It is obvious that one may apply our method to find higher order modular equations $F_p(X, Y)$ for p $\geq 17$.

TABLE 1. The modular equations $F_p(X,Y)$ of $U(\tau)$ of levels $2,3,5,7,11$ and $13$

| $p$ | the modular equations $F_p(X,Y)$ of $U(\tau)$ |
|---|---|
| 2 | $X^2+Y^2-X^2Y-Y+2XY$ |
| 3 | $X^3-X^3Y+X^3Y^3-Y+Y^2-Y^3+3XY-3X^2Y^2$ |
| 5 | $(X^5-Y)(XY^5-1)$<br>$+5(XY^5-2X^3Y^4+2X^2Y^3+X^2Y-XY^4-XY+2X^4Y^3-X^5Y^2-2X^3Y^3$<br>$+X^4Y^5-X^5Y^5+X^5)$ |
| 7 | $(X^7-Y)(XY^7-1)$<br>$+7(2X^2Y+XY^5-XY^3+XY^7-2XY^6+XY^2+4X^6Y^5+4X^2Y^3$<br>$-4X^2Y^2+X^7Y-X^2Y^5+2X^6Y^7-XY+5X^4Y^5-2X^7Y^2-X^2Y^7$<br>$+5X^4Y^3-4X^3Y^6+X^3Y^7+X^7Y^3-4X^5Y^2-4X^6Y^6-X^5Y^5$<br>$+4X^6Y^2+X^5Y^3+X^5Y-X^5Y^7-X^6Y^3-X^6Y+X^5Y^6-5X^5Y^4$<br>$+X^3Y^2-X^3Y+X^7Y^6-X^7Y^7-X^3Y^3-X^7Y^5+X^3Y^5-5X^3Y^4+4X^2Y^6)$ |
| 11 | $(X^{11}-Y)(X-Y^{11})$<br>$+11(X^2Y-10X^2Y^2+69X^7Y^3-33X^7Y^2-5XY^5+33X^{10}Y^7-4XY^3+69X^9Y^5$<br>$-48X^{10}Y^8+73X^9Y^3-X^{10}Y+7XY^4+5X^{11}Y^5-88X^8Y^8-X^{11}Y^2-X^{11}Y$<br>$+133X^8Y^4+7X^8Y^{11}-99X^8Y^3-33X^{10}Y^5+4X^{11}Y^3+84X^6Y^6+48X^4Y^{10}+69X^5Y^9$<br>$-136X^5Y^5+64X^5Y^7-69X^5Y^3+92X^5Y^4-5X^5Y+33X^5Y^2+133X^4Y^8-99X^4Y^9$<br>$+92X^4Y^5-92X^4Y^7+4X^3Y^{11}+7X^4Y+99X^3Y^4-69X^3Y^5-X^2Y^{11}-136X^7Y^7$<br>$-34X^2Y^9+16X^2Y^{10}-7X^8Y+48X^2Y^8+33X^2Y^5-5X^7Y^{11}+34X^2Y^3-48X^2Y^4$<br>$+4X^9Y+34X^{10}Y^9+4XY^9+5XY^7-7XY^8-4X^9Y^{11}-34X^{10}Y^3+48X^{10}Y^4+92X^8Y^7$<br>$-5X^{11}Y^7+7X^{11}Y^8-XY^{11}-XY^{10}-92X^7Y^4-48X^4Y^2-88X^4Y^4+99X^4Y^3-99X^3Y^8$<br>$-34X^3Y^{10}+73X^3Y^9+X^{10}Y^{11}+99X^9Y^8-34X^9Y^2+34X^3Y^2-93X^3Y^3+92X^7Y^8$<br>$+99X^8Y^9+34X^9Y^{10}-7X^{11}Y^4-92X^5Y^8-48X^8Y^{10}-10X^{10}Y^{10}+XY^2+5X^7Y$<br>$-99X^9Y^4+5X^5Y^{11}+33X^7Y^{10}+48X^8Y^2-33X^2Y^7-7X^4Y^{11}-33X^5Y^{10}-93X^9Y^9$<br>$+16X^{10}Y^2-4X^{11}Y^9-69X^9Y^7-92X^8Y^5+69X^3Y^7+X^{11}Y^{10}+64X^7Y^5$<br>$-4X^3Y-69X^7Y^9)$ |
| 13 | $(X^{13}-Y)(X-Y^{13})$<br>$+13(339X^{11}Y^{10}-90X^8Y^{12}-487X^9Y^{11}-10X^2Y^7+XY^2+70X^7Y^4+6X^{13}Y^7$<br>$+493X^9Y^3-993X^9Y^9-90X^{12}Y^8-6X^9Y^7-90X^6Y^2+652X^9Y^8-794X^9Y^4$<br>$-6X^5Y^7+101X^2Y^{10}-302X^3Y^8+6XY^6-396X^8Y^8+486X^{10}Y^6-409X^4Y^{11}$<br>$-106X^4Y^2+838X^5Y^9-64X^{11}Y^7+70X^7Y^{10}+116X^2Y^8-10X^7Y^2-147X^2Y^9$<br>$-642X^6Y^9-14XY^5+362X^8Y^{11}-X^3Y^{13}-409X^3Y^{10}-2XY^{13}-16XY^8$<br>$-106X^2Y^4-556X^4Y^6-37X^{12}Y^3-37X^2Y^{11}+486X^4Y^8-396X^6Y^6+671X^6Y^8$<br>$-10X^7Y^{12}+6XY^7-64X^3Y^7+70X^4Y^7-10X^{12}Y^7+116X^6Y^{12}-60X^7Y^6+15X^9Y$<br>$-X^{11}Y+116X^8Y^2+204X^{11}Y^3-168X^{11}Y^{11}-642X^8Y^5+4X^{12}Y-37X^3Y^{12}-64X^7Y^{11}$<br>$-5X^3Y+6X^{13}Y^8-64X^7Y^3+47X^2Y^3-556X^6Y^4+362X^6Y^3-147X^5Y^{12}+493X^5Y^{11}$<br>$-6X^7Y^5+6X^7Y-5X^{13}Y^{11}-7X^4Y^{13}+101X^4Y^{12}-794X^4Y^9+789X^5Y^4-7XY^{10}$<br>$-106X^{10}Y^{12}-794X^5Y^{10}+4X^{13}Y^2-3X^{12}Y^2-587X^4Y^4+789X^4Y^5-5X^{11}Y^{13}$<br>$+204X^3Y^{11}+339X^4Y^3+47X^3Y^2+362X^3Y^6+493X^3Y^9+739X^4Y^{10}-302X^6Y^{11}$<br>$-3X^2Y^{12}+142X^2Y^5-302X^8Y^3-106X^{12}Y^{10}-11X^2Y^2+X^2Y+671X^8Y^6+15XY^9$<br>$-90X^2Y^6+838X^9Y^5-642X^9Y^6+6X^8Y^{13}-60X^8Y^7+652X^5Y^6-60X^6Y^7-16X^8Y$<br>$-168X^3Y^3+739X^{10}Y^4+12X^4Y-642X^5Y^8-14X^{13}Y^9-16X^{13}Y^6+15X^{13}Y^5-7X^{13}Y^4$<br>$+47X^{12}Y^{11}-11X^{12}Y^{12}+142X^{12}Y^9+101X^{12}Y^4-487X^{11}Y^9+362X^{11}Y^8+493X^{11}Y^5$<br>$-409X^{11}Y^4-37X^{11}Y^2+12X^{10}Y^{13}+339X^{10}Y^{11}-587X^{10}Y^{10}+789X^{10}Y^9-556X^{10}Y^8$<br>$+70X^{10}Y^7+15X^5Y^{13}+101X^{10}Y^2-409X^{10}Y^3-7X^{10}Y+142X^9Y^{12}+652X^6Y^5+6X^6Y$<br>$-993X^5Y^5-487X^5Y^3+116X^{12}Y^6-147X^9Y^2-147X^{12}Y^5-X^{13}Y^3-487X^3Y^5$<br>$+789X^9Y^{10}-2X^{13}Y+12X^{13}Y^{10}+X^{12}Y^{13}+12XY^4-6X^7Y^9+X^{13}Y^{12}-60X^7Y^8$<br>$+339X^3Y^4-14X^5Y+142X^5Y^2+6X^7Y^{13}-136X^7Y^7-794X^{10}Y^5-XY^{11}$<br>$-556X^8Y^{10}-14X^9Y^{13}+652X^8Y^9-302X^{11}Y^6+47X^{11}Y^{12}-16X^6Y^{13}-5XY^3$<br>$+486X^6Y^{10}+486X^8Y^4+4X^2Y^{13}+4XY^{12})$ |

From now on, we let $n$ be a positive integer with $(n,6)=1$. We will find the Kronecker congruence relations for the modular equations of $U(\tau)$ and $U(n\tau)$. If $\sigma_a\in\mathrm{SL}_2(\mathbb{Z})$ satisfies $\sigma_a\equiv\left(\begin{smallmatrix}a^{-1}&0\\0&a\end{smallmatrix}\right)\pmod{12}$ for any integer $a$ with $(a,6)=1$, then we have

$$\Gamma_1(12)\begin{pmatrix}1&0\\0&n\end{pmatrix}\Gamma_1(12)=\bigcup_{\substack{a>0\\a\mid n}}\ \bigcup_{\substack{0\le b<n/a\\(a,b,\frac{n}{a})=1}}\Gamma_1(12)\sigma_a\begin{pmatrix}a&b\\0&n/a\end{pmatrix},$$

in which the right-hand side is a disjoint union [14, Proposition 3.36]. Note that

$$d := \left|\Gamma_1(12)\backslash\Gamma_1(12)\begin{pmatrix}1 & 0\\ 0 & n\end{pmatrix}\Gamma_1(12)\right| = n\prod_{p|n}\left(1+\frac{1}{p}\right).$$

Since $\sigma_a$ depends only on $a$ modulo 12, we choose $\sigma_a$ as

$$\sigma_{\pm 1} = \pm\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\ \sigma_{\pm 5} = \pm\begin{pmatrix}5 & 12\\ 12 & 29\end{pmatrix}.$$

From the transformation formulas (K0)–(K2), we get

$$U\circ\sigma_{\pm 1} = U \quad\text{and } U\circ\sigma_{\pm 5} = \frac{1}{U}.$$

For convenience, let $\alpha_{a,b} = \sigma_a\left(\begin{smallmatrix}a & b\\ 0 & n/a\end{smallmatrix}\right)$ for such $a, b$ with $0 < a \mid n, 0 \leq b < n/a$ and $(a, b, n/a) = 1$. We now consider the following polynomial $\Psi_n(X,\tau)$ with the indeterminate $X$:

$$\Psi_n(X,\tau) = \prod_{\substack{a>0\\ a|n}}\prod_{\substack{0\leq b<n/a\\ (a,b,\frac{n}{a})=1}}\left[X - (U\circ\alpha_{a,b})(\tau)\right].$$

Since all the coefficients of $\Psi_n(X,\tau)$ are the elementary symmetric functions of $U\circ\alpha_{a,b}$, they are invariant under $\Gamma_1(12)$, i.e., $\Psi_n(X,\tau)\in\mathbb{C}(U(\tau))[X]$, and we may write $\Psi_n(X,U(\tau))$ instead of $\Psi_n(X,\tau)$.

**Theorem 3.8.** *With the notations as above, for a positive integer $n > 1$ with $(n,6) = 1$ we define*

$$F_n(X,U(\tau)) = U(\tau)^{r_n}\Psi_n(X,U(\tau)),$$

*that is, $F_n(X,Y) = Y^{r_n}\Psi_n(X,Y)$ with the nonnegative integer*

$$r_n = -\sum_{s\in S_{1,\infty}\cap S_{2,0}}\mathrm{ord}_s U(\tau).$$

*Here we assume that $r_n = 0$ if $S_{1,\infty}\cap S_{2,0}$ is empty. Then we obtain the following assertions:*

(1) *$F_n(X,Y)\in\mathbb{Z}[X,Y]$ and $\deg_X F_n(X,Y) = n\prod_{p|n}(1+1/p)$.*
(2) *$F_n(X,Y)$ is irreducible both as a polynomial in $X$ over $\mathbb{C}(Y)$ and as a polynomial in $Y$ over $\mathbb{C}(X)$.*
(3) *Let $d = n\prod_{p|n}(1+1/p)$. Then*

$$\begin{cases} F_n(X,Y) = F_n(Y,X) & \text{if } n\equiv\pm 1 \pmod{12},\\ F_n(X,Y) = (-1)^{r_n}Y^d F_n(1/Y,X) & \text{if } n\equiv\pm 5 \pmod{12}.\end{cases}$$

*Moreover, if $p$ is a prime number congruent to $\pm 5 \pmod{12}$, then $r_p = p$ and*

$$F_p(X,Y) = -Y^{p+1}F_p(1/Y,X).$$

(4) *If $n$ is not a square, then $F_n(X,X)$ is a polynomial of degree $> 1$ whose leading coefficient is $\pm 1$.*

*Proof.* Since $U(\tau) = q + O(q^2)$, we may let $U(\tau) = \sum_{m=1}^{\infty} c_m q^m$ with $c_m\in\mathbb{Z}$. We further let $\psi_k\in\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ be such that $\psi_k(\zeta_n) = \zeta_n^k$ for some integer $k$ with $(k,n) = 1$. Then $\psi_k$ induces an automorphism of $\mathbb{Q}(\zeta_n)((q^{1/n}))$ through the action on the coefficients. We denote the induced automorphism by the same notation $\psi_k$. Since

$$U \circ \begin{pmatrix} a & b \\ 0 & n/a \end{pmatrix} (\tau) = U\left(\frac{a^2}{n}\tau + \frac{ab}{n}\right) = \sum_{m=1}^{\infty} c_m \zeta_n^{abm} \left(q^{1/n}\right)^{a^2 m},$$

we obtain that

$$\psi_k\left(U \circ \begin{pmatrix} a & b \\ 0 & n/a \end{pmatrix} (\tau)\right) = \sum_{m=1}^{\infty} c_m \zeta_n^{abkm} \left(q^{1/n}\right)^{a^2 m}.$$

Let $b'$ be the unique integer such that $0 \le b' < n/a$ and $b' \equiv bk \pmod{n/a}$. Then

$$\psi_k\left(U \circ \begin{pmatrix} a & b \\ 0 & n/a \end{pmatrix} (\tau)\right) = U \circ \begin{pmatrix} a & b' \\ 0 & n/a \end{pmatrix} (\tau)$$

because $\zeta_n^{abk} = \zeta_n^{ab'}$. Since $U \circ \sigma_a = U$ or $1/U$, we have $\psi_k(U \circ \alpha_{a,b}) = U \circ \alpha_{a,b'}$, and so all the coefficients of $\Psi_n(X, U(\tau))$ are contained in $\mathbb{Q}((q^{1/n}))$. Hence by observing the fact $\Psi_n(X, U(\tau)) \in \mathbb{C}(U(\tau))[X]$ we see that $\Psi_n(X, U(\tau)) \in \mathbb{Q}(U(\tau))[X]$.

For each $\alpha_{a,b}$, we have $\Gamma_1(12) \cdot \alpha_{a,b} \subset \Gamma_1(12) \left(\begin{smallmatrix} 1 & 0 \\ 0 & n \end{smallmatrix}\right) \Gamma_1(12)$, and there exist $\gamma, \gamma'$ and $\gamma_{a,b} \in \Gamma_1(12)$ such that

$$\gamma \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} \gamma_{a,b} = \gamma' \alpha_{a,b},$$

i.e., $\left(\begin{smallmatrix} 1 & 0 \\ 0 & n \end{smallmatrix}\right) \gamma_{a,b} \alpha_{a,b}^{-1} \in \Gamma_1(12)$. We consider an embedding $\xi_{a,b}$ of $\mathbb{C}\left(U(\tau/n), U(\tau)\right)$ over $\mathbb{C}(U(\tau))$ defined by

$$\xi_{a,b}(h) = h \circ \gamma_{a,b}.$$

In fact, $\xi_{a,b}(U) = U \circ \gamma_{a,b} = U$ and

$$\xi_{a,b}\left(U\left(\frac{\tau}{n}\right)\right) = \xi_{a,b}\left(U \circ \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}\right)(\tau) = U \circ \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} \gamma_{a,b}(\tau) = U \circ \alpha_{a,b}(\tau).$$

When $\alpha_{a,b} \neq \alpha_{a',b'}$, $U \circ \alpha_{a,b} \neq U \circ \alpha_{a',b'}$. This means that

$$[\mathbb{C}(U(\tau/n), U(\tau)) : \mathbb{C}(U(\tau))] = d.$$

So, $\Psi_n(X, U(\tau))$ is irreducible over $\mathbb{C}(U(\tau))$.

With the notation as in Lemma 3.3, we let $f_1(\tau) = U(\tau)$ and $f_2(\tau) = U(n\tau)$ for $(n, 6) = 1$. Assume that $a/c \in S_{1,0} \cup S_{1,\infty}$. In other words, $a \equiv \pm 1$ or $\pm 5 \pmod{12}$ and $c \equiv 0 \pmod{12}$. Since $(n, 6) = 1$,

$$\frac{n}{m} \equiv \pm 1, \pm 5 \pmod{12}$$

for $m = (c, n)$. Note that

$$\frac{na}{c} = \frac{(n/m)a}{c/m} = \frac{a'}{c'}$$

such that $a' \equiv \pm 1$ or $\pm 5 \pmod{12}$ and $c' \equiv 0 \pmod{12}$. Hence $f_2(a/c) = U(na/c) = U(a'/c') = 0$ or $\infty$ and $S_{1,0} \cup S_{1,\infty} \subset S_{2,0} \cup S_{2,\infty}$. Similarly, one can prove the reverse inclusion and we have

$$S_{1,\infty} \cup S_{1,0} = S_{2,\infty} \cup S_{2,0}.$$

If we let

$$r = r_n = -\sum_{s\in S_{1,\infty}\cap S_{2,0}} \mathrm{ord}_s U(\tau), \qquad s = \sum_{s\in S_{1,0}\cap S_{2,0}} \mathrm{ord}_s U(\tau),$$

$$r' = -\sum_{s\in S_{2,\infty}\cap S_{1,0}} \mathrm{ord}_s U(n\tau), \qquad s' = \sum_{s\in S_{2,0}\cap S_{1,0}} \mathrm{ord}_s U(n\tau),$$

then $F(X,Y)$ in Lemma 3.3 is written as the form

$$F(X,Y) = C_{d_n,r}X^{d_n}Y^r + C_{r',d_1}X^{r'}Y^{d_1} + C_{s',0}X^{s'} + C_{0,s}Y^s + \sum_{\substack{0<i<d_n\\0<j<d_1}} C_{i,j}X^iY^j,$$

where $d_1$ (respectively, $d_n$) is the total degree of poles of $U(\tau)$ (respectively, $U(n\tau)$) and $C_{d_n,r}, C_{r',d_1}, C_{s',0}, C_{0,s}$ are nonzero. Since $F(X,U(\tau))$ is an irreducible polynomial of $U(\tau/n)$ over $\mathbb{C}(U(\tau))$ and $F(U(\tau/n),Y)$ is also an irreducible polynomial of $U(\tau)$ over $\mathbb{C}(U(\tau/n))$, we know that

$$C_{d_n,r}\cdot U(\tau)^r\Psi_n(X,U(\tau)) = F(X,U(\tau))$$

and $F_n(X,Y) = Y^r\Psi_n(X,Y)$ is a polynomial in $X$ and $Y$ which is irreducible both as a polynomial in $Y$ over $\mathbb{C}(X)$ and a polynomial in $X$ over $\mathbb{C}(Y)$. Since $U(\tau)^r\Psi_n(X,U(\tau)) \in \mathbb{Q}[X,U(\tau)]$ and all the Fourier coefficients of the coefficients of the $\Psi_n(X,U(\tau))$ are algebraic integers, it turns out that $U(\tau)^r\Psi(X,U(\tau)) \in \mathbb{Z}[X,U(\tau)]$, i.e., $F_n(X,Y)\in\mathbb{Z}[X,Y]$. Hence (1) and (2) follow.

(3) We first consider the case $n\equiv \pm 1 \pmod{12}$. By (2), $F_n(X,U(\tau))$ is an irreducible polynomial in $X$ over $\mathbb{C}(U(\tau))$ with root $U(\tau/n)$. Since $\Psi_n(U(n\tau),U(\tau)) = 0$, i.e., $\Psi_n(U(\tau),U(\tau/n)) = 0$, $U(\tau/n)$ is a root of the polynomial $F_n(U(\tau),X) \in \mathbb{Z}[X,U(\tau)]$. So we derive that

$$F_n(U(\tau),X) = g(X,U(\tau))F_n(X,U(\tau))$$

for some polynomial $g(X,U(\tau)) \in \mathbb{Z}[X,U(\tau)]$ by the Gauss lemma on the irreducibility of polynomials. Since

$$F_n(X,U(\tau)) = g(U(\tau),X)F_n(U(\tau),X),$$
$$F_n(U(\tau),X) = g(X,U(\tau))g(U(\tau),X)F_n(U(\tau),X)$$

implies $g(X,U(\tau)) = \pm 1$. If $g(X,U(\tau)) = -1$, then

$$F_n(U(\tau),U(\tau)) = -F_n(U(\tau),U(\tau));$$

hence $U(\tau)$ is a root of $F_n(X,U(\tau))$, which is a contradiction to the irreducibility of $F_n(X,U(\tau))$ over $\mathbb{C}(U(\tau))$. Therefore we have

$$F_n(X,U(\tau)) = F_n(U(\tau),X).$$

Next, we consider the case $n\equiv \pm 5 \pmod{12}$. $U(\tau/n)$ is a root of the polynomial $U(\tau)^dF_n(1/U(\tau),X) \in \mathbb{Z}[X,U(\tau)]$ because we see that

$$X - \frac{1}{U(n\tau)} = X - U\circ\sigma_n\begin{pmatrix} n & 0\\ 0 & 1\end{pmatrix}(\tau) = X - (U\circ\alpha_{n,0})(\tau)$$

is one of the factors of

$$\Psi_n(X,U(\tau)), \quad \Psi_n\left(1/U(n\tau),U(\tau)\right) = 0$$

and

$$U(\tau)^dF_n(1/U(\tau),U(\tau/n)) = \Psi_n\left(1/U(\tau),U(\tau/n)\right) = 0.$$

In a similar way as the case $n \equiv \pm 1 \pmod{12}$, we have

$$U(\tau)^d F_n\left(\frac{1}{U(\tau)}, X\right) = g(X, U(\tau)) F_n(X, U(\tau)) \tag{3.2}$$

for some $g(X, U(\tau)) \in \mathbb{Z}[X, U(\tau)]$ with the Gauss lemma because $F_n(X, U(\tau))$ is an irreducible polynomial in $X$ with root $U(\tau/n)$ by (2). We write (3.2) as

$$Y^d F_n\left(\frac{1}{Y}, X\right) = g(X, Y) F_n(X, Y).$$

We note that

$$\deg_X F_n(X, Y) + \deg_X g(X, Y) = \deg_X Y^d F_n\left(\frac{1}{Y}, X\right) = \deg_Y F_n(X, Y)$$

and

$$\deg_Y F_n(X, Y) + \deg_Y g(X, Y) = \deg_Y Y^d F_n\left(\frac{1}{Y}, X\right) = d = \deg_X F_n(X, Y)$$

because

$$\begin{aligned} Y^d F_n\left(\frac{1}{Y}, X\right) &= \frac{1}{C_{d_n, r}}\Big(C_{d_n, r} X^{r_n} Y^{d-d_n} + C_{0,s} X^s Y^d \\ &\quad + C_{r', d_1} X^{d_1} Y^{d-r'} + C_{s',0} Y^{d-s'} + (\text{lower degree terms})\Big) \end{aligned}$$

with nonzero coefficient $C_{0,s}$. So $g(X, Y)$ is a constant and

$$\deg_X F_n(X, Y) = \deg_Y F_n(X, Y) = d.$$

Since $F_n(X, Y)$ is a primitive polynomial, we have $g := g(X, Y) = \pm 1$. By using that $F_n(Y^{-1}, X) = g \cdot Y^{-d} F_n(X, Y)$,

$$\begin{aligned} F_n(X, Y) &= g \cdot X^d F_n\left(Y, \frac{1}{X}\right) \\ &= g \cdot X^d \left(Y^d X^{-r} + (\text{other terms})\right) \\ &= g \cdot X^{d-r} Y^d + (\text{other terms}) \end{aligned}$$

and the coefficient of $X^{d-r} Y^d$ in $F_n(X, Y)$ is $g$.

On the other hand, since $\Psi_n(X, U(\tau))$ is equal to

$$\begin{aligned} &\prod_{\substack{a>0, a \mid n \\ a \equiv \pm 1 \pmod{12}}} \prod_{\substack{0 \le b < \frac{n}{a} \\ \left(a, b, \frac{n}{a}\right) = 1}} (X - \zeta_n^{ab} q^{a^2/n} + \cdots) \\ &\times \prod_{\substack{a>0, a \mid n \\ a \equiv \pm 5 \pmod{12}}} \prod_{\substack{0 \le b < \frac{n}{a} \\ \left(a, b, \frac{n}{a}\right) = 1}} (X - \zeta_n^{-ab} q^{-a^2/n} + \cdots), \end{aligned}$$

we see that the coefficient of $X^{d-r} Y^d$ in $F_n(X, Y)$ is equal to

$$\prod_{\substack{a>0, a \mid n \\ a \equiv \pm 5 \pmod{12}}} \prod_{\substack{0 \le b < \frac{n}{a} \\ \left(a, b, \frac{n}{a}\right) = 1}} (-\zeta_n^{-ab}) = \epsilon \prod_{\substack{a>0, a \mid n \\ a \equiv \pm 5 \pmod{12}}} \prod_{\substack{0 \le b < \frac{n}{a} \\ \left(a, b, \frac{n}{a}\right) = 1}} \zeta_n^{-ab}, \tag{3.3}$$

where

$$\epsilon = \prod_{\substack{a>0, a \mid n \\ a \equiv \pm 5 \pmod{12}}} \prod_{\substack{0 \le b < \frac{n}{a} \\ \left(a, b, \frac{n}{a}\right) = 1}} (-1).$$

In fact, we note $\epsilon = (-1)^{r_n}$ and the other factor of the right-hand side of (3.3) is $\prod\prod \zeta_n^{-ab} = 1$ by the elementary lemma in [1, Lemma 6.7]; if $m > 0$ is an odd integer with $k \mid m$, then $\prod_{0\le b<m,(b,k)=1} \zeta_m^{-b} = 1$. Therefore $g = (-1)^{r_n}$.

Now, assume that $p$ is a prime with $p \equiv \pm 5 \pmod{12}$. Clearly, $d = p\prod(1+1/p) = p+1$. In the proof of Theorem 3.7, $S_{1,\infty} \cap S_{2,0} = \{5/12\}$ and $r_p = -\mathrm{ord}_{5/12} U(\tau) = p$. Hence $F_p(X,Y) = -Y^{p+1}F_p(1/Y, X)$.

(4) Assume that $n$ is not a square. Since

$$U(\tau) - (U \circ \alpha_{a,b})(\tau) = \begin{cases} q - \zeta_n^{ab} q^{a^2/n} + \cdots & \text{if } a \equiv \pm 1 \pmod{12}, \\ -\zeta_n^{-ab} q^{-a^2/n} + q + \cdots & \text{if } a \equiv \pm 5 \pmod{12}, \end{cases}$$

the coefficient of its lowest degree term is

$$\begin{cases} 1 & \text{if } a^2 > n \text{ and } a \equiv \pm 1 \pmod{12}, \\ -\zeta_n^{-ab} & \text{otherwise.} \end{cases}$$

Therefore the coefficient of the lowest degree term in $F_n(U(\tau), U(\tau))$ is a product of $-\zeta_n^{-ab}$ for $a, b$ where $a$ is a positive divisor of $n$ such that $a \equiv \pm 5 \pmod{12}$ or $a \equiv \pm 1 \pmod{12}$ with $a^2 < n$ and $b$ is a nonnegative integer with $0 \le b < n/a$ and $(a, b, n/a) = 1$. By (1), $F_n(X, X)$ has the integral leading coefficient, which should be $\pm 1$. □

*Proof of Theorem* 1.3. Let $p$ be an odd prime. For any $g(\tau)$ and $h(\tau) \in \mathbb{Z}[\zeta_p]((q^{\frac{1}{p}}))$ and $\alpha \in \mathbb{Z}[\zeta_p]$, we write

$$g(\tau) \equiv h(\tau) \pmod{\alpha}$$

if $g(\tau) - h(\tau) \in \alpha\mathbb{Z}[\zeta_p]((q^{\frac{1}{p}}))$.

Since $U(\tau) = q + \sum_{m=2}^{\infty} c_m q^m$ with $c_m \in \mathbb{Z}$, we have that

$$\begin{aligned} (U \circ \alpha_{1,b})(\tau) &= \zeta_p^b q^{\frac{1}{p}} + \sum_{m=2}^{\infty} c_m \zeta_p^{bm} (q^{\frac{1}{p}})^m \\ &\equiv q^{\frac{1}{p}} + \sum_{m=2}^{\infty} c_m (q^{\frac{1}{p}})^m \equiv (U \circ \alpha_{1,0})(\tau) \pmod{1-\zeta_p}, \end{aligned}$$

for any $b = 0, \ldots, p-1$. Note that

$$\begin{aligned} (U \circ \alpha_{1,0})(\tau)^p &= \left( \zeta_p q^{\frac{1}{p}} + \sum_{m=2}^{\infty} c_m \zeta_p^m q^{\frac{m}{p}} \right)^p \\ &\equiv q + \sum_{m=2}^{\infty} c_m^p q^m \pmod{1-\zeta_p} \\ &\equiv q + \sum_{m=2}^{\infty} c_m q^m \pmod{1-\zeta_p} \\ &= U(\tau). \end{aligned}$$

Suppose that $p \equiv \pm 1 \pmod{12}$. By observing that

$$(U \circ \alpha_{p,0})(\tau) = (U \circ \sigma_p)(p\tau) = U(p\tau) = q^p + \sum_{m=2}^{\infty} c_m q^{pm}$$

and $c_m^p \equiv c_m \pmod{p}$, we see that

$$(U \circ \alpha_{p,0})(\tau) \equiv U(\tau)^p \pmod{p}$$

and

$$(U \circ \alpha_{p,0})(\tau) \equiv U(\tau)^p \pmod{1-\zeta_p}.$$

Since $r_p = -\sum_{s\in S_{1,\infty}\cap S_{2,0}} \mathrm{ord}_s U(\tau) = 0$, we have

$$\begin{aligned} F_p(X, U(\tau)) &= \Psi_p(X, U(\tau)) = (X - (U\circ\alpha_{p,0})(\tau)) \prod_{0\le b<p} (X - (U \circ \alpha_{1,b})(\tau)) \\ &\equiv (X - U(\tau)^p)(X - (U\circ\alpha_{1,0})(\tau))^p \\ &\equiv (X - U(\tau)^p)(X^p - (U\circ\alpha_{1,0})(\tau)^p) \\ &\equiv (X - U(\tau)^p)(X^p - U(\tau)) \pmod{1-\zeta_p}. \end{aligned}$$

Let $F_p(X, U(\tau)) - (X - U(\tau)^p)(X^p - U(\tau)) = \sum_\nu \psi_\nu(U(\tau))X^\nu$, where $\psi_\nu(U(\tau)) \in \mathbb{Z}[U(\tau)]$. Since all the Fourier coefficients of $\psi_\nu(U(\tau))$ are rational integers and divisible by $1-\zeta_p$ in $\mathbb{Z}[\zeta_p]$, we see that $\psi_\nu(U(\tau)) \in p\mathbb{Z}[U(\tau)]$. Hence we have

$$F_p(X, U(\tau)) \equiv (X^p - U(\tau))(X - U(\tau)^p) \pmod{p\mathbb{Z}[X, U(\tau)]}$$

when $p \equiv \pm 1 \pmod{12}$ as desired.

Now assume that $p \equiv \pm 5 \pmod{12}$. Since $(U\circ\alpha_{p,0})(\tau) = (U\circ\sigma_p)(p\tau) = 1/U(p\tau)$ and $U(p\tau) \equiv U(\tau)^p \pmod p$, we get that $(U\circ\alpha_{p,0})(\tau) \equiv 1/U(\tau)^p \pmod p$. In other words,

$$(U\circ\alpha_{p,0})(\tau) \equiv \frac{1}{U(\tau)^p} \pmod{1-\zeta_p}.$$

Note that $r_p = -\sum_{s\in S_{1,\infty}\cap S_{2,0}} \mathrm{ord}_s U(\tau) = p$. So we get

$$\begin{aligned} F_p(X, U(\tau)) &= U(\tau)^p \Psi_p(X, U(\tau)) \\ &= U(\tau)^p (X - (U\circ\alpha_{p,0})(\tau)) \prod_{0\le b<p} (X - (U\circ\alpha_{1,b})(\tau)) \\ &\equiv U(\tau)^p (X - 1/U(\tau)^p)(X - (U\circ\alpha_{1,0})(\tau))^p \\ &\equiv (XU(\tau)^p - 1)(X^p - (U\circ\alpha_{1,0})(\tau)^p) \\ &\equiv (XU(\tau)^p - 1)(X^p - U(\tau)) \pmod{1-\zeta_p}. \end{aligned}$$

With the same argument as in the case $p \equiv \pm 1 \pmod{12}$, we get that

$$F_p(X, U(\tau)) \equiv (X^p - U(\tau))(XU(\tau)^p - 1) \mod p\mathbb{Z}[X, U(\tau)].$$

□

## 4. Ray class fields and evaluation of $U(\tau)$

In this section, we focus on finding the value $U(\tau)$ and the extension field generated by the value $U(\tau)$.

**Lemma 4.1.** *Let $K$ be an imaginary quadratic field with discriminant $d_K$ and $\tau \in K \cap \mathfrak{H}$ be a root of the primitive equation $ax^2 + bx + c = 0$ in $\mathbb{Z}[x]$ such that $b^2 - 4ac = d_K$, and let $\Gamma'$ be any congruence subgroup such that $\Gamma(N) \subset \Gamma' \subset \Gamma_1(N)$. Suppose that $(N, a) = 1$. Then the field generated over $K$ by all the values $h(\tau)$, where $h \in A_0(\Gamma')_{\mathbb{Q}}$ is defined and finite at $\tau$, is the ray class field modulo $N$ over $K$.*

*Proof.* See [2, Corollary 5.2]. □

The previous lemma gives us the ray class field generated by $U(\tau)$ and the proof of Theorem 1.4.

*Proof of Theorem* 1.4. If $\Gamma'$ be the congruence subgroup such that $\mathbb{Q}(U(\tau)) = A_0(\Gamma')_{\mathbb{Q}}$, then $\Gamma(12) \subset \Gamma' \subset \Gamma_1(12)$ because $\Gamma' = \Gamma_1(12)$ by Theorem 1.1(1). For an imaginary quadratic field $K$ with discriminant $d_K$, consider $\tau \in K \cap \mathfrak{H}$ satisfying $a\tau^2 + b\tau + c = 0$ such that $b^2 - 4ac = d_K$, $(a, 6) = 1$ and $a, b, c \in \mathbb{Z}$. Since $U$ is defined and finite at this $\tau$, $K(U(\tau))$ is the ray class field modulo 12 over $K$ by Lemma 4.1. □

*Proof of Corollary* 1.5. Assume that $\mathbb{Z}[\tau]$ is the ring of integers in $K$. If $a\tau^2 + b\tau + c = 0$ with $a, b, c \in \mathbb{Z}$ and $(a, b, c) = 1$, then $a$ should be 1. Hence $K(U(\tau))$ is the ray class field modulo 12 over $K$. □

By definition, a *modular unit* $h(\tau)$ over $\mathbb{Z}$ is a modular function of some level $N$ which is rational over $\mathbb{Q}(\zeta_N)$ such that $h(\tau)$ and $1/h(\tau)$ are integral over $\mathbb{Z}[j(\tau)]$, where $j(\tau)$ is the classical elliptic modular function.

**Lemma 4.2.** *Let $h(\tau)$ be a modular function of some level $N$ rational over $\mathbb{Z}(\zeta_N)$ for which $h(\tau)$ has neither zeros nor poles on $\mathfrak{H}$. If for every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ the Fourier expansion of $h \circ \gamma$ has algebraic integer coefficients and the coefficient of the term of lowest degree is a unit, then $h(\tau)$ is a modular unit over $\mathbb{Z}$.*

*Proof.* See [8, Chapter 2, Lemma 2.1]. □

Let $h(\tau)$ be a modular unit over $\mathbb{Z}$ and $K$ be an imaginary quadratic field. Since it is well known that $j(\tau)$ is an algebraic integer for every $\tau \in K - \mathbb{Q}$, we can derive that for such $\tau$, $h(\tau)$ is an algebraic integer which is a unit. By observing this fact, we derive the property of $U(\tau)$.

*Proof of Theorem* 1.6. It is enough to prove that $U(\tau)$ is a modular unit over $\mathbb{Z}$. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Then $U(\tau)$ is written in the product of Klein forms as

$$U(\tau) = \zeta_{12}^{-1} \prod_{j=0}^{11} \frac{K_{\left(\frac{1}{12}, \frac{j}{12}\right)}(\tau)}{K_{\left(\frac{5}{12}, \frac{j}{12}\right)}(\tau)}.$$

By (K1) in Section 2, the action $\gamma$ on $U(\tau)$ is

$$U(\gamma\tau) = \zeta_{12}^{-1} \prod_{j=0}^{11} \frac{K_{\left(\frac{a+cj}{12}, \frac{b+dj}{12}\right)}(\tau)}{K_{\left(\frac{5a+cj}{12}, \frac{5b+dj}{12}\right)}(\tau)}.$$

If we replace the Klein forms by the $q$-products in (K4) and expand the products as a series, then the series is the Fourier expansion of $U(\gamma\tau)$. Since we want to use Lemma 4.2 to prove that $U(\gamma\tau)$ has Fourier coefficients which are algebraic integers and the coefficient of the lowest degree term is a unit, we may assume that

$$0 \le (a + cj)/12, (5a + cj)/12 \le 1$$

by (K2). If we assume these, then the only term we should consider in (K4) is

$$1 - q_z = 1 - \zeta_{12}^{b+dj} q^{\frac{a+cj}{12}} \text{ or } 1 - \zeta_{12}^{5b+dj} q^{\frac{5a+cj}{12}}.$$

Put $c' = (c, 6)$. First, assume that $c' \ne 1$. Then $a$ is relatively prime to $c'$ and $a + cj \equiv a \not\equiv 0$, $5a + cj \equiv 5a \not\equiv 0 \pmod{c'}$; thus, the exponents $(a + cj)/12$ and $(5a + cj)/12$ of $q$ are not integers and $1 - q_z$ cannot be complex numbers, namely it has algebraic integer coefficients with the lowest coefficient 1, and the series expansion of $U(\gamma\tau)$ has the desired properties.

Now assume that $c' = 1$. There exist unique integers $j_1, j_2 \in \{0, \dots, 11\}$ such that

$$a + c \cdot j_1 \equiv 0 \pmod{12} \text{ and } 5a + c \cdot j_2 \equiv 0 \pmod{12}. \tag{4.1}$$

Hence, the coefficient of the lowest degree term of $U(\gamma\tau)$ is

$$(1 - \zeta_{12}^{b+d\cdot j_1})/(1 - \zeta_{12}^{5b+d\cdot j_2})$$

up to a unit. Since

$$\begin{aligned}\left(\frac{1}{12}, \frac{j_1}{12}\right) &= \left(\frac{a + c\cdot j_1}{12}, \frac{b + d\cdot j_1}{12}\right)\begin{pmatrix} d & -b \\ -c & a\end{pmatrix} \\ &= \left(\frac{(a + c\cdot j_1)d - (b + d\cdot j_1)c}{12}, *\right)\end{aligned}$$

and

$$\begin{aligned}\left(\frac{5}{12}, \frac{j_2}{12}\right) &= \left(\frac{5a + c\cdot j_2}{12}, \frac{5b + d\cdot j_2}{12}\right)\begin{pmatrix} d & -b \\ -c & a\end{pmatrix} \\ &= \left(\frac{(5a + c\cdot j_2)d - (5b + d\cdot j_2)c}{12}, *\right),\end{aligned}$$

we know that

$$1 = (a + c\cdot j_1)d - (b + d\cdot j_1)c \equiv -(b + d\cdot j_1)c \pmod{12}$$

and

$$5 = (5a + c\cdot j_2)d - (b + d\cdot j_2)c \equiv -(5b + d\cdot j_2)c \pmod{12}$$

by (4.1). Hence neither $\zeta_{12}^{b+d\cdot j_1}$ nor $\zeta_{12}^{5b+d\cdot j_2}$ is 1.

Let $c_0$ be an integer such that $-c_0 \cdot c \equiv 1 \pmod{12}$. Then

$$\begin{aligned}\frac{1 - \zeta_{12}^{b+d\cdot j_1}}{1 - \zeta_{12}^{5b+d\cdot j_2}} &= \frac{1 - (\zeta_{12}^{-c_0\cdot c})^{b+d\cdot j_1}}{1 - (\zeta_{12}^{-c_0\cdot c})^{5b+d\cdot j_2}} = \frac{1 - \zeta_{12}^{c_0}}{1 - \zeta_{12}^{5c_0}} = \frac{1 - \zeta_{12}^{25c_0}}{1 - \zeta_{12}^{5c_0}} \\ &= 1 + \zeta_{12}^{5c_0} + \zeta_{12}^{10c_0} + \zeta_{12}^{15c_0} + \zeta_{12}^{20c_0} \in \mathbb{Z}[\zeta_{12}], \\ \frac{1 - \zeta_{12}^{5b+d\cdot j_2}}{1 - \zeta_{12}^{b+d\cdot j_1}} &= \frac{1 - \zeta_{12}^{5c_0}}{1 - \zeta_{12}^{c_0}} = 1 + \zeta_{12}^{c_0} + \zeta_{12}^{2c_0} + \zeta_{12}^{3c_0} + \zeta_{12}^{4c_0} \in \mathbb{Z}[\zeta_{12}].\end{aligned}$$

This means that $(1 - \zeta_{12}^{b+d\cdot j_1})/(1 - \zeta_{12}^{5b+d\cdot j_2})$ is a unit. $\square$

**Corollary 4.3.** *Let $g(\tau) = \eta(\tau)^{-1}\eta(3\tau)^3\eta(4\tau)\eta(12\tau)^{-3}$ and let $K$ be an imaginary quadratic field. Then $g(\tau)$ is an algebraic integer for every $\tau \in K - \mathbb{Q}$.*

*Proof.* For any $\tau \in K - \mathbb{Q}$, $U(\tau)$ and $1/U(\tau)$ are algebraic integers by Theorem 1.6. By Theorem 1.1 (4), $g(\tau) = U(\tau) + 1/U(\tau)$, so $g(\tau)$ is also an algebraic integer. $\square$

*Proof of Theorem* 1.7. Suppose that the value $U(\tau)$ is expressed in terms of radicals. To write $U(r\tau)$ in terms of radicals as well, we need to factorize $r$.

Assume that $r = a/b$ with $a, b \in \mathbb{Z}_{>0}$ and $(a, b) = 1$. Find all solutions $s_1, \dots, s_t$ of the equation $F_a(U(\tau), x)$, where $F_a(X, Y)$ is the modular equation obtained by Theorem 1.2. Then write them in terms of radicals. For a sufficiently large $N$, compare the obtained solutions $s_1, \dots, s_t$ with

$$v := q^a \prod_{n=1}^{N} \frac{(1 - q^{a(12n-1)})(1 - q^{a(12n-11)})}{(1 - q^{a(12n-5)})(1 - q^{a(12n-7)})}$$

and $q = e^{2\pi i\tau}$. We choose $s_j$ to be the value closest to $v$, then we take $s_j$ to be the explicit value of $U(a\tau)$.

In a similar way, we get the value $U(r\tau)$ in terms of radicals by using the modular equation $F_b(X,Y)$ and an approximation

$$w := q^{a/b} \prod_{n=1}^{N} \frac{(1-q^{a(12n-1)/b})(1-q^{a(12n-11)/b})}{(1-q^{a(12n-5)/b})(1-q^{a(12n-7)/b})}$$

for a sufficiently large $N$. $\square$

In [10, Theorem 5.1] one can find 12 values $U(\tau)$ if $\tau = ix/2$ with $x = 1, \sqrt{3}, \sqrt{5}, \sqrt{7}, \sqrt{13}, \sqrt{17}, 1/\sqrt{3}, 1/\sqrt{5}, 1/\sqrt{7}, \sqrt{5/3}, \sqrt{7/3}$ and $\sqrt{11/3}$. We find more values in the following.

**Example 4.4.**

(1)

$$U(i) = \frac{\sqrt{2}\left(\sqrt{2}-\sqrt{3t_1+\sqrt[4]{27}\sqrt{t_1}}\right)}{\left(\sqrt[8]{27}\sqrt[4]{t_1}+1\right)^2} \quad \text{for } t_1 = 2-\sqrt{3},$$

(2)

$$U(\sqrt{3}i) = \frac{\sqrt{2t_3+4}\left(\sqrt{2}-\sqrt{3t_3}\sqrt{\sqrt[3]{2}+1/\sqrt{t_3(t_3+2)}}\right)}{\left(\sqrt[4]{9t_3}+\sqrt[4]{t_3+2}\right)^2} \quad \text{for } t_3 = \sqrt[3]{2}-1,$$

(3)

$$U(\sqrt{5}i) = \frac{2-\sqrt{3}\sqrt{(1-t_5)(1+3t_5)}}{(\sqrt{3t_5}+1)^2} \quad \text{for } t_5 = \sqrt{14\sqrt{5}+8\sqrt{15}-18\sqrt{3}-31},$$

(4)

$$U(\sqrt{7}i) = \frac{4\sqrt{24+t_7}-2\sqrt{3}\sqrt{t_7+4\sqrt{2}\sqrt{24+t_7}}}{(\sqrt[4]{288}+\sqrt[4]{96+4t_7})^2}$$

for $t_7 = 4\sqrt{21} + \sqrt{\sqrt{21}-3}(3\sqrt{14}+5\sqrt{6})$.

*Solution.* The values of $U(ix/2)$ are as follows [10, Theorem 5.1]:

(1) $U(i/2) = \frac{\sqrt[4]{6\sqrt{3}-9}-1}{\sqrt[4]{6\sqrt{3}-9}+1}$,

(2) $U(\sqrt{3}i/2) = \frac{\sqrt[4]{-108+108\sqrt[3]{2}}-\sqrt[4]{12+12\sqrt[3]{2}}}{\sqrt[4]{-108+108\sqrt[3]{2}}+\sqrt[4]{12+12\sqrt[3]{2}}}$,

(3) $U(\sqrt{5}i/2) = \frac{\sqrt[4]{126\sqrt{5}+72\sqrt{15}-162\sqrt{3}-279}-1}{\sqrt[4]{126\sqrt{5}+72\sqrt{15}-162\sqrt{3}-279}+1}$,

(4) $U(\sqrt{7}i/2) = \frac{\sqrt{12\sqrt{2}}-\sqrt[4]{32+\sqrt{5+\sqrt{21}}(\sqrt{5+\sqrt{21}}+\sqrt{\sqrt{21}-3})^3}}{\sqrt{12\sqrt{2}}+\sqrt[4]{32+\sqrt{5+\sqrt{21}}(\sqrt{5+\sqrt{21}}+\sqrt{\sqrt{21}-3})^3}}$.

We will show case (1) and the rest of the cases are obtained by exactly the same process.

The modular equation of level 2 is

$$F_2(X,Y) = X^2 - X + 2XY - X^2Y + Y^2.$$

Then the zeros of $F_2(U(i/2), x) = 0$ are

$$\frac{2+\sqrt{12-6\sqrt{3}+2\sqrt{6\sqrt{3}-9}}}{(\sqrt[4]{6\sqrt{3}-9}+1)^2} \approx 0.9171526117$$

and

$$\frac{2-\sqrt{12-6\sqrt{3}+2\sqrt{6\sqrt{3}-9}}}{(\sqrt[4]{6\sqrt{3}-9}+1)^2} \approx 0.001863955375.$$

By letting $q_0 = e^{-2\pi}$ and finding the approximation of

$$q_0 \prod_{n=1}^{2000} \frac{(1-q^{12n-1})(1-q^{12n-11})}{(1-q^{12n-7})(1-q^{12n-5})} \approx 0.001863955388,$$

we get the value

$$U(i) = \frac{2-\sqrt{12-6\sqrt{3}+2\sqrt{6\sqrt{3}-9}}}{(\sqrt[4]{6\sqrt{3}-9}+1)^2} = \frac{\sqrt{2}\left(\sqrt{2}-\sqrt{3t_1+\sqrt[4]{27}\sqrt{t_1}}\right)}{\left(\sqrt[8]{27}\sqrt[4]{t_1}+1\right)^2}$$

for $t_1 = 2-\sqrt{3}$. □

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