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H^1-SUPERCONVERGENCE OF A DIFFERENCE FINITE ELEMENT METHOD BASED ON THE $P_1 - P_1$-CONFORMING ELEMENT ON NON-UNIFORM MESHES FOR THE 3D POISSON EQUATION

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ABSTRACT. In this paper, a difference finite element (DFE) method is presented for the 3D Poisson equation on non-uniform meshes by using the P_1-P_1-conforming element. This new method consists of combining the finite difference discretization based on the P_1-element in the z-direction with the finite element discretization based on the P_1-element in the (x, y)-plane. First, under the regularity assumption of $u \in H^3(\Omega) \cap H_0^1(\Omega)$ and $\partial_{zz} f \in L^2((0, L_3); H^{-1}(\omega))$, the H^1-superconvergence of the discrete solution u_τ in the z-direction to the first-order interpolation function $I_\tau u$ is obtained, and the H^1-superconvergence of the second-order interpolation function $I_{2\tau}^2 u_\tau$ in the z-direction to u is then provided. Moreover, the H^1-superconvergence of the DFE solution u_h to the H^1-projection $R_h u_\tau$ of u_τ is deduced and the H^1-superconvergence of the second-order interpolation function $I_{2\tau}^2 I_{2h}^2 u_h$ to u in the $((x, y), z)$-space is also established. Finally, numerical tests are presented to show the H^1-superconvergence results of the DFE method for the 3D Poisson equation under the above regularity assumption.

1. INTRODUCTION

The finite element (FE) and finite difference (FD) methods are two standard and important methods for the discretization of partial differential equations (PDEs) and are often used in scientific computing. For the literature, the reader is referred to [4–7, 10, 13, 16–22, 24, 29, 31, 33, 38, 40–43, 46, 48, 49, 52, 53, 55, 61]. Usually, the FD methods are easier to implement than the FE methods, but the FE methods are more easily adapted to the general geometry of the underlying domain on which a differential problem is formulated and to the practical treatment of inhomogeneous physical properties of media.

In this paper, we present a difference finite element (DFE) method based on the $P_1 - P_1$-conforming element for the 3D Poisson equation and provide an explicit computational formula of a coefficient matrix for solving numerically the 3D Poisson equation. The DFE method consists of combining the finite difference discretization

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based on the P_1-element in the z-direction and the finite element discretization based on the P_1-element in the (x, y)-plane. In this method a numerical solution of the 3D Poisson equation is obtained by a combination of numerical solutions of a series of 2D elliptic equations. Hence, a very interesting fact is that a coefficient matrix from the DFE method is easier to be computed than those from the FE methods for solving the 3D Poisson equation, since we only need to compute the coefficient matrix in a 2D domain ω.

The research of superconvergence for finite elements can be traced to [11, 14, 15, 23, 32, 34–36, 50, 51, 62] in the 1970s and 1980s. Later, superconvergence results of the P_2-element were also obtained by Zhu [60] in 1981. In the middle 1990s the local symmetry theory developed by Schatz et al. [44] verified that on quasi-uniform meshes, a finite element solution has superconvergence at the mesh symmetry points far from a boundary. Furthermore, Babuska et al. [3] found that the mesh symmetry points on the whole domain are derivative superconvergence points for the P_1-element in the case of four typical quasi-uniform mesh patterns: Regular mesh, Criss-Cross mesh, Chevron mesh and Union Jack mesh. In recent years, the superconvergence of the FE methods has been a subject of active research due to its strong relevance with a posterior error estimates for the adaptive FE methods. Under the above mentioned four mesh patterns, Lin and Zhang in [37] showed that the mesh symmetry points are "almost" all superconvergence points for linear and higher order finite elements. In [16, 60], the authors proved that the mesh symmetry points are all convergence points for Chevron and Criss-Cross triangular P_1-elements. Moreover, further superconvergence on the FE methods was studied by many authors. For example, see [4–6, 12, 17, 33, 38, 45, 56–59, 61].

In 1986, Manteuffel and White [39] showed second-order convergence in both vertex-centered finite difference (CFD) and cell-centered finite difference (CCFD) schemes on non-uniform meshes for some scalar problems. These results were extended to full tensor coefficients and triangular and logically rectangular grids by Arbogast et al. [1, 2]. Recently, Barbeiro et al. [8, 9] studied the convergence properties of the CCFD schemes for second-order elliptic equations with variable coefficients on a union of rectangular domains. They proved that the FD schemes on non-uniform meshes are second-order convergent although not even being consistent. The convergence was studied with the aid of an appropriate negative norm, namely a discrete analogue of a standard negative norm. Moreover, Ferreira et al. [25–27] studied the convergence properties of a finite difference discretization on non-uniform rectangular grids for the solution of a second-order elliptic equation with mixed derivatives and variable coefficients in polygonal domains subject to general boundary conditions. They showed that the above finite difference scheme is equivalent to a fully discrete linear FE approximation with quadrature and proved that the FD scheme on non-uniform grids exhibits the phenomenon of supraconvergence. In 2014, Feng and He studied the H^1-superconvergence for the CFD method based on the P_1-conforming element with non-uniform meshes for an elliptic equation with variable coefficients in [24]. For the uniform meshes, Hannukainen et al. provided a nodal $O(h^4)$-superconvergence result by using averaged piecewise linear, bilinear, and trilinear finite element approximations for the 3D Poisson equation in [30].

In this paper, we first recall the H^1-superconvergence results for the FE methods for the 2D Poisson equation studied by Chen and Huang [17], Chen [16], Lin and Yan

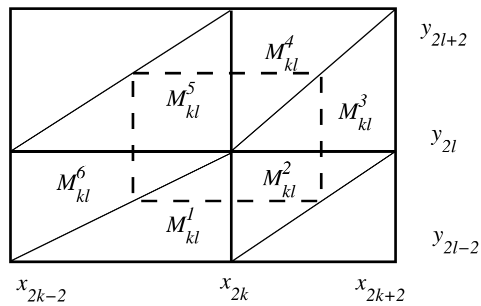


FIGURE 1. The coarse mesh J_{2h} in 2D domain

[38], Shi and Liang [45], and Zhu and Lin [61]. Then we represent the CFD method based on the P_1-element for the semi-discretization of the 3D Poisson equation in the z-direction and show the H^1-superconvergence of the discrete solution u_τ in the z-direction to the first-order interpolation function $I_\tau u$ of the exact solution u and the H^1-superconvergence of the second-order interpolation function $I^2_{2\tau}u_\tau$ in the z-direction to u. Next, we obtain the H^1-superconvergence of the DFE solution u_h based on the $P_1 - P_1$-element to the H^1-projection $R_h u_\tau$ of u_τ in the (x, y)-plane and the H^1-superconvergence of the second-order interpolation function $I^2_{2\tau}I^2_{2h}u_h$ in the $((x, y), z)$-space of the DFE solution u_h to u. Finally, numerical tests are presented to show the H^1-superconvergence results of the second-order interpolation function $I^2_{2\tau}I^2_{2h}u_h$ of the DFE solution u_h to the exact solution u for the 3D Poisson equation.

The paper is structured as follows: The H^1-superconvergence results for the FE methods for the 2D Poisson equation are recalled in Section 2. The DFE method based on the P_1-element for the z-direction discretization of the 3D Poisson equation and the H^1-superconvergence of the second-order interpolation function $I^2_{2\tau}u_\tau$ of u_τ to the exact solution u are presented in Section 3. The H^1-superconvergence of the second-order interpolation function $I^2_{2\tau}I^2_{2h}u_h$ of the DFE solution u_h to u is established in Section 4. In Section 5, numerical experiments are presented to check the theoretical analysis for the DFE method for the 3D Poisson equation. Finally, conclusions are drawn in the last section, Section 6.

2. FINITE ELEMENT METHOD BASED ON THE P_1-ELEMENT FOR THE 2D POISSON EQUATION

In this section, we recall the finite element method based on the P_1-conforming element for the 2D Poisson equation with the Dirichlet boundary condition

$$-\Delta u(x, y) = f(x, y), \qquad (x, y) \in \omega, \tag{2.1}$$

$$u = 0, \qquad (x, y) \in \partial\omega, \tag{2.2}$$

in a bounded domain $\omega = (0, L_1) \times (0, L_2)$, where $\Delta = \partial_{xx} + \partial_{yy}$. It is well-known that the weak formulation of (2.1)–(2.2) is to find $u \in X$ such that

$$(\nabla u, \nabla v)_\omega = (f, v)_\omega \quad \forall v \in X, \tag{2.3}$$

where $X = H^1_0(\omega)$ and $\nabla = (\partial_x,\ \partial_y)^\top$.

In this section, we construct a coarse mesh

$$J_{2h} = \{(M^6_{kl}, M^1_{kl});\ k = 1, \dots, l_1/2,\ l = 1, \dots, l_2/2\},$$

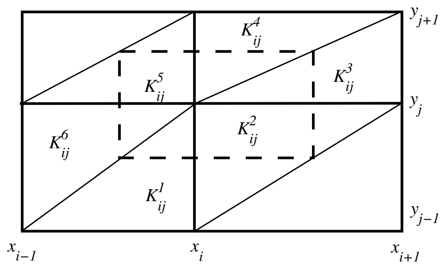


FIGURE 2. The fine mesh J_h in 2D domain

where l_1 and l_2 are even positive numbers, $(0, L_1)$ is divided into $x_0 = 0 < x_2 < \cdots < x_{l_1} = L_1$ and $(0, L_2)$ is divided into $y_0 = 0 < y_2 < \cdots < y_{l_2} = L_2$ with $M_{kl} = [x_{2k-2}, x_{2k}] \times [y_{2l-2}, y_{2l}] = M_{kl}^6 \cup M_{kl}^1$ as in Figure 1.

By setting $x_{2k-1} \in (x_{2k-2}, x_{2k})$ and $y_{2l-1} \in (y_{2l-2}, y_{2l})$, we construct a fine mesh

$$J_h = \{(K_{ij}^6, K_{ij}^1);\ i = 1, \dots, l_1,\ j = 1, \dots, l_2\},$$

as in Figure 2. Here we denote $h = \max_{1 \le i \le l_1, 1 \le j \le l_2}\{x_i - x_{i-1}, y_j - y_{j-1}\}$.

For each point $(x_i, y_j) \in \omega$ with $i = 1, \dots, l_1 - 1,\ j = 1, \dots, l_2 - 1$, we construct a patch $\tilde{K}_{ij} = \bigcup_{\nu=1}^{6} K_{ij}^{\nu} \subset [x_{i-1}, x_{i+1}] \times [y_{j-1}, y_{j+1}]$ (see Figure 2). So, we can define the P_1-basis functions $\phi_{ij}(x, y)$ such that supp $\phi_{ij} \subset \tilde{K}_{ij}$ and

$$\phi_{ij}(x_i, y_j) = 1,\ \phi_{ij}(x_k, y_l) = 0 \quad \text{for} \quad (k, l) \neq (i, j).$$

Then we establish the following finite element space based on the P_1-element:

$$X_h = \text{span}\{\phi_{ij}(x, y); i = 1, \dots, l_1 - 1,\ j = 1, \dots, l_2 - 1\}.$$

For each function $v(x, y) \in H^2(\omega) \cap X$, we define the interpolation function $I_h v(x, y)$ on X_h as follows:

$$I_h v(x, y) = \sum_{j=1}^{l_2-1} \sum_{i=1}^{l_1-1} v(x_i, y_j)\phi_{ij}(x, y).$$

Based on the finite element space X_h, the finite element approximation of u is to find u_h such that $u_h(x, y) = \sum_{j=1}^{l_2-1} \sum_{i=1}^{l_1-1} u_{ij}\phi_{ij}(x, y) \in X_h$ satisfies

$$(\nabla u_h, \nabla v_h)_\omega = (f, v_h)_\omega \quad \forall v_h \in X_h, \tag{2.4}$$

where u_{ij} is an approximation of $u(x_i, y_j)$ for $i = 1, \dots, l_1 - 1,\ j = 1, \dots, l_2 - 1$.

Recalling [17, 22, 38, 41, 45], the interpolation operator I_h has the important results as follows.

Theorem 2.1. *Interpolation operator I_h satisfies the following approximation properties:*

$$\|u - I_h u\|_{0,\omega} + h\|\nabla(u - I_h u)\|_{0,\omega} \le ch^2\|\Delta u\|_{0,\omega}, \tag{2.5}$$

for $u \in H^2(\omega) \cap X$.

Moreover, assume that $u \in H^3(\omega) \cap X$ satisfies (2.1)–(2.2) and $u_h \in X_h$ satisfies (2.4). Then, there holds

$$\|\nabla(u_h - I_h u)\|_{0,\omega} \le ch^2\|u\|_{3,\omega}. \tag{2.6}$$

Also, we define the projection operator $R_h : X \to X_h$ such that for each $u \in X$,

$$(\nabla R_h u, \nabla v_h)_\omega = (\nabla u, \nabla v_h)_\omega \quad \forall v_h \in X_h. \tag{2.7}$$

It is easy to show that the projection operator R_h satisfies the following approximation properties.

Theorem 2.2. *Projection operator R_h satisfies*

$$\|\nabla R_h u\|_{0,\omega} \le \|\nabla u\|_{0,\omega} \quad \forall u \in X,$$

$$\|u - R_h u\|_{0,\omega} + h\|\nabla(u - R_h u)\|_{0,\omega} \le ch^2\|\Delta u\|_{0,\omega} \quad \forall u \in H^2(\omega) \cap X, \tag{2.8}$$

and

$$\|\nabla(I_h u - R_h u)\|_{0,\omega} \le ch^2\|u\|_{3,\omega} \quad \forall u \in H^3(\omega) \cap X. \tag{2.9}$$

Proof. It is well known [41, 54] that (2.8) holds. Moreover, set $f := -\Delta u$, then the finite element solution u_h satisfies

$$(\nabla u_h, \nabla v_h)_\omega = (f, v_h)_\omega = (\nabla u, \nabla v_h)_\omega = (\nabla R_h u, \nabla v_h)_\omega \quad \forall v_h \in X_h. \tag{2.10}$$

Thus, $u_h = R_h u$. By using (2.6), we obtain

$$\|\nabla(R_h u - I_h u)\|_{0,\omega} = \|\nabla(u_h - I_h u)\|_{0,\omega} \le ch^2\|u\|_{3,\omega},$$

which yields (2.9). □

In order to obtain a H^1-superconvergence approximation of u by using the finite element solution u_h, we try to establish an second-order interpolation function $I_{2h}^2 u_h$ of u_h. Based on the coarse mesh J_{2h}, we give the definition of $I_{2h}^2 u$ on M_{kl}. We divide M_{kl} into two elements M_{kl}^1 and M_{kl}^6 as shown in Figure 1. For M_{kl}^1, we define three P_1-shape functions as follows:

$$\lambda_{k-1,l-1}(x,y) = \frac{x - x_{2k}}{x_{2k-2} - x_{2k}},$$
$$\lambda_{k,l-1}(x,y) = 1 - \frac{x - x_{2k}}{x_{2k-2} - x_{2k}} - \frac{y - y_{2l-2}}{y_{2l} - y_{2l-2}},$$
$$\lambda_{k,l}(x,y) = \frac{y - y_{2k-2}}{y_{2k} - y_{2k-2}}.$$

Moreover, we define six P_2-shape functions on M_{kl}^1 as follows:

$$\psi_{2k-2,2l-2}(x,y) = \lambda_{k-1,l-1}(x,y)(2\lambda_{k-1,l-1}(x,y) - 1),$$
$$\psi_{2k,2l}(x,y) = \lambda_{k,l}(x,y)(2\lambda_{k,l}(x,y) - 1),$$
$$\psi_{2k,2l-2}(x,y) = \lambda_{k,l-1}(x,y)(2\lambda_{k,l-1}(x,y) - 1),$$
$$\psi_{2k,2l-1}(x,y) = 4\lambda_{k,l-1}(x,y)\lambda_{k,l}(x,y),$$
$$\psi_{2k-1,2l-1}(x,y) = 4\lambda_{k-1,l-1}(x,y)\lambda_{k,l}(x,y),$$
$$\psi_{2k-1,2l-2}(x,y) = 4\lambda_{k-1,l-1}(x,y)\lambda_{k,l-1}(x,y),$$

and for M_{kl}^6, we also define three P_1-shape functions as follows:

$$\lambda_{k-1,l-1}(x,y) = \frac{y - y_{2l}}{y_{2l-2} - y_{2l}},$$
$$\lambda_{k-1,l}(x,y) = 1 - \frac{x - x_{2k-2}}{x_{2k} - x_{2k-2}} - \frac{y - y_{2l}}{y_{2l-2} - y_{2l}},$$
$$\lambda_{k,l}(x,y) = \frac{x - x_{2k-2}}{x_{2k} - x_{2k-2}}.$$

Moreover, we also define six P_2-shape functions on M^6_{kl} as follows:

$$\begin{aligned}
&\psi_{2k-2,2l-2}(x,y) = \lambda_{k-1,l-1}(x,y)(2\lambda_{k-1,l-1}(x,y)-1),\\
&\psi_{2k,2l}(x,y) = \lambda_{k,l}(x,y)(2\lambda_{k,l}(x,y)-1),\\
&\psi_{2k-2,2l}(x,y) = \lambda_{k-1,l}(x,y)(2\lambda_{k-1,l}(x,y)-1),\\
&\psi_{2k-1,2l-1}(x,y) = 4\lambda_{k-1,l-1}(x,y)\lambda_{k,l}(x,y),\\
&\psi_{2k-1,2l}(x,y) = 4\lambda_{k-1,l}(x,y)\lambda_{k,l}(x,y),\\
&\psi_{2k-2,2l-1}(x,y) = 4\lambda_{k-1,l-1}(x,y)\lambda_{k-1,l}(x,y).
\end{aligned}$$

Now, we define the finite element space X^2_{2h} based on the coarse mesh J_{2h} as follows:

$$X^2_{2h} = \{v_h \in X; v_h|_{M^1_{lk}} \in P_2,\ v_h|_{M^6_{lk}} \in P_2,\ k=1,\ldots,l_1/2,\ l=1,\ldots,l_2/2\}.$$

Here, we define the interpolation operator $I^2_{2h} : H^2(\omega) \to X^2_{2h}$ such that

$$\begin{aligned}
I^2_{2h}u|_{M^1_{kl}} = {}& u(x_{2k-2},y_{2l-2})\psi_{2k-2,2l-2}(x,y) + u(x_{2k-1},y_{2l-2})\psi_{2k-1,2l-2}(x,y)\\
&+ u(x_{2k},y_{2l-2})\psi_{2k,2l-2}(x,y) + u(x_{2k},y_{2l-1})\psi_{2k,2l-1}(x,y)\\
&+ u(x_{2k},y_{2l})\psi_{2k,2l}(x,y) + u(x_{2k-1},y_{2l-1})\psi_{2k-1,2l-1}(x,y),
\end{aligned}$$

and

$$\begin{aligned}
I^2_{2h}u|_{M^6_{kl}} = {}& u(x_{2k-2},y_{2l-2})\psi_{2k-2,2l-2}(x,y) + u(x_{2k-1},y_{2l-1})\psi_{2k-1,2l-1}(x,y)\\
&+ u(x_{2k},y_{2l})\psi_{2k,2l}(x,y) + u(x_{2k-1},y_{2l})\psi_{2k-1,2l}(x,y)\\
&+ u(x_{2k-2},y_{2l})\psi_{2k-2,2l}(x,y) + u(x_{2k-2},y_{2l-1})\psi_{2k-2,2l-1}(x,y).
\end{aligned}$$

It is well-known [16, 17, 38, 45] that the interpolation operator I^2_{2h} satisfies the following important properties.

Lemma 2.3. *The interpolation operator I^2_{2h} satisfies*

$$I^2_{2h}I_h v = I^2_{2h}v, \tag{2.11}$$

$$\|v - I^2_{2h}v\|_{0,\omega} + h\|\nabla(v - I^2_{2h}v)\|_{0,\omega} \le ch^2\|\Delta v\|_{0,\omega} \tag{2.12}$$

for each $v \in H^2(\omega) \cap X$ and

$$\|\nabla(v - I^2_{2h}v)\|_{0,\omega} \le ch^2\|v\|_{3,\omega} \tag{2.13}$$

for each $v \in H^3(\omega)$, and

$$\|\nabla I^2_{2h}v_h\|_{0,\omega} \le c\|\nabla v_h\|_{0,\omega} \tag{2.14}$$

for each $v_h \in X_h$.

3. Finite difference discretization in the direction of z for the 3D Poisson equation

In this section, we present the formulation of a finite difference method in the direction of z for the boundary value problem of the 3D Poisson equation:

$$-\partial_{zz}u - \Delta u = f, \qquad (x,y,z) \in \Omega, \tag{3.1}$$

$$u = 0, \qquad (x,y,z) \in \partial\Omega, \tag{3.2}$$

where $\Omega = \omega \times (0, L_3) \subset R^3$ and $L_3 \in (0,\infty)$. For convenience, we denote $v(x,y,z)$ by $v(z)$ for any function $v(x,y,z)$.

We first consider the finite difference discretization of (3.1)–(3.2) in the direction of z. Here we need to construct a coarse mesh in the direction of z:

$$z_0 = 0 < z_2 < \cdots < z_{2m-2} < z_{2m} < \cdots < z_{l_3} = L_3,$$

where l_3 is an even positive number. By setting $z_{2m-1} \in (z_{2m-2}, z_{2m})$, we establish a fine mesh on $(0, L_3)$:

$$0 = z_0 < z_1 < z_2 \cdots < z_{2m-2} < z_{2m-1} < z_{2m} < \cdots < z_{l_3-2} < z_{l_3-1} < z_{l_3} = L_3,$$

or

$$0 = z_0 < z_1 < z_2 \cdots < z_k < \cdots < z_{l_3-1} < z_{l_3} = L_3$$

Now, we define a finite difference solution u_τ such that

$$u_\tau(x,y,z) = \sum_{k=1}^{l_3-1} u^k(x,y)\phi_k(z),$$

where $\{\phi_k(z)\}_1^{l_3-1}$ is the piecewise linear basis function system in the z-direction such that $\phi_k(z) = \frac{z-z_{k-1}}{z_k-z_{k-1}}$ for $z \in [z_{k-1}, z_k]$, $\phi_k(z) = \frac{z_{k+1}-z}{z_{k+1}-z_k}$ for $z \in [z_k, z_{k+1}]$ and $\phi_k(z) = 0$ for $z \notin [z_{k-1}, z_{k+1}]$ with $k = 1, \ldots, l_3 - 1$. Here $u^k = u^k(x,y) \in H^2(\omega)$ is defined such that

$$-d_{zz}u^k - \Delta u^k = f(z_k), \tag{3.3}$$

for $k = 1, \ldots, l_3 - 1$ with $u^k|_{\partial\omega} = 0$ and $u^0(x,y) = u^{l_3}(x,y) = 0$ for $(x,y) \in \omega$, where

$$d_{zz}u^k = \frac{2}{z_{k+1} - z_{k-1}}(d_z u^{k+1} - d_z u^k),\ d_z u^k = \frac{u^k - u^{k-1}}{z_k - z_{k-1}},\quad z_{k-\frac{1}{2}} = \frac{1}{2}(z_k + z_{k-1}).$$

It follows from (3.3) that

$$\begin{aligned}\frac{\tau_{k+1}+\tau_k}{2}\|d_{zz}u^k\|_{0,\omega}^2 + \frac{\tau_{k+1}+\tau_k}{2}\|\Delta u^k\|_{0,\omega}^2 + 2(d_z u^{k+1} - d_z u^k, \Delta u^k)_\omega \\ = \frac{\tau_{k+1}+\tau_k}{2}\|f(z_k)\|_{0,\omega}^2,\end{aligned} \tag{3.4}$$

where $\tau_k = z_k - z_{k-1}$ and $\tau = \max_{1\le k\le l_3} \tau_k$.

In this paper, we will often use the following discrete Green formulas:

$$-\sum_{k=1}^{l_3-1}(a^{k+1} - a^k, b^k)_\omega = \sum_{k=1}^{l_3}(a^k, b^k - b^{k-1})_\omega, \tag{3.5}$$

provided $a^k,\ b^k \in L^2(\omega)$ with $k = 1, \ldots l_3$ and $b^0 = b^{l_3} = 0$, and

$$-\sum_{k=1}^{l_3-1}(\nabla a^{k+1} - \nabla a^k, \nabla b^k)_\omega = \sum_{k=1}^{l_3}(\nabla a^k, \nabla b^k - \nabla b^{k-1})_\omega, \tag{3.6}$$

provided $a^k,\ b^k \in H^1(\omega)$ with $k = 1, \ldots, l_3$ and $b^0 = b^{l_3} = 0$.

Summing (3.4) from $k = 1$ to $k = l_3 - 1$ and using (3.6), we obtain

$$\begin{aligned}2\sum_{k=1}^{l_3}\tau_k\|\nabla d_z u^k\|_{0,\omega}^2 + \sum_{k=1}^{l_3-1}\frac{\tau_{k+1}+\tau_k}{2}(\|d_{zz}u^k\|_{0,\omega}^2 + \|\Delta u^k\|_{0,\omega}^2) \\ = \sum_{k=1}^{l_3-1}\frac{\tau_{k+1}+\tau_k}{2}\|f(z_k)\|_{0,\omega}^2 \le (2\|f\|_{0,\Omega}^2 + \tau^2\|\partial_z f\|_{0,\Omega}^2).\end{aligned} \tag{3.7}$$

Here and hereafter, we always assume that $u \in H^3(\Omega) \cap H_0^1(\Omega)$ and $\partial_{zz} f \in L^2((0, L_3); H^{-1}(\omega))$ in this paper. Now, in order to analyse the approximation properties of u^k with respect to $u(z_k)$, we consider $z = z_k$ in (3.1) and set

$$E^k = \partial_{zz} u(z_k) - d_{zz} u(z_k), \qquad G^k = \frac{1}{\tau_k} \int_{z_{k-1}}^{z_k} (z - z_k)^2 \partial_{zzz} u(z) dz,$$

to obtain

$$-\frac{2}{z_{k+1} - z_{k-1}} \Big(d_z u(z_{k+1}) - d_z u(z_k) \Big) - \Delta u(z_k) = f(z_k) + E^k, \tag{3.8}$$

for $k = 1, \ldots, l_3 - 1$.

Using the integration by parts, we obtain

$$\begin{aligned} E^k &= \partial_{zz} u(z_k) - \frac{2}{z_{k+1} - z_{k-1}} \Big(d_z u(z_{k+1}) - d_z u(z_k) \Big) \\ &= \partial_{zz} u(z_k - \frac{2}{z_{k+1} - z_{k-1}} \Big(\frac{1}{\tau_{k+1}} \int_{z_k}^{z_{k+1}} \partial_z u(z) dz - \frac{1}{\tau_k} \int_{z_{k-1}}^{z_k} \partial_z u(z) dz \Big) \\ &= \partial_{zz} u(z_k) + \frac{2}{z_{k+1} - z_{k-1}} \Big(\frac{1}{\tau_{k+1}} \int_{z_k}^{z_{k+1}} (z - z_{k+1}) \partial_{zz} u(z) dz \\ &\qquad - \frac{1}{\tau_k} \int_{z_{k-1}}^{z_k} (z - z_{k-1}) \partial_{zz} u(z) dz \Big) \\ &= -\frac{1}{z_{k+1} - z_{k-1}} \Big(\frac{\int_{z_k}^{z_{k+1}} (z - z_{k+1})^2 \partial_{zzz} u(z) dz}{\tau_{k+1}} \\ &\qquad - \frac{\int_{z_{k-1}}^{z_k} (z - z_{k-1})^2 \partial_{zzz} u(z) dz}{\tau_k} \Big). \end{aligned} \tag{3.9}$$

Furthermore,

$$\begin{aligned} E^k = &- \frac{1}{z_{k+1} - z_{k-1}} (G^{k+1} - G^k) \\ &+ \frac{2}{z_{k+1} - z_{k-1}} \int_{z_{k-1}}^{z_k} (z - z_{k-\frac{1}{2}}) \partial_{zzz} u(z) dz \end{aligned} \tag{3.10}$$

for $k = 1, \ldots, l_3 - 1$. Setting $e^k = u(z_k) - u^k$, subtracting (3.3) from (3.7) and using (3.10), we obtain the following error equations:

$$\begin{aligned} -\frac{2}{z_{k+1} - z_{k-1}} \Big(d_z e^{k+1} - d_z e^k \Big) - \Delta e^k = &-\frac{2}{z_{k+1} - z_{k-1}} (G^{k+1} - G^k) \\ &+ \frac{2}{z_{k+1} - z_{k-1}} \int_{z_{k-1}}^{z_k} (z - z_{k-\frac{1}{2}}) \partial_{zzz} u(z) dz. \end{aligned} \tag{3.11}$$

Taking the $L^2(\omega)$-inner product (3.11) with $(\tau_k+\tau_{k+1})e^k$ and using (3.1) and the Young inequalities, we deduce

$$\begin{aligned}
&-2(d_z e^{k+1}-d_z e^k,e^k)_\omega+(\tau_k+\tau_{k+1})\|\nabla e^k\|^2_{0,\omega}+2(G^{k+1}-G^k,e^k)_\omega\\
&\quad=2\Big(\int_{z_{k-1}}^{z_k}(z-z_{k-\frac12})\partial_{zzz}u(z)dz,e^k\Big)_\omega\\
&\quad=\Big(\int_{z_{k-1}}^{z_k}\Big((\frac{\tau_k}{2})^2-(z-z_{k-\frac12})^2\Big)\Big(\nabla\Delta^{-1}\partial_{zz}f(z)+\nabla\partial_{zz}u(z)\Big)dz,\nabla e^k\Big)_\omega\\
&\quad\le\frac{\tau_k+\tau_{k+1}}{2}\|\nabla e^k\|^2_{0,\omega}+\frac12\tau^4\int_{z_{k-1}}^{z_k}(\|\nabla\partial_{zz}u(z)\|^2_{0,\omega}\\
&\qquad\qquad+\|\nabla\Delta^{-1}\partial_{zz}f(z)\|^2_{0,\omega})dz.
\end{aligned}\tag{3.12}$$

Summing ((3.12)) from $k=1$ to $k=l_3-1$, and using (3.5) and the Young inequalities, we obtain

$$\begin{aligned}
&\sum_{k=1}^{l_3}\tau_k\|d_z e^k\|^2_{0,\omega}+\sum_{k=1}^{l_3-1}\frac{\tau_k+\tau_{k+1}}{2}\|\nabla e^k\|^2_{0,\omega}\\
&\qquad\le\tau^4\|\nabla\Delta^{-1}\partial_{zz}f(z)\|^2_{0,\Omega}+\sum_{k=1}^{l_3}\tau_k\|G^k\|^2_{0,\omega}\\
&\qquad\le c\tau^4(\|\nabla\Delta^{-1}\partial_{zz}f(z)\|^2_{0,\Omega}+\|\nabla\partial_{zz}u\|^2_{0,\omega}+\|\partial_{zzz}u\|^2_{0,\Omega}).
\end{aligned}\tag{3.13}$$

Next, taking the $L^2(\omega)$-inner product (3.11) with $-(\tau_k+\tau_{k+1})\Delta e^k$ and using the Young inequalities, we deduce

$$-2(\nabla d_z e^{k+1}-\nabla d_z e^k,\nabla e^k)_\omega+\frac{\tau_k+\tau_{k+1}}{2}\|\Delta e^k\|^2_{0,\omega}\le\frac{\tau_k+\tau_{k+1}}{2}\|E^k\|^2_{0,\omega}.$$

the above inequalities from $k=1$ to $k=l_3-1$ and using (3.6)–(3.9) yields

$$\begin{aligned}
\sum_{k=1}^{l_3}\tau_k\|\nabla d_z e^k\|^2_{0,\omega}+\sum_{k=1}^{l_3-1}\frac{\tau_k+\tau_{k+1}}{2}\|\Delta e^k\|^2_{0,\omega}&\le\sum_{k=1}^{l_3-1}\frac{\tau_k+\tau_{k+1}}{2}\|E^k\|^2_{0,\omega}\\
&\le c\tau^2\|\partial_{zzz}u\|^2_{0,\Omega}.
\end{aligned}\tag{3.14}$$

Now, define the general interpolation operator I_τ such that for each

$$\begin{aligned}
&v\in H_0^1((0,L_3);L^2(\omega))\\
&I_\tau v(x,y,z)=\sum_{k=1}^{l_3-1}v(x,y,z_k)\phi_k(z).
\end{aligned}\tag{3.15}$$

From the above definition of the finite difference solution u_τ and (3.13)–(3.14), we have the following superconvergence results in the H^1-norm.

Lemma 3.1. *If* $u(x,y,z) \in H^3(\Omega) \cap H_0^1(\Omega)$ *satisfies* (3.1)–(3.2) *and* $\partial_{zz} f \in L^2((0,L_3);H^{-1}(\omega))$*, then there holds*

$$\begin{aligned}\|\nabla(I_\tau u - u_\tau)\|_{0,\Omega}^2 =& \sum_{k=1}^{l_3} \int_{z_{k-1}}^{z_k} \|\nabla(I_\tau u - u_\tau)\|_{0,\omega}^2 dz \\ \le& \frac{2}{3} \sum_{k=1}^{l_3} \tau_k (\|\nabla e^k\|_{0,\omega}^2 + \|\nabla e^{k-1}\|_{0,\omega}^2) \\ \le& c\tau^4 (\|\nabla \Delta^{-1} \partial_{zz} f(z)\|_{0,\Omega}^2 + \|u\|_{3,\Omega}^2),\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}\|\partial_z(I_\tau u - u_\tau)\|_{0,\Omega}^2 =& \sum_{k=1}^{l_3} \tau_k \|d_z e^k\|_{0,\omega}^2 \\ \le& c\tau^4 \Big(\int_0^{L^3} \|\nabla \Delta^{-1} \partial_{zz} f(z)\|_{0,\omega}^2 dz + \|u\|_{3,\Omega}^2 \Big),\end{aligned} \tag{3.17}$$

$$\|\nabla \partial_z(I_\tau u - u_\tau)\|_{0,\Omega}^2 = \sum_{k=1}^{l_3} \tau_k \|\nabla d_z e^k\|_{0,\omega}^2 \le C\tau^2 \|u\|_{3,\Omega}^2. \tag{3.18}$$

By some simple calculations, there hold the following properties of I_τ.

Lemma 3.2. *If* $v(x,y,z) \in H_0^1((0,L_3);L^2(\omega))$*, then there holds*

$$\|v - I_\tau v\|_{0,\Omega} \le \tau \|\partial_z(v - I_\tau v)\|_{0,\Omega}, \quad \|\partial_z I_\tau v\|_{0,\Omega} \le \|\partial_z v\|_{0,\Omega}. \tag{3.19}$$

Furthermore, if $v(x,y,t) \in H^2((0,L_3);L^2(\omega)) \cap H_0^1((0,L_3);L^2(\omega))$*, then there holds*

$$\|\partial_z(v - I_\tau v)\|_{0,\Omega} \le \tau \|\partial_{zz} v\|_{0,\Omega}. \tag{3.20}$$

Proof. First, we derive from (3.15) and the definition of $\phi_k(z)$ that

$$\begin{aligned} I_\tau v|_{[z_{k-1},z_k]} =& v(z_{k-1})\phi_{k-1}(z) + v(z_k)\phi_k(z), \\ I_\tau v(z_{k-1}) =& v(z_{k-1}), \quad I_\tau v(z_k) = v(z_k). \end{aligned} \tag{3.21}$$

Hence, using integration by parts and (3.21), we have

$$\begin{aligned} \int_{z_{k-1}}^{z_k} \|v(z) - I_\tau v(z)\|_{0,\omega}^2 dz &= -2 \int_{z_{k-1}}^{z_k} (z - z_{k-\frac{1}{2}}) \Big(v - I_\tau v, \partial_z(v - I_\tau v) \Big)_\omega dz \\ &\le \tau \Big(\int_{z_{k-1}}^{z_k} \|v - I_\tau v\|_{0,\omega}^2 dz \Big)^{\frac{1}{2}} \Big(\int_{z_{k-1}}^{z_k} \|\partial_z(v - I_\tau v)\|_{0,\omega}^2 dz \Big)^{\frac{1}{2}}, \\ \int_{z_{k-1}}^{z_k} \|\partial_z I_\tau v(z)\|_{0,\omega}^2 dz &= \int_{z_{k-1}}^{z_k} \Big\| \frac{v(z_k) - v(z_{k-1})}{\tau_k} \Big\|_{0,\omega}^2 dz \\ &= \frac{1}{\tau_k} \| \int_{z_{k-1}}^{z_k} \partial_z v dz\|_{0,\omega}^2 \le \int_{z_{k-1}}^{z_k} \|\partial_z v\|_{0,\omega}^2 dz. \end{aligned}$$

Summing the above inequalities from $k = 1$ to $k = l_3$ yields (3.19). Next, using again integration by parts and (3.21), there holds

$$\begin{aligned} \int_{z_{k-1}}^{z_k} \|\partial_z \Big(v(z) - I_\tau v(z) \Big)\|_{0,\omega}^2 dz &= - \int_{z_{k-1}}^{z_k} \Big(v - I_\tau v, \partial_{zz}(v - I_\tau v) \Big)_{0,\omega} dz \\ &\le \Big(\int_{z_{k-1}}^{z_k} \|v - I_\tau v\|_{0,\omega}^2 dz \Big)^{\frac{1}{2}} \Big(\int_{z_{k-1}}^{z_k} \|\partial_{zz} v\|_{0,\omega}^2 dz \Big)^{\frac{1}{2}}. \end{aligned} \tag{3.22}$$

Summing the above inequality from $k = 1$ to $k = l_3$ yields (3.20). ☐

Although Lemma 3.1 shows that u_τ is a superconvergence approximation to $I_\tau u$ in H^1-norm, we cannot obtain that u_τ is the second-order approximation to u by Lemma 3.2 and the decomposition $u - u_\tau = (u - I_\tau u) + (I_\tau u - u_\tau)$.

Hence, in order to obtain the second-order numerical approximation of u, we introduce the second-order general interpolation function $I^2_{2\tau} u_\tau$ as in references [17, 24, 38, 45]. For this purpose, we set the macro-element $\tilde{I}_k = [z_{2k-2}, z_{2k}]$ for $k = 1, \ldots, l_3/2$. Define the interpolation operator $I^2_{2\tau}$ such that for each

$$v \in H^1((0,T); L^2(\omega))$$

$$I^2_{2\tau} v|_{\tilde{I}_k} = v(x, y, z_{2k-2})\psi_{2k-2}(z) + v(x, y, z_{2k-1})\psi_{2k-1}(z) + v(x, y, z_{2k})\psi_{2k}(z), \tag{3.23}$$

where

$$\psi_{2k-2}(z) = \frac{(z - z_{2k-1})(z - z_{2k})}{(z_{2k-2} - z_{2k-1})(z_{2k-2} - z_{2k})},$$
$$\psi_{2k-1}(z) = \frac{(z - z_{2k-2})(z - z_{2k})}{(z_{2k-1} - z_{2k-2})(z_{2k-1} - z_{2k})},$$
$$\psi_{2k}(z) = \frac{(z - z_{2k-2})(z - z_{2k-1})}{(z_{2k} - z_{2k-2})(z_{2k} - z_{2k-1})}.$$

Next, we will establish the following properties of the interpolation operator $I^2_{2\tau}$.

Lemma 3.3. *If $v(x, y, z) \in H^1_0((0, L_3); L^2(\omega))$, then there holds*

$$I^2_{2\tau} I_\tau v = I^2_{2\tau} v, \tag{3.24}$$

$$\|v - I^2_{2\tau} v\|_{0,\Omega} \le \tau \|\partial_z (v - I^2_{2\tau} v)\|_{0,\Omega},$$
$$\|\partial_z I^2_{2\tau} v\|_{0,\Omega} \le \sqrt{\frac{7}{3}} \|\partial_z v\|_{0,\Omega}. \tag{3.25}$$

Furthermore, if $v(x, y, z) \in H^2((0, L_3); L^2(\omega)) \cap H^1_0((0, L_3); L^2(\omega))$, then there holds

$$\|\partial_z (v - I^2_{2\tau} v)\|_{0,\Omega} \le \tau \|\partial_{zz} (v - I^2_{2\tau} v)\|_{0,\Omega}, \quad \|\partial_{zz} I^2_{2\tau} v\|_{0,\Omega} \le \sqrt{\frac{8}{3}} \|\partial_{zz} v\|_{0,\Omega} \tag{3.26}$$

and

$$\|\partial_{zz} (v - I^2_{2\tau} v)\|_{0,\Omega} \le \tau \|\partial_{zzz} v\|_{0,\Omega}. \tag{3.27}$$

Proof. First, (3.24) is obvious by using (3.21) and (3.23). Next, we derive from (3.23) and the definition of $\psi_l(z)$ that

$$I^2_{2\tau} v(z_l) = v(z_l), \ l = 2k-2, \ 2k-1, \ 2k, \tag{3.28}$$

$$\partial_z I^2_{2\tau} v(z) = \frac{v(z_{2k}) - v(z_{2k-1})}{z_{2k} - z_{2k-1}} \frac{2z - z_{2k-1} - z_{2k-2}}{z_{2k} - z_{2k-2}} - \frac{v(z_{2k-1}) - v(z_{2k-2})}{z_{2k-1} - z_{2k-2}} \frac{2z - z_{2k} - z_{2k-1}}{z_{2k} - z_{2k-2}}, \tag{3.29}$$

$$\partial_{zz} I^2_{2\tau} v(z) = d_{zz} v(z_{2k-1}). \tag{3.30}$$

Hence, using integration by parts and (3.28)–(3.29), we have

$$\begin{aligned}
&\int_{z_{2k-2}}^{z_{2k}} \|v(z) - I_{2\tau}^2 v(z)\|_{0,\omega}^2 dz \\
&\quad = -2\int_{z_{2k-1}}^{z_{2k}} (z - z_{2k-\frac{1}{2}})\Big(v - I_{2\tau}^2 v, \partial_z(v - I_{2\tau}^2 v)\Big)_\omega dz \\
&\quad = -2\int_{z_{2k-2}}^{z_{2k-1}} (z - z_{2k-\frac{3}{2}})\Big(v - I_{2\tau}^2 v, \partial_z(v - I_{2\tau}^2 v)\Big)_\omega dz \\
&\quad \le \tau\Big(\int_{z_{2k-2}}^{z_{2k}} \|v - I_{2\tau}^2 v\|_{0,\omega}^2 dz\Big)^{\frac{1}{2}}\Big(\int_{z_{2k-1}}^{z_{2k}} \|\partial_z(v - I_{2\tau}^2 v)\|_{0,\omega}^2 dz\Big)^{\frac{1}{2}}, \\
&\quad \int_{z_{2k-2}}^{z_{2k}} \|\partial_z I_{2\tau}^2 v(z)\|_{0,\omega}^2 dz \\
&\quad \le 2\int_{z_{2k-2}}^{z_{2k}} \Big(\frac{2z - z_{2k} - z_{2k-1}}{z_{2k} - z_{2k-2}}\Big)^2 \Big\|\frac{v(z_{2k-1}) - v(z_{2k-2})}{z_{2k-1} - z_{2k-2}}\Big\|_{0,\omega}^2 dz \\
&\qquad + 2\int_{z_{2k-2}}^{z_{2k}} \Big(\frac{2z - z_{2k-1} - z_{2k-2}}{z_{2k} - z_{2k-2}}\Big)^2 \Big\|\frac{v(z_{2k}) - v(z_{2k-1})}{z_{2k} - z_{2k-1}}\Big\|_{0,\omega}^2 dz \\
&\quad \le \frac{7}{3}\int_{z_{2k-2}}^{z_{2k}} \|\partial_z v\|_{0,\omega}^2 dz.
\end{aligned}$$

Summing the above inequalities from $k = 1$ to $k = l_3/2$ yields (3.25).

Furthermore, using integration by parts again and (3.28)–(3.30), there holds

$$\begin{aligned}
&\int_{z_{2k-2}}^{z_{2k}} \|\partial_z\Big(v(z) - I_{2\tau}^2 v(z)\Big)\|_{0,\omega}^2 dz \\
&\quad = -\int_{z_{2k-2}}^{z_{2k}} \Big(v - I_{2\tau}^2 v, \partial_{zz}(v - I_{2\tau}^2 v)\Big)_{0,\omega} dz \\
&\quad \le \Big(\int_{z_{2k-2}}^{z_{2k}} \|v - I_{2\tau}^2 v\|_{0,\omega}^2 dz\Big)^{\frac{1}{2}}\Big(\int_{z_{2k-2}}^{z_{2k}} \|\partial_{zz}(v - I_{2\tau}^2 v)\|_{0,\omega}^2 dz\Big)^{\frac{1}{2}}
\end{aligned}$$

and

$$\begin{aligned}
&\int_{z_{2k-2}}^{z_{2k}} \|\partial_{zz} I_{2\tau}^2 v(z)\|_{0,\omega}^2 dz \\
&\quad = \frac{4}{z_{2k} - z_{2k-2}}\Big\|\frac{v(z_{2k}) - v(z_{2k-1})}{z_{2k} - z_{2k-1}} - \frac{v(z_{2k-1}) - v(z_{2k-2})}{z_{2k-1} - z_{2k-2}}\Big\|_{0,\omega}^2 \\
&\quad = \frac{4}{z_{2k} - z_{2k-2}}\Big\|\frac{1}{z_{2k} - z_{2k-1}}\int_{z_{2k-1}}^{z_{2k}} \partial_z v dz - \frac{1}{z_{2k-1} - z_{2k-2}}\int_{z_{2k-2}}^{z_{2k-1}} \partial_z v dz\Big\|_{0,\omega}^2 \\
&\quad = \frac{4}{z_{2k} - z_{2k-2}}\Big\|\frac{\int_{z_{2k-1}}^{z_{2k}} (z - z_{2k})\partial_{zz} v dz}{z_{2k} - z_{2k-1}} - \frac{\int_{z_{2k-2}}^{z_{2k-1}} (z - z_{2k-2})\partial_{zz} v dz}{z_{2k-1} - z_{2k-2}}\Big\|_{0,\omega}^2 \\
&\quad \le \frac{8}{3}\int_{z_{2k-2}}^{z_{2k}} \|\partial_{zz} v\|_{0,\omega}^2 dz.
\end{aligned} \tag{3.31}$$

Summing the above inequalities from $k = 1$ to $k = l_3/2$ and using (3.25) yields (3.26).

Finally, by using integration by parts and (3.30), we have

$$\begin{aligned}\int_{z_{2k-2}}^{z_{2k}} &\|\partial_{zz}\Big(v(z) - I_{2\tau}^2 v(z)\Big)\|_{0,\omega}^2 dz \\ &\le 2\int_{z_{2k-2}}^{z_{2k}} \|\partial_{zz}v(z) - \partial_{zz}v(z_{2k-1})\|_{0,\omega}^2 dz \\ &\quad + 2(z_{2k} - z_{2k-2})\|\partial_{zz}v(z_{2k-1}) - d_{zz}v(z_{2k-1})\|_{0,\omega}^2,\end{aligned}$$

$$\begin{aligned}2\int_{z_{2k-2}}^{z_{2k}} &\|\partial_{zz}v(z) - \partial_{zz}v(z_{2k-1})\|_{0,\omega}^2 dz \\ &= -4\int_{z_{2k-1}}^{z_{2k}} (z - z_{2k})\Big(\partial_{zz}v(z) - \partial_{zz}v(z_{2k-1}), \partial_{zzz}v(z)\Big)_\omega dz \\ &\quad - 4\int_{z_{2k-2}}^{z_{2k-1}} (z - z_{2k-2})\Big(\partial_{zz}v(z) - \partial_{zz}v(z_{2k-1}), \partial_{zzz}v(z)\Big)_\omega dz \\ &\le \int_{z_{2k-2}}^{z_{2k}} \|\partial_{zz}v(z) - \partial_{zz}v(z_{2k-1})\|_{0,\omega}^2 dz + 4\tau^2\int_{z_{2k-2}}^{z_{2k}} \|\partial_{zzz}v(z)\|_{0,\omega}^2 dz\end{aligned}$$

and

$$\begin{aligned}\|\partial_{zz}&v(z_{2k-1}) - d_{zz}v(z_{2k-1})\|_{0,\omega}^2 \\ &= \Big\|\partial_{zz}v(z_{2k-1}) - \frac{2}{z_{2k} - z_{2k-2}}\Big(\frac{\int_{z_{2k-1}}^{z_{2k}} \partial_z v(z)dz}{z_{2k} - z_{2k-1}} - \frac{\int_{z_{2k-2}}^{z_{2k-1}} \partial_z v(z)dz}{z_{2k-1} - z_{2k-2}}\Big)\Big\|_{0,\omega}^2 \\ &= \Big\|\partial_{zz}v(z_{2k-1}) + \frac{2}{z_{2k} - z_{2k-2}}\Big(\frac{\int_{z_{2k-1}}^{z_{2k}} (z - z_{2k})\partial_{zz}v(z)dz}{z_{2k} - z_{2k-1}} \\ &\quad - \frac{\int_{z_{2k-2}}^{z_{2k-1}} (z - z_{2k-2})\partial_{zz}v(z)dz}{z_{2k-1} - z_{2k-2}}\Big)\Big\|_{0,\omega}^2 \\ &= \frac{1}{(z_{2k} - z_{2k-2})^2}\Big\|\frac{\int_{z_{2k-1}}^{z_{2k}} (z - z_{2k})^2\partial_{zzz}v(z)dz}{z_{2k} - z_{2k-1}} \\ &\quad - \frac{\int_{z_{2k-2}}^{z_{2k-1}} (z - z_{2k-2})^2\partial_{zzz}v(z)dz}{z_{2k-1} - z_{2k-2}}\Big\|_{0,\omega}^2 \\ &\le \frac{2}{5(z_{2k} - z_{2k-2})}\tau^2\int_{z_{2k-2}}^{z_{2k}} \|\partial_{zzz}v(z)\|_{0,\omega}^2 dz.\end{aligned}$$

Combining the above inequalities gives

$$\int_{z_{2k-2}}^{z_{2k}} \|\partial_{zz}\Big(v(z) - I_{2\tau}^2 v(z)\Big)\|_{0,\omega}^2 dz \le c\tau^2\int_{z_{2k-2}}^{z_{2k}} \|\partial_{zzz}v(z)\|_{0,\omega}^2 dz.$$

Summing the above inequality for $k = 1$ to $k = l_3/2$, we derive (3.27). □

Now, we can give the following convergence result on $I_{2\tau}^2 u_\tau$.

Theorem 3.4. *Under the assumptions of Lemma* 3.1*, the interpolation function* $I_{2\tau}^2 u_\tau$ *satisfies*

$$\|\partial_z(u - I_{2\tau}^2 u_\tau)\|_{0,\Omega} \le c\tau^2(\|\Delta^{-\frac{1}{2}}\partial_{zz}f\|_{0,\Omega} + \|u\|_{3,\Omega}), \tag{3.32}$$

$$\|\nabla(u - I_{2\tau}^2 u_\tau)\|_{0,\Omega} \le c\tau^2(\|\Delta^{-\frac{1}{2}}\partial_{zz}f\|_{0,\Omega} + \|u\|_{3,\Omega}). \tag{3.33}$$

Proof. Using (3.24), we can write

$$u - I_{2\tau}^2 u_\tau = u - I_{2\tau}^2 u + I_{2\tau}^2 (I_\tau u - u_\tau). \tag{3.34}$$

First, using Lemma 3.1 and Lemma 3.3, we have

$$\begin{aligned}
\|\partial_z(u - I_{2\tau}^2 u_\tau)\|_{0,\Omega}^2 \le & 2\|\partial_z(u - I_{2\tau}^2 u)\|_{0,\Omega}^2 + 2\|\partial_z I_{2\tau}^2 (I_\tau u - u_\tau)\|_{0,\Omega}^2 \\
\le & 2\|\partial_z(u - I_{2\tau}^2 u)\|_{0,\Omega}^2 + \|\partial_z (I_\tau u - u_\tau)\|_{0,\Omega}^2 \\
\le & c\tau^4 \|\partial_{zzz} u\|_{0,\Omega}^2 + c\tau^4 \|u\|_{3,\Omega}^2, \qquad (3.35)\\
\|\nabla(u - I_{2\tau}^2 u_\tau)\|_{0,\Omega}^2 \le & 2\|\nabla(u - I_{2\tau}^2 u)\|_{0,\Omega}^2 + 2\|I_{2\tau}^2 \nabla (I_\tau u - u_\tau)\|_{0,\Omega}^2 \\
\le & c\tau^4 \|\partial_{zz} \nabla u\|_{0,\Omega}^2 + 4\|I_{2\tau}^2 \nabla (I_\tau u - u_\tau) - \nabla(I_\tau u - u_\tau)\|_{0,\Omega}^2 \\
& + 4\|\nabla(I_\tau u - u_\tau)\|_{0,\Omega}^2 \\
\le & c\tau^4 \|\partial_{zz} \nabla u\|_{0,\Omega} + c\tau^2 \|\nabla \partial_z \Big(I_\tau u(t) - u_\tau(t)\Big)\|_{0,\Omega}^2 \\
& + 4\|\nabla(I_\tau u - u_\tau)\|_{0,\Omega}^2 \le c\tau^4 (\|\Delta^{-\frac{1}{2}} \partial_{zz} f\|_{0,\Omega}^2 + \|u\|_{3,\Omega}^2), \qquad (3.36)
\end{aligned}$$

which are (3.32) and (3.33). □

4. Difference finite element method for the 3D Poisson equation

In this section, we first design a difference finite element(DFE) method based on $P_1 - P_1$-confroming element for solving the 3D Poisson equation and derive a H^1-superconvergence of the DFE solution u_h to $R_h u_\tau$. Furthermore, using the interpolation operator $I_{2\tau}^2 I_{2h}^2$, we prove that the numerical solution $I_{2\tau}^2 I_{2h}^2 u_h$ has a second-order approximation to u in the H^1-norm if $u \in H^3(\Omega) \cap H_0^1(\Omega)$, $\partial_{zz} f \in L^2((0, L_3); H^{-1}(\omega))$ and $u_\tau \in H^3(\Omega) \cap H_0^1(\Omega)$.

Now, we define the DFE solution for the 3D Poisson equation as follows:

$$u_h(x, y, z) = \sum_{k=1}^{l_3 - 1} u_h^k(x, y) \phi_k(z),$$

where $u_h^k(x, y) \in X_h$ is the finite element approximation of $u^k(x, y)$ and is defined by

$$u_h^k(x, y) = \sum_{j=1}^{l_2 - 1} \sum_{i=1}^{l_1 - 1} u_{ij}^k \phi_{ij}(x, y) \quad \text{or} \quad u_h(x, y, z) = \sum_{k=1}^{l_3 - 1} \sum_{j=1}^{l_2 - 1} \sum_{i=1}^{l_1 - 1} u_{ij}^k \phi_{ij}(x, y) \phi_k(z)$$

such that (cf. (3.3))

$$-(d_{zz} u_h^k, v_h)_\omega + (\nabla u_h^k, \nabla v_h)_\omega = (f(z_k), v_h)_\omega \qquad \forall v_h \in X_h, \tag{4.1}$$

where

$$d_{zz} u_h^k = \frac{2}{z_{k+1} - z_{k-1}} (d_z u_h^{k+1} - d_z u_h^k), \ d_z u_h^k = \frac{u_h^k - u_h^{k-1}}{z_k - z_{k-1}}.$$

Also, we set that $\phi_1(x, y), \ldots, \phi_M(x, y)$ is a set of basis functions in ω such that

$$X_h = \text{span}\{\phi_1, \ldots, \phi_M\},$$

with $l = i + (l_1 - 1)j$ and $M = (l_1 - 1)(l_2 - 1)$.

That is, to find u_l^k with $l = 1, \ldots, M$ and $k = 1, \ldots, l_3 - 1$ such that

$$\mathbf{A}\mathbf{U} = \mathbf{F}, \tag{4.2}$$

where

$$\mathbf{U} = (U^1, U^2, \ldots, U^{l_3-1})^\top,\ U^k = (u_1^k, \ldots, u_M^k)^\top,$$

$$\mathbf{F} = (F^1, F^2, \ldots, F^{l_3-1})^\top,\ F^k = \Big((f(z_k), \phi_1)_\omega, \ldots, (f(z_k), \phi_M)_\omega\Big)^\top,$$

$$B = (b_{ml})_{M\times M},\ b_{ml} = (\nabla\phi_l, \nabla\phi_m)_\omega,$$

$$C = (c_{ml})_{M\times M},\ c_{ml} = (\phi_l, \phi_k)_\omega,\ C_0 = \frac{2}{z_{k+1} - z_{k-1}}\Big(\frac{1}{z_{k+1} - z_k} + \frac{1}{z_k - z_{k-1}}\Big)C,$$

$$C_{-1} = \frac{2}{z_{k+1} - z_{k-1}}\frac{1}{z_k - z_{k-1}}C,\ C_1 = \frac{2}{z_{k+1} - z_{k-1}}\frac{1}{z_{k+1} - z_k}C$$

and

$$\mathbf{A} = \begin{pmatrix} B + C_0 & -C_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -C_{-1} & B + C_0 & -C_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -C_{-1} & B + C_0 & -C_1 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & -C_{-1} & B + C_0 & -C_1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -C_{-1} & B + C_0 \end{pmatrix}_{(l_3-1)\times(l_3-1)} \tag{4.3}$$

Note that B and C denote the standard stiffness matrix and mass matrix defined on the two-dimensional domain ω, respectively. Moreover, C_i $(i = -1, 0, 1)$ are also the mass matrixes. Since B and C are two irreducible diagonally dominant matrices and C and C_i are positive matrixes, the matrix A is also a irreducible diagonally dominant matrix and is non-singular. So we can use the classical iterative methods for the resulting linear systems.

Furthermore, we define the discrete Laplace operator $-\Delta_h$ by

$$(-\Delta_h v_h, w_h)_\omega = (\nabla v_h, \nabla w_h)_\omega \quad \forall w_h \in X_h,$$

for each $v_h \in X_h$. Thus, we can obtain the following discrete H^2-norm bounds of u_h^k:

$$\begin{aligned} 2\sum_{k=1}^{l_3} \tau_k \|\nabla d_z u_h^k\|_{0,\omega}^2 + \sum_{k=1}^{l_3-1} \frac{\tau_{k+1} + \tau_k}{2}(\|d_{zz} u_h^k\|_{0,\omega}^2 + \|\Delta u_h^k\|_{0,\omega}^2) \\ \le (2\|f\|_{0,\Omega}^2 + \tau^2\|\partial_z f\|_{0,\Omega}^2). \end{aligned} \tag{4.4}$$

Setting $e_h^k = R_h u^k - u_h^k$, we deduce from (3.3) and (4.1) that

$$\begin{aligned} &-\frac{2}{z_{k+1} - z_k}(d_z e_h^{k+1} - d_z e_h^k, v_h)_\omega + (\nabla e_h^k, \nabla v_h)_\omega \\ &= \frac{2}{z_{k+1} - z_k}\Big((I - R_h)d_z u^{k+1} - (I - R_h)d_z u^k, v_h\Big)_\omega \qquad \forall v_h \in X_h. \end{aligned} \tag{4.5}$$

Taking $v_h = (\tau_{k+1} + \tau_k)e_h^k$ in (4.5) yields

$$-2(d_z e_h^{k+1} - d_z e_h^k, e_h^k)_\omega + (\tau_k + \tau_{k+1})\|\nabla e_h^k\|_{0,\omega}^2 = 2\Big((I - R_h)(d_z u^{k+1} - d_z u^k), \tag{4.6}$$

Summing (4.6) from $k = 1$ to $k = l_3 - 1$ and using (3.5), we obtain

$$\sum_{k=1}^{l_3} \tau_k \|d_z e_h^k\|_{0,\omega}^2 + \sum_{k=1}^{l_3-1} (\tau_k + \tau_{k+1}) \|\nabla e_h^k\|_{0,\omega}^2 \quad \le \quad ch^4 \sum_{k=1}^{l_3} \|\Delta d_z u^k\|_{0,\omega}^2. \tag{4.7}$$

Now, taking $v_h = -(\tau_k + \tau_{k+1})\Delta_h e_h^k$ in (4.5) yields

$$\begin{aligned} &-2(\nabla d_z e_h^{k+1} - \nabla d_z e_h^k, e_h^k)_\omega + (\tau_k + \tau_{k+1}) \|\Delta_h e_h^k\|_{0,\omega}^2 \\ &\qquad = 2\Big(\nabla(d_z u^{k+1} - R_h d_z u^{k+1}) - \nabla(d_z u^k - R_h d_z u^k), e_h^k\Big)_\omega. \end{aligned} \tag{4.8}$$

Summing (4.8) from $k = 1$ to $k = l_3 - 1$ and using (3.6) and the Young inequality, we obtain

$$\begin{aligned} &\sum_{k=1}^{l_3} \tau_k \|\nabla d_z e_h^k\|_{0,\omega}^2 + \sum_{k=1}^{l_3-1} (\tau_k + \tau_{k+1}) \|\Delta_h e_h^k\|_{0,\omega}^2 \\ &\quad = 2\sum_{k=1}^{l_3} \Big(\nabla(d_z u^k - R_h d_z u^k), \nabla(e_h^{k-1} - e_h^k)\Big)_\omega \\ &\quad \le \sum_{k=1}^{l_3} \tau_k \|\nabla(d_z u^k - R_h d_z u^k)\|_{0,\omega}^2 \le ch^2 \sum_{k=1}^{l_3} \tau_k \|\Delta d_z u^k\|_{0,\omega}^2 \\ &\quad \le ch^2 \|\Delta \partial_z u_\tau\|_{0,\Omega}^2. \end{aligned} \tag{4.9}$$

Using (4.7), (4.9), and Lemma 2.3, we obtain the following H^1-superconvergence results of u_h to $R_h u_\tau$.

Lemma 4.1. *Assume that $u_\tau(x, y, z) \in H^3(\Omega) \cap H_0^1(\Omega)$, then there hold*

$$\begin{aligned} \|\nabla(R_h u_\tau - u_h)\|_{0,\Omega}^2 &= \sum_{k=1}^{l_3} \int_{z_{k-1}}^{z_k} \|\nabla(R_h u_\tau - u_h)\|_{0,\omega}^2 dz \\ &= \sum_{k=1}^{l_3} \int_{z_{k-1}}^{z_k} \|\nabla e_h^{k-1}\phi_{k-1}(z) + \nabla e_h^k \phi_k(z)\|_{0,\omega}^2 dz \\ &\le ch^4 \sum_{k=1}^{l_3} \tau_k \|\Delta d_z u^k\|_{0,\omega}^2 \le ch^4 \|u_\tau\|_{3,\Omega}^2 \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} \|\partial_z(R_h u_\tau - u_h)\|_{0,\Omega}^2 &= \sum_{k=1}^{l_3} \int_{z_{k-1}}^{z_k} \|\partial_z(R_h u_\tau - u_h)\|_{0,\omega}^2 dz \\ &= \sum_{k=1}^{l_3} \tau_k \|d_z e_h^k\|_{0,\omega}^2 \le Ch^4 \|u_\tau\|_{3,\Omega}^2, \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} \|\nabla\partial_z(R_h u_\tau - u_h)\|_{0,\Omega}^2 &= \sum_{k=1}^{l_3} \int_{z_{k-1}}^{z_k} \|\nabla(R_h \partial_z u_\tau - \partial_z u_h)\|_{0,\omega}^2 dz \\ &= \tau \sum_{k=1}^{l_3} \|\nabla d_z e_h^k\|_{0,\omega}^2 \le Ch^2 \|u_\tau\|_{3,\Omega}^2. \end{aligned} \tag{4.12}$$

Now, we can obtain the following error estimates of $I_{2h}^2 u_h$.

Theorem 4.2. *Under the assumptions of Lemma* 3.1 *and Lemma* 4.1*, the interpolation function* $I_{2\tau}^2 I_{2h}^2 u_h$ *of* u_h *defined by* (4.1) *satisfies*

$$\|\nabla(u - I_{2\tau}^2 I_{2h}^2 u_h)\|_{0,\Omega} \le c(h^2+\tau^2)(\|\Delta^{-\frac{1}{2}}\partial_{zz} f\|_{0,\Omega} + \|u\|_{3,\Omega} + \|u_\tau\|_{3,\Omega}), \tag{4.13}$$

$$\|\partial_z(u - I_{2\tau}^2 I_{2h}^2 u_h)\|_{0,\Omega} \le c(h^2+\tau^2)(\|\Delta^{-\frac{1}{2}}\partial_{zz} f\|_{0,\Omega} + \|u\|_{3,\Omega} + \|u_\tau\|_{3,\Omega}). \tag{4.14}$$

Proof. We write $u - I_{2\tau}^2 I_{2h}^2 u_h$ into

$$\begin{aligned} u - I_{2\tau} I_{2h}^2 u_h &= (u - I_{2\tau}^2 u_\tau) + (I_{2\tau}^2 u_\tau - I_{2\tau}^2 I_{2h}^2 u_\tau) \\ &\quad + I_{2\tau}^2 I_{2h}^2 (I_h u_\tau - R_h u_\tau) + I_{2\tau}^2 I_{2h}^2 (R_h u_\tau - u_h) \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{4.15}$$

Using Lemma 2.3, Theorem 2.2, Lemma 3.1, Lemma 3.3 and Theorem 3.4, there holds

$$\|\nabla I_1\|_{0,\Omega}^2 = \|\nabla(u - I_{2\tau}^2 u_\tau)\|_{0,\Omega}^2 \le c\tau^4(\|\Delta^{-\frac{1}{2}}\partial_{zz} f\|_{0,\Omega}^2 + \|u\|_{3,\Omega}^2), \tag{4.16}$$

$$\begin{aligned} \|I_2\|_{0,\Omega}^2 &= \|I_{2\tau}^2 \nabla(u_\tau - I_{2h}^2 u_\tau) - \nabla(u_\tau - I_{2h}^2 u_\tau) + \nabla(u_\tau - I_{2h}^2 u_\tau)\|_{0,\Omega}^2 \\ &\le c\tau^2 \|\partial_z \nabla(u_\tau - I_{2h}^2 u_\tau)\|_{0,\Omega}^2 + 2\|\nabla(u_\tau - I_{2h}^2 u_\tau)\|_{0,\Omega}^2 \\ &= c\tau^2 \sum_{k=1}^{l_3} \int_{z_{k-1}}^{z_k} \|\partial_z \nabla(u_\tau - I_{2h}^2 u_\tau)\|_{0,\omega}^2 dz \\ &\quad + 2\sum_{k=1}^{l_3} \int_{z_{k-1}}^{z_k} \|\nabla(u_\tau - I_{2h}^2 u_\tau)\|_{0,\omega}^2 dz \\ &\le c(\tau^2 h^2 + h^4)\|u_\tau\|_{3,\Omega}^2, \end{aligned} \tag{4.17}$$

$$\begin{aligned} \|\nabla I_3\|_{0,\Omega}^2 &= \|I_{2\tau}^2 \nabla I_{2h}^2 (I_h u_\tau - R_h u_\tau) - \nabla I_{2h}^2 (I_h u_\tau - R_h u_\tau) \\ &\qquad + \nabla I_{2h}^2 (I_h u_\tau - R_h u_\tau)\|_{0,\Omega}^2 \\ &\le c\tau^2 \|\nabla I_{2h}^2 (I_h \partial_z u_\tau - R_h \partial_z u_\tau)\|_{0,\Omega}^2 + 2\|\nabla I_{2h}^2 (I_h u_\tau - R_h u_\tau)\|_{0,\Omega}^2 \\ &\le c\tau^2 \sum_{k=1}^{l_3} \int_{z_{k-1}}^{z_k} \|\nabla(I_h \partial_z u_\tau - R_h \partial_z u_\tau)\|_{0,\omega}^2 dz \\ &\quad + c\sum_{k=1}^{l_3} \int_{z_{k-1}}^{z_k} \|\nabla(I_h u_\tau - R_h u_\tau)\|_{0,\omega}^2 dz \\ &\le c\sum_{k=1}^{l_3} \int_{z_{k-1}}^{z_k} (\tau^2 h^2 \|\Delta \partial_z u_\tau\|_{0,\omega}^2 + h^4 \|u_\tau\|_{3,\omega}^2) dz \\ &\le c(\tau^2 h^2 + h^4)\|u_\tau\|_{3,\Omega}^2, \end{aligned} \tag{4.18}$$

$$
\begin{aligned}
\|\nabla I_4\|_{0,\Omega}^2 &= \|I_{2\tau}^2 \nabla I_{2h}^2(R_h u_\tau - u_h) - \nabla I_{2h}^2(R_h u_\tau - u_h) \\
&\quad + \nabla I_{2h}^2(R_h u_\tau - u_h)\|_{0,\Omega}^2 \\
&\le c\tau^2 \|\partial_z \nabla I_{2h}^2(R_h u_\tau - u_h)\|_{0,\Omega}^2 + c\|\nabla I_{2h}^2(R_h u_\tau - u_h)\|_{0,\Omega}^2 \\
&\le \sum_{k=1}^{l_3} \int_{z_{k-1}}^{z_k} \Big(c\tau^2 \|\nabla I_{2h}^2 \partial_z (R_h d_z u^k - d_z u_h^k)\|_{0,\omega}^2 \\
&\quad + 2\|\nabla I_{2h}^2(R_h u_\tau - u_h)\|_{0,\omega}^2 \Big) dz \\
&\le c\sum_{k=1}^{l_3} \int_{z_{k-1}}^{z_k} \Big(\tau^2 \|\nabla \partial_z (R_h d_z u^k - d_z u_h^k)\|_{0,\omega}^2 \\
&\quad + \|\nabla (R_h u_\tau - u_h)\|_{0,\omega}^2 \Big) dz \\
&\le c(\tau^2 h^2 + h^4)\|u_\tau\|_{3,\Omega}^2,
\end{aligned} \tag{4.19}
$$

Combining (4.15) with (4.16)–(4.19) yields (4.13).

Next, using Lemma 2.3, Theorem 2.2, Lemma 3.3, and Theorem 3.4, there hold

$$
\begin{aligned}
\|\partial_z I_1\|_{0,\Omega}^2 &= \|\partial_z(u - I_{2\tau}^2 u_\tau)\|_{0,\Omega}^2 \le c\tau^4 (\|\Delta^{-\frac{1}{2}} \partial_{zzzz} u\|_{0,\Omega}^2 + \|u\|_{3,\Omega}^2), \\
\|\partial_z I_2\|_{0,\Omega}^2 &= \|\partial_z I_{2\tau}^2 (u_\tau - I_{2h}^2 u_\tau)\|_{0,\Omega}^2 \le c\|\partial_z (u_\tau - I_{2h}^2 u_\tau)\|_{0,\Omega}^2 \\
&= c\|\partial_z u_\tau - I_{2h}^2 \partial_z u_\tau\|_{0,\Omega} = c\sum_{k=1}^{l_3} \int_{z_{k-1}}^{z_k} \|\partial_z u_\tau - I_{2h}^2 \partial_z u_\tau\|_{0,\omega}^2 dz \\
&\le ch^4 \sum_{k=1}^{l_3} \int_{z_{k-1}}^{z_k} \|\Delta \partial_z u_\tau\|_{0,\omega}^2 dz \le ch^4 \|u_\tau\|_{3,\Omega}^2, \\
\|\partial_z I_3\|_{0,\Omega}^2 &= \|\partial_z I_{2\tau}^2 I_{2h}^2 (I_h u_\tau - R_h u_\tau)\|_{0,\Omega}^2 \le c\|I_{2h}^2 (I_h \partial_z u_\tau - R_h \partial_z u_\tau)\|_{0,\Omega}^2 \\
&\le ch^2 \sum_{k=1}^{l_3} \int_{z_{k-1}}^{z_k} \|\nabla (I_h \partial_z u_\tau - R_h \partial_z u_\tau)\|_{0,\omega}^2 \\
&\quad + c\sum_{k=1}^{l_3} \int_{z_{k-1}}^{z_k} \|I_h \partial_z u_\tau - R_h \partial_z u_\tau\|_{0,\omega}^2 dz \\
&\le ch^4 \sum_{k=1}^{l_3} \int_{z_{k-1}}^{z_k} \|\Delta \partial_z u_\tau\|_{0,\omega}^2 dz \le ch^4 \|u_\tau\|_{3,\Omega}^2, \\
\|\partial_z I_4\|_{0,\Omega}^2 &= \|\partial_z \Big(I_{2\tau}^2 I_{2h}^2 (R_h u_\tau - u_h) \Big)\|_{0,\Omega}^2 \\
&\le c\|I_{2h}^2 (R_h \partial_z u_\tau - \partial_z u_h) - (R_h \partial_z u_\tau - \partial_z u_h) + (R_h \partial_z u_\tau - \partial_z u_h)\|_{0,\Omega}^2 \\
&\le c\sum_{k=1}^{l_3} \int_{z_{k-1}}^{z_k} \Big(h^2 \|\nabla (R_h \partial_z u_\tau - \partial_z u_h)\|_{0,\omega}^2 + \|(R_h \partial_z u_\tau - \partial_z u_h)\|_{0,\omega}^2 \Big) dz \\
&= c(h^2 \|\nabla (R_h \partial_z u_\tau - \partial_z u_h)\|_{0,\Omega}^2 + \|(R_h \partial_z u_\tau - \partial_z u_h)\|_{0,\Omega}^2) \le ch^4 \|u_\tau\|_{3,\Omega}^2,
\end{aligned}
$$

Combining the above inequalities with (4.15) yields (4.14). □

5. Numerical experiments

In this section, we present two test problems on the uniform mesh and non-uniform mesh, respectively, to illustrate the theoretical results obtained in previous section. Here we use the conjugate gradient method as an iterative solver with initial guess ($u_0 = 0.0$) and the stopping criterion is that the norm of residual vectors is less than $1.0e{-}12$.

Example 1. We solve the 3D Poisson equation on the unit cube domain with the exact solution

$$u = \sin^\alpha(\pi x)\sin^\alpha(\pi y)\sin^\alpha(\pi z), \qquad (x, y, z) \in \Omega = (0, 1)^3,$$

where the real parameter $\alpha > 0$ is a constant. Note that the exact solution is a symmetrical and completely variable separation function. We will prove that u satisfies different kinds of smoothness conditions if the suitable choices of α are given.

Let $\psi(x) = \sin^\alpha(\pi x)$, then $u(x, y, z) = \psi(x)\psi(y)\psi(z)$. It is obvious that there holds $u|_{\partial\Omega} = 0$. By some simple computations, there holds

$$\begin{aligned}
||\psi(x)||_0^2 &= \int_0^1 \sin^{2\alpha}(\pi x)dx \le 1, \quad \alpha > 0,\\
||\psi^{(1)}(x)||_0^2 &= \int_0^1 (\alpha\pi)^2 \sin^{2\alpha-2}(\pi x)\cos^2(\pi x)dx = \begin{cases} \infty, & 0 < \alpha \le \frac{1}{2},\\ c, & \alpha > \frac{1}{2}.\end{cases}\\
||\psi^{(2)}(x)||_0^2 &= \int_0^1 \pi^4\alpha^2\Big((\alpha-1)\sin^{\alpha-2}(\pi x)\cos^2(\pi x) - \sin^\alpha(\pi x)\Big)^2 dx\\
&= \begin{cases} \infty, & 0 < \alpha \le \frac{3}{2} \text{ and } \alpha \ne 1,\\ c, & \text{others}.\end{cases}\\
||\psi^{(3)}(x)||_0^2 &= \int_0^1 \pi^6\alpha^2\cos^2(\pi x)\Big((\alpha-1)(\alpha-2)\sin^{\alpha-3}(\pi x)\cos^2(\pi x)\\
&\qquad - (3\alpha-2)\sin^{\alpha-1}(\pi x)\Big)^2 dx\\
&= \begin{cases} \infty, & 0 < \alpha \le \frac{5}{2} \text{ and } \alpha \ne 1, 2,\\ c, & \text{others}.\end{cases}
\end{aligned}$$

Similarly, for all $n \ge 4$, we can easily see that

$$||\psi^{(n)}(x)||_0^2 \;=\; \begin{cases} \infty, & 0 < \alpha \le \frac{2n-1}{2} \text{ and } \alpha \ne 1, 2, \ldots,\ n-1,\\ c, & \text{others}.\end{cases}$$

Here $c > 0$ denotes the different bounded constant. From the above relations, we can give some suitable choices of α such that u satisfies the following smoothness conditions.

Theorem 5.1. *If we take $0 < \alpha \le \frac{1}{2}$, then $u \in L^2(\Omega)$ and $u \notin H_0^1(\Omega)$. If we take $\frac{2n-1}{2} < \alpha \le \frac{2n+1}{2}$ and $\alpha \ne 1, 2, \cdots,\ n$, then $u \in H^n(\Omega) \cap H_0^1(\Omega)$ and $u \notin H^{n+1}(\Omega) \cap H_0^1(\Omega)$, (for all $n \ge 1$). If we take $\alpha = m$, then $u \in H^n(\Omega) \cap H_0^1(\Omega)$, (for all $n \ge 1, m = 1, 2, \ldots$).*

TABLE 1. Numerical results of $u \in L^2(\Omega)$ but $u \notin H_0^1(\Omega)$ on uniform mesh with $\alpha = \frac{1}{2}$.

Mesh	$\|\partial_z E^1_{\tau,h}\|_0$	$\|\nabla E^1_{\tau,h}\|_0$	$\|E^1_{\tau,h}\|_0$	$\|\partial_z E^2_{2\tau,2h}\|_0$	$\|\nabla E^2_{2\tau,2h}\|_0$	$\|E^2_{2\tau,2h}\|_0$
(τ,h)	9.3572e-1	1.2491e-0	1.9295e-1	3.1838e-1	6.1959e-1	1.3396e-1
$(\tau,h)/2$	8.2388e-1	1.0539e-0	1.0332e-1	3.1456e-1	4.7576e-1	7.9645e-2
$(\tau,h)/4$	8.0864e-1	9.7279e-1	6.1138e-2	3.2727e-1	4.5135e-1	5.2405e-2
$(\tau,h)/8$	8.1194e-1	9.4335e-1	3.9011e-2	3.3878e-1	4.4714e-1	3.5844e-2
$(\tau,h)/16$	8.1677e-1	9.3317e-1	2.6096e-2	3.4651e-1	4.4626e-1	2.4957e-2

TABLE 2. Numerical results of $u \in L^2(\Omega)$ but $u \notin H_0^1(\Omega)$ on non-uniform mesh with $\alpha = \frac{1}{2}$.

Mesh	$\|\partial_z E^1_{\tau,h}\|_0$	$\|\nabla E^1_{\tau,h}\|_0$	$\|E^1_{\tau,h}\|_0$	$\|\partial_z E^2_{2\tau,2h}\|_0$	$\|\nabla E^2_{2\tau,2h}\|_0$	$\|E^2_{2\tau,2h}\|_0$
(τ,h)	1.0149e-0	1.4879e-0	2.2402e-1	1.4130e-0	9.5190e-0	6.5978e-1
$(\tau,h)/2$	8.4563e-1	1.1756e-0	1.0661e-1	3.2454e-1	5.4350e-1	6.9577e-2
$(\tau,h)/4$	8.1714e-1	1.0249e-0	5.7672e-2	3.3355e-1	4.7696e-1	4.5133e-2
$(\tau,h)/8$	8.1613e-1	9.6265e-1	3.4878e-2	3.4325e-1	4.5842e-1	3.0662e-2
$(\tau,h)/16$	8.1894e-1	9.3927e-1	2.2635e-2	3.4934e-1	4.5069e-1	2.1208e-2

Next we consider five cases in numerical experiments by choosing different values of α.

(i) If $\alpha = \frac{1}{2}$, then $u \in L^2(\Omega)$, but $u \notin H_0^1(\Omega)$.
(ii) If $\alpha = \frac{3}{2}$, then $u \in H_0^1(\Omega)$, but $u \notin H^2(\Omega) \cap H_0^1(\Omega)$.
(iii) If $\alpha = \frac{5}{2}$, then $u \in H^2(\Omega) \cap H_0^1(\Omega)$, but $u \notin H^3(\Omega) \cap H_0^1(\Omega)$.
(iv) If $\alpha = \frac{7}{2}$, then $u \in H^3(\Omega) \cap H_0^1(\Omega)$, but $u \notin H^4(\Omega) \cap H_0^1(\Omega)$.
(v) If $\alpha = 1$, then $u \in H^n(\Omega) \cap H_0^1(\Omega)$, for all $n \geq 2$.

Denote two error functions $E^1_{\tau,h} = u - I_\tau I_h u_h$ and $E^2_{2\tau,2h} = u - I^2_{2\tau} I^2_{2h} u_h$. To verify the convergence rate, we compute the errors of $\|\partial_z E^1_{\tau,h}\|_0$, $\|\partial_z E^2_{2\tau,2h}\|_0$, $\|\nabla E^1_{\tau,h}\|_0$ and $\|\nabla E^2_{2\tau,2h}\|_0$, respectively. Furthermore, the errors of $\|E^1_{\tau,h}\|_0$ and $\|E^2_{2\tau,2h}\|_0$ are also computed.

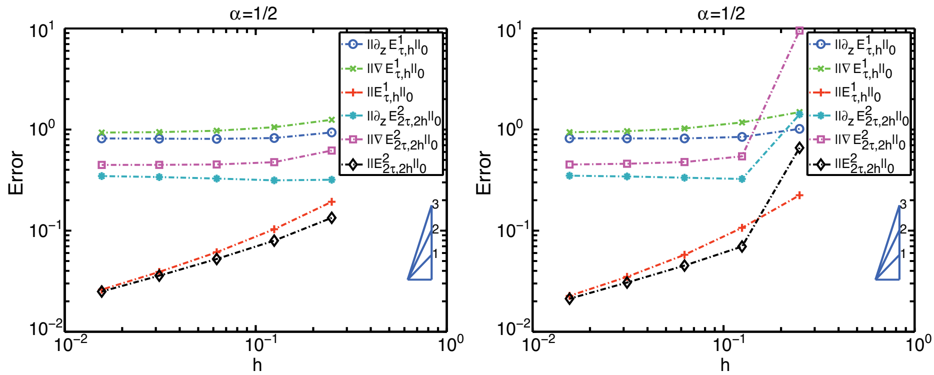
We first consider numerical approximations on the uniform mesh and take $\tau = h = 1/4$, and then, the numerical solutions are also investigated on the non-uniform mesh produced by random numbers generated from uniform distribution on the interval [0, 1]. Let us rewrite (3.3) with the non-equidistant step as follows:

$$-\frac{2}{\tau_{k+1}+\tau_k}\Big(\frac{u^{k+1}}{\tau_{k+1}} - (\frac{1}{\tau_{k+1}} + \frac{1}{\tau_k})u^k + \frac{u^{k-1}}{\tau_k}\Big) - \Delta u^k = f(x, y, z_k).$$

Here we adopt the above scheme for the numerical approximations on the following non-uniform mesh:

$$T_{hx} = \{0,\ 0.12991,\ 0.56882,\ 0.93401,\ 1\},$$
$$T_{hy} = \{0,\ 0.011902,\ 0.33712,\ 0.46939,\ 1\},$$
$$T_{hz} = \{0,\ 0.16218,\ 0.31122,\ 0.79428,\ 1\}.$$

We also denote $E^1_{\tau,h} = u - I_\tau I_h u_h$ and $E^2_{2\tau,2h} = u - I^2_{2\tau} I^2_{2h} u_h$ on the non-uniform mesh; here $\tau = \max_{1\leq k\leq 4} \tau_k,\ h = \max_{1\leq i\leq 4, 1\leq j\leq 4}\{x_i - x_{i-1}, y_j - y_{j-1}\}$.



(a) (b)

FIGURE 3. $u \in L^2(\Omega)$ but $u \notin H_0^1(\Omega)$: (a) Convergence rates on uniform mesh, (b) Convergence rates on non-uniform mesh.

TABLE 3. Numerical results of $u \in H_0^1(\Omega)$ but $u \notin H^2(\Omega) \cap H_0^1(\Omega)$ on uniform mesh with $\alpha = \frac{3}{2}$.

Mesh	$\|\partial_z E_{\tau,h}^1\|_0$	$\|\nabla E_{\tau,h}^1\|_0$	$\|E_{\tau,h}^1\|_0$	$\|\partial_z E_{2\tau,2h}^2\|_0$	$\|\nabla E_{2\tau,2h}^2\|_0$	$\|E_{2\tau,2h}^2\|_0$
(τ,h)	3.4490e-1	5.9195e-1	5.4002e-2	2.2932e-1	4.7987e-1	4.3003e-2
$(\tau,h)/2$	1.8761e-1	3.3347e-1	1.6017e-2	6.4671e-2	1.7069e-1	7.6691e-3
$(\tau,h)/4$	1.0070e-1	1.7786e-1	4.3667e-3	2.2908e-2	5.5689e-2	1.6913e-3
$(\tau,h)/8$	5.3736e-2	9.3206e-2	1.1808e-3	1.0449e-2	2.0246e-2	5.0253e-4
$(\tau,h)/16$	2.8480e-2	4.8514e-2	3.2763e-4	5.1560e-3	8.6452e-3	1.7395e-4

The convergence rates of different values of α are presented respectively in Tables 1 and 10 in terms of six error functions ($\partial_z E_{\tau,h}^1$, $\nabla E_{\tau,h}^1$, $E_{\tau,h}^1$, $\partial_z E_{2\tau,2h}^2$, $\nabla E_{2\tau,2h}^2$ and $E_{2\tau,2h}^2$). Of course, we can choose other values of α for numerical experiments.

From Tables 1 and 2, we can see that only the L^2-norm errors of two functions $E_{\tau,h}^1$ and $E_{2\tau,2h}^2$ with $\alpha = \frac{1}{2}$ are decreased as mesh refinement, but the convergence rate is less than 1. From Figure 3, we can observe that the convergence rates cannot be obtained for other four error functions, since $u \in L^2(\Omega)$ and $u \notin H_0^1(\Omega)$.

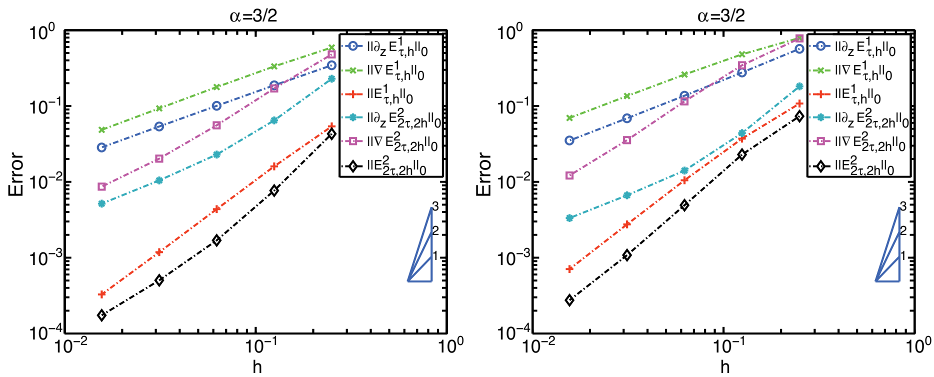
From Tables 3 and 4, we can see that the convergence rates of the L^2-norm errors of two functions $\partial_z E_{\tau,h}^1$ and $\nabla E_{\tau,h}^1$ are optimal, but for other four functions, we can see in Figure 4 that the convergence rates are less than quadratic, since $u \in H_0^1(\Omega)$ and $u \notin H^2(\Omega) \cap H_0^1(\Omega)$.

From Tables 5 and 6, we can see that the convergence rates of the L^2-norm errors of three functions $\partial_z E_{\tau,h}^1$, $\nabla E_{\tau,h}^1$ and $E_{\tau,h}^1$ are optimal. But for the other two functions $\partial_z E_{2\tau,2h}^2$ and $\nabla E_{2\tau,2h}^2$, we can see in Figure 5 that the convergence rates are less than quadratic, since $u \in H^2(\Omega) \cap H_0^1(\Omega)$ and $u \notin H^3(\Omega) \cap H_0^1(\Omega)$. In addition, for the error function $E_{2\tau,2h}^2$, we can see in Figure 5 that the convergence rate is also less than $\frac{5}{2}$.

From Tables 7 and 8, we can see that the convergence rates of the L^2-norm errors of five functions are optimal except the function $E_{2\tau,2h}^2$. For the L^2-norm errors of function $E_{2\tau,2h}^2$, the convergence rate is about $\frac{5}{2}$. Moreover, from Figure 6, we can

TABLE 4. Numerical results of $u \in H_0^1(\Omega)$ but $u \notin H^2(\Omega) \cap H_0^1(\Omega)$ on non-uniform mesh with $\alpha = \frac{3}{2}$.

Mesh	$\|\partial_z E^1_{\tau,h}\|_0$	$\|\nabla E^1_{\tau,h}\|_0$	$\|E^1_{\tau,h}\|_0$	$\|\partial_z E^2_{2\tau,2h}\|_0$	$\|\nabla E^2_{2\tau,2h}\|_0$	$\|E^2_{2\tau,2h}\|_0$
(τ, h)	5.6798e-1	7.9124e-1	1.0801e-1	1.8179e-1	7.8065e-1	7.3466e-2
$(\tau, h)/2$	2.7692e-1	4.8075e-1	3.7165e-2	4.3984e-2	3.4492e-1	2.2910e-2
$(\tau, h)/4$	1.3683e-1	2.6032e-1	1.0487e-2	1.4084e-2	1.1518e-1	4.9171e-3
$(\tau, h)/8$	6.8969e-2	1.3549e-1	2.7507e-3	6.6505e-3	3.5471e-2	1.0806e-3
$(\tau, h)/16$	3.5062e-2	6.9626e-2	7.0605e-4	3.3312e-3	1.2121e-2	2.7504e-4



(a) (b)

FIGURE 4. $u \in H_0^1(\Omega)$ but $u \notin H^2(\Omega) \cap H_0^1(\Omega)$: (a) Convergence rates on uniform mesh, (b) Convergence rates on non-uniform mesh.

TABLE 5. Numerical results of $u \in H^2(\Omega) \cap H_0^1(\Omega)$ but $u \notin H^3(\Omega) \cap H_0^1(\Omega)$ on uniform mesh with $\alpha = \frac{5}{2}$.

Mesh	$\|\partial_z E^1_{\tau,h}\|_0$	$\|\nabla E^1_{\tau,h}\|_0$	$\|E^1_{\tau,h}\|_0$	$\|\partial_z E^2_{2\tau,2h}\|_0$	$\|\nabla E^2_{2\tau,2h}\|_0$	$\|E^2_{2\tau,2h}\|_0$
(τ, h)	3.9779e-1	6.2244e-1	5.7967e-2	2.7458e-1	5.6482e-1	5.6157e-2
$(\tau, h)/2$	1.8905e-1	3.3104e-1	1.6606e-2	2.4834e-2	1.8078e-1	7.9324e-3
$(\tau, h)/4$	9.2751e-2	1.6917e-1	4.3275e-3	7.9630e-3	5.1265e-2	1.4588e-3
$(\tau, h)/8$	4.6134e-2	8.5096e-2	1.0942e-3	2.4621e-3	1.3432e-2	3.0339e-4
$(\tau, h)/16$	2.3037e-2	4.2615e-2	2.7443e-4	6.4598e-4	3.4477e-3	7.1243e-5

TABLE 6. Numerical results of $u \in H^2(\Omega) \cap H_0^1(\Omega)$ but $u \notin H^3(\Omega) \cap H_0^1(\Omega)$ on non-uniform mesh with $\alpha = \frac{5}{2}$.

Mesh	$\|\partial_z E^1_{\tau,h}\|_0$	$\|\nabla E^1_{\tau,h}\|_0$	$\|E^1_{\tau,h}\|_0$	$\|\partial_z E^2_{2\tau,2h}\|_0$	$\|\nabla E^2_{2\tau,2h}\|_0$	$\|E^2_{2\tau,2h}\|_0$
(τ, h)	6.1038e-1	7.7793e-1	9.9059e-2	2.3363e-1	8.4120e-1	9.3355e-2
$(\tau, h)/2$	2.9753e-1	5.2245e-1	4.1139e-2	2.7997e-2	4.4097e-1	2.7759e-2
$(\tau, h)/4$	1.4616e-1	2.7510e-1	1.1657e-2	5.8420e-3	1.3311e-1	5.5702e-3
$(\tau, h)/8$	7.2184e-2	1.3988e-1	3.0255e-3	1.3953e-3	3.6939e-2	1.0620e-3
$(\tau, h)/16$	3.5956e-2	7.0257e-2	7.6391e-4	3.2408e-4	9.5691e-3	2.2972e-4

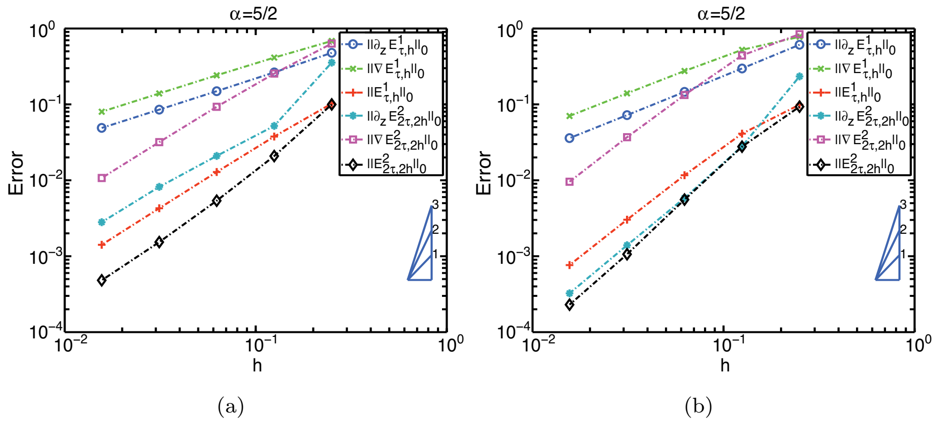


FIGURE 5. $u \in H^2(\Omega) \cap H_0^1(\Omega)$ but $u \notin H^3(\Omega) \cap H_0^1(\Omega)$: (a) Convergence rates on uniform mesh, (b) Convergence rates on non-uniform mesh.

TABLE 7. Numerical results of $u \in H^3(\Omega) \cap H_0^1(\Omega)$ but $u \notin H^4(\Omega) \cap H_0^1(\Omega)$ on uniform mesh with $\alpha = \frac{7}{2}$.

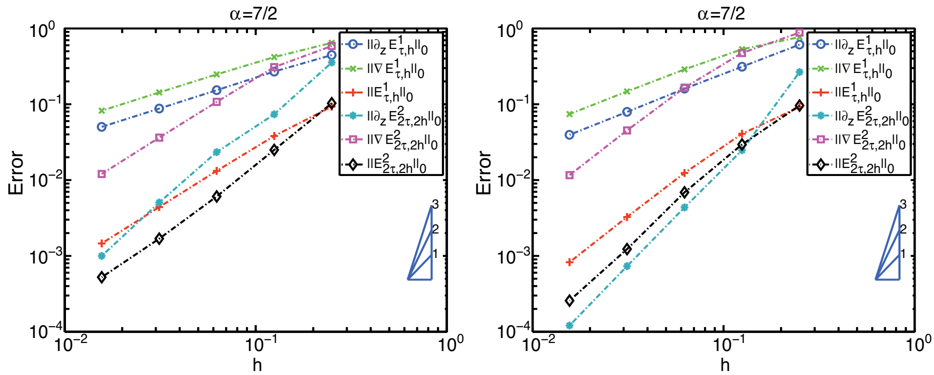
Mesh	$\|\partial_z E^1_{\tau,h}\|_0$	$\|\nabla E^1_{\tau,h}\|_0$	$\|E^1_{\tau,h}\|_0$	$\|\partial_z E^2_{2\tau,2h}\|_0$	$\|\nabla E^2_{2\tau,2h}\|_0$	$\|E^2_{2\tau,2h}\|_0$
(τ, h)	3.6464e-1	5.7872e-1	5.1500e-2	2.7410e-1	5.1377e-1	5.8417e-2
$(\tau, h)/2$	1.9471e-1	3.3751e-1	1.6859e-2	3.8479e-2	2.3041e-1	9.9351e-3
$(\tau, h)/4$	9.5925e-2	1.7454e-1	4.4768e-3	9.1457e-3	6.1517e-2	1.6964e-3
$(\tau, h)/8$	4.7720e-2	8.8034e-2	1.1365e-3	1.3487e-3	1.5838e-2	3.4219e-4
$(\tau, h)/16$	2.3828e-2	4.4114e-2	2.8523e-4	1.7770e-4	3.9902e-3	7.9012e-5

TABLE 8. Numerical results of $u \in H^3(\Omega) \cap H_0^1(\Omega)$ but $u \notin H^4(\Omega) \cap H_0^1(\Omega)$ on non-uniform mesh with $\alpha = \frac{7}{2}$.

Mesh	$\|\partial_z E^1_{\tau,h}\|_0$	$\|\nabla E^1_{\tau,h}\|_0$	$\|E^1_{\tau,h}\|_0$	$\|\partial_z E^2_{2\tau,2h}\|_0$	$\|\nabla E^2_{2\tau,2h}\|_0$	$\|E^2_{2\tau,2h}\|_0$
(τ, h)	6.1255e-1	7.6897e-1	9.2037e-2	2.6625e-1	8.8128e-1	9.6098e-2
$(\tau, h)/2$	3.1297e-1	5.3096e-1	4.0852e-2	2.4752e-2	4.7463e-1	2.9486e-2
$(\tau, h)/4$	1.6034e-1	2.8836e-1	1.2395e-2	4.3614e-3	1.6543e-1	6.8935e-3
$(\tau, h)/8$	7.9315e-2	1.4752e-1	3.2602e-3	7.3287e-4	4.5216e-2	1.2306e-3
$(\tau, h)/16$	3.9509e-2	7.4214e-2	8.2590e-4	1.2061e-4	1.1618e-2	2.5700e-4

see that the convergence rate is about $\frac{5}{2}$ which is greater than the theoretical result for the error function $\partial_z E^2_{2\tau,2h}$.

Furthermore, the convergence rates of the L^2-norm errors of six functions with α=1 are shown in Tables 9 and 10. Similar results can be obtained as $\alpha = \frac{7}{2}$ from Figure 7. Only the L^2-norm errors of function $E^2_{2\tau,2h}$ cannot reach optimal results. From above numerical results, we can see that the L^2-norm error obtained on the uniform mesh is smaller than that on the non-uniform mesh for six functions.



(a) (b)

FIGURE 6. $u \in H^3(\Omega) \cap H_0^1(\Omega)$ but $u \notin H^4(\Omega) \cap H_0^1(\Omega)$: (a) Convergence rates on uniform mesh, (b) Convergence rates on non-uniform mesh.

TABLE 9. Numerical results of $u \in H^n(\Omega) \cap H_0^1(\Omega)$ (for all $n \geq 1$) on uniform mesh with $\alpha = 1$.

Mesh	$\|\partial_z E^1_{\tau,h}\|_0$	$\|\nabla E^1_{\tau,h}\|_0$	$\|E^1_{\tau,h}\|_0$	$\|\partial_z E^2_{2\tau,2h}\|_0$	$\|\nabla E^2_{2\tau,2h}\|_0$	$\|E^2_{2\tau,2h}\|_0$
(τ, h)	2.9061e-1	5.9448e-1	6.0314e-2	8.3639e-2	3.4704e-1	2.3641e-2
$(\tau, h)/2$	1.3098e-1	3.0551e-1	1.5760e-2	1.7527e-2	9.3293e-2	3.1643e-3
$(\tau, h)/4$	6.3602e-2	1.5384e-1	3.9817e-3	3.1862e-3	2.3765e-2	4.3870e-4
$(\tau, h)/8$	3.1560e-2	7.7060e-2	9.9801e-4	5.6633e-4	5.9695e-3	7.1492e-5
$(\tau, h)/16$	1.5749e-2	3.8547e-2	2.4966e-4	1.0024e-4	1.4942e-3	1.4519e-5

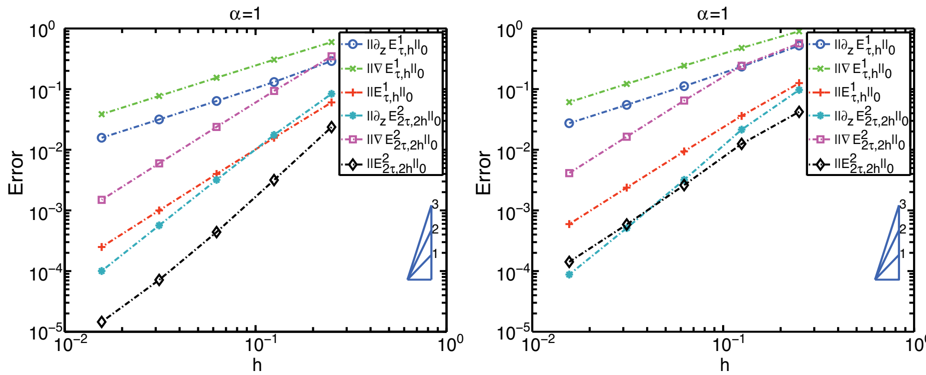
TABLE 10. Numerical results of $u \in H^n(\Omega) \cap H_0^1(\Omega)$ (for all $n \geq 1$) on non-uniform mesh with $\alpha = 1$.

Mesh	$\|\partial_z E^1_{\tau,h}\|_0$	$\|\nabla E^1_{\tau,h}\|_0$	$\|E^1_{\tau,h}\|_0$	$\|\partial_z E^2_{2\tau,2h}\|_0$	$\|\nabla E^2_{2\tau,2h}\|_0$	$\|E^2_{2\tau,2h}\|_0$
(τ, h)	5.2087e-1	8.9833e-1	1.2555e-1	9.6771e-2	5.6791e-1	4.2160e-2
$(\tau, h)/2$	2.3447e-1	4.7686e-1	3.6099e-2	2.1465e-2	2.4184e-1	1.2506e-2
$(\tau, h)/4$	1.1163e-1	2.4258e-1	9.3942e-3	3.1848e-3	6.4617e-2	2.6115e-3
$(\tau, h)/8$	5.5000e-2	1.2183e-1	2.3727e-3	5.1122e-4	1.6414e-2	5.8591e-4
$(\tau, h)/16$	2.7394e-2	6.0986e-2	5.9471e-4	8.7652e-5	4.1203e-3	1.4154e-4

Example 2. We consider the 3D Poisson equation with the following exact solution:

$$u = (1-x)(1-y)(1-z)\sin(\pi xyz)\exp(x+2z), \quad (x,\ y,\ z) \in \Omega = [0,1]^3.$$

Further the right-hand side function is generated by (3.1). Here the exact solution is not a symmetrical and variable separation function. Similarly, we use the same uniform mesh as in Example 1 and the non-uniform mesh produced by the random



(a) (b)

FIGURE 7. $u \in H^n(\Omega) \cap H_0^1(\Omega)$ (for all $n \geq 1$): (a) Convergence rates on uniform mesh, (b) Convergence rates on non-uniform mesh.

TABLE 11. Numerical results of the second example on uniform mesh.

Mesh	$\|\partial_z E^1_{\tau,h}\|_0$	$\|\nabla E^1_{\tau,h}\|_0$	$\|E^1_{\tau,h}\|_0$	$\|\partial_z E^2_{2\tau,2h}\|_0$	$\|\nabla E^2_{2\tau,2h}\|_0$	$\|E^2_{2\tau,2h}\|_0$
(τ, h)	1.6341e-1	1.9140e-1	2.2846e-2	1.8114e-2	1.1264e-1	9.0524e-3
$(\tau, h)/2$	8.1133e-2	9.6849e-2	6.1223e-3	2.1975e-3	3.0319e-2	1.4428e-3
$(\tau, h)/4$	4.0393e-2	4.8544e-2	1.5559e-3	3.5180e-4	7.7582e-3	2.6271e-4
$(\tau, h)/8$	2.0172e-2	2.4286e-2	3.9056e-4	6.2264e-5	1.9518e-3	5.7191e-5
$(\tau, h)/16$	1.0083e-2	1.2145e-2	9.7737e-5	1.1156e-5	4.8873e-4	1.3701e-5

numbers as follows to test the theoretical result:

$$T_{hx} = \{0,\ 0.15238,\ 0.22898,\ 0.91334,\ 1\},$$
$$T_{hy} = \{0,\ 0.53834,\ 0.82582,\ 0.99613,\ 1\},$$
$$T_{hz} = \{0,\ 0.078176,\ 0.10665,\ 0.44268,\ 1\}.$$

The numerical results are presented in Tables 11 and 12 in terms of six error functions. For the error functions $E^1_{\tau,h}$, we can obtain that the convergence rates are optimal for the L^2-norm error of $\partial_z E^1_{\tau,h}$, $\nabla E^1_{\tau,h}$ and $E^1_{\tau,h}$, respectively, on uniform and non-uniform meshes in Figure 8. Namely, the convergence rates are 1, 1 and 2, respectively. Moreover, the error obtained on the uniform mesh is smaller than that on the non-uniform mesh.

On the other hand, for the error functions $E^2_{2\tau,2h}$, the theoretical convergence rate of error is 2. We can obtain the superconvergence results for the L^2-norm error of two functions $\partial_z E^2_{2\tau,2h}$ and $\nabla E^2_{2\tau,2h}$, respectively, on uniform and non-uniform meshes in Figure 8, which confirm the theoretical result. Particularly, the convergence rate is about $\frac{5}{2}$ for the L^2-norm error of function $\partial_z E^2_{2\tau,2h}$ both on the uniform mesh and the non-uniform mesh. However, the convergence rate is about $\frac{5}{2}$ and 2 for the L^2-norm error of function $E^2_{2\tau,2h}$ on the uniform mesh and the non-uniform mesh, respectively. So we cannot obtain the optimal convergence rate of the L^2-norm error of function $E^2_{2\tau,2h}$.

TABLE 12. Numerical results of the second example on non-uniform mesh.

Mesh	$\|\partial_z E^1_{\tau,h}\|_0$	$\|\nabla E^1_{\tau,h}\|_0$	$\|E^1_{\tau,h}\|_0$	$\|\partial_z E^2_{2\tau,2h}\|_0$	$\|\nabla E^2_{2\tau,2h}\|_0$	$\|E^2_{2\tau,2h}\|_0$
(τ, h)	3.0657e-1	3.3787e-1	6.6852e-2	4.1378e-2	3.5245e-1	3.3689e-2
$(\tau, h)/2$	1.7751e-1	1.8628e-1	2.4408e-2	4.0079e-3	1.2275e-1	1.0275e-2
$(\tau, h)/4$	8.9755e-2	9.3809e-2	6.6931e-3	4.6364e-4	3.3408e-2	1.5456e-3
$(\tau, h)/8$	4.4867e-2	4.6929e-2	1.7108e-3	6.1552e-5	8.5745e-3	2.3655e-4
$(\tau, h)/16$	2.2428e-2	2.3466e-2	4.3005e-4	9.6900e-6	2.1593e-3	4.3286e-5

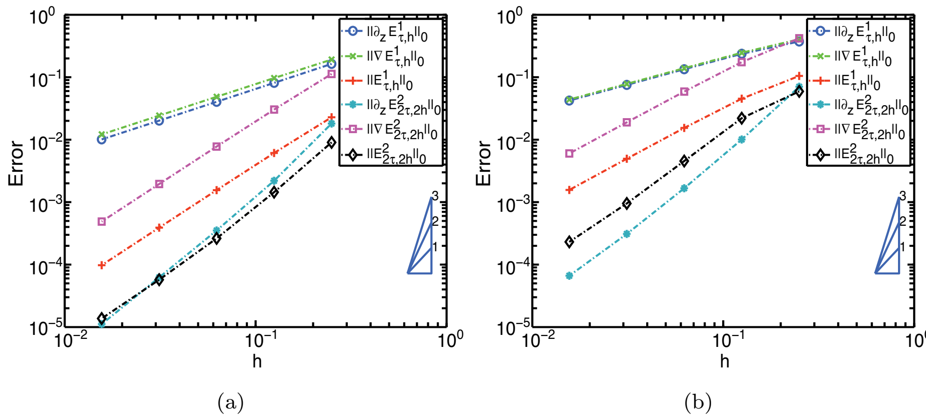


FIGURE 8. The second example: (a) Convergence rates on uniform mesh, (b) Convergence rates on non-uniform mesh.

6. CONCLUSIONS

In this work, we proposed a difference finite element method (DFE) based on the $P_1 - P_1$-conforming element for the 3D Poisson equation. The main idea of the DFE method is that a numerical solution of the 3D Poisson equation can be rewritten into a combination of numerical solutions of a series of 2D elliptic equations. The H^1-superconvergence of the second-order interpolation function $I^2_{2\tau} I^2_{2h} u_h$ in the $((x, y), z)$-direction to u is provided. Some numerical tests are presented to show the H^1-superconvergence results of the DFE method for the 3D Poisson equation. Furthermore, the DFE method can be extended to the solution of a general elliptic equation in a higher-dimensional domain, and its numerical analysis and numerical tests are similar. For example, the DFE method for a 4D general elliptic equation consists of combining the FD discretization based on the P_1-element in the t-direction and the FE discretization based on the P_1-element in the (x, y, z)-domain, or the FD discretization based on the P_1-element in the (z, t)-direction and the FE discretization based on the P_1-element in the (x, y)-domain.

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