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COMPUTING ANNIHILATORS OF CLASS GROUPS FROM DERIVATIVES OF L-FUNCTIONS

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Abstract. We computationally verify that certain group ring elements obtained from the first derivatives of abelian L-functions at the origin annihilate ideal class groups. In our test cases, these ideal class groups are connected with cyclic extensions of degree 6 over real quadratic fields.

1. Introduction

Let K/F be a finite Galois extension of number fields with abelian Galois group $G = \mathrm{Gal}(K/F)$. Let S contain the set of places of F that ramify in K, together with the infinite places of F, and let S_K denote the set of places of K that lie above those in S. Then there is an associated (imprimitive) equivariant L-function $\Theta(s) = \Theta^S_{K/F}(s)$ of the complex variable s, with values in the complex group ring $\mathbb{C}[G]$. When F is totally real, and K is totally complex, then typically $\Theta(0) \neq 0$, and the Brumer-Stark conjecture (see [GRT], [Ta]) posits that multiplying $\Theta(0)$ by any annihilator in $\mathbb{Z}[G]$ of the group of roots of unity in K yields a non-trivial element in the integral group ring $\mathbb{Z}[G]$ which annihilates the ideal class group Cl_K of K. Stickelberger's Theorem implies that this conjecture is true for $K/\mathbb{Q}$ where $\mathbb{Q}$ is the field of rational numbers, and [Sa1] then shows that it holds for the same field K and set S_K, when F is taken to be an intermediate field between $\mathbb{Q}$ and K. The conjecture holds for G of order 2 by [Ta], for G non-cyclic of order 4 by [Sa2], and for most cases with G of order 6 by [GRT]. See these works and [GrPo] for additional results and references.

Extending the Brumer-Stark conjecture, relatively recent results (see [Buc1], [Buc2], and [Bur]) suggest that when $\Theta(s)$ has order of vanishing equal to r at the origin, one should consider the rth derivative $\Theta^{(r)}(0)$, and that when multiplied by appropriate factors, this can still provide non-trivial elements in the integral group ring $\mathbb{Z}[G]$ which annihilate the S_K-ideal class group Cl^S_K of K. These appropriate factors are composed of "rationalizing factors" involving the $\mathbb{Z}[G]$-module structure of the group of S_K-units U^S_K in the ring of S_K-integers $\mathcal{O}^S_K$ of K, and "integralizing factors" involving the $\mathbb{Z}[G]$-module structure of the group of roots of unity μ_K in K. An additional idea is that idempotents in the rational group ring $\mathbb{Q}[G]$, when multiplied by the order $|G|$ of the group G, should provide another type of "integralizing element" that also leads to annihilators of the S_K-ideal class group. Results of these types are obtained for K abelian over $\mathbb{Q}$ in [Bur], as a consequence of the proof in [BG] and [Fl] of the Equivariant Tamagawa Number Conjecture

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for that case. Similarly, [MC] obtains results of this form for arbitrary relative quadratic extensions K/F, and for certain composites of such extensions.

Our object here is to use PARI/GP [GP] to conduct a computational test of these predictions in a setting that lies outside the realm of current theoretical results and can provide further insight into this phenomenon. Specifically, we arrange to investigate the case when the order of vanishing at the origin is $r = 1$, F is real quadratic, and the degree of the extension is $[K : F] = |G| = 6$, with K unramified outside of one finite prime and one infinite prime of F.

2. Annihilation of class groups

The question we wish to investigate computationally uses much of the formalism of the principal Stark conjecture of [Ta], but extends that conjecture from a rationality statement to an integrality statement and an annihilation statement. In this sense, it is similar to the Brumer-Stark conjecture. As in that conjecture, one assumes that the finite Galois extension K/F is abelian, but now rather than just the value $\Theta(0)$, which may be zero, one considers the derivative $\Theta'(0)$, or a higher derivative in order to obtain a non-zero value. See [Ta] for a more detailed introduction to these conjectures.

2.1. **The G-module X_K^S.** First define Y_K^S as usual to be the free abelian group on the set S_K of places of K that lie above those in S, and X_K^S to be the subgroup of Y_K^S consisting of those elements whose coefficient sum is 0. In other words, X_K^S is the kernel of the augmentation map from Y_K^S to the group of integers $\mathbb{Z}$. Here the abelian Galois group G acts on the set S_K on the left as usual: if $\sigma \in G$ and $|\ |_v$ is the normalized absolute value associated with the place v, then the normalized absolute value associated with $\sigma \cdot v$ is characterized by the property that $|\sigma(a)|_{\sigma \cdot v} = |a|_v$ for all $a \in K$. Using this natural action of G, X_K^S becomes a $\mathbb{Z}[G]$-submodule of Y_K^S. The module X_K^S is closely related to the module U_K^S of S_K-units of K, namely those elements of K whose valuation is trivial at all places of K not in S_K. Perhaps a more natural description of U_K^S is as the multiplicative group of units of the ring $\mathcal{O}_K^S$ of elements of K whose valuation is non-negative at each prime not corresponding to a place in S_K. The torsion subgroup of U_K^S is the group of roots of unity μ_K in K.

2.2. **Regulators.** If R is a ring containing $\mathbb{Z}$, then a $\mathbb{Z}[G]$-module M gives rise by extension of scalars to an $R[G]$-module $RM = R \otimes_{\mathbb{Z}} M = R[G] \otimes_{\mathbb{Z}[G]} M$. Letting $\mathbb{Q}$ denote the field of rational numbers and $\mathbb{R}$ denote the field of real numbers, Stark's principal conjecture calls for the choice of a $\mathbb{Q}[G]$-module isomorphism between $\mathbb{Q}U_K^S$ and $\mathbb{Q}X_K^S$. According to Herbrand's theorem in representation theory, such an isomorphism is known to exist since there is a logarithmic map $\lambda = \lambda_K$ that gives an isomorphism $\mathbb{R}U_K^S \to \mathbb{R}X_K^S$. To be specific, λ is the $\mathbb{R}$-linear extension of the map that sends $u \in U_K^S$ to $\sum_{v \in S_K} \log(|u|_v) \cdot v$. In order to pose an integrality question, we note that a $\mathbb{Q}[G]$-module isomorphism from $\mathbb{Q}U_K^S$ to $\mathbb{Q}X_K^S$ induces a $\mathbb{Z}[G]$-module homomorphism from the finitely generated $\mathbb{Z}[G]$-module U_K^S to a finitely generated $\mathbb{Z}[G]$-submodule of $\mathbb{Q}X_K^S$. Hence some non-zero integer multiple of this homomorphism gives a $\mathbb{Z}[G]$-module homomorphism

$$f : U_K^S \to X_K^S,$$

where the free $\mathbb{Z}$-module X_K^S may be viewed as a subset of $\mathbb{Q}X_K^S$. Since f maps a $\mathbb{Q}$-basis of $\mathbb{Q}U_K^S$ to a $\mathbb{Q}$-basis of $\mathbb{Q}X_K^S$, its kernel is finite and its image is of finite index. Note that the same reasoning can be applied to obtain an injective homomorphism from X_K^S to U_K^S/μ_K, since X_K^S has no finite subgroups.

The annihilation question that we test calls for the choice of such an f: a $\mathbb{Z}[G]$-module homomorphism from U_K^S to X_K^S with finite kernel and cokernel. Extending scalars from $\mathbb{Z}$ to $\mathbb{R}$ gives an $\mathbb{R}[G]$-module isomorphism $f_{\mathbb{R}} : \mathbb{R}U_K^S \to \mathbb{R}X_K^S$, and hence one gets an $\mathbb{R}[G]$-module automorphism $f_{\mathbb{R}} \circ \lambda^{-1} : \mathbb{R}X_K^S \to \mathbb{R}X_K^S$. Since $\mathbb{R}[G]$ is a semisimple Artinian ring, and hence $\mathbb{R}X_K^S$ is a projective module, the $\mathbb{R}[G]$-determinant

$$R(f) = \det_{\mathbb{R}[G]}(f_{\mathbb{R}} \circ \lambda^{-1}) \in \mathbb{R}[G]$$

is defined. This can be accomplished by extending $f_{\mathbb{R}} \circ \lambda^{-1}$ by the identity on a free complement of $\mathbb{R}X_K^S$ and taking the determinant of the extended map, which is an endomorphism of a free $\mathbb{R}[G]$-module. For the purposes of computation, one can take the determinant character by character, as follows. This approach requires extending $f_{\mathbb{R}}$ and λ $\mathbb{C}$-linearly to obtain maps $f_{\mathbb{C}}$ and $\lambda_{\mathbb{C}}$. This extension of scalars does not change the determinant, of course. For each ψ in the character group $\widehat{G}$ of the abelian group G, let $e_\psi := \frac{1}{|G|}\sum_{\sigma\in G}\psi(\sigma)\sigma^{-1} \in \mathbb{C}[G]$ be the associated idempotent. Then

$$\mathbb{C}[G] \cong \bigoplus_{\psi\in\widehat{G}} \mathbb{C}e_\psi,$$

so

$$\begin{aligned} R(f) &= R(f)\sum_{\psi\in\widehat{G}} e_\psi = \sum_{\psi\in\widehat{G}} R(f)e_\psi = \sum_{\psi\in\widehat{G}} \det_{\mathbb{C}[G]}(f_{\mathbb{C}} \circ \lambda_{\mathbb{C}}^{-1})e_\psi \\ &= \sum_{\psi\in\widehat{G}} \det_{\mathbb{C}e_\psi}(f_{\mathbb{C}} \circ \lambda_{\mathbb{C}}^{-1}|_{e_\psi\mathbb{C}X_K^S}). \end{aligned}$$

Since $\mathbb{C}[G]e_\psi = \mathbb{C}e_\psi \cong \mathbb{C}$ is a field, evaluating this last sum involves only linear algebra. For each ψ, we set $R(f,\psi) = \det_{\mathbb{C}e_\psi}(f_{\mathbb{C}} \circ \lambda_{\mathbb{C}}^{-1}|_{e_\psi\mathbb{C}X_K^S})$, so that

$$R(f,\psi) = R(f)e_\psi$$

and

$$R(f) = \sum_{\psi\in\widehat{G}} R(f,\psi) = \sum_{\psi\in\widehat{G}} R(f)e_\psi\,.$$

2.3. **Equivariant L-values.** Similarly, the values of the imprimitive equivariant L-function $\Theta_{K/F}^S(s)$ lie in $\mathbb{C}[G]$ and can thus be decomposed character by character. For a prime ideal $\mathfrak{q}$ of the ring of integers $\mathcal{O}_F$ of F not corresponding to a place in S, we denote the associated Frobenius automorphism in G by $\sigma_{\mathfrak{q}}$, and the absolute norm by $N\mathfrak{q}$. Then

$$\Theta_{K/F}^S(s) = \prod_{\mathfrak{q}\notin S}(1 - N\mathfrak{q}^{-s}\sigma_{\mathfrak{q}}^{-1})^{-1}.$$

The product converges for $\Re(s) > 1$ to an analytic function with values in $\mathbb{C}[G]$ in the sense that each component function is analytic. For a character ψ (always assumed irreducible) of G, the S-imprimitive L-function is

$$L_{K/F}^S(s,\psi) = \prod_{\mathfrak{q}\notin S}(1 - \psi(\sigma_{\mathfrak{q}})N\mathfrak{q}^{-s})^{-1}.$$

The function $L^S_{K/F}(s,\psi)$ has a meromorphic continuation to the whole complex plane and indeed can only have a pole when $s=1$ and ψ is the trivial character ψ_0, in which case there is a simple pole. At $s=0$, $L^S_{K/F}(s,\psi)$ has a zero of order

$$\begin{aligned}(2.1)\quad r^S(\psi) &= \dim_{\mathbb{C}}(e_\psi \mathbb{C} X^S_K) \\ &= |\{v \in S : v \text{ splits completely in the fixed field of the kernel of } \psi\}| - \delta_\psi,\end{aligned}$$

with $\delta_{\psi_0}=1$, and $\delta_\psi=0$ otherwise. So if S contains a finite place and we let r be the number of (infinite) places of S that split completely in K/F, then $r^S(\psi) \geq r$ for all characters ψ of G.

Now

$$\Theta^S_{K/F}(s) = \sum_{\psi \in \widehat{G}} L^S_{K/F}(s, \psi^{-1}) e_\psi$$

and one considers

$$\Theta^{S,(r)}_{K/F}(0) = \lim_{s\to 0} \frac{\Theta^S_{K/F}(s)}{s^r} = \lim_{s \to 0} \sum_{\psi \in \widehat{G}} \frac{L^S_{K/F}(s,\psi^{-1})}{s^r} e_\psi = \sum_{\psi\in\widehat{G}} L^{S,(r)}_{K/F}(0,\psi^{-1}) e_\psi\ .$$

Under our assumption that the Galois group G of K/F is abelian, the principal Stark conjecture for the set S and all characters ψ of G for which $r^S(\psi)=r$ is equivalent to the statement that

$$\Phi(f) = \Phi^{S,(r)}_{K/F}(f) = \Theta^{S,(r)}_{K/F}(0) R(f) \in \mathbb{Q}[G]$$

for any choice of f as in subsection 2.2. In other words, $R(f)$ is conjectured to be a rationalizing factor for $\Theta^{S,(r)}_{K/F}(0)$, and we define $\Phi(f)$ to be the product. The rationality of $\Phi(f)$ in turn is equivalent to the statement that

$$\Phi(f)\sum_\psi e_\psi = \sum_\psi \Theta^{S,(r)}_{K/F}(0) R(f) e_\psi = \sum_\psi L^{S,(r)}_{K/F}(0,\psi^{-1}) R(f,\psi)$$

lies in $\mathbb{Q}[G]$ when summed over all algebraic conjugates ψ of any single character ψ_1 for which $r^S(\psi_1)=r$. We refer to this as a sum over a conjugacy class of characters. The condition that $\Phi(f)\sum_\psi e_\psi$ lies in $\mathbb{Q}[G]$ for any such sum is what we actually verify computationally. We now proceed from the question of rationality to questions of integrality and annihilation of ideal class groups.

Question 2.1. Assume the principal Stark conjecture for the abelian extension K/F, the set S, and all characters ψ with $r^S(\psi)=r$. For each $\nu \in \operatorname{Ann}_{\mathbb{Z}[G]}(\mu_K)$, does $\nu\Phi^{S,(r)}_{K/F}(f)$ lie in $\mathbb{Z}[G]$ and annihilate Cl_K?

Remark 2.2.
1. Question 1.1 of David Burns in [MC] asks this for all choices of f, with Cl^S_K in place of Cl_K and allows for larger choices of S as well. Macias Castillo gives evidence for a positive answer to Burns's question based on the Equivariant Tamagawa Number Conjecture (ETNC). Indeed, Theorem 1.2 of [MC] shows that the ETNC for K/F would imply a positive answer to the p-part of Burns's question for primes p at which the p-part of μ_K is G-cohomologically trivial. For our computations, we choose extensions K/F for which the ETNC is not yet proved, and thus our results may be seen as additional support for the ETNC. Since $|G|=6$ and $|\mu_K|=2$ in our examples, μ_K is G-cohomologically trivial at all primes p except $p=2$. However, the prime $p=2$ does not create any exceptions in our computational results.

2. One may ask further how closely the $\mathbb{Z}[G]$-Fitting ideal of Cl_K^S or of Cl_K is approximated by the ideal in $\mathbb{Z}[G]$ generated by the elements $\nu\Phi_{K/F}^{S,(r)}(f)$ as ν ranges over $\mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)$ and f ranges over $\mathrm{Hom}_{\mathbb{Z}[G]}(U_K^S, X_K^S)$. We will comment on this in the course of discussing our computations.

Results of Burns and Buckingham in [Bur], [Buc1], and [Buc2] relate to the following question. Here, rather than consider all characters ψ for which $r^S(\psi) = r$, one considers a subset consisting of all algebraic conjugates of a single character.

Question 2.3. Assume the principal Stark conjecture for the abelian extension K/F, the set S, and all algebraic conjugates of a fixed character ψ_1 with $r^S(\psi_1) = r$. Define the idempotent $e = \sum_\psi e_\psi$, where the sum is over all algebraic conjugates ψ of ψ_1, so that $e \in \mathbb{Q}[G]$. Then does $\Phi_{K/F}^{S,(r)}(f)|G|^r e$ lie in $\mathbb{Z}[G]$ and annihilate Cl_K?

All of our computations produce a positive answer to Question 2.1. With our choices of $r = 1$, $|G| = 6$ and $|\mu_K| = 2$, this positive answer also implies a positive answer to Question 2.3. Indeed $|G|e \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)$ for each possible e in Question 2.3. So one may use $\nu = |G|e$ in Question 2.1 to deduce an affirmative answer to Question 2.3.

Theorem 2.4. *Let $p = 7$ (respectively, $p = 19$). Let F be one of the 11 (respectively, 15) real quadratic fields of discriminant $d_F < 2000$ with the following properties:*
i) *The prime p splits in F.*
ii) *There is an abelian extension K of F of relative degree 6 which is ramified only at one infinite prime $\mathfrak{p}_\infty^{(2)}$ of F and one prime $\mathfrak{p}$ above p in F.*
iii) *The class number h_K of K is strictly larger than 1.*

For each such F and K, put $S = \{\mathfrak{p}, \mathfrak{p}_\infty^{(1)}, \mathfrak{p}_\infty^{(2)}\}$ and $G = \mathrm{Gal}(K/F) = \langle\sigma\rangle$. The $\mathbb{Z}[G]$-module homomorphism $f : U_K^S \to X_K^S$ that we construct via (3.1) has an associated integer invariant $N = N(f)$ given in Table 1 (respectively, Table 2) such that $NX_K^S \subseteq f(U_K^S)$.

Then:
1) *The principal Stark conjecture holds numerically for K/F and for each of the three characters ψ of G with $r^S(\psi) = 1$.*
2) *The element $2\Phi_{K/F}^{S,(1)}(f)$ in $\mathbb{R}[G]$ is numerically indistinguishable from an element of $\mathbb{Z}[G](1-\sigma^3)$ to an accuracy of at least 10^{-30}. The refined abelian Stark conjecture for K/F then implies that $2\Phi_{K/F}^{S,(1)}(f) \in \mathbb{Z}[G](1-\sigma^3)$.*
3) *Assuming $2\Phi_{K/F}^{S,(1)}(f) \in \mathbb{Z}[G]$, Question 2.1 has an affirmative answer for each $\nu \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)$. Thus,*

$$\mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)\Phi_{K/F}^{S,(1)}(f) \subseteq \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}_K) \cap (1-\sigma^3),$$

and furthermore,

$$(2N)(\mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}_K) \cap (1-\sigma^3)) \subseteq \mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)\Phi_{K/F}^{S,(1)}(f).$$

Proof. The proof is computational and the example given in subsection 4.1 shows in detail all of the steps required to give a full confirmation in each case. □

See the tables in subsection 4.2 for more explicit results. For instance, the factor $2N$ can often be replaced by N. There were 18 total examples computed with respect to the prime $p = 7$ and 22 total examples computed with respect to the prime $p = 19$.

3. Investigating the questions

3.1. Choosing the field extensions K/F. As a class of examples that can provide some new insight while keeping computations feasible, we seek to take K to be an abelian extension of relative degree 6 over a real quadratic field F and take $r = 1$. Since much has already been proved for K abelian over $\mathbb{Q}$, we want K to be non-abelian over $\mathbb{Q}$. To simplify the S-unit computations, we wish to have a single finite prime in S_K, hence only one finite prime may ramify in K/F, and it must not split at all. In order to be able to generate such an extension, we would like to use the rank one abelian Stark conjecture (see [ST], whose notation and conventions we follow) with one infinite prime of F splitting completely in K. These considerations lead us to proceed as follows.

To a given real quadratic field F of discriminant d_F we associate a canonically defined polynomial $f[d_F]$ as follows:

$$f[d_F] = \begin{cases} x^2 - d_F/4 & \text{if} \quad d_F \equiv 0 \bmod (4), \\ x^2 - x - (d_F - 1)/4 & \text{if} \quad d_F \equiv 1 \bmod (4). \end{cases}$$

If $\theta \in \overline{\mathbb{Q}}$ is a root of $f[d_F]$, then $F = \mathbb{Q}(\theta)$ and $[1, \theta]$ is an integral basis for the ring of integers $\mathcal{O}_F$. A given $f[d_F]$ always has one positive real root, denoted by $\theta^{(1)}$, and one negative real root, denoted by $\theta^{(2)}$. This convention allows us to fix the two real embeddings of F into $\mathbb{R}$ as follows:

$$\begin{aligned} &e_1 : F \hookrightarrow \mathbb{R} \quad \text{is defined by the map} \quad a + b\theta \mapsto a + b\theta^{(1)}, \ (a, b \in \mathbb{Q}), \\ &e_2 : F \hookrightarrow \mathbb{R} \quad \text{is defined by the map} \quad a + b\theta \mapsto a + b\theta^{(2)}. \end{aligned}$$

The two infinite primes corresponding to the two real embeddings of F are denoted by $\mathfrak{p}_\infty^{(1)}$ and $\mathfrak{p}_\infty^{(2)}$, respectively. Let p be a small rational prime congruent to 1 modulo 6 which splits into a product of two distinct prime ideals $\mathfrak{p}$ and $\mathfrak{p}'$ in $\mathcal{O}_F$. We want K to be an abelian extension of F of relative degree 6 which is ramified at $\mathfrak{p}$ and at the infinite prime $\mathfrak{p}_\infty^{(2)}$, and at no other primes. The set S is then chosen to be $S = \{\mathfrak{p}, \mathfrak{p}_\infty^{(1)}, \mathfrak{p}_\infty^{(2)}\}$.

To construct an extension field K of this type, we first compute the ray class group $H(\widetilde{\mathfrak{m}})$ of F modulo the generalized modulus $\widetilde{\mathfrak{m}} = \mathfrak{p}\mathfrak{p}_\infty^{(2)}$, which we assume from now on to be of order divisible by 6 since an appropriate example of K does not exist otherwise. Let $\widehat{H(\widetilde{\mathfrak{m}})}$ denote the set of all ray class characters modulo $\widetilde{\mathfrak{m}}$, i.e., the set of all homomorphisms from $H(\widetilde{\mathfrak{m}})$ to $\mathbb{C}^\times$. An element $\chi \in \widehat{H(\widetilde{\mathfrak{m}})}$ is defined on classes of ideals in F but it is useful to think of χ as defined on individual ideals relatively prime to $\mathfrak{p}$ as well. An appropriate example of K will only exist if there is a character $\chi \in \widehat{H(\widetilde{\mathfrak{m}})}$ having the following properties:

(i) χ is of order 6,
(ii) the conductor $\mathfrak{f}(\chi)$ of χ is precisely equal to $\widetilde{\mathfrak{m}}$.

We note that the infinite prime $\mathfrak{p}_\infty^{(2)}$ appears in the conductor $\mathfrak{f}(\chi)$ if and only if $\chi((\beta)) \neq 1$, where the principal ideal (β) is generated by any algebraic integer $\beta \in \mathcal{O}_F$ satisfying the following three conditions: $\beta \equiv 1 \pmod{\mathfrak{p}}$, $\beta^{(1)} > 0$, and $\beta^{(2)} < 0$ (it is easy to show that such an integer β always exists). If χ has properties

(i) and (ii), then the corresponding abelian L-function

$$L_{\widetilde{\mathfrak{m}}}(s,\chi)=\sum\frac{\chi(\mathfrak{a})}{\mathrm{N}\mathfrak{a}^s},$$

with the sum running over all integral ideals $\mathfrak{a}\subseteq\mathcal{O}_F$ relatively prime to $\mathfrak{p}$, vanishes to precisely first order at $s=0$ upon analytic continuation (see [ST], pp. 254–255). The L-functions $L_{\widetilde{\mathfrak{m}}}(s,\chi^3)$ and $L_{\widetilde{\mathfrak{m}}}(s,\chi^{-1})$ also have first order zeros at $s=0$ and $\mathfrak{f}(\chi^3)=\mathfrak{f}(\chi^{-1})=\widetilde{\mathfrak{m}}$ as well. On the other hand, the L-functions $L_{\widetilde{\mathfrak{m}}}(s,\chi^0)$, $L_{\widetilde{\mathfrak{m}}}(s,\chi^2)$, and $L_{\widetilde{\mathfrak{m}}}(s,\chi^4)$ all have at least second order zeros at $s=0$ (actually, $L_{\widetilde{\mathfrak{m}}}(s,\chi^0)$ vanishes to precisely second order at $s=0$ since $|S|=3$). By class field theory, there exists a Galois extension field K of F such that $G=\mathrm{Gal}(K/F)$ is cyclic of order 6 and a prime ideal $\mathfrak{q}\subset\mathcal{O}_F$ with $\mathfrak{q}\neq\mathfrak{p}$ splits completely in K if and only if $\chi(\mathfrak{q})=1$. This characterization of the primes splitting completely in a Galois extension K of F defines K uniquely by a theorem of Bauer (see [Ja], Cor. 5.5 on p. 136) and for this reason we refer to K as "the class field corresponding to χ". Note, however, that no explicit construction of K is provided by class field theory. The remarkable feature of Stark's rank one abelian conjecture is that a fast and efficient algorithm is made available for the explicit generation of K over F using only the values of the non-zero first derivatives $L'_{\widetilde{\mathfrak{m}}}(0,\chi^j)$, $j=1,3,5$. By the conductor-discriminant formula (see [Ha]), the relative discriminant $d(K/F)$ will be equal to $\mathfrak{p}^5$ if $\mathfrak{p}$ appears in the conductor $\mathfrak{f}(\chi^2)$, and will be $\mathfrak{p}^3$ otherwise. In the former case, the ramification index $e_{\mathfrak{p}}$ of $\mathfrak{p}$ is 6 and so only one prime ideal $\mathfrak{P}\subset\mathcal{O}_K$ lies over $\mathfrak{p}$, as desired. In the latter case, the ray class character χ^2 has trivial conductor and so its primitive version $(\chi^2)_{pr}$ takes on a non-zero value on the ideal $\mathfrak{p}$. If $(\chi^2)_{pr}(\mathfrak{p})\neq 1$, then $e_{\mathfrak{p}}=2$, the inertial degree $f_{\mathfrak{p}}$ is 3, and again only one prime $\mathfrak{P}$ lies over $\mathfrak{p}$ (in none of our examples did we have $(\chi^2)_{pr}(\mathfrak{p})=1$ which implies that in all of our examples both $L_{\widetilde{\mathfrak{m}}}(s,\chi^2)$ and $L_{\widetilde{\mathfrak{m}}}(s,\chi^4)$ vanish to precisely second order at $s=0$). The infinite prime $\mathfrak{p}_\infty^{(1)}$ will split completely in the extension K/F (this simply means that every embedding of K into $\mathbb{C}$ extending the embedding $e_1: F\hookrightarrow\mathbb{R}$ defined above is a real embedding), and $\mathfrak{p}_\infty^{(2)}$ ramifies. The algorithm used for obtaining the minimal polynomial $f_\varepsilon(x)\in\mathcal{O}_F[x]$ of the conjectured Stark unit $\varepsilon\in U_K$ is described in detail on pages 258–259 of [ST]. According to Stark's conjecture (see Theorem 1 on p. 66 of [Sta]), the field K described above is obtained by adjoining a root ρ of $f_\varepsilon(x)$ to F. Even if the polynomial $f_\varepsilon(x)$ is secured via an unproven conjecture, an independent verification can be made that $F(\rho)$ is in fact equal to the field K uniquely described as above by class field theory (see subsection 4.1 for further details on how this independent verification is actually carried out in practice). Note that K will not be a Galois extension of $\mathbb{Q}$ since half of its embeddings into $\mathbb{C}$ will be real and the other half non-real.

The match-up between the abelian L-functions $L_{\widetilde{\mathfrak{m}}}(s,\chi^j)$, $0\le j\le 5$, and the L-functions $L^S_{K/F}(s,\psi)$ for $\psi\in\widehat{G}$, introduced in subsection 2.3, is made via the Artin map. From the generating ray class character χ possessing properties (i) and (ii) above, we find a first degree prime ideal $\mathfrak{q}\subset\mathcal{O}_F$ such that $\chi(\mathfrak{q})=e^{2\pi i/6}$. The corresponding Frobenius automorphism $\sigma_{\mathfrak{q}}\in G$ is a generator for G and we can identify χ as an element of $\widehat{G}$ by setting $\chi(\sigma_{\mathfrak{q}})=e^{2\pi i/6}$. With this identification, $L_{\widetilde{\mathfrak{m}}}(s,\chi^j)=L^S_{K/F}(s,\chi^j)$ for $0\le j\le 5$. We will be investigating Question 2.1 when $r=1$ and the characters $\psi\in\widehat{G}$ with $r^S(\psi)=1$ are $\psi\in\{\chi,\chi^3,\chi^5\}$.

3.2. **Choosing the homomorphism f.** Assume an abelian extension K/F of relative degree six has been singled out as in subsection 3.1. Since the extension K/F is cyclic of degree 6, there is an intermediate field K^+ of degree 3 over F, and $\mathfrak{p}_\infty^{(2)}$ necessarily splits completely in this extension of odd degree so that K^+ is a totally real field. Let $H = \mathrm{Gal}(K/K^+)$, of order 2. The set S_K of primes in K above those in S consists of six real primes which are conjugates of a prime $\mathfrak{P}_\infty^{(1)}$ above $\mathfrak{p}_\infty^{(1)}$, three complex primes which are conjugates of a prime $\mathfrak{P}_\infty^{(2)}$ above $\mathfrak{p}_\infty^{(2)}$, and one finite prime $\mathfrak{P}$ above $\mathfrak{p}$. Then Y_K^S is the free abelian group on these 10 generators, and X_K^S is the subgroup of elements whose coefficient sum is zero. Since $\mathfrak{P}$, being the unique prime above $\mathfrak{p}$ in K, is necessarily stable under $G = \mathrm{Gal}(K/F)$, while $\mathfrak{P}_\infty^{(2)}$ is stable under H, one finds that

$$X_K^S = \mathbb{Z}[G] \cdot (\mathfrak{P}_\infty^{(1)} - \mathfrak{P}) \oplus \mathbb{Z}[G] \cdot (\mathfrak{P}_\infty^{(2)} - \mathfrak{P}) \cong \mathbb{Z}[G] \oplus \mathbb{Z}[G/H].$$

We will find it useful to identify X_K^S with $\mathbb{Z}[G] \oplus \mathbb{Z}[G/H]$ via this isomorphism of $\mathbb{Z}[G]$-modules.

Next we consider the $\mathbb{Z}[G]$ module U_K^S of S_K-units of K. Since K has real embeddings, the group of roots of unity in K is just $\mu_K = \pm 1$. Thus $U_K^S/\{\pm 1\}$ must contain a submodule of finite index isomorphic to X_K^S, and hence to the module $\mathbb{Z}[G] \oplus \mathbb{Z}[G/H]$ which we are identifying with X_K^S. Our goal is to specify a $\mathbb{Z}[G]$-module homomorphism $f : U_K^S \to X_K^S$ with finite kernel, but we will do so by first choosing a monomorphism between $\mathbb{Z}[G] \oplus \mathbb{Z}[G/H]$ and a submodule of $U_K^S/\{\pm 1\}$. Since $U_K^S/\{\pm 1\}$ contains a module isomorphic to $\mathbb{Z}[G]$, we can select an element $u \in U_K^S$ whose six G-conjugates are independent over $\mathbb{Z}$. We test elements of a set of generators for U_K^S (i.e. fundamental S-units) until we find such a u. Likewise, $U_K^S/\{\pm 1\}$ contains a module isomorphic to $\mathbb{Z}[G/H]$ which is complementary to the one generated by u. It is necessarily generated by an element $\overline{v}$ of $U_K^S/\{\pm 1\}$ that is fixed by H. Equivalently, $\overline{v}$ is represented by an element $v \in U_K^S$ whose square is fixed by H and hence lies in K^+. We consider elements of a set of generators for U_K^S until we find such a v whose three G-conjugates are $\mathbb{Z}$-independent and together with the conjugates of u generate a free abelian group of rank 9. Then we define

$$g : \mathbb{Z}[G] \oplus \mathbb{Z}[G/H] \to U_K^S/\{\pm 1\}$$

by

$$g(\gamma, \rho) = \overline{u^\gamma v^\rho}.$$

By our choices of u and v, g is injective and its image is a free $\mathbb{Z}$-module of rank 9, hence is of finite index. We set N equal to the exponent of the finite abelian group $(U_K^S/\{\pm 1\})/\,\mathrm{im}(g)$. In other words, N is the index of the subgroup generated by -1 and the conjugates of u and v in U_K^S. Then we can choose f to be the composite of the $\mathbb{Z}[G]$-module maps

$$U_K^S \xrightarrow{N} (U_K^S)^N \twoheadrightarrow (U_K^S/\{\pm 1\})^N \hookrightarrow \mathrm{im}(g) \xrightarrow{g^{-1}} \mathbb{Z}[G] \oplus \mathbb{Z}[G/H] \cong X_K^S. \tag{3.1}$$

3.3. **Computing the regulator $R(f)$.** With f defined as above, we set

$$T = \lambda \circ f_{\mathbb{R}}^{-1}$$

so that

$$R(f) = \det_{\mathbb{R}[G]}(f_{\mathbb{R}} \circ \lambda^{-1}) = \det_{\mathbb{R}[G]}(T^{-1}) = \det_{\mathbb{R}[G]}(T)^{-1}.$$

It then suffices to compute

$$\det_{\mathbb{R}[G]}(T) = \det_{\mathbb{R}[G]}(\lambda \circ f_{\mathbb{R}}^{-1}) = \det_{\mathbb{R}[G]}(\frac{1}{N}\lambda \circ g_{\mathbb{R}}).$$

Here we are identifying $\mathbb{R}X_K^S$ with $\mathbb{R}[G] \oplus \mathbb{R}[G/H]$. Then $(1,\bar{0})$ and $(0,\bar{1})$ generate this $\mathbb{R}[G]$-module and it suffices to determine $T(1,\bar{0})$ and $T(0,\bar{1})$. Now

$$\text{(3.2)}\quad T(1,\bar{0}) = \lambda(f_{\mathbb{R}}^{-1}(1,\bar{0})) = \lambda(g(1,\bar{0})^{1/N}) = \frac{1}{N}\lambda(u) = \frac{1}{N}\sum_{v\in S_K}\log(|u|_v)\cdot v$$

$$= \frac{1}{N}\left[\sum_{\tau\in G}\log(|u^{\tau^{-1}}|_{\mathfrak{P}_\infty^{(1)}})\,\tau\cdot\mathfrak{P}_\infty^{(1)} + \sum_{\tau\in G/H}\log(|u^{\tau^{-1}}|_{\mathfrak{P}_\infty^{(2)}})\,\tau\cdot\mathfrak{P}_\infty^{(2)} + \log|u|_{\mathfrak{P}}\mathfrak{P}\right]$$

$$= \alpha\cdot(\mathfrak{P}_\infty^{(1)} - \mathfrak{P}) + \bar{\beta}\cdot(\mathfrak{P}_\infty^{(2)} - \mathfrak{P})$$

in $\mathbb{R}X_K^S$, with $(\alpha,\bar{\beta}) \in \mathbb{R}[G]\oplus\mathbb{R}[G/H]$. Hence under our identification of $\mathbb{R}X_K^S$ with $\mathbb{R}[G]\oplus\mathbb{R}[G/H]$, we have

$$T(1,\bar{0}) = (\alpha,\bar{\beta})$$

and the coefficients of $\alpha\in\mathbb{R}[G]$ and $\bar{\beta}\in\mathbb{R}[G/H]$ may be read off from the coefficients $\frac{1}{N}\log(|u^{\tau^{-1}}|_{\mathfrak{P}_\infty^{(1)}})$ and $\frac{1}{N}\log(|u^{\tau^{-1}}|_{\mathfrak{P}_\infty^{(2)}})$, respectively. Similarly,

$$T(0,\bar{1}) = (\alpha',\bar{\beta}').$$

In this case, $\bar{1}\in\mathbb{R}[G/H]$ is necessarily H-invariant, hence so are α' and $\bar{\beta}'$.

A generating character χ of $\widehat{G}$ was nailed down at the end of subsection 3.1 and with it a generating automorphism $\sigma = \sigma_{\mathfrak{q}}$ for G such that $\chi(\sigma) = e^{2\pi i/6}$. We also saw there that the characters $\psi\in\widehat{G}$ with $r^S(\psi) = 1$ are precisely $\psi\in\{\chi,\chi^3,\chi^5\}$. We refer to these three characters as the odd characters and the remaining three as the even characters. So the odd characters are non-trivial on $H = \mathrm{Gal}(K/K^+)$, and the even characters are trivial on H. Let ψ be an odd character. Then $r^S(\psi) = 1$ is equal to the dimension over $\mathbb{C}$ of $e_\psi\mathbb{C}X_K^S \cong e_\psi(\mathbb{C}[G]\oplus\mathbb{C}[G/H])$ and indeed we see that $e_\psi\mathbb{C}[G] = \mathbb{C}e_\psi$ and $e_\psi\mathbb{C}[G/H] = 0$. Extend ψ by $\mathbb{C}$-linearity to a $\mathbb{C}$-valued function on $\mathbb{C}[G]$. From $T(1,\bar{0}) = (\alpha,\bar{\beta})$, we see that the restriction T_ψ of T to the ψ-component is determined on this 1-dimensional $\mathbb{C}$-space by $T(e_\psi,0) = (e_\psi(\alpha),0) = (\psi(\alpha)e_\psi,0)$. Hence for each odd character ψ, we have $\det(T_\psi) = \psi(\alpha)e_\psi$ and

$$R(f,\psi) = R(f)e_\psi = \det(T_\psi)^{-1}e_\psi = (1/\psi(\alpha))e_\psi.$$

Finally, with $r = 1$, we have

$$\text{(3.3)}\quad \Phi(f) = \Theta_{K/F}^{S,(1)}(0)R(f) = \Theta_{K/F}^{S,(1)}(0)R(f)\sum_{\psi\in\widehat{G}}e_\psi = \sum_{\psi\in\widehat{G}}\Theta_{K/F}^{S,(1)}(0)R(f)e_\psi$$

$$= \sum_{\psi\in\widehat{G}}L_{K/F}^{S,(1)}(0,\psi^{-1})R(f)e_\psi = \sum_{\psi\ \text{odd}}\frac{L_{K/F}^{S,(1)}(0,\psi^{-1})}{\psi(\alpha)}e_\psi\,.$$

The idempotents for $r = 1$ in Question 2.3, are $e_1 = e_\chi + e_{\chi^{-1}}$ and $e_3 = e_{\chi^3}$. The first is the sum of the idempotents for the two conjugates of χ, and the second is the idempotent for the only conjugate of χ^3. We have

$$\Phi(f)e_1 = \frac{L^{S,(1)}_{K/F}(0,\chi^{-1})}{\chi(\alpha)}e_\chi + \frac{L^{S,(1)}_{K/F}(0,\chi)}{\chi^{-1}(\alpha)}e_{\chi^{-1}} \tag{3.4}$$
$$= \sum_{j=0}^{5} 2\Re\left(\frac{L^{S,(1)}_{K/F}(0,\chi)}{\chi^{-1}(\alpha)}\chi(\sigma)^j\right)\frac{1}{6}\sigma^j = \frac{1}{6}\sum_{j=0}^{5} 2\Re\left(\frac{L^{S,(1)}_{K/F}(0,\chi)}{\chi^{-1}(\alpha)}e^{2\pi i j/6}\right)\sigma^j,$$

and since $\chi^3(\sigma) = -1$,

$$\Phi(f)e_3 = \frac{L^{S,(1)}_{K/F}(0,\chi^3)}{\chi^3(\alpha)}e_{\chi^3} = \frac{1}{6}\frac{L^{S,(1)}_{K/F}(0,\chi^3)}{\chi^3(\alpha)}\sum_{j=0}^{5}(-1)^j\sigma^j. \tag{3.5}$$

Finally, since $L^{S,(1)}_{K/F}(0,\psi) = 0$ for ψ even,

$$\Phi(f) = \Phi(f)e_1 + \Phi(f)e_3. \tag{3.6}$$

These formulas, (3.4) through (3.6), are what we use to compute $\Phi(f)$.

4. A detailed example and some tabular data

4.1. **Example.** In this section we present a detailed example that illustrates the main points we encountered in our computations. We make full use of the notation introduced in the earlier sections of this paper without further comment. All computations were carried out using the PARI/GP [GP] software package.

Let $d_F = 253$ and let $\theta \in \overline{\mathbb{Q}}$ be a root of $f[253] = x^2 - x - 63$. Our basefield is the real quadratic field $F = \mathbb{Q}(\theta)$. The prime 19 splits into a product of two distinct prime ideals in $\mathcal{O}_F$. Let $\mathfrak{p}$ denote the prime ideal in $\mathcal{O}_F$ lying over 19 that has Hermite normal form equal to $[19, 16; 0, 1]$ with respect to the ordered integral basis $[1, \theta]$ of $\mathcal{O}_F$. The ray class group $H(\widetilde{\mathfrak{m}})$ of F modulo $\widetilde{\mathfrak{m}} = \mathfrak{p}\mathfrak{p}_\infty^{(2)}$ is a cyclic group of order 6 and the field K is the corresponding ray class field. For our generating character $\chi \in \widehat{G}$ (a preliminary check is made using PARI that $\mathfrak{f}(\chi) = \widetilde{\mathfrak{m}}$), we have

$$L^{S,(1)}_{K/F}(0,\chi) = 7.9515677980422774160151\ldots - i\ 18.153418744976319567593\ldots$$

and

$$L^{S,(1)}_{K/F}(0,\chi^3) = 13.635741502568012881693\ldots.$$

The relative discriminant $d(K/F)$ is equal to $\mathfrak{p}^5$ since $\mathfrak{p}$ appears in the conductor of the character χ^2. The unique polynomial singled out by the algorithm based upon Stark's conjecture and satisfied by the Stark units in U_K is

$$\begin{aligned} f_\varepsilon(x) = {} & x^6 - (122389 + 16421\,\theta)x^5 + (2739273249 + 367540329\,\theta)x^4 \\ & -(11234100459463 + 1507328628111\,\theta)x^3 + (2739273249 + 367540329\,\theta)x^2 \\ & -(122389 + 16421\,\theta)x + 1. \end{aligned}$$

For a given root ρ of $f_\varepsilon(x)$, we still need to check that $F(\rho) = K$. The relative number field extension commands in PARI allow us to verify that $\mathfrak{p}_\infty^{(1)}$ splits completely in the extension $F(\rho)/F$ and that $\mathfrak{p}_\infty^{(2)}$ ramifies in this relative extension. We also verify that the relative discriminant $d(F(\rho)/F)$ is precisely equal to $\mathfrak{p}^5$. Using the PARI command **nfgaloisconj**, we also find that $F(\rho)/F$ is a relative Galois extension with abelian Galois group, which confirms that $K = F(\rho)$ (the possibility that the conductor of the extension $F(\rho)/F$ is $\mathfrak{p}^2\mathfrak{p}_\infty^{(2)}$ is ruled out by the fact that the ray class group of F modulo $\mathfrak{p}^2\mathfrak{p}_\infty^{(2)}$ is also a cyclic group of order 6). The unique prime ideal $\mathfrak{P} \subset \mathcal{O}_K$ lying over $\mathfrak{p}$ is a principal ideal and so Cl_K and Cl_K^S are identical; in this case, they are both cyclic groups of order 13.

A defining polynomial for K with relatively small coefficients is found to be

$$p_K(x) = x^{12} - 23x^{10} - 43x^8 + 2316x^6 + 3458x^4 - 22401x^2 - 13851.$$

So we may assume $K = \mathbb{Q}[x]/(p_K(x))$. We let η represent the root $\overline{x}$ of $p_K(x)$ in K. Since $p_K(x)$ is an even polynomial, the unique automorphism of K of order two sends η to $-\eta$.

The action of $\mathrm{Gal}(K/F)$ on Cl_K naturally corresponds to a homomorphism from $\mathrm{Gal}(K/F)$ to $\mathrm{Aut}(\mathrm{Cl}_K)$ that we determine precisely. A first degree prime ideal $\mathfrak{q} \subset \mathcal{O}_F$ such that $\chi(\mathfrak{q}) = e^{2\pi i/6}$ (see the last part of subsection 3.1) lies over the prime 31 and has Hermite normal form equal to $[31, 18; 0, 1]$ with respect to the ordered integral basis $[1, \theta]$ of $\mathcal{O}_F$. The corresponding Frobenius automorphism $\sigma = \sigma_\mathfrak{q}$ generating $\mathrm{Gal}(K/F)$ is given by

$$\begin{aligned}(4.1)\quad \sigma(\eta) = &-\frac{1855730}{9134914851}\eta^{11} + \frac{43229776}{9134914851}\eta^9 + \frac{53424980}{9134914851}\eta^7 \\ &- \frac{449616606}{1014990539}\eta^5 - \frac{4975444921}{9134914851}\eta^3 + \frac{7433128427}{3044971617}\eta.\end{aligned}$$

We compute that $\sigma^3(\eta) = -\eta$, in accord with the theory. Applying σ to an ideal representing a generator of Cl_K reveals that σ acts on Cl_K as the automorphism which raises each ideal class to the 4th power. So σ corresponds to $\overline{4}$ in the identification of $\mathrm{Aut}(\mathrm{Cl}_K) \cong \mathrm{Aut}(\mathbb{Z}/13\mathbb{Z})$ with $(\mathbb{Z}/13\mathbb{Z})^\times$, which is cyclic of order 12. Note that $\overline{4}$ has order 6 in $(\mathbb{Z}/13\mathbb{Z})^\times$. Since σ is a generator of $\mathrm{Gal}(K/F)$ of order 6, we see that $\mathrm{Gal}(K/F)$ injects into $\mathrm{Aut}(\mathrm{Cl}_K)$ in this example.

Next, we obtain a set of fundamental S_K-units for K (we will just say S-units from now on). We test each S-unit in this set until we find one whose 6 conjugates over F are independent. Such an S-unit u is guaranteed to exist by the theory we have described, but in general, we may need to use combinations of the fundamental S-units to get one. In this case, as in several others, we are able to use the last of eight fundamental units provided by PARI, namely

$$\begin{aligned}(4.2)\quad u = &\frac{42367153}{164428467318}\eta^{11} - \frac{4172215}{9134914851}\eta^{10} - \frac{383767237}{82214233659}\eta^9 + \frac{59256560}{9134914851}\eta^8 \\ &- \frac{2062927913}{82214233659}\eta^7 + \frac{494687602}{9134914851}\eta^6 + \frac{11908104422}{27404744553}\eta^5 - \frac{1540043035}{3044971617}\eta^4 \\ &+ \frac{283941384893}{164428467318}\eta^3 - \frac{25919729621}{9134914851}\eta^2 - \frac{1668724147}{9134914851}\eta - \frac{3986411931}{2029981078}.\end{aligned}$$

Since $H = \langle\sigma^3\rangle$, we obtain a suitable $v \in U_K^S$ by considering combinations of fundamental S-units until we find one whose square is fixed by σ^3, and whose 3 independent Galois conjugates, along with the 6 Galois conjugates of u, provide 9 independent S-units of K. In this case, we find that

$$\begin{aligned}(4.3)\quad v = &\frac{1687522}{82214233659}\eta^{11} - \frac{49704455}{82214233659}\eta^9 + \frac{366962453}{82214233659}\eta^7 + \frac{350070643}{27404744553}\eta^5 \\ &- \frac{24198750739}{82214233659}\eta^3 - \frac{1270777661}{9134914851}\eta\end{aligned}$$

is a generator of $\mathfrak{P}$ that satisfies these conditions. Indeed, since v is an odd polynomial in η and $\sigma^3(\eta) = -\eta$, one clearly sees that $\sigma^3(v) = -v$.

The 9 independent S-units, together with -1 now generate a subgroup of finite index in U_K^S. We determine the exponent N of the corresponding quotient group by using the Smith normal form of the matrix whose kth column gives the exponents needed to express the kth independent S-unit as ± 1 times a product of powers of fixed fundamental S-units. This is a diagonal matrix whose diagonal entries are the invariant factors of the quotient group in question, and the largest of these is the value N. In general, we try different choices of u and v to minimize N, or to obtain two values of N whose greatest common divisor is minimal. In this particular example, as is often the case, we find that we can take u to be one of the fundamental units, and v can be a generator for the unique ramified (principal) prime ideal $\mathfrak{P}$ of K over F. With these choices we happily obtain $N = 1$ in this example, so that our 9 independent S-units are actually fundamental S-units and

$$U_K^S/\{\pm 1\} \cong \mathbb{Z}[G] \oplus \mathbb{Z}[G/H]$$

in this case.

Having established u and N, we fix an embedding corresponding to the infinite prime $\mathfrak{P}_\infty^{(1)}$ by sending $\eta \mapsto -4.181547496284723285851807...$. Recall that α was defined in equation (3.2) as $\alpha = \frac{1}{N}\sum_{\tau \in G} \log(|u^{\tau^{-1}}|_{\mathfrak{P}_\infty^{(1)}})\,\tau$. We find in the present example that

(4.4)
$$\alpha = 4.731085453563581445985493478\ldots - \sigma \cdot 1.6056628346177633858278031\ldots$$
$$+ \sigma^2 \cdot 2.60114402583156061784042498\ldots - \sigma^3 \cdot 1.22271862156683774631477\ldots$$
$$+\sigma^4 \cdot 3.25007575583982696813743934\ldots - \sigma^5 \cdot 0.2250548111484427175877855\ldots .$$

Using $\chi(\sigma) = e^{2\pi i/6}$, this gives us

$$\chi(\alpha) = 2.112835361411622347603536985\ldots - i\, 1.757632984501997366967240261\ldots$$

and

$$\chi^3(\alpha) = 13.63574150256801288169371665\ldots .$$

Hence,

$$\frac{L_{K/F}^{S,(1)}(0,\chi^3)}{\chi^3(\alpha)} = \frac{13.6357415025680128816937166\ldots}{13.6357415025680128816937166\ldots} = 1.00000000000000000000\ldots,$$

and

$$(4.5)\quad \frac{L_{K/F}^{S,(1)}(0,\chi)}{\chi^{-1}(\alpha)}$$
$$= \frac{7.9515677980422774160151\ldots - i\, 18.153418744976319567593\ldots}{2.112835361411622347603536985\ldots + i\, 1.757632984501997366967240261\ldots}$$
$$= -1.99999999999999999999999999\ldots - i\, 6.92820323027550917419785366\ldots,$$

and

$$(4.6)\quad \chi(\sigma)\frac{L_{K/F}^{S,(1)}(0,\chi)}{\chi^{-1}(\alpha)}$$
$$= 5.00000000000000000000000000\ldots - i\, 5.196152422706631880582339023\ldots,$$

and

$$(4.7)\quad \chi^2(\sigma)\frac{L^{S,(1)}_{K/F}(0,\chi)}{\chi^{-1}(\alpha)}$$
$$= 6.99999999999999999999999999\ldots + i\ 1.73205080756887729352744634 2\ldots.$$

Thus from equations (3.4) and (3.5), we see that, up to the accuracy of our computations (at least 30 decimal digits),

$$\Phi(f)e_1 = \frac{1}{6}(1-\sigma^3)(-4+10\sigma+14\sigma^2)$$

and

$$\Phi(f)e_3 = \frac{1}{6}(1-\sigma^3)(1-\sigma+\sigma^2)$$

so that, by equation (3.6),

$$\Phi(f) = \Phi(f)(e_1+e_3) = \frac{1}{6}(1-\sigma^3)(-3+9\sigma+15\sigma^2) = \frac{1}{2}(1-\sigma^3)(-1+3\sigma+5\sigma^2).$$

The rank one abelian Stark conjecture implies that these equalities must indeed hold exactly (see [ST]).

Now consider Question 2.1. The choices $\nu = 2$ and $\nu = 1+\sigma$ generate

$$\mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K) = \mathrm{Ann}_{\mathbb{Z}[G]}(\pm 1).$$

First, both $2\Phi(f) = (1-\sigma^3)(-1+3\sigma+5\sigma^2)$ and $(1+\sigma)\Phi(f) = (1-\sigma^3)(-3+\sigma+4\sigma^2)$ lie in $\mathbb{Z}[G]$. Second, since σ acts as 4, they act as $(1-4^3)(-1+3(4)+5(4)^2) = (-63)(91)$ and $(1-4^3)(-3+4+4(4^2)) = (-63)(65)$, respectively. As both are multiples of $13 = |\mathrm{Cl}_K|$, they annihilate Cl_K and the question has an affirmative answer for all choices of ν. Furthermore, the two annihilators $2\Phi(f)$ and $(1+\sigma)\Phi(f)$ generate $13(1-\sigma^3)$ and $(\sigma-4)(1-\sigma^3)$ in $\mathbb{Z}[G]$, and hence the full annihilator of Cl_K in $\mathbb{Z}[G](1-\sigma^3)$. Indeed,

$$13(1-\sigma^3) = (8-11\sigma+7\sigma^2)\Phi(f),$$

with $(8-11\sigma+7\sigma^2) \in \mathrm{Ann}_{\mathbb{Z}[G]}(\pm 1)$, and

$$(\sigma-4)(1-\sigma^3) = (-3+4\sigma-3\sigma^2)\Phi(f),$$

with $(-3+4\sigma-3\sigma^2) \in \mathrm{Ann}_{\mathbb{Z}[G]}(\pm 1)$. So

$$\mathrm{Ann}_{\mathbb{Z}[G]}(\mu(K))\Phi(f) = \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}_K) \cap (1-\sigma^3)$$

in this case.

In this example we also have $\mathrm{Cl}_K = \mathrm{Cl}_K^S$, as $\mathfrak{P}$ is principal. More generally, under our hypotheses, $\mathrm{Cl}_K^S = \mathrm{Cl}_K/\langle[\mathfrak{P}]\rangle$ and we note that $\mathfrak{P}$ is fixed by σ^3. Otherwise $\mathfrak{p}$ is unramified in the unique quadratic extension of F in K, so only $\mathfrak{p}_\infty^{(2)}$ can ramify there. However, class field theory shows that a single infinite prime cannot be the conductor of an abelian extension. So this relative quadratic extension is unramified at both infinite primes, which are consequently both unramified in K/F. But we have specifically required $\mathfrak{p}_\infty^{(2)}$ to ramify in K/F. So $(\mathfrak{P}^{1-\sigma^3}) = (1)$. Thus $\mathrm{Cl}_K^{(1-\sigma^3)}$ is a quotient of Cl_K^S, and multiplying any annihilator of Cl_K^S by $(1-\sigma^3)$ yields an annihilator of Cl_K. Each $\Phi(f)$ in our computations is necessarily a multiple of $1-\sigma^3$ in $\mathbb{Q}[G]$. In every case, we find that $\nu\Phi(f)$ annihilates Cl_K for $\nu \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)$, yielding a positive answer to both Questions. If we only knew that $\nu\Phi(f)$ annihilated Cl_K^S, it would follow that $\nu\Phi(f)(1-\sigma^3) = 2\nu\Phi(f)$

annihilates Cl_K, and thus indeed $\nu\Phi(f)$ annihilates the odd part of Cl_K. Our tables contain a few examples where h_K is even, and in all of them, $\nu\Phi(f)$ annihilates all of Cl_K.

4.2. **Tabular data.** For $p = 7$ and $p = 19$, we consider each real quadratic field F with discriminant $d_F < 2000$ in which p splits as $\mathfrak{p}\mathfrak{p}'$. Setting $S = \{\mathfrak{p}, \mathfrak{p}_\infty^{(1)}, \mathfrak{p}_\infty^{(2)}\}$, we then determine those examples for which the ray class group $H(\mathfrak{p}\mathfrak{p}_\infty^{(2)})$ has a character χ of order 6 with $r^S(\chi) = 1$. When the class field K corresponding to χ has a non-trivial ideal class group Cl_K, we proceed to choose a homomorphism f as in (3.1) and consider the annihilation of this ideal class group by $\nu\Phi(f)$ for $\nu = 2$ and $\nu = 1+\sigma$, which generate $\mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)$. In all of these cases, our computations verify that $\mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)\Phi(f)$ lies in $\mathbb{Z}[G]$ and annihilates Cl_K. We note that Cl_K is a cyclic group in most of the examples, and is a cyclic $\mathbb{Z}[G]$-module in all of them, so that the Fitting ideal $\mathrm{Fitt}_{\mathbb{Z}[G]}(\mathrm{Cl}_K)$ of Cl_K over $\mathbb{Z}[G]$ equals $\mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}_K)$.

The tables give the abelian group structure of Cl_F, of $H(\mathfrak{p}\mathfrak{p}_\infty^{(2)})$, of Cl_K, and of Cl_K^S, by listing their invariant factors in square brackets. Then χ is a fixed element of order 6 in the character group of $H(\mathfrak{p}\mathfrak{p}_\infty^{(2)})$. As this character group has the same invariant factors as $H(\mathfrak{p}\mathfrak{p}_\infty^{(2)})$, χ is identified by recording the list of exponents required to express it as a product of powers of the standard generators corresponding to the invariant factors.

The chosen generator σ of the cyclic Galois group G of K over F acts as an automorphism of Cl_K, and this automorphism is determined by its effect on the generators of Cl_K. When Cl_K is cyclic and generated by $\mathfrak{c}$, $\sigma(\mathfrak{c}) = \mathfrak{c}^k$ for some integer k, and we record this k. When Cl_K has generators $\mathfrak{c}_1$ and $\mathfrak{c}_2$ corresponding to the pair of invariant factors $[n_1, n_2]$, we have $\sigma(\mathfrak{c}_1) = \mathfrak{c}_1^{a_{1,1}}\mathfrak{c}_2^{a_{2,1}}$ and $\sigma(\mathfrak{c}_2) = \mathfrak{c}_2^{a_{1,2}}\mathfrak{c}_2^{a_{2,2}}$, for some integer exponents, which we record as $(a_{1,1}, a_{1,2}; a_{2,1}, a_{2,2})$. To preserve readability, the choices of S-units we arrived at to define each homomorphism f are not recorded here. We do record the value N associated with f, and note that this $N = N(f)$ is a multiple of the exponent of $X_K^S/f(U_K^S)$. We list two choices of f in a single box when we could not find a single choice of f that would lead to an ideal $(\mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)\Phi(f))$ containing such ideals for all other choices of f (this explains the missing horizontal lines in certain boxes; here we have one example with two choices of f being used). The resulting group ring element $2\Phi(f)$ is recorded in the next column, with the notation $d(c_1, c_2, c_3)$ indicating that $2\Phi(f) = d(c_1 + c_2\sigma + c_3\sigma^2)(1-\sigma^3)$. We note that (to the accuracy of the computation) this lies in $\mathbb{Z}[G]$ in every case. Furthermore, with the given action of σ and the structure of Cl_K, one may verify that $2\Phi(f)$ and $(1+\sigma)\Phi(f)$ annihilate Cl_K in every case. Finally, the last column records the exponent m of $(\mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}_K)\cap(1-\sigma^3))/(\mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)\Phi(f)) = (\mathrm{Fitt}_{\mathbb{Z}[G]}(\mathrm{Cl}_K) \cap (1-\sigma^3))/(\mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)\Phi(f))$. We note that this is always found to equal N or $2N$. When two choices of f for the same example result in relatively prime values of m, the ideal generated by $\mathrm{Ann}_{\mathbb{Z}[G]}(\mu_K)\Phi(f)$ for the different choices of f is then equal to $\mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}_K) \cap (1-\sigma^3)$.

TABLE 1. Annihilation of ideal class groups. Case of $p = 7$.

d_F	Cl_F	$H(\mathfrak{p}\mathfrak{p}_\infty^{(2)})$	χ	Cl_K	Cl_K^S	σ on Cl_K	N	$2\Phi(f)$	m
632	[1]	[6]	[1]	[7]	[7]	5	1	$(3, 3, -1)$	1
764	[1]	[6]	[1]	[13]	[13]	4	1	$(5, 1, -3)$	1
856	[1]	[6]	[1]	[3]	[3]	-1	39	$39(-1, 1, 1)$	39
856	[1]	[6]	[1]	[3]	[3]	-1	93	$93(-1, 1, 1)$	93
1129	[9]	[54]	[1]	[3]	[3]	1	7	$7(1, 1, 1)$	7
1429	[5]	[30]	[5]	[5]	[5]	1	7	$7(1, 1, 1)$	7
1429	[5]	[30]	[5]	[5]	[5]	1	4	$4(0, 0, -1)$	4
1436	[3]	[6]	[1]	[3]	[3]	-1	21	$7(-1, 1, 5)$	21
1436	[3]	[6]	[1]	[3]	[3]	-1	93	$31(1, 1, 5)$	93
1537	[2]	[6,2]	[1,0]	[8]	[8]	-1	2	$4(0, -1, 1)$	2
1537	[2]	[6,2]	[1,1]	[2,2]	[2,2]	(1,0;1,1)	1	$2(0, 0, -1)$	2
1597	[1]	[6]	[1]	[3]	[3]	-1	3	$3(1, 1, 1)$	3
1597	[1]	[6]	[1]	[3]	[3]	-1	3	$(-1, 1, 5)$	3
1772	[3]	[6,3]	[1,0]	[15]	[5]	4	1	$(1, -1, 3)$	1
1772	[3]	[6,3]	[1,1]	[15]	[5]	4	2	$2(3, -1, 1)$	2
1772	[3]	[6,3]	[1,1]	[15]	[5]	4	7	$7(3, -1, 1)$	7
1772	[3]	[6,3]	[3,1]	[15]	[5]	4	1	$(3, -1, 1)$	1
1772	[3]	[6,3]	[1,0]	[3]	[1]	1	2	$2(1, 1, -1)$	2
1772	[3]	[6,3]	[1,1]	[3]	[1]	1	2	$2(-1, 1, 1)$	2
1772	[3]	[6,3]	[3,1]	[3]	[1]	1	1	$2(-1, 1, 1)$	2
1793	[1]	[6]	[1]	[7]	[7]	-1	1	$(3, -1, 3)$	1
1864	[2]	[6,2]	[1,1]	[4]	[4]	-1	2	$4(0, 1, 0)$	2
1864	[2]	[6,2]	[2,1]	[2,2]	[2,2]	(1,0;1,1)	1	$2(0, 0, 1)$	2

TABLE 2. Annihilation of ideal class groups. Case of $p = 19$.

d_F	Cl_F	$H(\mathfrak{p}\mathfrak{p}_\infty^{(2)})$	χ	Cl_K	Cl_K^S	σ on Cl_K	N	$2\Phi(f)$	m
253	[1]	[6]	[1]	[13]	[13]	4	1	$(-1,3,5)$	1
365	[2]	[18,2]	[3,0]	[4]	[4]	−1	1	$(0,-2,2)$	1
365	[2]	[18,2]	[6,1]]	[2,2]	[2,2]	(0,1;1,0)	2	$(0,4,0)$	2
461	[1]	[6]	[1]	[7]	[7]	3	2	$2(-1,3,3)$	2
617	[1]	[6]	[1]	[3]	[3]	−1	21	$21(-1,1,1)$	21
617	[1]	[6]	[1]	[3]	[3]	−1	93	$93(-1,1,1)$	93
785	[6]	[6,2]	[2,1]	[2]	[2]	1	2	$2(-1,1,1)$	2
785	[6]	[6,2]	[1,1]	[2]	[2]	1	1	$(-1,1,1)$	1
985	[6]	[6,2]	[1,1]	[6]	[6]	−1	3	$2(2,1,2)$	6
985	[6]	[6,2]	[2,1]	[2]	[2]	1	2	$2(1,1,1)$	2
1165	[2]	[6,2]	[1,1]	[26]	[26]	17	1	$2(2,1,-2)$	2
1165	[2]	[6,2]	[2,1]	[2]	[2]	1	2	$2(-1,1,1)$	2
1297	[11]	[66]	[11]	[66]	[66]	23	3	$(5,1,-1)$	3
1309	[2]	[6,2]	[1,0]	[4]	[4]	−1	2	$4(1,0,0)$	2
1309	[2]	[6,2]	[1,1]	[2,2]	[2,2]	(1,0;1,1)	13	$26(0,1,0)$	13
1309	[2]	[6,2]	[1,1]	[2,2]	[2,2]	(1,0;1,1)	7	$14(1,0,0)$	7
1601	[7]	[42]	[7]	[14,2]	[2,2]	(8,7;1,1)	2	$2(-1,1,1)$	2
1657	[1]	[6]	[1]	[7]	[7]	5	1	$(3,3,-1)$	1
1673	[1]	[6]	[1]	[7]	[7]	3	1	$(3,3,1)$	1
1765	[6]	[6,2]	[1,1]	[2]	[2]	1	2	$2(1,1,-1)$	2
1765	[6]	[6,2]	[2,1]	[2]	[2]	1	1	$2(0,0,1)$	2
1949	[1]	[6]	[1]	[3]	[3]	−1	3	$(-1,1,5)$	3
1985	[2]	[6,2]	[1,0]	[2,2]	[2,2]	(1,0;0,1)	1	$2(0,1,0)$	1
1985	[2]	[6,2]	[2,1]	[4]	[4]	−1	2	$4(1,0,1)$	4
1985	[2]	[6,2]	[2,1]	[4]	[4]	−1	7	$14(1,0,1)$	14

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