AMERICAN MATHEMATICAL SOCIETY

Current Events Bulletin

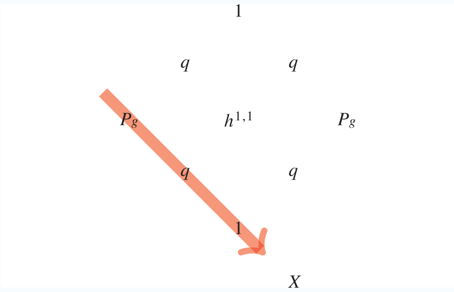
Friday, April 8, 2022
2:00–6:00 pm

2:00 pm

Thomas Scanlon
University of California, Berkeley

Tame geometry for Hodge theory

How can it be that deep analytic questions can be attacked from model theory?

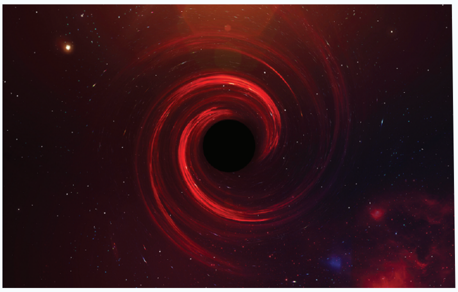


3:00 pm

Elena Giorgi
Columbia University

The stability of black holes with matter

The “ringing” and the “pictures” of black holes are exciting; mathematics has the potential to unlock the secrets.



4:00 pm

Anup Rao
University of Washington

Sunflowers: From soil to oil

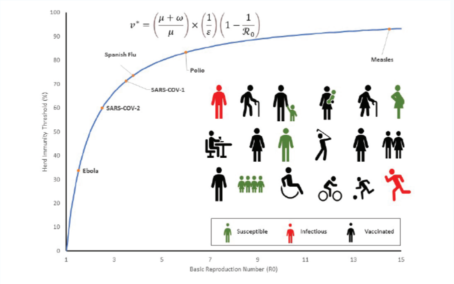
A beautiful way in which order appears out of randomness.

5:00 pm

Elamin Elbasha
Merck & Co., Inc.

Mathematics and the quest for vaccine-induced herd immunity threshold

Will this be the way we finally emerge from the pandemic?



Organized by **David Eisenbud**, *University of California, Berkeley*

AMS AMERICAN MATHEMATICAL SOCIETY

jmm 2022 Joint Mathematics Meetings
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Introduction to the Current Events Bulletin

Will the Riemann Hypothesis be proved this week? What is the Geometric Langlands Conjecture about? How could you best exploit a stream of data flowing by too fast to capture? I think we mathematicians are provoked to ask such questions by our sense that underneath the vastness of mathematics is a fundamental unity allowing us to look into many different corners -- though we couldn't possibly work in all of them. I love the idea of having an expert explain such things to me in a brief, accessible way. And I, like most of us, love common-room gossip.

The Current Events Bulletin Session at the Joint Mathematics Meetings, begun in 2003, is an event where the speakers do not report on their own work, but survey some of the most interesting current developments in mathematics, pure and applied. The wonderful tradition of the Bourbaki Seminar is an inspiration, but we aim for more accessible treatments and a wider range of subjects. I've been the organizer of these sessions since they started, but a varying, broadly constituted advisory committee helps select the topics and speakers. Excellence in exposition is a prime consideration.

A written exposition greatly increases the number of people who can enjoy the product of the sessions, so speakers are asked to do the hard work of producing such articles. These are made into a booklet distributed at the meeting. Speakers are then invited to submit papers based on them to the *Bulletin of the AMS*, and this has led to many fine publications.

I hope you'll enjoy the papers produced from these sessions, but there's nothing like being at the talks -- don't miss them!

David Eisenbud, Organizer
Mathematical Sciences Research Institute
de@msri.org

For PDF files of talks given in prior years, see
http://www.ams.org/ams/current-events-bulletin.html.
The list of speakers/titles from prior years may be found at the end of this booklet.

TAME GEOMETRY FOR HODGE THEORY

THOMAS SCANLON

ABSTRACT. O-minimality, a condition on the simplicity of the definable sets in ordered structures, has proven to be a versatile and powerful formalization of the idea of tame geometry. This o-minimal tame geometry, and especially the o-minimal avatar of Chow's theorem on the algebraicity of complex analytic subspaces of projective spaces, has a played a central role in a series of recent works on algebraicity conjectures in Hodge theory. We survey some of these theorems, paying special attention to the use of o-minimality in their proofs.

1. INTRODUCTION

The concept of o-minimality was isolated by van den Dries [50] for structures on the real numbers and then by Pillay and Steinhorn [36] more generally as the abstract context for studying real geometry to situations possibly extending semi-algebraic geometry, but retaining its tameness. While o-minimality is rooted in mathematical logic it has become an indispensable tool in many other branches of mathematics. Just over a decade ago, I presented another lecture [40] at the AMS meeting in New Orleans on the then novel application of o-minimality to problems in diophantine geometry, and especially on Pila's proof [32] of the André-Oort conjecture for products of modular curves. At that time, the most important technical contribution of o-minimality to the solution of these geometric problems had been the theorem of Pila and Wilkie [35] on counting rational points in definable sets. Since then, geometers have adopted o-minimality, have refined it for its use in geometry, and have then employed it to resolve some deep and long standing problems.

O-minimality should be understood as a tameness condition on the class of functions allowed for geometric constructions. Specialized to the case of o-minimal expansions of the field of real numbers, an o-minimal structure $\mathbb{R}_{\mathcal{F}}$ on $\mathbb{R}$ is given by the choice of a collection $\mathcal{F}$ of functions $f : \mathbb{R}^n \to \mathbb{R}$ containing all of the functions defined by real polynomials so that every subset of $\mathbb{R}$ which is first-order definable in $\mathbb{R}$ with the functions in $\mathcal{F}$ considered as distinguished functions is a finite union of points and intervals. Many people first encountering o-minimality are struck by two conflicting impressions. First, it would be remarkable that there are any nontrivial examples of o-minimal structures $\mathbb{R}_{\mathcal{F}}$. Second, the condition on definable subsets of the real line seems to be too weak to have any useful consequences.

Contrary to that first impression, there are very rich o-miminal structures. It follows from Tarski's work on elementary geometry [48] that if $\mathcal{F}$ consists of all the functions $\mathbb{R}^n \to \mathbb{R}$ given by real polynomials, then $\mathbb{R}_{\mathcal{F}}$ is o-minimal. Much more complicated structures are known to be o-minimal, including the structure $\mathbb{R}_{\text{an,exp}}$ in which $\mathcal{F}$ consists of the real polynomial functions together with the real exponential function $\exp : \mathbb{R} \to \mathbb{R}$ and all restricted analytic functions, that is, functions $f : \mathbb{R}^n \to \mathbb{R}$ which on $[0,1]^n$ are the restrictions of some real analytic function on a neighborhood of the box and are zero outside of the box. The second impression could not be more wrong. O-minimality implies a very strong form of geometric tameness for the definable sets in all dimensions. I will make this more explicit in Section 2.

The concept of first-order definability underlies the strength of o-minimality. This basic idea from logic is not as well known in mathematics in general as I believe it should be. A fair proportion of Section 2 will be devoted to an explication of definability. On the other hand, it is heartening to see that people coming to this subject from geometry and number theory are engaging with and extending the theory of definability in o-minimal structures. Especially notable is the work of Bakker, Brunebarbe, and Tsimerman [5] in which a theory of definable analytic spaces and of definable sites is developed in order to prove a conjecture of Griffiths on the algebraicity of the images of period mappings.

The paper of Bakker, Brunebarbe, and Tsimerman is just one of several recent publications and preprints in which o-minimality is used an essential way to solve problems in Hodge theory. My principal aim will be to explain some of these applications. At this point I should warn the reader that I possess at best an outsider's comprehension of Hodge theory. Thus, my presentation of that theory will be rather pedestrian and almost certainly misses some important ideas.

This paper is organized as follows. In Section 2, I recount some of the theory of o-minimality paying special attention to definability. Since several surveys on o-minimality (including my old Current Events Bulletin report mentioned above) are already available, this will be brief. I describe in Section 3 the basics of Hodge theory, at least as far as I understand them. Special attention will be paid to the objects from Hodge theory, such as period mappings, Hodge loci, and period domains, which appear in the applications. I describe in Section 4 some of the theorems in Hodge theory which have been proven using methods from o-minimality. Finally, in Section 4.3 I discuss some of the other recent notable appearances of o-minimality in algebraic and diophantine geometry.

2. O-MINIMAL GEOMETRY

2.1. **Basics of o-miniality.** Since o-minimality is exposed in a textbook [51] and several survey articles [54, 38, 43, 41, 42, 57], this introduction will be brief. Indeed, a very clear, much more detailed account of the topics described in this paper is available in the exposition of Bakker and Tsimerman in [8].

O-minimality, short for "order minimality" is the property of a totally ordered structure $(R, <, \ldots)$ that the only sets definable in one dimension are those which are definable in every ordered structure: finite unions of points and intervals. To unpack this definition we need to discuss what is meant by "totally ordered structure" (and, in particular, what those ellipses are hiding) and by the notion of "definable".

In some other accounts of o-minimality intended for readers from outside of mathematical logic, the terms *structure* and *definable set* are defined simultaneously. I will first recall this presentation and then compare it to the definition based on formal languages.

Definition 2.1. A *structure* $\mathfrak{R}$ consists of a non-empty set R together with a sequence $\langle \mathcal{D}_n \rangle_{n=0}^{\infty}$ where $\mathcal{D}_n$ is a set of subsets of R^n satisfying the following list of closure properties.

- `BA` Each $\mathcal{D}_n$ is a Boolean subalgebra of $\mathcal{P}(R^n)$, the power set of R^n.
- `diagonal` The diagonal $\Delta = \{\langle a, b \rangle \in R^2 \ : \ a = b\}$ is an element of $\mathcal{D}_2$.
- `product` If $A \in \mathcal{D}_n$, then $A \times R \in \mathcal{D}_{n+1}$.
- `permutation` For each permutation $\sigma \in \mathrm{Sym}(n)$, if $f_\sigma : R^n \to R^n$ is the map given by $\langle a_1, \ldots, a_n \rangle \mapsto \langle a_{\sigma(1}, \ldots, a_{\sigma(n)} \rangle$ and $A \in \mathcal{D}_n$, then $f_\sigma(A) \in \mathcal{D}_n$.
- `projection` If $\rho : R^{n+1} \to R^n$ is the projection map $\langle a_1, \ldots, a_{n+1} \rangle \mapsto \langle a_1, \ldots, a_n \rangle$ and $A \in \mathcal{D}_{n+1}$, then $\rho(A) \in \mathcal{D}_n$.

The elements of $\mathcal{D}_n$ are called the *definable* subsets of R^n.

Imposing further restrictions on the class of definable sets, we obtain the notion an o-minimal structure.

Definition 2.2. A structure $\mathfrak{R} = (R, \langle \mathcal{D}_n \rangle_{n=0}^{\infty})$ is o-minimal if there is a total order $<$ on R so that the following conditions hold.

`order` The set $\{\langle a, b \rangle \in R^2 : a < b\}$ is an element of $\mathcal{D}_2$.

`singleton` For each $a \in R$ the singleton set $\{a\}$ is definable. That is, $\{a\} \in \mathcal{D}_1$.

`interval` The definable subsets of R are finite unions of points and intervals relative to the distinguished order $<$.

Convention 2.3. For the purposes of this exposition, all structures $\mathfrak{R} = (R, \langle \mathcal{D}_n \rangle_{n=0}^{\infty})$ will have $R = \mathbb{R}$ and when considering o-minimal structures on $\mathbb{R}$, the distinguished order will be the usual ordering of real numbers.

Example 2.4. It follows from Tarski's quantifier elimination theorem for real closed fields, that $\mathbb{R}$ given together with the semialgebraic sets, that is, Boolean combinations of sets of the form $\{\langle a_1, \ldots, a_n \rangle \in R^n : f(a_1, \ldots, a_n) \geq 0\}$ for $f \in \mathbb{R}[x_1, \ldots, x_n]$ a real polynomial, is an o-minimal structure.

Example 2.5. No structure on the real numbers for which the graph of the sine function is definable can be o-minimal. Indeed, suppose that $G := \{\langle a, b \rangle \in \mathbb{R}^2 : b = \sin(a)\} \in \mathcal{D}_2$. By the condition on singleton sets, $\{0\} \in \mathcal{D}_1$. By the condition on products, $\{0\} \times \mathbb{R} \in \mathcal{D}_2$. It then follows from the condition on permutations that $\mathbb{R} \times \{0\} \in \mathcal{D}_2$. Applying closure of the class of definable sets under intersections and projections, it then follows that $\mathbb{Z}\pi = \rho(\mathbb{Z}\pi \times \{0\}) = \rho(G \cap (\mathbb{R} \times \{0\})) \in \mathcal{D}_1$. This violates the condition `interval` in Definition 2.2.

In practice, we do not actually use arguments like those appearing in Example 2.5 to understand definable sets in a structure. Indeed, we almost never present the class of definable sets in a structure by fully specifying all of the sets $\mathcal{D}_n$. Instead, we list some distinguished definable sets, and then the class of all definable sets is taken to be the smallest collection of sets containing those distinguished sets and closed under the conditions in Definition 2.1. For example, to describe the structure on the real numbers coming from semialgebaic geometry in Example 2.4 we might say instead that this is the smallest structure on the real numbers for which the graphs of addition and multiplication, the order relation, and each singleton are definable. Even this way of presenting a structure is still somewhat artificial. In model theory, a structure is given by a nonempty set together with interpretations of some distinguished relation, function, and constant symbols. The definable sets are then those sets which may be defined by a formula in the associated language of first-order logic. When demonstrating that some set is definable in a given structure and when analyzing the structure of the definable sets, it is usually this approach that is taken.

Specializing the discussion to the structures seen in applications of o-minimality, we would consider only signatures having one binary relation symbol $<$, constant symbols r indexed by the real numbers, and a set $\mathcal{F}_n$ of function symbols of arity n for each natural number n indexed by a set of functions $f : \mathbb{R}^n \to \mathbb{R}$. We then form the structure (in the sense of first-order logic, see [19]; not yet in the sense of Definition 2.1) $\mathbb{R}_{\mathcal{F}}$ by interpreting each of these distinguished symbols by the corresponding relating, element, or function. With these choices of non-logical symbols, we may form a first-order language by allowing (unnested) atomic formulae of the form $x_i = x_j$, $x_i = r$, $x_i < x_j$, and $f(x_{i_1}, \ldots, x_{i_n}) = x_j$ for any choice of variables with indices i, j, and i_k taken from the natural numbers and $f \in \mathcal{F}_n$. Each of these formulae defines a subset of $\mathbb{R}^m$ as long as m is greater than all of the indices listed. For example, if $i = 0$, $j = 1$, and we use the variables x_0 and x_1 as the standard coordinates on $\mathbb{R}^2$, then $x_0 = x_1$ defines the diagonal Δ. From the atomic formulae we form the full first-order language by closing off under the Boolean operations of $\neg$ ("not"), $\vee$ ("or"), and $\wedge$ ("and"), and existential quantification over elements of the structure (i.e., if ϕ is a formula, then so is $(\exists x_i)\phi$). The interpretations of these formulae in $\mathbb{R}_{\mathcal{F}}$ follow the usual mathematical practice, so that, for example, the set defined by $\neg\phi$ is the complement of the set defined by ϕ. If we take $\mathcal{D}_n$ to be the set of subsets of $\mathbb{R}^n$ which are defined by some formula in this language, then $(\mathbb{R}, \langle \mathcal{D}_n \rangle_{n=0}^{\infty})$ is a structure in the sense of Definition 2.1. The various closure conditions in that definition correspond to logical operations in the syntax. For example, if a set $A \in \mathcal{D}_{n+1}$ is defined by the formula ϕ, then the projection $\rho(A)$ is defined by $(\exists x_n)\phi$.

The specific o-minimal structure on the real numbers which has appeared most often in the applications to geometry is $\mathbb{R}_{\text{an,exp}}$, which has the form $\mathbb{R}_{\mathcal{F}}$ where $\mathcal{F}$ consists of all polynomial functions, the real exponential function, and all *restricted analytic* functions.

Definition 2.6. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be *restricted analytic* if there is an open superset $U \supset [0,1]^n$ of the unit box and a real analytic function $g : U \to \mathbb{R}$ so that $f \restriction [0,1]^n = g \restriction [0,1]^n$ and

$f \upharpoonright (\mathbb{R}^n \smallsetminus [0,1]^n) \equiv 0$. That is, f is the restriction of a real analytic function on the unit box $[0,1]^n$ and is constantly zero off the box.

The o-minimality of $\mathbb{R}_{\text{an,exp}}$ was established by van den Dries and Miller [53], building on Wilkie's proof of the o-minimality of the expansion of the ordered field of real numbers by the real exponential function [56] and a quantifier simplification theorem of Denef and van den Dries [12] (or a corresponding geometric theroem of Garbrielov [15]) which gave the o-minimality of the expansion $\mathbb{R}_{\text{an}}$ of the ordered field of real numbers by the set of all restricted analytic functions. Van den Dries, Macintrye, and Marker [52] subsequently offered a more algebraic proof the o-minimality $\mathbb{R}_{\text{an,exp}}$ based on their construction of generalized series models for asymptotic expansions inspired by the theory of transseries. The key to the proof of o-minimality of $\mathbb{R}_{\text{an}}$ is the Weierstrass division theorem which permitted a reduction of the problem to Tarski's theorem on semialgebraic geometry. The o-minimality of the further expansion including the real exponential function was accomplished through a study of Pfaffian differential equations, which at the first stage allow one to express the derivative of a function as a polynomial in the function itself.

To date, $\mathbb{R}_{\text{an,exp}}$ has served as a sufficiently rich o-minimal structure for the applications to geometry. Other, potentially more complicated, structures have been shown to be o-minimal, As a generalization of Wilkie's theorem on Pfaffian functions, Speissegger [46] has shown that for any o-minimal structure on the real numbers, its Pfaffian closure is also o-minimal. Basically, this means that in this Pfaffian closure all of the functions definable in the original o-minimal structure are definable and if $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable and for each $j \leq n$ there is some definable function G_j so that $\frac{\partial f}{\partial x_j} = G_j(f, x_1, \ldots, x_n)$, then f is also definable. In a different direction, Rolin, Speissegger, and Wilkie [39] have shown that expansions of the real numbers by quasianalytic classes of functions are also o-minimal. One could imagine that the functions appeing in the Pfaffian closure of $\mathbb{R}_{\text{an,exp}}$ may be needed for some geometric arguments. For example, it may be necessary to integrate certain definable functions. To my knowledge, this added flexibility has not yet been employed. Likewise, it is hoped that the o-minimality of quasianalytic classes may be useful for the analysis of Hilbert's Sixteenth Problem [20].

The utility of the condition that a structure be o-minimal comes from the geometrically tame properties of its definable sets. Even though these structures are constructed using methods from analysis, we may work with the definable functions in these structures as we do in a first-year calculus course, that is, using geometric reasoning, rather than the more complicated arguments from general topology and functional analysis we have come to expect when studying functions of several real variables. Van den Dries suggested in [51] that o-minimality may fit the bill for Grothendieck's topologie moderée [18], or what we might prefer to call tame geometry. As a practical matter, the applications of o-minimality over the past two decades have borne out this suggestion.

The tameness of o-minimal geometry manifests itself through properties of the definable sets and definable functions. For example, the condition `interval` in Definition 2.2 may be rephrased as saying that the definable subsets of $\mathbb{R}$ are precisely those sets having finitely many connected components. O-minimality of a structure implies that for all natural numbers n and all definable sets $A \in \mathcal{D}_n$ the set A has only finitely many connected components. This fact is a consequence of the fundamental theorem on cell decomposition in o-minimal structures.

The class of cells in an o-minimal structure on the real numbers is defined recursively. In the following definition, to for ease of readability I will write $\mathbf{a}$ for the tuple $\langle a_1, \ldots, a_n \rangle$ and I identify $\mathbb{R}^n \times \mathbb{R}$ with $\mathbb{R}^{n+1}$.

Definition 2.7. Each singleton and open interval is a cell in $\mathbb{R}$. If $X \subseteq \mathbb{R}^n$ is a cell and $f : X \to \mathbb{R}$ is a definable continuous function, then the graph of f on X, $\Gamma_X(f) := \{\langle \mathbf{a}, b \rangle \in \mathbb{R}^{n+1} \ : \ \mathbf{a} \in X \,\&\, f(\mathbf{a}) = b\}$ is a cell in $\mathbb{R}^{n+1}$. If $g : X \to \mathbb{R}$ is another continuous definable function on X such that $f(\mathbf{a}) < g(\mathbf{a})$ for all $\mathbf{a} \in X$, then the band $(f, g)_X := \{\langle \mathbf{a}, b \rangle \in \mathbb{R}^{n+1} \ : \ \mathbf{a} \in X \,\&\, f(\mathbf{a}) < b < g(\mathbf{a})\}$ is a cell in $\mathbb{R}^{n+1}$ as are the bands $(-\infty, f)_X := \{\langle \mathbf{a}, b \rangle \in \mathbb{R}^{n+1} \ : \ \mathbf{a} \in X \,\&\, b < f(\mathbf{a})\}$ and $(g, \infty)_X = \{\langle \mathbf{a}, b \rangle \in \mathbb{R}^{n+1} \ : \ \mathbf{a} \in X \,\&\, g(\mathbf{a}) < b\}$.

The fundamental theorem of o-minimality is that every definable set admits a cell decomposition.

Theorem 2.8 (van den Dries [50] for o-minimal structures on the real numbers; Knight, Pillay, and Steinhorn [25] in general)**.** *Given $A_1, \ldots, A_m \in \mathcal{D}_n$ a finite collections of definable subsets of $\mathbb{R}^n$, there is a partition Π of $\mathbb{R}^n$ into finitely many cells so that each A_i is also partitioned by Π. That is, for each $C \in \Pi$ and $i \leq m$, either $C \subseteq A_i$ or $C \cap A_i = \varnothing$.*

It is fairly easy to see that each cell is homeomorphic to $\mathbb{R}^m$ for a suitable choice of m. As such, the finiteness of the number of connected components of a definable set follows easily from Theorem 2.8. Likewise, it is easy to see that if $C \subseteq \mathbb{R}^n$ is a cell, then C is a submanifold of the ambient space. Thus, the cell decomposition theorem implies immediately that every definable set may be expressed as a finite disjoint union of submanifolds.

The proof of Theorem 2.8 is interwoven with the proof of a regularity theorem for definable functions: every definable function is piecewise continuous in the sense that if $f : \mathbb{R}^n \to \mathbb{R}$ is definable, then there is a partition Π of $\mathbb{R}^n$ into cells so that for each $C \in \Pi$ the restriction of f to C is continuous. Consequently, every definable function is continuous except possibly on a finite union of lower dimensional submanifolds. In fact, even stronger regularity properties hold. Provided that addition and multiplication are definable, every definable function is piecewise continuously differentiable, and in the structure $\mathbb{R}_{\text{an,exp}}$ this may be upgraded to the assertion that every definable function is piecewise real analytic. It then follows that in the definition of cells for $\mathbb{R}_{\text{an,exp}}$ one may require that each of the functions f and g be definable and real analytic rather than merely continuous.

Another tameness principle which comes out of the proof of the proof of the cell decomposition theorem is that all one-sided limits exist. That is, if $f : (0,1) \to \mathbb{R}^n$ is a definable, bounded function, then $\lim_{x \to 0^+} f(x)$ exists. If one does not mind taking $\pm\infty$ as the value of some of the coordinates of the limit, then the condition that f is bounded may be dropped.

The cell decomposition theorem implies uniform versions of itself from which strong uniformities of definable sets may be deduced. For example, if $f : X \to Y$ is a definable function from some definable set X to another definable set Y, then there is a bound B, depending just on these data, so that for any $y \in Y$ the fiber $X_y := f^{-1}\{y\}$ has at most B connected components. As a special case of this, if all of the fibers are finite, then there is a uniform bound on the size of these finite fibers.

The cell decomposition theorem also implies that all definable functions have definable sections. That is, as long as addition is definable, if $f : X \to Y$ is a definable function from the definable set X to the definable set Y, then there is definable function $g : Y \to X$ so that $f \circ g = \mathrm{id}_Y$. I will sketch a proof of this fact. Consider the case that $X \subseteq \mathbb{R}$. The general case may be proven by induction. Let Z be the converse of the graph of f. That is, $Z = \{\langle \mathbf{b}, a \rangle \ : \ a \in X \ \& \ \mathbf{b} \in$

Y & $f(a) = \mathbf{b}\}$. By the cell decomposition theorem, Z may be expressed as a finite disjoint union of cells. If such a cell C has the form $C = \Gamma(h)_W$ where W is a cell and $h : W \to \mathbb{R}$ is a continuous definable function, then because $C \subseteq Z$, $W \subseteq Y$ is a cell in Y and $f \circ h = \mathrm{id}_W$, If the cell C has the form $(k, \ell)_W$ where k and ℓ are continuous definable functions on W with $k < \ell$, then the function $h : W \to \mathbb{R}$ defined by $h(\mathbf{b}) := \frac{1}{2}(k(\mathbf{b} + \ell(\mathbf{b}))$ satisfies $f \circ h = \mathrm{id}_W$. Likewise, if $C = (-\infty, k)_W$, then the function $h : W \to \mathbb{R}$ defined by $h(\mathbf{b}) := k(\mathbf{b}) - 1$ is a right inverse to f on W. Refining the decomposition of Y appropriately, the various h so constructed may be amalgamated to give g.

The Pila-Wilkie counting theorem is one of the highlights of o-minimal geometry and has been a central player in many of the applications of o-minimality to geometry. As I have written about this theorem in several other surveys, I will limit the discussion here to the mere statement of the theorem.

Recall that the multiplicative height of a rational number is defined by $H(0) = 0$ and if p and q are coprime integers, then $H(\frac{p}{q}) = \max\{|p|, |q|\}$. For an n-tuple of rational numbers, $H(\langle a_1, \dots, a_n \rangle) = \max H(a_i)$. For a subset $X \subseteq \mathbb{R}^n$, the transcendental part X^{tr} of X is defined to be the the complement in X of all infinite, connected, semialgebraic subsets of X. That is,

$$X^{\mathrm{tr}} = X \setminus \bigcup_{\substack{\mathrm{Y} \subseteq X,\ \text{semialgebraic,} \\ \text{connected, and infinite}}} Y\,.$$

Theorem 2.9 (Pila and Wilkie [35]). *For $X \subseteq \mathbb{R}^n$ a definable set in an o-minimal structure on the real numbers and $\epsilon > 0$ there is a constant $C = C(X, \epsilon)$ so that for all $t > 0$, the inequality $\#\{\mathbf{a} \in X \cap \mathbb{Q}^n : H(\mathbf{a}) < t\} \leq Ct^{\epsilon}$ holds.*

2.2. **O-minimal complex analysis.** Many of the applications of o-minimality involve complex analytic functions and sets. For example, in Pila's proof of the André-Oort conjecture for products of modular curves, the first step was to verify that in some suitable sense Klein's analytic j-function is definable in some o-minimal structure, and in $\mathbb{R}_{\mathrm{an,exp}}$ in particular. There are at least two issues which must be resolved. First, $j : \mathfrak{h} \to \mathbb{C}$ is a *complex* analytic function from the upper half plane $\mathfrak{h} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ to the complex numbers. As such, it is not immediately accessible to the o-minimal structure

$\mathbb{R}_{\text{an,exp}}$. Second, the fibers of j are countably infinite. As a general rule no definable set in an o-minimal structure on the real numbers may be countably infinite.

The solution to the first issue of how $\mathbb{R}_{\text{an,exp}}$ can see the complex analytic function $j : \mathfrak{h} \to \mathbb{C}$ is fairly simple. Using the decomposition into real and imaginary parts, each complex number may be identified with a pair of real numbers. The solution to the second issue is subtler. The simplest soluton is to observe that it may be enough to know that j is definable when restricted to some set F which contains a fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ acting on the upper half plane through fractional linear transformations. For this one might take $F = \{z \in \mathfrak{h} : |z| \geq 1 \;\&\; |\operatorname{Re}(z)| \leq \frac{1}{2}\}$. The conditions defining F are clearly semialgebraic in the real and imaginary parts of z and F contains one of the usual fundamental domains for the modular group. To check that the restriction of j to F is definable, one may see from the the syntactic presentation of the notion of a structure, that if $f : \mathbb{R}^n \to \mathbb{R}$, $g_0 : \mathbb{R}^m \to \mathbb{R}$, $\ldots$, and $g_{n-1} : \mathbb{R}^m \to \mathbb{R}$ are definable functions (meaning that their graphs are definable sets), then the function $\mathbb{R}^m \to \mathbb{R}$ given by $\mathbf{a} \mapsto f(g_0(\mathbf{a}), \ldots, g_{n-1}(\mathbf{a}))$ is also definable. Thus, it would suffice to represent $j \upharpoonright F$ as the composition of some explicitly definable functions. The complex exponential function $\exp : \mathbb{C} \to \mathbb{C}^\times$ is not definable (since its fibers are countably infinite), but its restrictions to strips of the form $\mathbb{R} \times [a, b]i$ are definable in $\mathbb{R}_{\text{an,exp}}$ since the sine and cosine when restricted to the interval $[a, b]$ are definable. Indeed, if we let g be the restriction of the function $\sin(bx + a(1 - x))$ to the interval $[0, 1]$, then g an explicitly definable function as a restricted analytic function and the restriction of the sine function to the interval $[a, b]$ is the composition of g with the linear function $x \mapsto \frac{x-a}{b-a}$ which is definable. Using this observation, we see that the map $\tau \mapsto q = \exp(2\pi i \tau)$ is definable on F and maps F into the disk of radius $e^{-\sqrt{3}\pi}$ around the origin. When expressed with its q-expansion, the j-function may be expressed as a meromorphic function on the unit disk with a simple pole. The restriction of this meromorphic function to the smaller compact disk of radius $e^{-\sqrt{3}\pi}$ is then definable and its composition with the definable (on F) map $\tau \mapsto \exp(2\pi i \tau)$ gives the restriction of j to F. This kind of reasoning is used in most of the proofs of definability of the functions appearing in the applications of o-minimality.

While it is possible to interpret the complex numbers in the real numbers and thereby reduce complex analysis to real analysis, it is well known that complex analysis has a markedly different character from its real counterpart. In the series of papers [27, 28, 29] Peterzil and Starchenko studied the extent to which a robust theory of complex analysis may be developed from the assumption that the complex numbers are interpreted in an o-minimal structure. For them, the challenge was to build this theory for arbitrary o-minimal structures, and not merely those on the real numbers. In particular, they could not simply quote the classical theorems of complex analysis. An even more serious difficulty with this setting is that in every other such structure the underlying order topology is totally disconnected rendering arguments based on such ideas as analytic continuation along paths invalid.

That Peterzil and Starchenko were motivated by the problem of understanding the extent to which complex analysis would make sense relative to an arbitrary o-minimal structure rather than applications to actual complex analytic sets may explain the tepid initial reception of their work. Indeed, the paper [28] was rejected for publication at the *Duke Mathematical Journal* on account of a referee report [1] in which the lack of applications was highlighted. To be fair, the referee noted that such applications may in fact exist but objected that they were not spelled out in that paper. Over time, the importance of the Peterzil-Starchenko approach to o-minimal complex analysis has been recognized, as evidenced, for example, by their invitation to speak about this topic at the International Congress of Mathematicians [30].

Remarkably, when specialized to the case of complex analysis in o-minimal expansions of the real numbers, many of the Peterzil-Starchenko theorems give stronger conclusions than do the classical theorems. For the purposes of the applications, the most important of these theorems is what has come to be called the *definable Chow theorem*. Chow proved [11] that if $X \subseteq \mathbb{P}^n_{\mathbb{C}}$ is a closed complex analytic subvariety of complex projective space, then X is necessarily algebraic. With the definable Chow theorem, the condition that the ambient space is projective may be dropped provided that X is definable.

Theorem 2.10 (Peterzil and Starchenko). *Let $X \subseteq \mathbb{C}^n$ be a complex analytic subvariety of complex affine space which is definable in some o-minimal structure on the ordered field of real numbers. Then X is an algebraic variety.*

The proof of Theorem 2.10 passes through an o-minimal version of a theorem of Remmert and Stein [37] on the analyticity of closures of analytic sets.

Theorem 2.11 (Peterzil and Starchenko)**.** *Let M be a definable complex analytic manifold and $E \subseteq M$ a complex analytic subset of M. If $X \subseteq M \smallsetminus E$ is both complex analytic and definable, then the closure of X is a complex analytic subset of M.*

Theorem 2.10 follows from Theorem 2.11 by taking M to be the projective space $\mathbb{P}^n_{\mathbb{C}}$, E to be the divisor at infinity, and applying the classical Chow theorem to the closure of X.

Alternative proofs of Theorem 2.10 making use of other classical theorems on complex analysis have been given. For example, Brosnan [9] proves Theorem 2.10 by proving a volume estimate for definable sets in o-minimal structures and then applying a theorem of Stoll [47] that the only complex analytic sets which satisfy this estimate are the algebraic varieties.

Chow's theorem is an important result on the structure of complex analytic sets, but for purposes of modern algebraic geometry in which schemes replace set theoretically defined algebraic varieties and cohomological methods are central, Serre's GAGA ("géométrie algébrique et géométrie analytique") theorems [45] are required. Roughly speaking, Serre's GAGA theorem asserts the category of coherent complex algebraic sheaves on a complex projective algebraic variety is equivalent to the category of coherent complex analytic sheaves on that variety via the natural analytification functor. It is notable that even though the statement of the GAGA principle is rather abstract, Serre's proof involves a reduction to the case of projective space itself, some concrete computations in this case, and an invocation of Chow's theorem.

In the recent paper [5], Bakker, Brunebarbe, and Tsimerman develop a theory of definable sheaves over a definable complex analytic space and prove a version of the GAGA theorem in this context. In a related development, sophisticated theories of sheaves in o-minimal structures have been studied by Edmundo and Prelli [14].

3. Basics of Hodge theory

As I noted in the introduction, I am very far from being an expert on Hodge theory. To say that I am not an expert on Hodge theory is gross understatement. Unlike most of you readers, I have never properly learned differential geometry. Indeed, at some point during my graduate studies I had been assigned to assist in Yau's class on this subject but was promptly dismissed when my ignorance became apparent. That first appointment was replaced by an equally inappropriate appointment as an assistant in Bott's algebraic topology course, but somehow through a combination of his graciousness, the clarity of his perspective, and a pure trial by fire, I came through that experience with some appreciation for that subject without unduly confusing the students. I am conscious of the dual risks of presenting this subject as an outsider. On one hand, there will be some parts that I will not explain in the mistaken belief that even though they were not well known to me, most readers would find them to be well known and elementary. That is, you might not be one of my imagined readers who "properly understands" differential geometry inside and out and instead you share my relative ignorance. On the other hand, I am sure that there are important points that I have misunderstood, or even completely missed. Mindful of these potential pitfalls, I will say a bit more about the basic theory than may be necessary for you, will make some simplifying assumptions to cut down the technicalities, will ask for your patience, and will encourage you to seek out better accounts by true experts. For example, you would do well to consult Voisin's book [55] for a detailed, well motivated account of this subject.

3.1. **Hodge structures on the cohomology of smooth projective varieties.** For a smooth manifold M, de Rham's theorem gives a canonical isomorphism between the singular cohomology of M, defined purely topologically in terms of cocycles, and its de Rham cohomology, defined as the cohomology of the complex of differential forms on M with exterior differentiation given as the coboundary map. That is, for each natural number n, there is a pairing $H_n(M, \mathbb{R}) \times H^n_{\mathrm{dR}}(M) \to \mathbb{R}$ coming from the map which takes an n-cycle Y and an n-form α and returns $\int_Y \alpha$. De Rham's theorem says that this is a perfect pairing, so that $H^n_{\mathrm{dR}}(M) = H^n(X, \mathbb{R})$ via this identification.

If M is compact complex manifold, then the de Rham cohomology admits a refinement taking into account holomorphic and antiholomorphic components. For a pair of natural numbers p and q, we say that a differential form on M is a (p,q)-form if locally it may be expressed as a $\mathcal{C}^\infty(M)$-linear combination of forms expressed as $dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q}$ where $z_1, \ldots, z_m$ are complex coordinates on M, $1 \leq i_1 < \ldots < i_p \leq m$, and $1 \leq j_1 < \ldots < j_q \leq m$. The complex subspace of the complexified de Rham cohomology group $H^{p+q}_{\mathrm{dR}}(M)_{\mathbb{C}} := H^{p+q}_{\mathrm{dR}}(M) \otimes_{\mathbb{R}} \mathbb{C}$ (which by de Rham's theorem we can, and will, identify with $H^n(M, \mathbb{C})$) of classes represented by (p,q)-forms is denoted $H^{p,q}(M)$. Under the assumption that M is a smooth, projective, complex algebraic variety, these spaces give a particularly well behaved decomposition of the cohomology of M.

- $H^n(M,\mathbb{C}) = \bigoplus_{p+q} H^{p,q}(M)$
- $\overline{H^{p,q}(M)} = H^{q,p}(M)$
- There is a symmetric pairing $Q : H^n(M,\mathbb{Z}) \times H^n(M,\mathbb{Z}) \to \mathbb{Z}$ which then induces a Hermitian pairing on $H^n(M,\mathbb{C})$ via the rule $\langle \alpha, \beta \rangle := i^n Q(\alpha, \overline{\beta})$. With respect to this pairing, if $(p,q) \neq (p',q')$, then $H^{p,q}(M)$ and $H^{p',q'}(M)$ are orthogonal.
- The restriction of the pairing $\langle \cdot, \cdot \rangle$ to $H^{p,q}(M)$ is positive (respectively, negative) definite if n is even (respectively, odd).

These properties of the Hodge decomposition of the cohomology of a smooth, connected, projective, complex algebraic variety may be abstracted to give the notion of a Hodge structure.

Definition 3.1. Let A be a finite rank abelian group with a symmetric bilinear form $Q : A \times A \to \mathbb{Z}$ and let n be a natural number. A pure, effective, polarized Hodge structure of weight n on A is given by a finite sequence of complex subspace $H^{p,n-p}$ of $A_{\mathbb{C}}$ for $0 \leq p \leq n$ satisfying the following.

- $A_{\mathbb{C}} = \bigoplus_{p+q=n} H^{p,q}$
- $\overline{H^{p,q}} = H^{q,p}$
- For the Hermitian form defined by $\langle x, y \rangle = i^n Q(x, \overline{y})$, distinct $H^{p,q}$ and $H^{p',q}$ are orthogonal.
- The Hermitian form $\langle \cdot, \cdot \rangle$ is positive (respectively, negative) definite on $H^{p,q}$ if n is even (respectively, odd).

Remark 3.2. In Definition 3.1, the purity distinguishes these Hodge structures from *mixed* Hodge structures which arise in the study of not necessarily smooth algebraic varieties. There have been some important contributions of o-minimality to the theory of mixed Hodge structures, and I will mention some these in Section 4, but as pure Hodge structures are complicated enough, I will not go into the details here. The word *effective* refers to the restriction that p and q are nonnegative. For the general theory, there can be good reasons to drop this condition. Polarization refers to the conditions on the interaction between the components of the Hodge structure and the Hermitian form. Dropping the data of the form Q on A and thus also the last two conditions on the spaces yields the notion of a Hodge structure on A. I hope that the meaning of "weight n" is clear enough.

The data of a Hodge structure on the finite rank abelian group A may be packaged instead as a decreasing filtration $A_{\mathbb{C}} = F^0 \geq F^1 \geq$

$\cdots \geq F^n$ of subspaces of $A_{\mathbb{C}}$ by setting $F^p := \bigoplus_{r \geq p} H^{r,n-r}$. The individual components of the decomposition may be recovered from the equation $H^{p,q} = F^p \cap \overline{F^q}$. In general, given any such filtration of $A_{\mathbb{C}}$, the sequence of spaces defined by $H^{p,q} = F^p \cap \overline{F^q}$ gives a Hodge structure on A (possibly without the polarization condition) provided that $F^p \cap \overline{F^{n-p+1}} = 0$ for $0 \leq p \leq n$. On the face of it, because we can pass so easily from a Hodge structure as originally defined to one given as a filtration and back so easily, and especially since we need to go back to the space $H^{p,q}$ to check the polarization conditions, it would seem that nothing is gained by working with the alternative formulation. However, this way of viewing a Hodge structure is necessary for the study of variations of Hodge structure. With the natural way of parameterizing Hodge structures as sequences of spaces $H^{p,q}$, the period maps which describe the variation of the Hodge decomposition in families are not complex analytic, but they are when the Hodge structures are described by the sequence of spaces F^p.

3.2. **Variation of Hodge structure, period maps, and period domains.** Consider a family $\pi : X \to B$ of smooth, connected, projective, complex algebraic varieites over a smooth, connected base, by which I will mean that $\pi : X \to B$ is itself a regular map of smooth projective varieties all of whose fibers $X_b := \pi^{-1}\{b\}$ for $b \in B$ are smooth and connected. Treating π as a map of continuous manifolds, it is locally trivial. That is, for any given point $b_0 \in B$ we can find a neighborhood U of b_0 in B so that $X_U := \pi^{-1}U$ is homeomorphic to $X_{b_0} \times U$ over U. Passing to cohomology, this homeomorphism gives isomorphisms $H^n(X_b, \mathbb{Z}) \cong H^n(X_{b_0}, \mathbb{Z})$, and thus also, $H^n(X_b, \mathbb{C}) \cong H^n(X_{b_0}, \mathbb{C})$ for $b \in U$. Fiber by fiber, we have Hodge decompositions of each $H^n(X_b, \mathbb{C})$, but these isomorphisms coming from the local trivializations do not respect the Hodge decompositions. Since the Hodge structure is defined using the complex structure on X_b whereas the local trivialization of the family sees only the topology, this mismatch should not be surprising. The map which associates to each $b \in B$ the image of the Hodge structure of $H^n(X_b, \mathbb{C})$ in $H^n(X_{b_0}, \mathbb{C})$ is called the local period mapping and in many cases, this map describing the variation of Hodge structure serves as a proxy for the variation of the complex structure on these fibers.

Since the local period mapping and its globalization are central to the theorems I will describe in Section 4, a more detailed description of the map is in order. Let $\mathbf{G} := \mathrm{Aut}(H^n(X_{b_0}, \mathbb{Q}), Q)$ be the linear algebraic group defined over the rational numbers of linear symmetries of the rational cohomology of X_{b_0} preserving the intersection form mentioned above. Let $H^n(X_{b_0}, \mathbb{C}) = F_0^0 \geq F_0^1 \geq \cdots \geq F_0^n$ be the Hodge structure on $H^n(X_{b_0}, \mathbb{Z})$ defined by $F_0^p = \bigoplus_{r \geq p} H^{r,n-r}(X_{b_0})$. The space of $(n+1)$-tuples of subspaces V_j of $H^n(X_{b_0}, \mathbb{C})$ with $\dim V_j = \dim F_0^j$ for $0 \leq j \leq n$ and $V_0 \geq V_1 \geq \cdots \geq V_n$ is naturally an algebraic subvariety of a product of Grassmannian varieties. The orbit of $\langle F_0^0, \ldots, F_0^n \rangle$ under $\mathbf{G}(\mathbb{C})$ in this variety is itself an algebraic variety which we will denote as $\check{D}$. The orbit of $\langle F_0^0, \ldots, F_0^n \rangle$ under $G := \mathbf{G}(\mathbb{R})^+$ is an open domain D inside $\check{D}$ and each point in D corresponds to a Hodge structure on $H^n(X_{b_0}, \mathbb{Z})$. Such a set D is called a polarized period domain. For each point $b \in U$, the image of the Hodge structure on $H^n(X_b, \mathbb{Z})$ in $H^n(X_{b_0}, \mathbb{C})$ coming from the given diffeomorphism $X_b \cong X_{b_0}$ corresponds to a point in D. In this way, we obtain a map $\varphi : U \to D$. Griffiths showed that this local period mapping is holomorphic. Performing analytic continuation, the period map extends to all of B, but, of course, it is not well-defined. The ambiguity in the extension may be expressed through the monodromy of the local system of the integral cohomology of the fibers and this may be seen through a representation of the fundamental group of B, $\pi_1(B, b_0) \to \mathbf{G}(\mathbb{Q})$. If we write Γ for the image of this monodromy representation, then the period map may be considered as a holomorphic map $\varphi : B \to \Gamma \backslash D$ from B to what we will call the period space, the quotient of the period domain D by the action of this finitely generated group.

Just as the concept of a Hodge structure abstracts that of the Hodge decomposition on cohomology, abstract notions of a period map and of a variation of Hodge structure, neither of which is necessarily associated with a family of projective varieties, may be defined and appear in the theorems I will describe in Section 4. The extra generality these afford is important for the proofs, but the reader will not miss anything crucial by restricting attention to these variations of Hodge structure and period mappings coming from geometry. As I have already said too much about classical Hodge theory, I will refer the interested reader to the survey [8] or the original papers [6] or [23] for a fuller discussion.

3.3. **Hodge conjecture.** Of course, no introduction to Hodge theory is complete without some discussion of the Hodge conjecture. I will not report that o-minimality has been used to prove he Hodge conjecture, which, alas, remains very much open. However, some of the theorems described in Section 4 resolve positively certain highly nontrivial consequences of the Hodge conjecture.

If M is smooth, projective, connected, complex algebraic variety and $Z \subseteq M$ is a subvariety of codimension p, then Z naturally gives rise to a homology class in $H_{2p}(M, \mathbb{Z})$ which using duality may be seen as a class in $H^{2p}(M, \mathbb{Z})$. When further regarded as a class in $H^{2p}(M, \mathbb{C})$ this class actually belongs to $H^{p,p}(M)$. As orignally formulated, the Hodge conjecture predicted that every class in $H^{2p}(M, \mathbb{Z}) \cap H^{p,p}(M)$ comes from an algebraic p-cycle, that is, a formal $\mathbb{Z}$-linear combination of the classes of algebraic varieties of codimension p. Counterexamples to this version of the conjecture were observed by Atiyah and Hirzebruch [2] and the Hodge conjecture now takes the form that every *rational Hodge class*, that is, element of $H^{2p}(M, \mathbb{Q}) \cap H^{p,p}(M)$, comes from a rational p-cycle, that is, a formal $\mathbb{Q}$-linear combination of the classes of algebraic varieties of codimension p.

4. O-MINIMAL PROOFS OF THEOREMS IN HODGE THEORY

Several recent works have applied methods from o-minimal geometry to Hodge theory producing some very striking results. In retrospect, this development should have been expected. O-minimal geometry, especially o-minimal complex analysis, is all about extending the methods of semialgebraic geometry to analytic geometry and it supplies general methods for recognizing algebraicity. The constructions in Hodge theory are transcendental, but they arise from alghebraic geometry in an explicit manner. One might expect them to be definable in some o-minimal structure. Moreover, the spectacular theorems proven using o-minimality around the André-Oort conjecture may be understood in Hodge theoretic terms. Predictable or not (I did not see them coming), these are impressive advances. Let us discuss a few of the highlights.

4.1. **Definability of period maps.** We have defined the period mapping $\varphi : B \to \Gamma \backslash D$ associated to a family $X \to B$ of sooth projective varieities (and a choice a number n so that D parameterizes Hodge structures on $H^n(X_{b_0}, \mathbb{Z})$ for some base point $b_0 \in B$) in Section 3, but it is a highly nontrivial theorem, proven by Bakker, Klingler, and Tsimerman in [6], that, properly interpreted, these functions are definable in the o-minimal structure $\mathbb{R}_{\text{an,exp}}$. In a later paper [4], Bakker, Brunebarbe, Klingler, and Tsimerman extend this result to period mapping variations of mixed Hodge structues.

What do I mean by "properly interpreted"? The variety B is clearly a definable complex manifold in the sense that it admits a covering by finitely many open sets, say, coming from the intersections with the usual charts on projective space, so that each of the open sets is biholomorphic with a definable open subset of $\mathbb{C}^m$ for some m and the transition maps are definable. The same is true of the quotient space $\Gamma\backslash D$ (as long as we allow for singularities), though this result is not immediate. It is shown that $\Gamma\backslash D$ is definable, already in semi-algebraic geometry, by using the theory of Siegel sets to find an open semialgebraic set $F \subseteq D$ so that every orbit of Γ in D is represented by some element of F and so that $\Gamma_0 := \{\gamma \in \Gamma \ : \ \gamma \cdot F \cap F \neq \varnothing\}$ is finite. In this way, $\Gamma\backslash D$ may be represented as the quotient of the definable set F by the definable equivalence relation defined by $x \sim y :\Longleftrightarrow (\exists \gamma \in \Gamma_0)\gamma \cdot x = y$. It is a general theorem that in any o-minimal expansion of an ordered field, the quotient of a definable set by a definable equivalence relation may be realized as a definable set.

Proving that the period map itself is definable is more complicated, but the key ideas may be be understood as an elaboration of the proof of the definability of the restriction of the j-function to its fundamental domain combined with the difficult nilpotent orbit theorem of Schmid [44].

In what follows, we will write $\Delta := \{z \in CC \ : \ |z| \leq 1\}$ for the closed unit ball and $\Delta^* := \{z \in \Delta \ : \ z \neq 0\}$ for the puncture unit disk.

If B were projective already, then the holomorphic map $\varphi : B \to \Gamma\backslash D$ would already be definable using restricted analytic functions. Indeed, we could cover B by finitely many definable sets U each of which is definably, biholomorphic with Δ^n (where $n = \dim B$) so that the restriction of φ to U takes values in one of the coordinate charts of $\Gamma\backslash D$. On each such U, the real and imaginary parts of the components of φ are explicitly restricted analytic functions. When B is not compact, then we should find a compactification $\overline{B}$ so that $\overline{B} \smallsetminus B$ is a divisor with normal crossings and then take a finite cover of $\overline{B}$ by coordinate neighborhoods. When intersected with B, these neighborhoods are definably biholomorphic with $(\overline{\Delta}^*)^\ell \times \overline{\Delta}^m$ where $\ell + m = \dim B$. The monodromy around the punctures is known to be quasiunipotent, by which I mean that their eigenvalues are roots of unity. Replacing this situation with a finite cover, we may arrange that the monodromy is actually unipotent. Recall that when checking the definability of a suitable restriction of the j-function we considered the map $\mathfrak{h} \to \{q \in \mathbb{C}' : |q| < '1\}$ given by $\tau \mapsto \exp(2\pi i)$ and then observed that if restricted to any strip of the form $S = \{\tau \in \mathfrak{h} : a \leq \mathrm{Re}(\tau) \leq b\}$, then this map is definable in $\mathbb{R}_{\text{an,exp}}$. Put another way, after sclaing this map so that its image contains Δ, we may cover Δ by finitely many pieces so that some inverse $L\Delta \to \mathfrak{h}$ of this is definable and holomorphic on each of these pieces. Let $\mathfrak{g}$ be the Lie algebra of G, the real Lie group with respect to which D is a homogeneous space. Then Schmid's nilpotent orbit theorem says that when read with respect to the coordinates on $(\overline{\Delta}^*)^\ell \times (\overline{\Delta})^m$, $\varphi(z_1, \ldots, z_\ell, w_1, \ldots, w_m) = \exp_G(\sum_{j=1}^{\ell} L(z_j)N_j) \cdot \psi(z, w)$ where ψ is a holomorphic function on some neighborhood of $\overline{\Delta}^{\ell+m}$, $N_1, \ldots, N_\ell$ and commuting nilpotent matrices appearing as the logarithms of the generators of the monodromy around the hyperplanes defined by $z_i = 0$, and $\exp_G : \mathfrak{g} \to G$ is the exponential from the Lie algebra of G back to G itself. From the Baker-Campbell-Hausdorff formula, the Lie exponential is usually given by an infinite series expansion. However, because the N_i's are nilpotent and commute, the components of a matrix representation of $\exp_G(\sum_{j=1}^{\ell} L(z_j)N_j)$ are polynomials in the $L(z_j)$. Of course, the action on G on D is definable, in that it is the restriction of an algebraic action. Thus, the overall expression for φ on the charts where each $L(z_i)$ is definable explicitly presents φ as a definable function.

Many strong consequences may be deduced from the definability of these period mappings. For example, the equivalence relation $\sim$ on B given by $x \sim y :\Longleftrightarrow \varphi(x) = \varphi(y)$ is definable and complex analytic. Hence, from the definable Chow theorem, this equivalence relation is algebraic. In general, the quotient of an algebraic variety by an algebraic equivalence relation may be identified with an algebraically constructible set. Thus, the image $\varphi(B) \subseteq \Gamma \backslash D$ is naturally algebraic, even though the complex analytic space $\Gamma \backslash D$ might not be algebraic itself. Bakker, Brunebarbe, and Tsimerman use their upgraded o-minimal GAGA theorem to strengthen the conclusion of this observation by proving a conjecture of Griffiths [5] that the quasiprojectivity of $\varphi(B)$ may be witnessed by the ampleness of the restriction of a natural line bundle on $\Gamma \backslash D$ to $\varphi(B)$.

4.2. **Algebraicity of Hodge loci.** In a paper entitled "Hodge's general conjecture is false for trivial reasons" [17], Grothendieck noted a strong consequence of the Hodge conjecture. (While the title might suggest that he disproved the Hodge conjecture itself, what he actually observed was that a certain generalization of the the Hodge conjecture required a correction.) If $\pi : X \to B$ is a family of smooth, connected, projective, complex algebraic varieties over a smooth, connected base B, then one may consider the *Hodge locus*, $\mathrm{HL}(X) \subseteq B$, the set of $b \in B$ for which there are more Hodge classes in $H^n(X_b)$ than what occurs generically. That is, $\mathrm{HL}_p(X) := \{b \in B : \dim_{\mathbb{Q}}(H^{2p}(X_b, \mathbb{Q}) \cap H^{p,p}(X_b)) > \min_{b' \in B} \dim_{\mathbb{Q}}(H^{2p}(X_{b'}, \mathbb{Q}) \cap H^{p,p}(X_{b'}))\}$. Grothendieck observed that if the Hodge conjecture holds, then the Hodge locus must be a countable union of algebraic subvarieties of B and that if the family $\pi : X \to B$ is defined over some subfield $K \subseteq \mathbb{C}$ of the complex numbers, then, in fact, each component of the Hodge locus is defined over the algebraic closure of K and each $\mathrm{Gal}(K^{\mathrm{alg}}/K)$-conjugate of a component of the Hodge locus is itself a component of the Hodge locus.

The algebraicity of the Hodge locus, in the sense that it is a countable union of algebraic varieties, was established by Cattani, Deligne, and Kaplan [10] through a careful study of the behavior of period mappings at infinity. Bakker, Klingler, and Tsimerman [6] offer an alternative proof using the definability of the period mappings. To be fair, the Cattani-Deligne-Kaplan and Bakker-Klingler-Tsimerman arguments both rely on the same analytic source, namely Schmid's work on period maps, but this new argument has the virtue of encapsulating Schmid's growth estimates in the statement of definability from which soft arguments may be deployed. In addition to the definability of the period map, the Bakker-Klingler-Tsimerman argument makes use of the observation that if $b \in B$ lies in the Hodge locus, then Hodge structure on $H^n(X_n, \mathbb{Z})$ will be have fewer symmetries so that $\varphi(b)$ will lie in the image in $\Gamma \backslash D$ of some D' which parameterizes those special Hodge structures (such domains are called Mumford-Tate domains), that there are only countably many such special spaces to consider, and that each of these images is complex analytic. By the definable Chow theorem, the preimage under φ of the image of D' is algebraic.

The second part of Grothendieck's conjecture on the algebraicity of the Hodge locus does not follow immediately from the definable Chow theorem because, on the face of it, the definitions of the components of the Hodge locus may use the full structure of $\mathbb{R}_{\mathrm{an,exp}}$ so that the parameters used in the definition of the algebraic varieties may be obtained by evaluating transcendental functions. Nevertheless, using the Tannakian formalism, Klingler, Otwinowska, and Urabnik [24] proved that each component of the Hodge locus which is "non-facteur", which basically means that it does not come from a family of special varieties, is defined over the algebraic closure of the field of definition of the variation of Hodge structure. Later, Urbanik [49] proved the Galois invariance of the Hodge locus (again restricted to the non-facteur components) using differential algebraic methods. Pila and I have found a simple way to express these arguments using Kolchin-style differential algebra and hope that this will give a stronger algebraicity theorem.

4.3. **Functional transcendence for the period map.** The period functions we have been considering are transcendental and generally we do not expect their values to satisfy any algebraic relations except for those they are forced to satisfy for geometric reasons. This expectation is taken in analogy to Schanuel's conjecture which predicts that if $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ and $\mathbb{Q}$-linearly independent complex numbers, then the transcendence degree of the field generated by these numbers and their exponentials is at least n. Schanuel's conjecture is open, but Ax proved a functional version [3] one formulation of which says that if $\gamma_i : \Delta^\ell \to \mathbb{C}$ for $1 \leq i \leq n$ is a finite sequence of complex analytic maps from the polydisk to the complex numbers for which no nontrivial linear combination is constant, then the transcendence degree over $\mathbb{C}$ of the field generated by these functions and their exponentials is at least n plus the rank of the Jacobian matric $\left(\frac{\partial \gamma_i}{\partial z_j}\right)$.

Variants of Ax's theorem have been proven for the Weierstrass $\wp$-function [22, 21], the j-function [34], and for covering maps of Shimura varieties [26]. Kirby's theorem on Weierstrass $\wp$-function was proven using methods from differential algebra, in much the same way as Ax had proven the functional Schanuel conjecture for the exponential function. Pila's theorem for the j-function and more generally the Mok-Pila-Tsimerman Ax-Schanuel theorem for Shimura varieties were proven using methods from o-minimality. Specifically, it was shown that the relevant maps were definable in $\mathbb{R}_{\text{an,exp}}$ and the Pila-Wilke counting theorem was used to show that the existence of additional algebraic relations must be explained by the presence of some homogeneous space for the action of an algebraic group. A recent theorem of Bakker and Tsimerman [7] giving an

Ax-Schanuel theorem for general period mappings follows a similar line of proof. Simplifying the statement somewhat, the Bakker-Tsimerman theorem says that if $\varphi : B \to \Gamma\backslash D$ is an injective period mapping with generic image (meaning that there is no proper period subspace containing the image), and $\gamma = (\alpha, \beta) : \Delta \to B \times D \subseteq B$ is a holomorphic map into the product of B with D such that $\alpha = \pi \circ \beta$ where $\pi : D \to \Gamma\backslash D$ is the quotient map, then either the image of α is contained in a proper Mumford-Tate domain, or the transcendence degree over $\mathbb{C}$ of the field generated by γ greater than $\dim B$.

CONCLUSION

No longer a novelty in diophantine geometry, o-minimal geometry has matured to the point that it is routinely employed and developed by mathematicians who were not originally trained in mathematical logic. Beyond the recent applications to Hodge theory I have described, o-minimal geometry has played an important role in several other applications to geometry and number theory. Some of the most notable recent successes include the resolution by Pila, Shankar, and Tsimerman of the full André-Oort conjecture for all Shimura varieties [33], the uniform Mordell conjecture proven Dimitrov, Gao, and Habegger [13] which uses o-minimality in a crucial way through Gao's proof of the Ax-Schanuel theorem for the universal abelian variety [16], and the equidistribution theorems for flows on nilmanifolds of Peterzil and Starchenko [31]. O-minimality is a mature subject which has earned its status as the avatar of tame geometry.

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UNIVERSITY OF CALFORNIA, BERKELEY, DEPARTMENT OF MATHEMATICS, EVANS HALL, BERKELEY, CA 94720-3840, USA

Email address: scanlon@math.berkeley.edu

THE STABILITY OF BLACK HOLES WITH MATTER

ELENA GIORGI

Abstract. Black holes are fundamental objects in our understanding of the universe. The mathematics behind them has surprising geometric properties, and their dynamics is governed by hyperbolic PDEs. A basic question one may ask is whether these solutions to the Einstein equation are stable under small perturbations, which is a typical requirement to be physically meaningful. We will see how the dispersion of gravitational waves plays a key role in the stability problem, illustrating the main conjectures and some recent theorems regarding the evolution of black holes and their interaction with matter fields.

1. Introduction

Black holes are astrophysical compact objects which are believed to be overwhelmingly present in the universe. Black holes embody the extreme nature of the theory of General Relativity, and the singularities hidden behind them are thought to be the point of contact of Einstein's theory of gravity with the theory of quantum mechanics, as quantum effects are believed to be relevant at those scales. A resolution for a quantum theory of gravity will most likely involve a better comprehension of black holes and their singularities.

In addition to the importance of black holes for theoretical physics, the recent breakthroughs involving the detection of gravitational waves emitted by the merger of binary black holes systems have opened up the possibilities of a more direct way of observing and studying these objects. The LIGO interferometers detected the first signal emitted by such an event in September 2015 [1], and their discovery was awarded the Nobel Prize in Physics 2017, just a little more than a year after the announcement. The gravitational waves, which were predicted by Albert Einstein precisely one hundred year before (in 1915) in his formulation of the theory of General Relativity, came from a collision between two black holes, which took place 1.3 billion light years away. The event can be described as follows. Two black holes rotate around each other; as they rotate they lose energy through the emission of gravitational waves, and in this process they get closer and closer to each other, while moving almost as fast as the speed of light, till they merge into one larger black hole. After the merger, the final black hole settles down to a stationary state, and in doing that it emits energy through gravitational waves.

What LIGO observed are precisely the waves emitted in this process, and they were detected as a wave-like signal by the two interferometers in Washington and Louisiana. Clearly, in order to extract any relevant information about the event (like the mass of the black holes, their rotations or their distance from us) from the raw detected signal, one needs a very precise model to compare it to. One of the crucial

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components in the LIGO discovery is in fact the statistical process of comparing the signal observed by the instrument to a catalogue of so-called "waveforms", which are numerically simulated signals as expected from the prediction of General Relativity. Remarkably, the first code able to simulate the merger of two black holes was only obtained in 2005 by Frans Pretorius [43], just 10 years before the actual detection of gravitational waves by LIGO.

LIGO has continued to observe mergers of black holes in recent years, and we are starting to have a much better understanding of the population of black holes in the universe, together with their channel formations and parameter properties. The numerical simulations involved in the process, extremely refined and complex on their own, rely on the mathematical understanding of black holes as solutions to the Einstein equation. In particular, the waves emitted in the merger events are the physical fingerprints of the particular dynamics associated to the Einstein equation: the dispersion properties of the wave equation. The mere detection of the gravitational waves emitted by those large perturbations of black holes, in addition to probe their existence, gives us a strong argument for the stability of these objects to perturbations, which is a question that has interested physicists and mathematicians alike in the past fifty years.

In this article, we will give an introduction to the mathematics of black holes, presenting the equations they satisfy, together with the most important explicit black hole solutions. We shall then introduce the problem of stability of black holes, together with the dynamics which govern their perturbations. We will consider the case of black holes which interact with matter fields, in particular electromagnetic radiation, and how such interaction changes the overall behavior of the perturbations.

2. The mathematics of black holes

In order to understand the mathematics of black holes, we have to go back to four hundred years ago, to the physics of Newtonian gravity.

According to Newton, the spacetime is given by a 3-dimensional flat space and a 1-dimensional absolute time. In such spacetime, not all observers agree on the value of the spatial coordinates or the velocity of an object (that depends on the choice of coordinates), but they agree on measurements of distances. In other words, the spatial separation

$$ds = \sqrt{dx^2 + dy^2 + dz^2},$$

which is the Euclidean distance, is conserved.

In 1905, Einstein challenged the Newtonian vision of spacetime, and in his Special Relativity he proposed a theory in which the spacetime is instead thought to be a 4-dimensional flat spacetime, with no absolute slices and no privileged family of observers. To each point of the spacetime, there is an associated cone, called the "light-cone", along whose edges light can travel, and everything else, massive objects like us, can only travel slower than the speed of light, therefore moving inside the light-cone. He proposed that the spacetime separation

$$ds = \sqrt{-c^2dt^2 + dx^2 + dy^2 + dz^2}$$

where c is the speed of light, is the conserved quantity.

This leads to the definition of the Minkowski spacetime, or the spacetime of Special relativity, as the flat metric on $\mathbb{R}^{3+1}$ with signature $(3,1)$ given by

$$g_m = -dt^2 + dx^2 + dy^2 + dz^2, \tag{2.1}$$

where from now on the speed of light has set to be $c = 1$. The Minkowski spacetime is the Lorentzian equivalent of the Euclidean space in Riemannian geometry, and its light-cones are all uniform and vertical with slope 1. The Minkowski spacetime represents the empty space, where no masses or objects are present.

It took Einstein about ten years to understand what happens in the presence of a massive object, for instance to describe the spacetime of a star, and this was the core of his theory of General Relativity. In Minkowski spacetime, an object which travels at constant speed would follow a straight line in spacetime. If a star is present instead, it will deform the spacetime so that the object, in following a straight line (i.e. a geodesics), would actually bend toward the star. This fact can be summarized in John Wheeler's words "Spacetime tells matter how to move; matter tells spacetime how to curve". The light-cones of the spacetime of a star are therefore not uniform, but they bend toward the star as they get closer to it.

An interesting phenomenon, which even escaped Einstein's comprehension for a long time, is that the geometry radically changes if the star becomes more and more massive and dense. If this happens, the spacetime gets distorted: the overall geometry of the light cones changes, and a region where not even light can escape forms. The light cones become tangent to a confined hypersurface, and in particular if a light signal happens to reach that hypersurface, it will in fact never be able to leave the region enclosed by that it. The hypersurface is called the "event horizon" and the region enclosed by it "black hole".

2.1. **The Einstein equation.** From a mathematical point of view, black holes are solutions to the main equation governing the theory of General Relativity, called the Einstein equation. According to General Relativity, a spacetime is a 4-dimensional manifold equipped with a Lorentzian metric g (i.e. with signature $(3,1)$), that satisfies the *Einstein equation*[1]:

$$\mathrm{Ric}(g) - \frac{1}{2}\,\mathrm{R}(g) g = \mathrm{T}, \tag{2.2}$$

where

- $\mathrm{Ric}(g)$ is the Ricci curvature of g,
- $\mathrm{R}(g)$ is the scalar curvature of g,
- T is called the stress-energy tensor, and contains information about the matter fields present in the spacetime.

The Einstein equation (2.2) is the mathematical representation of the fact that "spacetime tells matter how to move; matter tells spacetime how to curve": the left hand side of the equation is a particular combination of curvature of the spacetime, which encodes how curved the spacetime is, while the right hand side describes the behavior of the matter.

At first glance, equation (2.2) may seem like the definition of the right hand side T, but this is not how one should think about it. The unknown of the equation is the metric g, and even though the left hand side of (2.2) may seem easy to write, it is actually given by a system of second order partial differential equations in

[1]Here we set the speed of light and the gravitational constant to unity, i.e. $c = G = 1$

the metric, as in the definition of Ricci curvature. Those second order PDEs are sourced by the left hand side of the equation, which is a given expression containing all the information about the matter present in the universe. In its full generality, the Einstein equation is in fact very difficult to solve and study, and in what follows we will concentrate on two important cases.

The simplest scenario one can consider is the case where there are no matter fields present in the spacetime, i.e. $\mathrm{T} \equiv 0$. A vacuum spacetime is a spacetime satisfying the *Einstein vacuum equation*:

$$\mathrm{Ric}(g) = 0. \tag{2.3}$$

The Einstein vacuum equation is considered the benchmark of the study of General Relativity, and rightly so. In fact, solutions to the Einstein vacuum equation can be shown to have most of the geometrical properties of general spacetimes. Nevertheless, the analysis of the resulting equations describing the dynamics is simplified in the vacuum case, as the gravitational radiation is not sourced by other radiations and does not interact with matter fields.

If we assume that the gravitational field can interact with electromagnetic radiation (if we want to literally add some color to the equations!), we obtain an electrovacuum spacetime, which satisfies the *Einstein-Maxwell equation*:

$$\mathrm{Ric}(g) = 2\,\mathrm{F} \cdot \mathrm{F} - \frac{1}{2} |\,\mathrm{F}\,|^2 g, \tag{2.4}$$

where F is a 2-form, called the *electromagnetic tensor*, satisfying the Maxwell equations:

$$d\,\mathrm{F} = 0, \qquad \mathrm{div}\,\mathrm{F} = 0. \tag{2.5}$$

In this case, the Ricci curvature is not identically zero, but rather sourced by a quadratic expression in terms of the electromagnetic field, which satisfies itself the equations of electrodynamics (2.5).

2.2. **Why studying non-vacuum solutions?** In considering the Einstein-Maxwell equation (2.4)-(2.5) describing the interaction of gravitational and electromagnetic radiation, we clearly add difficulties to the problem, so it is good to stop and ask ourself why the understanding of those complications would be relevant to the study of black hole solutions.

As a first motivation in this direction, we can go back to the LIGO interferometers and its detection of the merger of two black hole solutions that we mentioned previously. In August 2017, LIGO made another breakthrough discovery when it detected the first merger of two neutron stars [2]. Neutron stars are the collapsed core of massive supergiant stars, mostly composed by neutrons and supported against their collapse by neutron degeneracy pressure. Just two seconds after the gravitational wave signal was detected by LIGO, a flash of gamma-rays was detected by the FERMI satellite on space, coming from the same tiny corner of the cosmos. This observation started an entire new field of gravitational astronomy, called multi-messenger astronomy, where the same cosmic event is observed through two different signals: one in gravitational waves (detected by LIGO) and one in electromagnetic rays (detected by the FERMI satellite).

Moreover, the merger of two neutrons stars is expected to collapse into a rotating and charged black hole [41]. In fact, even though the magnetic field potentially surrounding a black hole is expected to disperse very fast, in the case of a rotating collapse the momentum of the magnetic field produces a current of electric charge which is expected to remain after the magnetic field has dispersed.

In addition to that, there have been recent works [8][9] pointing out that even the merger of two black holes could be charged! In fact, the first detection in 2015 (which is still the strongest one observed by LIGO) is compatible with a charge-to-mass ration as high as 0.3, which makes the presence of the charge in analysis of these scenarios not negligible.

2.3. **Solutions to the Einstein equation.** We now present some particular solutions to the Einstein vacuum and Einstein-Maxwell equations and their properties.

The simplest solution to the Einstein vacuum equation is the Minkowski spacetime g_m, given by the metric (2.1), which can also be written in standard spherical coordinates (t, r, θ, ϕ) as

$$g_m = -dt^2 + dr^2 + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2 \right). \tag{2.6}$$

As mentioned above, Minkowski is the spacetime of Special Relativity, with no presence of matter.

A more interesting solution was discovered by Karl Schwarzschild in 1916 [45], just a few months after Einstein wrote down his equation. Schwarzschild wanted to find an explicit solution to the Einstein vacuum equation (2.3), and he looked for one with a high degree of symmetry. By imposing a spherical symmetry on a solution (the spherical part $r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2 \right)$ also present in the Minkowski metric (2.6)), he reduced the problem to solving an ODE. The ODE can be solved explicitly in terms of the radial coordinate, and in the process of integrating for the solution one obtains a constant of integration M. The *Schwarzschild* solution is then a 1-parameter family of solutions, parametrized by $M \in \mathbb{R}$, given by

$$g_M = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2 \right). \tag{2.7}$$

One can immediately notice that if $M = 0$, the above metric reduces to the Minkowski metric (2.6). Even more interestingly, it was clear early on that for large r the Schwarzschild metric g_M gives the gravitational potential according to Newtonian theory of an isolated body of mass M. This means that the Schwarzschild solution gives an explicit formula for the equivalent in General Relativity to the spacetime of a star.

One can easily notice that the metric (2.7) has a singular behavior at $r = 2M$, where the coefficient $\left(1 - \frac{2M}{r}\right)$ vanishes. Nevertheless, it took long time before the nature of such singularity became clear. At $r = 2M$, there is no geometrical invariant which becomes singular, but it is rather a coordinate singularity. This means that there is a change of coordinates which modifies the metric (2.7) into one which is perfectly well-behaved at $r = 2M$. Still, the hypersurface $r = 2M$ has an interesting geometrical property: the light-cones of the manifold bend toward the region $r < 2M$ and become tangent to $r = 2M$. In particular, light rays traveling along the edges of those cones, as well as massive objects traveling in the interior of the light cones, are not able to leave the region $r \leq 2M$ once they enter it. It is a *black hole* region, and the hypersurface $r = 2M$ is an event horizon.

A region of spacetime where not even light can escape was a very difficult concept to grasp for the physics community at that time, and the Schwarzschild solution was indeed only considered to be valid outside the black region, i.e. for $r > 2M$. It was believed that the presence of the black hole region was an artifact of the symmetry of the solution: as in the real world there is no perfectly spherically symmetric massive object, so there would not be any black hole region. If one were to consider a more realistic star, such a region where not even light can escape would not form. Einstein himself wrote in 1939 [19] that the "Schwarzschild singularities [which is how they called the black hole region back then] do not exist in physical reality". The entire physics community was convinced that the black hole region was merely a mathematical property of the very symmetric Schwarzschild solution.

The general expectation about the existence of black hole solutions changed dramatically when Roy Kerr in 1963 [33] wrote down another explicit solution to the Einstein equation, parametrized by two real parameters M and a, where a is a rotation parameter, with $|a| \leq M$. The *Kerr* solution is given by

$$g_{M,a} = -\frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\phi\right)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left(adt - (r^2+a^2)d\phi\right)^2 \tag{2.8}$$

where

$$\Delta = r^2 - 2Mr + a^2, \qquad \rho^2 = r^2 + a^2\cos^2\theta,$$

and for $a = 0$ it reduces to the Schwarzschild metric (2.7). The Kerr solution represents a much more realistic case of a massive object of mass M which rotates around an axis of symmetry with angular momentum Ma.

Analyzing its geometrical properties, one can see that the hypersurface $\Delta = 0$ is still an event horizon: the light-cones become tangent to it and the region enclosed by it is a black hole region. The derivation of such metric proved to be a game changer for the concept of black hole, as its existence proved that the black hole region was clearly not simply an artifact of symmetry and did not have to be just a mathematical property of an unrealistic physical object. It could in fact very well be real. The Kerr solution is now considered to be the most fundamental black hole solution, and it is believed to represent the astrophysical black holes present in the universe.

The Einstein-Maxwell equation has two explicit solutions which are directly related to Schwarzschild and Kerr. The *Reissner-Nordström* solution is parametrized by two parameters M and Q, where Q is the electric charge, with $|Q| \leq M$, and is given by

$$g_{M,Q} = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{2.9}$$

It represents a spherically symmetric charged black hole.

Finally, the *Kerr-Newman* solution [42] is parametrized by three parameters M, a and Q with $a^2 + Q^2 \leq M^2$, and is given by the same expression as in (2.8), where

$$\Delta = r^2 - 2Mr + a^2 + Q^2, \qquad \rho^2 = r^2 + a^2\cos^2\theta.$$

The Kerr-Newman metric is the most general known explicit black hole solution to the Einstein-Maxwell equation, and it is a 3-parameter family which describes the gravitational field around an isolated rotating charged black hole of mass M, angular momentum Ma and electric charge Q. The Kerr-Newman metric generalizes the Reissner-Nordström solution (for $a = 0$), the Kerr (for $Q = 0$) and the Schwarzschild metric (for $Q = a = 0$). As such, the Kerr-Newman spacetime plays a fundamental role in describing the final state of evolution in General Relativity.

2.4. **The dynamics of black holes.** The Kerr-Newman solutions are extremely useful to perform explicit computations, such as calculations of geodesics, curvature invariants, etc. Nevertheless, one can immediately notice that the metric coefficients appearing in (2.8) do not depend on the time variable t. We call such solutions stationary. In particular, stationary solutions cannot describe events such as the merger of black hole binaries that LIGO observed. In those events, the black holes rotate around each other and continuously change their configuration, in contrast with a black hole described by a stationary solution such as (2.8), which sits still forever.

In order to describe events which are not stationary, we need a mechanism to study the dynamics of black holes as solutions to the Einstein equation. This tool is precisely given by Choquet-Bruhat's theorem [20] and the formulation of the Einstein equation as an initial value problem.

Theorem 2.1 (Choquet-Bruhat, 1952)**.** *The Einstein equation in wave coordinates is given by a hyperbolic system of PDEs, of the form*

$$\Box_g g = N(g, \partial g) \tag{2.10}$$

where $\Box_g = g^{\mu\nu} D_\mu D_\nu$ is the D'Alembertian operator associated to the metric g, D is the covariant derivative associated to g and $N(g, \partial g)$ denote non-linear terms in g and its first derivative, with initial data given by $(g|_{\Sigma_0}, k|_{\Sigma_0})$, i.e. the metric and its second fundamental form on a spacelike hypersurface Σ_0, which satisfy some compatibility conditions, called the constraint equations.

The D'Alembertian operator associated to the Minkowski metric is given by the standard wave operator,

$$\Box_{g_m} = -\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2,$$

and the most important properties satisfied by solutions to the standard wave equation, such as finite speed of propagation, can be proved to be valid for solutions to the general covariant wave operator $\Box_g$ for any Lorentzian metric g.

In particular, Theorem 2.1 implies local well-posedness and continuous dependence on the initial data for the Einstein equation. More precisely, given a set of compatible initial data for the Einstein equation on some initial time slice Σ_0, one can uniquely solve locally the Einstein equation in the future of Σ_0. This gives a mechanism to study the dynamics of the Einstein equation. Suppose we are given an initial data set which describes two black holes which rotate around each other. Then we can use Choquet-Bruhat's theorem to solve locally for the Einstein equation, and describe the behavior of the solution after a very short period of time.

What about the *global behavior* of the solution, i.e. the behavior of the solution after a long time from the initial time slice? This is a much more difficult question that Theorem 2.1 cannot really answer. For example, consider a much simpler case than the Einstein equation: a non-linear scalar wave equation of the form

$$\Box_{g_m} \phi = (\partial_t \phi)^2. \tag{2.11}$$

If we consider equation (2.11) with initial data $\phi|_{t=0} = \partial_t \phi|_{t=0} = 0$, then, by uniqueness, the solution is given by $\phi(t) = 0$ for all $t \geq 0$, and it is therefore global and bounded in time. If instead we consider equation (2.11) with initial data $\phi|_{t=0} = \partial_t \phi|_{t=0} = \epsilon$, for some $\epsilon > 0$, there exists a finite time T such that

$\phi \to \infty$ for $t \to T$. The solution is then not defined globally in time and in fact blows up at finite time.

We say that the trivial solution to equation (2.11) is not *stable under small perturbations*: if the initial data changes of size ϵ, for any small ϵ, then the behavior of the solution changes dramatically, passing from one which is bounded for all times to one which blows up in finite time.

Since the Einstein equation (2.10), as given by Choquet-Bruhat's theorem, has in principle the same schematic structure as equation (2.11) (albeit more complicated as it involves tensorial quantities and quasi-linear terms), one may worry that perturbations of even the trivial solution (i.e. the Minkowski spacetime) blow up in finite time. Surprisingly, this does not happen, as proved by Christodoulou and Klainerman in 1993 [11].

Theorem 2.2 (Christodoulou-Klainerman, 1993)**.** *The Minkowski spacetime g_m is globally non-linearly stable as solution to the Einstein vacuum equation.*

One of the underlying reasons that the Einstein equation behaves differently than equation (2.11), whose trivial solution is not stable under small perturbations, is the absence of non-linear terms of the form $(\partial_t \phi)^2$ on the right hand side. This is called the Klainerman's *null condition*. The monumental proof of Christodoulou-Klainerman's theorem does not depend on coordinates, but is rather geometrical as it lays open the structures of the spacetimes which are constructed as perturbations of Minkowski spacetime.

Christodoulou-Klainerman's theorem answers affirmatively to the question of stability of the trivial solution to the Einstein equation. What about the global behavior of perturbations of *non-trivial* solutions to the Einstein equation, like black holes? If we consider initial data given in a black hole background and a pulse of radiation at a given time, then we know by Choquet-Bruhat's theorem that it will propagate with finite speed as a wave. We can expect some of the radiation to fall inside the black hole, some to disperse towards the far-away region. What about the bulk of radiation in between?

One of the main differences between the Minkowski space and the black hole solutions is the presence of a hypersurface, called the *photon sphere*, outside the event horizon, where null geodesics (i.e. geodesics which travel along the edges of the light-cones) tend to concentrate. Those are called trapped null geodesics, as they are trapped on this bounded region for a long time. In Schwarzschild spacetime, the photon sphere occurs at $r = 3M$, outside the event horizon $r = 2M$.

Because of the presence of the photon sphere, one may worry that the radiation in perturbations of a black hole will tend to accumulate on the photon sphere region, creating a concentration of energy which will cause the whole spacetime to explode, and blow up in finite time.

Surprisingly, once again, this does not happen, as the trapped null geodesics which concentrate around the photon sphere outside the black hole solutions are unstable: in particular, they tend to scatter off and disperse after some time in the region. This is one of the important properties which are used to prove the stability of the wave equation on black holes.

3. The stability problem for the Einstein equation

The stability problem for black hole solutions concerns the long-time behavior of solutions to the Einstein equation which are perturbations of the known family of black hole solutions. We now present the stability problem as one aspect in the bigger context of the Final State conjecture, which aims to describe the state of the evolution of any reasonable initial data for the Einstein equation.

Conjecture 3.1 (Final State conjecture [34]). *"Reasonable" initial data for the Einstein equation evolve asymptotically in time to a finite number of Kerr-Newman black holes, moving away from each other.*

The Final State conjecture as stated here is very general and its proof is out of reach in the foreseeable future. Nevertheless, this conjecture implies three sub-conjectures, each of which is interesting in its own right and for which progress has been made.

(1) **Collapse conjecture**. The Final State conjecture predicts that black holes are the state of evolution of any initial data for the Einstein equation. But how do black holes form in the first place? The Collapse conjecture states that there is a mechanism for which large initial data give rise to the formation of a black hole.[2]
(2) **Rigidity conjecture**. The Final State conjecture predicts that initial data for the Einstein equation evolve into a very special form of black holes, i.e. the Kerr-Newman family given by the metric (2.8) with $\Delta = r^2 - 2Mr + a^2 + Q^2$. One may ask why one should expect this form of the metric, and not any other one (which we may not have discovered yet!), to be precisely the final state of evolution. The Rigidity conjecture states that Kerr-Newman is in fact the only family of stationary solutions to the Einstein equation. The Rigidity conjecture is proved in the case of analytic solutions, and is known as the "no-hair theorem".
(3) **Stability conjecture**. The Final State conjecture predicts that the evolution of initial data settles down to a member of the Kerr-Newman family, in particular implying that all the extra radiation will not concentrate in the process and instead disperse to infinity. This implies the Stability conjecture, which states that the Kerr-Newman family is stable under small perturbations of the initial data.

We now focus on the Stability conjecture.

Conjecture 3.2 (Stability of the Kerr-Newman family conjecture). *Initial data for the Einstein equation which are sufficiently close to a Kerr-Newman black hole evolve asymptotically in time to another member of the Kerr-Newman family.*

Observe that the conjecture does not claim that one member of the Kerr-Newman family is stable, but it is instead the whole family to be stable. Physically, this corresponds to the fact that when perturbing a black hole of a certain mass, rotation and charge, one expects the perturbation to modify its mass, rotation or charge. Nevertheless the overall structure of its metric is expected to be unchanged.

We now translate the Stability conjecture in PDEs language. Let the Einstein equation be represented by the non-linear operator

$$\mathcal{P}(\phi) = 0, \tag{3.1}$$

and let ϕ_λ be a family of stationary solutions (the Kerr-Newman family) parametrized by some parameter λ (given by M, a, Q), i.e. $\mathcal{P}(\phi_\lambda) = 0$.

[2]Small initial data do not, as implied by Christodoulou-Klainerman's theorem of the stability of Minkowski space.

Proving that the family of solutions ϕ_λ is stable under small perturbations as a solution to (3.1) boils down to proving that a solution with initial data close to ϕ_λ converges asymptotically in time to $\phi_{\lambda'}$ with λ' close to λ.

There are various levels of increasing difficulty for the stability problem:

(1) Consider the linearized equation

$$(d\mathcal{P})|_{\phi_\lambda}(\psi) = 0 \tag{3.2}$$

and prove that
- separated solutions of (3.2), of the form

$$\psi(t, r, \theta, \phi) = e^{-i\omega t} e^{im\phi} R(r) S(\theta) \tag{3.3}$$

for $\omega \in \mathbb{C}$, $m \in \mathbb{Z}$, do not exponentially grow in time: this is called *mode stability.*
- all solutions of (3.2) decay in time: this is the *full linear stability.*

(2) Prove that all solutions to the fully non-linear equation (3.1) decay in time: this is the *full non-linear stability.*

Why is the Stability of the Kerr-Newman family conjecture believed to be true? There have been clues in the past fifty years which have been pointing toward the expectation for the conjecture to hold. The clues concern the following three aspects of the problem.

- Mode stability: there are no exponentially growing modes for separated mode solutions of the form (3.3) to the Einstein equation in Schwarzschild (Regge-Wheeler [44], Bardeen-Press [5]), in Reissner-Nordström (Moncrief [40], Chandrasekhar [10]), and in Kerr (Teukolsky [49], Chandrasekhar [10], Whiting [50], Shlapentokh-Rothman [46]). No mode stability has been proved for Kerr-Newman spacetime, and we will come back to this anomaly in Section 4.3.
- Stability for the scalar wave equation on black hole backgrounds: general solutions arising from regular initial data to

$$\Box_g \psi = 0 \tag{3.4}$$

remain bounded and decay in time in Schwarzschild and Reissner-Nordström (Kay-Wald [32], Blue-Soffer [7], Dafermos-Rodnianski [17]), in Kerr and Kerr-Newman (Dafermos-Rodnianski [17], Tataru-Tohaneanu [48], Andersson-Blue [3], Dafermos-Rodnianski-Shlapentokh-Rothman [18]).

  Extensive progress has been obtained in the last fifteen years which allowed to go beyond the mode analysis, tackling the full linear stability (b) for the linear wave equation. A robust geometric interpretation of the red-shift effect [15], a physical space analysis of the trapping region and the superradiance [17], a hierarchy of r-weighted decay [16] all contributed to a complete understanding of the boundedness of solutions to the linear wave equation.
- Non-linear stability of Minkowski spacetime: Solutions to the fully non-linear Einstein vacuum equation which are small perturbations of Minkowski give rise to a complete spacetime which converges to Minkowski space. (Christodoulou-Klainerman [11], Lindblad-Rodnianski [38], Bieri [6])

3.1. **Known results in stability for the Einstein equation.** We now collect here some of the known results for the full linear and non-linear stability of black hole solutions to the Einstein vacuum and Einstein-Maxwell equation.

The Schwarzschild spacetime has been proved to be linearly stable to gravitational perturbations (Dafermos-Holzegel-Rodnianski [14], Keller-Hung-Wang [30], Johnson [31], Hung [29]), and non-linearly stable under the symmetry class of axially symmetric polarized perturbations (Klainerman-Szeftel [36]), and up to a codimension-3 submanifold of moduli space (Dafermos-Holzegel-Rodnianski-Taylor [12]).

The Kerr spacetime has been proved to be linearly stable using Newman-Penrose formalism (Andersson-Bäckdahl-Blue-Ma [4]) and using harmonic gauge (Häfner-Hintz-Vasy [28]), in both cases for $|a| \ll M$. Other important results also concern the proof of boundedness and decay for solutions to the Teukolsky equation for $|a| \ll M$ (Dafermos-Holzegel-Rodnianski [13], Ma [39]) and for $|a| < M$ (Shlapentokh-Rothman-Teixeira da Costa [47]). The non-linear stability of the Kerr spacetime for $|a| \ll M$ has been partially obtained in a combination of results by Klainerman-Szeftel [35], Klainerman-Szeftel [37], and in an upcoming joint work with Klainerman and Szeftel.

The Reissner-Nordström spacetime has been proved to be linearly stable to electromagnetic-gravitational perturbations for $|Q| < M$ in our series of work [25], [22], [24], [21]. Boundedness and decay for solutions to the Teukolsky system in Kerr-Newman spacetime have been obtained for $|a| \ll M$ in our recent [23] [26].

We state here the statement of the linear stability of Reissner-Nordström spacetime, as obtained in our [25].

Theorem 3.3 (G, [25]). *All solutions to the linearized Einstein-Maxwell equations around Reissner-Nordström spacetime for $|Q| < M$ arising from regular initial data*

(1) *remain uniformly bounded on the exterior and*
(2) *decay to a linearized Kerr-Newman solution*

after adding a pure gauge solution which can itself be estimated by the initial data.

We emphasize here two instabilities appearing in the statement of the theorem: the linearized Kerr-Newman and the pure gauge solutions.

We already mentioned that upon perturbing a black hole solution, the parameters of the spacetime can change. In particular, when perturbing a Reissner-Nordström spacetime (i.e. a Kerr-Newman metric with parameter M, Q and $a = 0$), we should expect the final perturbations to converge to a Kerr-Newman metric with small angular momentum a. In linear theory, this corresponds to solutions to the linearized Einstein-Maxwell equation around Reissner-Nordström which decay to a linearized Kerr-Newman solution.

There is a more serious instability which is proper of the Einstein equation. Being a tensorial equation for the Ricci tensor, the equation is invariant under choice of coordinates or gauge. In particular, given any metric g solution to the Einstein equation, the pull-back of the metric through any diffeomorphism is also a solution to the same Einstein equation. For this reason, in order to prove any decay of the solution we will effectively have to pick a gauge and control the solution in that gauge. In linear theory, this corresponds to solutions to the linearized Einstein equation which decay in time only up to a pure gauge solution which represents the choice of gauge.

3.2. **The formalism.** We will briefly present here the formalism used to study the Einstein equation in most of the known results mentioned above.

Alternatively to Choquet-Bruhat's approach in wave coordinates, one can use the definition of "frames", i.e. vectorfields which are geometrically defined along the manifold. This approach has been introduced in the proof of non linear stability of Minkowski space [11], and has been widely used ever since.

Let (M, g) be a $3+1$-dimensional Lorentzian manifold solution to the Einstein equation, and let D be the covariant derivative associated to g. Suppose that (M, g) can be foliated by spacelike 2-surfaces $(S, \mathbf{g})$, where $\mathbf{g}$ is the pullback of the metric g to S. To each point of M, we can associate a null frame $\{e_3, e_4, e_a\}$, with $\{e_a\}_{a=1,2}$ being tangent vectors to $(S, \mathbf{g})$ such that

$$g(e_3, e_3) = 0 \quad , \quad g(e_4, e_4) = 0 \quad , \quad g(e_3, e_4) = -2$$
$$g(e_3, e_a) = 0 \quad , \quad g(e_4, e_a) = 0 \qquad g(e_a, e_b) = \mathbf{g_{ab}}$$

We denote $\nabla_3 = \nabla_{e_3}$ and $\nabla_4 = \nabla_{e_4}$ the projection to S of the spacetime covariant derivatives D_{e_3} and D_{e_4} respectively. We then project all geometric quantities along the null frame $\{e_3, e_4\}$, and obtain tensors on the sphere S. For the Ricci coefficients (or Christoffel symbols) we have for example

$$\chi_{ab} := g(D_a e_4, e_b), \qquad \underline{\chi}_{ab} := g(D_a e_3, e_b), \tag{3.5}$$

which can be interpreted as the second fundamental form of the embedding of the sphere with respect to the normal e_4 and e_3 respectively. The quantities χ and $\underline{\chi}$ are 2-tensors on the sphere. For the Weyl curvature, we have for example

$$\alpha_{ab} \ = W(e_a, e_4, e_b, e_4), \qquad \underline{\alpha}_{ab} := W(e_a, e_3, e_b, e_3),$$

which are the extreme null curvature components of the spacetime, and are 2-tensors on the sphere. For the electromagnetic tensor, the extreme null components are given by

$${}^{(F)}\beta_a := F(e_a, e_4), \qquad {}^{(F)}\underline{\beta}_a := F(e_a, e_3),$$

which are 1-forms on the sphere.

In order to study the Einstein equation in this formalism, one makes use of the fact that the Einstein equation is equivalent to the Bianchi identities for the Riemann curvature, and therefore by projecting those equations to the spheres, one obtains a large number of tensorial equations on the spheres. Unfortunately, those equations are a coupled system of transport, elliptic and hyperbolic equations with a complicated structure. One may wonder why such approach, based on null frames, would be more successful or convenient with respect to the one in wave coordinates.

The main reason to use null frames is that the Kerr-Newman family admits a special frame, called *principal null frame*, which diagonalizes the Weyl curvature. This means that for

$$e_3 = \frac{r^2+a^2}{\rho^2}\partial_t - \frac{\Delta}{\rho^2}\partial_r + \frac{a}{\rho^2}\partial_\phi, \qquad e_4 = \frac{r^2+a^2}{\Delta}\partial_t + \partial_r + \frac{a}{\Delta}\partial_\phi \tag{3.6}$$

the Weyl curvature becomes

$$W_{a4b4} = W_{a3b3} = W_{a343} = W_{a434} = 0, \qquad W_{abcd}, W_{3434} \neq 0,$$

with most vanishing components of the curvature. Since it is easier to linearize around zero, it is convenient to write the many equations on the spheres with respect to the principal null frame. In fact, they simplify dramatically and become tractable: more precisely, in vacuum the symmetric 2-tensors on the spheres α_{ab} and $\underline{\alpha}_{ab}$ satisfy a second order PDE which is wave-like and decouples from all the other components of the linearization.

3.3. **The Teukolsky equation.** The symmetric 2-tensor α satisfies the following wave-like equation, called Teukolsky equation, as discovered by Teukolsky in 1972 [49]

$$\mathcal{T}(\alpha) := \square_g \alpha + c_1(r,\theta)\nabla_{\partial_r}\alpha + c_2(r,\theta)\nabla_{\partial_t}\alpha + c_3(r,\theta)\nabla_{\partial_\phi}\alpha - V(r,\theta)\alpha = 0.$$

In the physics community, the Teukolsky equation is expressed in terms of a complex scalar of spin ± 2 (for gravitational perturbations) or of spin ± 1 (for electromagnetic perturbations).

The highest order terms of the Teukolsky equation are given by a D'Alembertian operator $\square_g$, followed by first order terms in ∂_r, ∂_t, ∂_ϕ, and a potential. One may therefore expect that the techniques developed for the standard wave equation (3.4) can be applied to the Teukolsky equation to obtain estimates for its general solutions.

Unfortunately, that is not the case. Recall that boundedness of the energy for $\square_{g_m}\psi = 0$ in Minkowski spacetime is obtained by multiplying the wave equation by $\partial_t\psi$ and integrating by parts. Schematically one obtains

$$\begin{aligned} 0 = \square_{g_m}\psi \cdot \partial_t\psi &= \big(- \partial_t^2\psi + \triangle\psi\big) \cdot \partial_t\psi \\ &= -\partial_t^2\psi \cdot \partial_t\psi - \partial_t\nabla\psi \cdot \nabla\psi + \text{boundary terms} \\ &= -\frac{1}{2}\partial_t(|\partial_t\psi|^2 + |\nabla\psi|^2) + \text{boundary terms}, \end{aligned}$$

which gives conservation of the energy density $|\partial_t\psi|^2 + |\nabla\psi|^2$. Similarly for a wave equation with a positive potential V, i.e. $\square_{g_m}\psi - V\psi = 0$, one obtains

$$\begin{aligned} 0 &= \big(\square_{g_m}\psi - V\psi\big) \cdot \partial_t\psi \\ &= -\frac{1}{2}\partial_t(|\partial_t\psi|^2 + |\nabla\psi|^2 + V|\psi|^2) + \text{boundary terms}, \end{aligned}$$

which gives conservation of the coercive energy density $|\partial_t\psi|^2 + |\nabla\psi|^2 + V|\psi|^2$. A wave equation of the form $\square_g\psi - V\psi = 0$ is called a *Regge-Wheeler equation.*

For a general Teukolsky equation instead, because of the presence of the first order terms $c_1\partial_r + c_2\partial_t + c_3\partial_\phi$ in the equation, one cannot directly obtain boundedness of the energy as for the standard wave equation.

3.4. **The Chandrasekhar transformation.** This issue appears already in the study of the Teukolsky equation in Schwarzschild spacetime, as one would like to pass from a Teukolsky equation of the form

$$\square_g\alpha - V(r)\alpha = c_1(r)\partial_r\alpha + c_2(r)\partial_\phi\alpha + c_3(r)\partial_t\alpha,$$

to a Regge-Wheeler equation of the form

$$\square_g\mathfrak{q} - V(r)\mathfrak{q} = 0.$$

Such a transformation was described by Chandrasekhar [10] in the setting of mode stability, and Dafermos-Holzegel-Rodnianski [14] introduced a physical-space version of the transformation in Schwarzschild, valid for general solutions to the linearized equation. Such transformation is crucial to prove boundedness for the solutions of the Teukolsky equation, by obtaining a Regge-Wheeler equation, to which one can apply the known techniques, such as the vectorfield method, to prove boundedness of the energy and integrated local decay estimates.

The Chandrasekhar transformation consists in taking two null derivatives along the incoming direction e_3 of the curvature component α. Schematically

$$\mathfrak{q} = f_1(r)\nabla_3\big(f_2(r)\nabla_3\alpha\big),$$

for carefully chosen functions f_1 and f_2.

This surprising transformation, consisting of taking two derivatives of a "bad" equation and expecting to obtain a better equation, relies on a hidden relation between the equation for curvature perturbations (the Teukolsky equation) and the equation for metric perturbations (the Regge-Wheeler equation).

3.5. **The generalized Regge-Wheeler equation in Kerr.** In passing from Schwarzschild to Kerr or Kerr-Newman, there is a crucial geometrical difference: the principal null frame (3.6) in Kerr(-Newman) is not integrable in Frobenius' sense. This means that the subspace of the tangent space which is orthogonal to e_3 and e_4 is not tangent to a surface, but rather a horizontal distribution or 2-plane field.

In particular, the Ricci coefficient χ_{ab} defined in (3.5) is not symmetric in its indices, but it has an antisymmetric part which we denote ${}^{(a)}\mathrm{tr}\chi$ in our joint work with Klainerman and Szeftel [27], i.e.

$$\chi_{ab} = \widehat{\chi}_{ab} + \frac{1}{2}\mathrm{tr}\,\chi\delta_{ab} + \frac{1}{2}\,{}^{(a)}\mathrm{tr}\chi \in_{ab} .$$

Because of the presence of the antisymmetric terms, it is convenient to complexify all the curvature and Ricci coefficients. The complexified curvature component $A = \alpha + i * \alpha$ satisfies the Teukolsky equation

$$\square_g A + c_1(r,\theta)\nabla_{\partial_r}A + c_2(r,\theta)\nabla_{\partial_t}A + c_3(r,\theta)\nabla_{\partial_\phi}A - V(r,\theta)A = 0.$$

Dafermos-Holzegel-Rodnianski [13] and Ma [39] defined the Chandrasekhar transformation in the case of linear perturbations of Kerr, as schematically

$$\mathfrak{q} = f_1(r,\theta)\nabla_3\big(f_2(r,\theta)\nabla_3 A\big),$$

for suitably chosen functions f_1 and f_2. The quantity $\mathfrak{q}$ satisfies the following generalized Regge-Wheeler (gRW) equation:

$$\begin{aligned}\square_g\mathfrak{q} - V(r,\theta)\mathfrak{q} - i\frac{4a\cos\theta}{\rho^2}\nabla_{\partial_t}\mathfrak{q} \quad &= \quad a\big[d_1(r,\theta)\nabla_{\partial_\phi}\nabla_3 A + d_2(r,\theta)\nabla_{\partial_\phi}A \\ &\qquad + d_3(r,\theta)\nabla_3 A + d_4(r,\theta)A\big],\end{aligned}$$

where $V(r,\theta)$ is a positive real potential. See also our [27] for the derivation of the gRW equation for non-linear perturbations of Kerr.

Despite the presence of the first order term $i\frac{4a\cos\theta}{\rho^2}\nabla_{\partial_t}\mathfrak{q}$, the left hand side of the gRW equation still has good divergence properties as the standard Regge-Wheeler equation in Schwarzschild, while the right hand side of the equation can be treated as lower order terms for $|a| \ll M$.

4. Black holes with matter fields

We now look at how the interaction of black holes with matter fields change the analysis of the equations governing the perturbations. As described in Section 2.1, the interaction between gravitational and electromagnetic fields in a spacetime is governed by the Einstein-Maxwell equation:

$$\mathrm{Ric}(g) = 2\,\mathrm{F}\cdot\mathrm{F} - \frac{1}{2}|\,\mathrm{F}\,|^2 g. \tag{4.1}$$

More precisely, the left hand side of the equation, involving the Ricci curvature, encodes information about the gravitational radiation, while the right hand side of the equation, involving the electromagnetic tensor F, encodes information about the electromagnetic radiation. The gravitational radiation is transported by the Weyl curvature $W_{\mu\nu\gamma\lambda}$ which is a 4-tensor whose extreme null component $\alpha_{ab} = W(e_a, e_4, e_b, e_4)$ is a 2-tensor on the horizontal structure. The electromagnetic radiation is transported by the electromagnetic tensor $F_{\mu\nu}$ which is a 2-form whose extreme null component ${}^{(F)}\beta_a = F(e_a, e_4)$ is a 1-tensor on the horizontal structure.

4.1. The equations.

The gravitational and electromagnetic fields, when taken independently, satisfy the Teukolsky equation of spin s, for $s = \pm 2, \pm 1$ respectively:

$$\begin{aligned} \mathcal{T}^{[2]}(\alpha) &= 0 \\ \mathcal{T}^{[1]}({}^{(F)}\beta) &= 0, \end{aligned}$$

where the operators $\mathcal{T}$ are schematically given by

$$\mathcal{T} := \Box_g + c_1(r,\theta)\nabla_{\partial_r} + c_2(r,\theta)\nabla_{\partial_t} + c_3(r,\theta)\nabla_{\partial_\phi} - V(r,\theta) = 0.$$

In electromagnetic-gravitational perturbations of a black hole instead, there is interaction between the gravitational and the electromagnetic radiation. Instead of having two independent Teukolsky equations, one would expect to have equations which couple the 2-tensor α_{ab} with the 1-tensor ${}^{(F)}\beta_a$, such as for example

$$\begin{aligned} \mathcal{T}^{[2]}(\alpha) &= Q\cdot\nabla\widehat{\otimes}{}^{(F)}\beta \\ \mathcal{T}^{[1]}({}^{(F)}\beta) &= Q\cdot \mathrm{div}\alpha, \end{aligned}$$

where Q is the charge of the black hole and

- $(\nabla\widehat{\otimes}\xi)_{ab} = \nabla_a\xi_b + \nabla_b\xi_a - \delta_{ab}(\mathrm{div}\xi)$ for a 1-tensor ξ,
- $(\mathrm{div}\theta)_a = \nabla^b\theta_{ab}$ for a symmetric traceless 2-tensor χ.

In fact, the operators $\nabla\widehat{\otimes}$ and div respectively transform a 2-tensor into a 1-tensor and viceversa.

But things are not quite as simple as one may expect. In fact, in a charged black hole with $Q \neq 0$, the 1-tensor ${}^{(F)}\beta$ is not *gauge-invariant*. Gauge-invariance is related to the instability due to the choice of gauge described above, and can be expressed in terms of null frames in the following way.

A general linear frame transformation of the null frame $\{e_3, e_4, e_a\}$ is of the form

$$\begin{aligned} e_4' &= \lambda(e_4 + \mu_a e_a) \\ e_3' &= \lambda^{-1}(e_3 + \underline{\mu}_a e_a) \\ e_a' &= e_a + \frac{1}{2}\underline{\mu}_a e_4 + \frac{1}{2}\mu_a e_3, \end{aligned} \tag{4.2}$$

with $\lambda = 1 + O(\epsilon)$, $\mu, \underline{\mu} = O(\epsilon)$, where ϵ is the size of the perturbation. In particular, neglecting terms of size ϵ^2, the transformed frame $\{e_3', e_4', e_a'\}$ defined by (4.2) is also a valid null frame. We say that a tensor Ψ on the horizontal structure is (quadratically) gauge-invariant if it only changes quadratically with a general linear frame transformation (4.2), i.e.

$$\Psi' = \Psi + O(\epsilon^2),$$

where Ψ' is the tensor evaluated in the transformed frame $\{e_3', e_4', e_a'\}$. We are interested in gauge-invariant quantities as they are believed to represent physical quantities, such as gravitational and electromagnetic waves, which should not depend on the chosen coordinates or gauge.

Even though α is still gauge-invariant for charged black holes, it is not as useful anymore in view of the equation it satisfies. We define [23] instead new gravitational and electromagnetic radiation quantities $\mathfrak{f}$ and $\mathfrak{b}$, respectively a 2-tensor and a 1-tensor, which are gauge-invariant, and related to the Weyl curvature α.

As a consequence of the Einstein-Maxwell equation, the 1-tensor $\mathfrak{b}$ satisfies a Teukolsky-type equation coupled with $\mathfrak{f}$, and the 2-tensor $\mathfrak{f}$ satisfies a Teukolsky-type equation coupled with $\mathfrak{b}$. We therefore have a system of the schematic form [23]

$$\begin{aligned} \mathcal{T}^{[1]}({}^{(F)}\beta) &= Q \cdot c_4(r,\theta)(\text{div}\mathfrak{f}), \\ \mathcal{T}^{[2]}(\mathfrak{f}) &= Q \cdot d_4(r,\theta)(\nabla\widehat{\otimes}\mathfrak{b}). \end{aligned}$$

It is tempting to ask ourself how to identify the gravitational and electromagnetic radiation from a physical point of view. The Einstein-Maxwell equation which governs the interaction between the two radiation does not clearly distinguish between them though, as the whole perturbation is governed by the coupled electromagnetic-gravitational perturbations. Historically, we know that gravitational radiation is a spin-2 quantity while the electromagnetic is spin-1. In the case of coupled radiation, the spin-2 quantity $\mathfrak{f}$ is also defined in terms of the electromagnetic tensor, and viceversa the spin-1 quantity $\mathfrak{b}$ is also defined in terms of the curvature. Both quantities encode part of the perturbations, and they satisfy the master equations describing the coupled perturbation.

We found that there exists a Chandrasekhar transformation in the case of charged black holes too. The Chandrasekhar-transformed quantities of $\mathfrak{b}$ and $\mathfrak{f}$, given schematically by

$$\begin{aligned} \mathfrak{p} &= f_1(r,\theta)\nabla_3\big(f_2(r,\theta)\mathfrak{b}\big) \\ \mathfrak{q} &= g_1(r,\theta)\nabla_3\big(g_2(r,\theta)\mathfrak{f}\big), \end{aligned}$$

for suitably chosen functions f_1, f_2, g_1, g_2, satisfy a symmetric system of

- *Regge-Wheeler equations* in Reissner-Nordström spacetime
- *generalized Regge-Wheeler equations* in Kerr-Newman spacetime.

Observe that in this case the transformation only involves one derivative in the ingoing null direction e_3.

4.2. **The case of Reissner-Nordström spacetime.** In Reissner-Nordström, we obtain the following symmetric system of Regge-Wheeler equations.

Theorem 4.1 (G., [25]). *The gauge-invariant quantities $\mathfrak{p}$ and $\mathfrak{q}$ representing electromagnetic and gravitational radiations for linear perturbations of Reissner-Nordström satisfy the following coupled system of wave equations:*

$$\Box_g \mathfrak{p} - V_1(r)\ \mathfrak{p} = \frac{Q}{r^2}\ div\ \mathfrak{q}$$
$$\Box_g \mathfrak{q} - V_2(r)\ \mathfrak{q} = \frac{Q}{r^2}\ \nabla\hat{\otimes}\mathfrak{p}$$

where $V_1(r) = \frac{1}{r^2}\left(1 - \frac{2M}{r} + \frac{6Q^2}{r^2}\right)$, $V_2(r) = \frac{4}{r^2}\left(1 - \frac{2M}{r} + \frac{3Q^2}{2r^2}\right)$ are positive potentials.

The system is symmetric, as the respective right hand sides of the equations are adjoint operators on the sphere, i.e. for a 1-tensor ξ and a 2-tensor θ

$$\int_S \xi \cdot (div\ \theta) = \int_S (\nabla\widehat{\otimes}\xi) \cdot \theta.$$

The symmetric structure on the right hand side of the system implies that it is possible to derive energy estimates for the system by summing the estimates for the two equations. Upon integration on the sphere, the coupling terms cancel out reducing to boundary terms.

In physical-space terms, this is equivalent to the fact that we can define a combined energy-momentum tensor for the system $\mathcal{Q}_{\mu\nu}[\mathfrak{p}, \mathfrak{q}]$ as

$$\mathcal{Q}_{\mu\nu}[\mathfrak{p}, \mathfrak{q}] \quad := \quad \mathcal{Q}_{\mu\nu}[\mathfrak{p}] + \mathcal{Q}_{\mu\nu}[\mathfrak{q}] - \frac{Q}{r^2}\left(\nabla\widehat{\otimes}\mathfrak{p} \cdot \mathfrak{q}\right) g_{\mu\nu},$$

where

$$\mathcal{Q}_{\mu\nu}[\psi] = \nabla_\mu\psi \cdot \nabla_\nu\psi - \frac{1}{2} g_{\mu\nu}\left(\nabla_\lambda\psi \cdot \nabla^\lambda\psi + V|\psi|^2\right)$$

is the energy-momentum tensor associated to the Regge-Wheeler equation $\Box_g\psi - V\psi = 0$. By applying the vectorfield method to the current $\mathcal{Q}_{\mu\nu}[\mathfrak{p}, \mathfrak{q}]X^\nu$ for a vectorfield X, we obtain [25]

- energy estimates, as above, for $X = \partial_t$,
- Morawetz estimates for $X = f(r)\partial_r$, with $f(r)$ vanishing at the photon sphere.

  In this case, one obtains in the bulk a quadratic form which can be proved to be positive definite in the exterior region for $|Q| < M$. This is done by separating the exterior region in subregions where the function $f(r)$ is defined differently.

4.3. **The mode stability of Kerr-Newman.** Recall that one of the clues to the validity of the Stability conjecture for the Kerr-Newman family is the proof of mode stability of Schwarzschild, Reissner-Nordström and Kerr as obtained by the physics community.

Quite strikingly, the Kerr-Newman solution stands up as genuinely different from the similar cases of Kerr or Reissner-Nordström in the simplest possible form of stability. As stated by Chandrasekhar in Section 111 of [10], "the methods that have proved to be so successful in treating the gravitational perturbations of the Kerr spacetime do not seem to be applicable (nor susceptible to easy generalizations) for treating the coupled electromagnetic-gravitational perturbations of the Kerr-Newman spacetime." The techniques applied in those early works, which relied on decomposition in frequency modes of perturbations of the solutions, failed to be extended to the case of Kerr-Newman spacetime.

The issue in the analysis of a coupled system comes from the decomposition in modes. In the mode stability analysis, one study solutions of the Teukolsky equation $\mathcal{T}^{[s]}(\psi^{[s]}) = 0$ of the separated form

$$\psi^{[s]}(t, r, \theta, \phi) \quad = \quad e^{-i\omega t} e^{im\phi} R^{[s]}(r) S^{[s]}_{m\ell}(a\omega, \cos\theta).$$

From the Teukolsky equation, one derives an angular ODE for S which defines the spin s-weighted spheroidal harmonics $S^{[s]}_{m\ell}(a\omega, \cos\theta)$, as eigenfunctions of the spin s-weighted laplacian

$$\triangle^{[s]} \quad = \quad \frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta) - \frac{m^2 + 2ms\cos\theta + s^2}{\sin^2\theta} + a^2\omega^2\cos^2\theta - 2a\omega s\cos\theta.$$

For $a = 0$, they reduce to the spherical harmonics $S^{[s]}_{m\ell}(0, \cos\theta) = Y^{[s]}_{m\ell}(\cos\theta)$. Spin-weighted spherical harmonics of different spins are simply related through the angular operators appearing on the right hand side of the coupled equations, and have the same eigenvalues. Schematically

$$\nabla\widehat{\otimes} Y^{[+1]}_{m\ell} = -\lambda Y^{[+2]}_{m\ell}, \qquad \text{div } Y^{[+2]}_{m\ell} = \lambda Y^{[+1]}_{m\ell}.$$

On the other hand, in the general axisymmetric case, the spin-weighted spheroidal harmonics of different spins are not simply related through those angular operators.

This in fact explains the "apparent indissolubility of the coupling between the spin-1 and spin-2 fields" [10] for electromagnetic-gravitational perturbations of Kerr-Newman, in contrast with Reissner-Nordström or Kerr. In treating the coupled electromagnetic-gravitational perturbations of Kerr-Newman spacetime, the decomposition in modes of the equations, which had the objective of simplifying the analysis of the perturbations, actually makes them unsolvable.

4.4. **The physical-space analysis in Kerr-Newman.** Our approach to solve this issue is to abandon the decomposition in modes, and perform a physical space analysis of the equations. Following the road map that mathematicians have taken in the last decade in interpreting in physical space the mode analysis done by the physics community, the Kerr–Newman solution may be the case where a physical space approach could succeed where the mode analysis in physics failed. Observe that our proof of boundedness of a general solution through a physical space analysis will in particular imply the absence of exponentially growing modes, therefore proving mode stability.

We derived [23] the following equations governing the linear stability of Kerr-Newman spacetime to coupled electromagnetic-gravitational perturbations.

Theorem 4.2 (G., [23]). *The gauge-invariant quantities $\mathfrak{p}$ and $\mathfrak{q}$ representing electromagnetic and gravitational radiations for linear perturbations of Kerr-Newman satisfy the following coupled system of wave equations:*

$$\begin{aligned} \square_g\mathfrak{p} - V_1(r,\theta)\mathfrak{p} - i\frac{2a\cos\theta}{\rho^2}\nabla_{\partial_t}\mathfrak{p} &= \frac{Q\overline{\Gamma}^3}{\rho^5}(div\ \mathfrak{q}) + a\nabla^{\leq 1}(\mathfrak{b}, \mathfrak{f}), \\ \square_g\mathfrak{q} - V_2(r,\theta)\mathfrak{q} - i\frac{4a\cos\theta}{\rho^2}\nabla_{\partial_t}\mathfrak{q} &= \frac{Q\Gamma^3}{\rho^5}\big(\nabla\widehat{\otimes}\mathfrak{p} - \frac{3}{2}(\eta - \underline{\eta})\widehat{\otimes}\mathfrak{p}\big) + a\nabla^{\leq 1}(\mathfrak{b}, \mathfrak{f}) \end{aligned}$$

where V_1, V_2 are positive real potentials and $\Gamma = r + ia\cos\theta$, $\overline{\Gamma} = r - ia\cos\theta$.

The system is symmetric, as the respective right hand sides of the equations are adjoint operators with respect to the spacetime integral, i.e. for a 1-tensor ξ and a 2-tensor θ

$$\xi \cdot (div\, \overline{\theta}) = (\nabla \widehat{\otimes} \xi) \cdot \overline{\theta} + ((\eta + \underline{\eta}) \widehat{\otimes} \xi) \cdot \overline{\theta} - D_\mu (\xi \cdot \overline{\theta})^\mu. \tag{4.3}$$

In the above η and $\underline{\eta}$ are Ricci coefficients which satisfy $\nabla \rho^2 = (\eta + \underline{\eta})\rho^2$. Because of this property, in deriving energy estimates for the system by summing the estimates of the two equations and integrating by parts in ∇, the interaction between the angular derivative of the function $\frac{\overline{\Gamma}^3}{\rho^5}$ and the lower order terms $-\frac{3}{2}(\eta - \underline{\eta})\widehat{\otimes}\mathfrak{p}$ in the system precisely cancels out [26], by giving rise to spacetime boundary terms, according to relation (4.3).

5. Conclusions

One of the fundamental problems in General Relativity is to understand the final state of evolution of initial data for the Einstein equation. Through gravitational collapse and dispersion of gravitational waves, the geometry to which solutions to the Einstein equation are expected to relax outside the event horizon of a black hole is the one given by the known stationary and axisymmetric explicit solutions: the Kerr and the Kerr-Newman black hole.

Rigorous analysis of the wave equation in black hole background by the mathematics community has allowed to go beyond the simplest form of stability, the mode stability, to have a better picture of the behavior of linear and non-linear perturbations of black hole solutions. In the case of the Einstein vacuum equation, the Teukolsky equation satisfied by a curvature component cannot be used directly to obtain boundedness of general solutions. The Chandrasekhar transformation allows to transform it into a well-behaved equation, the Regge-Wheeler equation, to which standard techniques for the wave equation can be applied.

In the case of the Einstein-Maxwell equation, some non-trivial coupling of the gravitational and electromagnetic radiations takes place, and it is not a priori clear what are the relevant quantities that one should consider. It turns out that new definitions of gauge-invariant quantities are needed to obtain a favorable system of generalized Regge-Wheeler equations, where the coupling terms appear in a symmetric fashion, allowing for derivation of spacetime estimates.

This is particularly important for the most general case of perturbations of the Kerr-Newman family, where in order to overcome the issue of "indissolubility of the coupling between spin-1 and spin-2 fields" [10] we necessarily need non-separated equations, which should be analyzed in physical space.

The equations governing perturbations of Kerr-Newman are a system of two coupled generalized Regge-Wheeler, obtained from the Teukolsky equations through the Chandrasekhar transformation, which crucially have:

- real potentials and first order terms of the form $i\partial_t$, (like in Kerr)
- adjoint coupling terms (like in Reissner-Nordström).

The decomposition in modes prevented to see this "good" structure of the equations. On the other hand, a physical space analysis of the above system allows to derive boundedness of the energy and spacetime energy decay estimates, giving a stronger result than mode stability, and therefore solving the long-standing problem of stability of the most general black hole solution.

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Department of Mathematics, Columbia University
Email address: elena.giorgi@columbia.edu

Sunflowers: from soil to oil

Anup Rao
University of Washington
anuprao@cs.washington.edu

Abstract

A *sunflower* is a collection of sets whose pairwise intersections are identical. In this article, we shall go sunflower-picking. We find sunflowers in several seemingly unrelated fields, before turning to discuss recent progress on the famous sunflower conjecture of Erdős and Rado, made by Alweiss, Lovett, Wu and Zhang.

1 Sunflowers

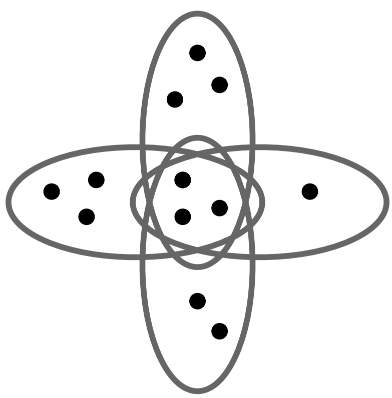

Figure 1: A sunflower with 4 petals.

The moral of Ramsey theory is that large systems can exhibit surprising structure. There are many examples of this kind, starting with the prototypical one: every graph on n vertices either contains a clique[1] on $(1/2) \cdot \log_2 n$ vertices, or an independent set[2] on $(1/2) \cdot \log_2 n$ vertices. Roth's theorem [12] proves that every subset of $\{1, \ldots, n\}$ of density $\Omega(1)$ must contain an arithmetic progression[3]. The Hales-Jewett theorem [7] and Ajtai and Szemeredi's Corner theorem [1] are other examples of this phenomenon.

A *sunflower* with p petals is a collection of p sets whose pairwise intersections are identical. The common intersection is called the *core*. In 1960, Erdős and Rado [4]

[1] Msutually adjacent vertices
[2] Mutually non-adjacent vertices
[3] Three numbers $a, a+d, a+2d$

proved a Ramsey theoretic result concerning sunflowers: every large collection of sets must contain a sunflower. They gave a simple inductive argument showing that every collection of more than $k! \cdot (w-1)^k$ sets of size at most k must contain a sunflower with w petals[4]. There are examples with $\Omega(w)^k$ sets that have no sunflowers, and they conjectured that the correct bound is $O(w)^k$.

The seeds were planted, and the search for sunflowers and sunflower lemmas began in earnest. We begin this article by taking a tour through various fields where sunflowers are essential. We shall see examples relevant to finding arithmetic progressions in sumsets, understanding models of computation such as monotone boolean circuits and data structures, and fundamental questions about the threshold of a monotone function. In each of these arenas, we skip details and zoom in to focus on the role played by sunflowers.

In 2019, Alweiss, Lovett, Wu and Zhang [2] made significant progress towards proving the sunflower conjecture. Subsequent refining by myself [10], Frankston, Kahn, Narayanan and Park [6] and Bell, Chueluecha and Warnke [3] led to the result that every collection of $O(w \log k)^k$ sets of size at most $O(k)$ must contain a sunflower with w petals. We shall return to the ideas that lead to this improved bound near the end of this article.

2 Arithmetic Progressions in Sumsets

In 1992, Erdős and Sárközy [5] used sunflowers to find arithmetic progressions in subset sums. Given a set $T \subseteq \{1, \ldots, n\}$, let $\mathsf{sum}(T)$ denote the quantity $\sum_{x \in T} x$. Given any set $S \subseteq \{1, \ldots, n\}$ of size $|S| \gg \log^2 n$, they proved that there are subsets $T_1, \ldots, T_{w+1} \subseteq S$, with $w \approx |S|/\log^2 n$, such that the sequence

$$\mathsf{sum}(T_1), \mathsf{sum}(T_2), \cdots, \mathsf{sum}(T_{w+1})$$

is an arithmetic progression. Much like the sunflower lemma, this is an example of finding structure in a large system. However, the structure we seek here is an arithmetic progression; what does this have to do with sunflowers? Erdős and Sárközy move between the two structures as follows. First, by counting the number of possible sums that can be obtained by subsets of S, and estimating a binomial coefficient, they show that some $(w \log n)^{\log n}$ subsets of S of size $\log n$ must attain the same sum. By the sunflower lemma, and the choice of parameters, this collection of sets is guaranteed to contain a sunflower. The proof is completed by the following claim, whose proof we leave as an exercise (Figure 2):

Claim 1. If $S_1, \ldots, S_w \subseteq \{1, \ldots, n\}$ is a sunflower with core C, $|S_1| = \cdots = |S_w|$, and $\mathsf{sum}(S_1) = \mathsf{sum}(S_2) = \cdots = \mathsf{sum}(S_w)$, then

$$\mathsf{sum}(C), \mathsf{sum}(S_1), \mathsf{sum}(S_1 \cup S_2), \ldots, \mathsf{sum}(S_1 \cup \cdots \cup S_w)$$

[4]Often the sunflower lemma is stated under the assumption that each set is of size *exactly* k rather than at most k. Here we use the more general form because many application rely on this form, and all of the ideas for proving the lemmas carry through.

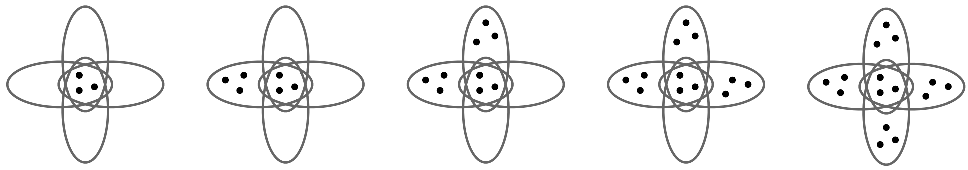

Figure 2: 4 petals induces an arithmetic progression of length 5.

is an arithmetic progression.

3 Monotone Circuit Lower Bounds

Sunflowers have had a huge impact in theoretical computer science. Perhaps the most well-known example is Razborov's [11] proof from 1985 that there are no small monotone circuits computing the *clique* function. Here, I will give a cartoon description of this clever argument.

A boolean circuit computes with the help of *gates* implementing boolean logic. These logic gates can compute the OR, AND or negation of their inputs. The inputs to the gates are either the outputs of other gates, or input variables. The size of the circuit is the number of wires used, which is the same as the number of connections made between gates. A *monotone* circuit is a boolean circuit that does not have any gates computing negations. The circuit computs a function if there is a gate whose value is equal to the value of the function, for every choice of the input variables.

For a graph G on n vertices, and a set S of vertices, define

$$\mathsf{clique}_S(G) = \begin{cases} 1 & \text{if } G \text{ contains a clique on the vertices of } S, \\ 0 & \text{otherwise.} \end{cases}$$

The function of interest for us is

$$\mathsf{clique}_k(G) = \bigvee_{S \subseteq \{1,\ldots,n\}, |S|=k} \mathsf{clique}_S(G),$$

which computes whether or not the graph contains a clique of size k. Razborov proves that this function requires exponentially large monotone circuits, if $k \approx n^{1/3}$. Razborov's result is one of the few examples where we are able to prove lower bounds on reasonable models of computation: it is a gem of theoretical computer science.

At a high level, sunflowers are used critically to show that any circuit computing clique_k can be used to obtain a smaller circuit with the same ability. Each such step involves a tiny error. We obtain a good approximation to the original circuit that is so simple that we can directly reason that it does not work. This proves that the original circuit does not work either.

Now, let us give a few more details. Let G be a graph on n vertices that contains a uniformly random clique of size k, and no other edges. Let H be a uniformly random

$(k-1)$-partite graph. G always contains a clique of size k, while H never contains a clique of size k. A monotone circuit computing the clique function would have to output 1 on G and 0 on H. An input variable to the circuit is the indicator for the presence of an edge, which can be thought of as clique_S for some set S of size 2.

Let us discuss how to approximate the circuit by a simpler circiut. First, we claim that $\mathsf{clique}_S \wedge \mathsf{clique}_T$ can be safely replaced by $\mathsf{clique}_{S \cup T}$. This is because by the choice of G,

$$\mathsf{clique}_S(G) \wedge \mathsf{clique}_T(G) \leq \mathsf{clique}_{S \cup T}(G),$$

and by the choice of H,

$$\mathsf{clique}_S(H) \wedge \mathsf{clique}_T(H) \geq \mathsf{clique}_{S \cup T}(H).$$

Thus, carrying out this approximation preserves the ability of the circuit to distinguish G from H, while reducing the size of the circuit.

Sunflowers play a key role in approximating OR gates, via the following claim:

Claim 2. If $S_1, \ldots, S_w$ form a sunflower with core C, and all sets S_i are of size at most $\sqrt{k}$, then

$$\mathsf{clique}_{S_1}(G) \vee \cdots \vee \mathsf{clique}_{S_w}(G) \leq \mathsf{clique}_C(G),$$

and with high probability over the choice of H,

$$\mathsf{clique}_{S_1}(H) \vee \cdots \vee \mathsf{clique}_{S_w}(H) \geq \mathsf{clique}_C(H).$$

When the input is G, the claim is trivial. When the input is H, the approximation causes a problem if there is a clique on C in H, yet none of the petals constitute a clique. This is extremely unlikely to happen: given that the core is a clique, the events that the petals are also cliques are independent, and the choice of parameters ensures that each occurs with probability $\Omega(1)$. So, one can argue that one of the petals will be a clique with probability $1 - 2^{-\Omega(w)}$.

Thus, if t is large enough, any expression of the type

$$\mathsf{clique}_{S_1} \vee \ldots \vee \mathsf{clique}_{S_t}$$

can be approximated by a smaller expression of the same type—use the sunflower lemma to find a sunflower among the sets and replace it by the core. Repeatedly applying these operations, one can show that any arbitrary small monotone circuit can be approximated by a circuit whose structure is so simple that it is trivial to verify that it cannot distinguish G from H.

4 Lower Bounds for Data Structures

Data structures are a fundamental concept in computer science. They are used to efficiently maintain an object so that the object can be quickly modified and queried. Our next example is a lower bound on the running time of data structures for the problem of maintaining a set and computing its minimum, from my work with Ramamoorthy [9].

We showed that any data structure that can maintain a subset of numbers $T \subseteq \{1, \ldots, n\}$ and can quickly and non-adaptively compute the minimum element of T must access $\Omega(\log n / \log \log n)$ locations for one of its operations. Our result is independent of the algorithm used to implement the data structure and the particular encoding of the data (namely T) used, the argument only relies on the sets of locations that the data structure reads and writes to.

A valid data structure for our purposes is one that encodes the set T as a vector $\mathsf{enc}(T) \in \{1, \ldots, n\}^m$. The data structure is associated with a family of subsets of the coordinates of the encoding $S_1, \ldots, S_n \subseteq \{1, \ldots, m\}$ and an algorithm for manipulating $\mathsf{enc}(T)$. For each i, the algorithm is able change $\mathsf{enc}(T)$ to either $\mathsf{enc}(T \cup \{i\})$ or $\mathsf{enc}(T - \{i\})$ and compute the new minimum of the set by reading and writing to the coordinates of $\mathsf{enc}(T)$ given by S_i. Under just these assumptions, we prove that some set S_i must be of size $\Omega(\log n / \log \log n)$.

If all of the sets $S_1, \ldots, S_n$ are of size $\ll \frac{\log n}{\log \log n}$, the choice of parameters implies that there is a sunflower, say $S_1, \ldots, S_w$, with $w \approx (\log n)^{100}$, and core C. Then the key claim is:

Claim 3. If $S_1, \ldots, S_w$ is a sunflower with core C, then every subset of $\{1, \ldots, w\}$ has an encoding as a vector $\mathsf{enc}(W) \in \{1, \ldots, n\}^{|C|}$.

This claim combined with a straightforward counting argument implies that $|C| \geq \Omega(\frac{\log n}{\log \log n})$, proving that one of the sets S_i must be large. To prove the claim, for any set $T \subseteq \{1, \ldots, w\}$, arrive at its encoding by deleting the elements of the set $\{1, \ldots, w\} - T$ from the encoding of $\{1, \ldots, w\}$. The claimed encoding corresponds to the contents of the core at this point, which is a string in $\{1, \ldots, n\}^{|C|}$. T can be recovered from the encoding by computing the minimum of T, then deleting the minimum, then computing the minimum and deleting it, and repeating these operations over and over until the entire set T has been recovered. Because each of these operations only interact on the coordinates of $\mathsf{enc}(T)$ that correspond to the core, the contents of the core are enough to simulate the entire process and determine T.

5 Estimating the Threshold of Monotone Functions

Suppose that we are given a parameter $0 \leq \epsilon \leq 1$ and sample a random graph by including each edge independently with probability ϵ. How can we estimate the probability that the graph contains a perfect matching?

This is a special case of a more general question. Let $X \in \{0,1\}^n$ be sampled by independently setting each coordinate:

$$X_i = \begin{cases} 1 & \text{with probability } \epsilon, \\ 0 & \text{with probability } 1 - \epsilon. \end{cases}$$

Let $f : \{0,1\}^n \to \{0,1\}$ be a *monotone* function, meaning that $x \geq y$ implies that $f(x) \geq f(y)$ ($x \geq y$ if $x_i \geq y_i$ for all i). Because f is monotone, $\mathbb{E}[f(X)]$ is increasing in ϵ. The *threshold* of f is the value of ϵ for which $\mathbb{E}[f(X)] = 1/2$. There are a couple of generic ways to bound $\mathbb{E}[f(X)]$, and these bounds induce other kinds of thresholds that capture something about the structure of f. These ideas were explored extensively by Kahn and Kalai [8], Talagrand [13] and Frankston, Kahn, Narayanan and Park [6].

Every monotone function f admits a minimal collection of binary strings F such that $f(x) = 1$ if and only if there is an element $y \in F$, with $y \leq x$. So, by the union bound:

$$\mathbb{E}[f(X)] \leq \sum_{y \in F} \mathbb{P}[y \leq X], \tag{5.1}$$

The *expectation-threshold* is the value of ϵ for which the right-hand-side of (5.1) is equal to $1/2$. By (5.1), the threshold is always at least as large as the expectation-threshold. When f computes whether or not a graph has a perfect matching, the threshold is $\approx \frac{\log n}{n}$, while the expectation threshold is $\approx \frac{1}{n}$. Kahn and Kalai conjectured that this is in fact the worst possible ratio: the threshold is always at most $O(\log n)$ times larger than the expectation-threshold.

In general, the union bound can be quite far from tight. It is not tight when the events $y \leq X$ have intersections of significant measure. There is a more sophisticated way to get upper bounds on $\mathbb{E}[f(X)]$, as observed by Talagrand [13]—it can be thought of as a fractional variant of the union bound. For $z \in \{0,1\}^n$, let $|z| = \sum_{i=1}^n z_i$. Suppose there is a probability distribution Z on $\{0,1\}^n$ and κ satisfying

$$f(x) \leq \kappa \cdot \mathop{\mathbb{E}}_{Z}[1_{Z \leq x} \cdot \epsilon^{-|Z|}],$$

for all x. Then we obtain the upper bound:

$$\mathop{\mathbb{E}}_{X}[f(X)] \leq \kappa \cdot \mathop{\mathbb{E}}_{X,Z}[1_{Z \leq X} \cdot \epsilon^{-|Z|}] = \kappa, \tag{5.2}$$

since for any fixed z, the probability that $z \leq X$ is exactly $\epsilon^{|z|}$.

The *fractional-expectation-threshold* is the value of ϵ for which there is a distribution Z with $\kappa = 1/2$. The union bound (5.1) can also be proved using (5.2), because if Z is sampled so that

$$\mathbb{P}[Z = z] = \begin{cases} \frac{\mathbb{P}[z \leq X]}{\sum_{w \in F} \mathbb{P}[w \leq X]} & \text{if } z \in F, \\ 0 & \text{otherwise,} \end{cases}$$

then we have

$$f(x) \leq \Big(\sum_{z \in F} \mathbb{P}[z \leq X] \Big) \cdot \mathop{\mathbb{E}}_{Z}[1_{Z \leq x} \cdot \epsilon^{-|Z|}] = \sum_{z \in F} \mathbb{P}[z \leq X],$$

for all x, proving (5.1). So, the bound given by (5.2) is certainly at least as good as the bound given by (5.1). In particular, this implies that the threshold is at least as large as the fractional-expectation-threshold, which in turn is at least the expectation-threshold. But how far apart can these numbers be?

Talagrand conjectured that the fractional-expectation-threshold is within a multiplicative factor of $O(\log n)$ from the threshold, and within an $O(1)$ factor of the expectation-threshold. Frankston, Kahn, Narayanan and Park [6] proved that the fractional-expectation-threshold is within $O(\log n)$ of the threshold, so resolving Talagrand's first conjecture. This allows to compute the threshold for many graph properties, such as perfect matchings, Hamiltonian circuits and bounded degree spanning trees. The ideas used to prove new sunflower lemmas play a key role in their proof, as we shall see below.

Talagrand made an important observation that is ultimately useful to understanding the gap between the threshold and the fractional-expectation-threshold. Suppose that κ is the smallest number for which there is a distribution Z establishing (5.2). Then by von-Neumann's minimax theorem, there is a distribution U on $\{0,1\}^n$ such that for every choice of z,

$$\mathop{\mathbb{E}}_{U}[f(U)] \geq \kappa \cdot \mathop{\mathbb{E}}_{U,z}[1_{z \leq U} \cdot \epsilon^{-|z|}]. \tag{5.3}$$

Without loss of generality, we may assume that U is supported on the min-terms of f, since we can always modify the distribution in this way and preserve the inequality. Moreover, the distribution of U can be thought of as the uniform distribution on a multiset $\mathcal{S}$, by equating binary strings with subsets of $\{1,\ldots,n\}$, and finding a close enough rational approximation to the distribution of U. So, after making these changes, we can rewrite (5.3) as:

$$\epsilon^{|z|}/\kappa \geq \mathop{\mathbb{E}}_{U,z}[1_{z \leq U}], \tag{5.4}$$

where here U samples a random element from a multi-set containing the min-terms of f. By (5.4), this multi-set has a very interesting property: It must be *spread*, in the sense that very few of these sets can all contain the same set. The fraction of min-terms containing z is at most $\mathbb{E}_{U,z}[1_{z \leq U}] \leq \epsilon^{|z|}/\kappa \leq r^{-|z|}$, for $r = \kappa/\epsilon$.

So, where are the sunflowers? This time, the sunflowers show up figuratively: our best known method for finding sunflowers involves understanding the probability that a random set contains an element of a spread family. To explain, let us put on hold our study of these thresholds and discuss the ideas needed to prove sunflower lemmas.

6 Finding Sunflowers (in Spread Families)

At last we return to the heart of the matter: how many sets of size k are sufficient to ensure the presence of a sunflower with w petals? Alweiss, Lovett, Wu and Zhang discovered an elementary counting argument that is surprisingly powerful to help answer this question.

Given a collection $\mathcal{S}$ of sets, we shall say that the sets are *r-spread* (for some parameter $r = O(w \log k)$) if for every set T of size at most $k-1$, the fraction of sets in the collection that contain T is at most $r^{-|T|}$. As we saw in the last section, this definition naturally arose in the work of Talagrand, but it is very directly applicable to proving a better sunflower lemma. Suppose $|\mathcal{S}| \geq r^k$. If $\mathcal{S}$ is not r-spread, then there is a set T such that the family $\mathcal{S}' = \{S \in \mathcal{S} : T \subseteq S\}$ has at least $r^{k-|T|}$ sets. In this case, we inductively find a sunflower in the family of sets of size at most $k - |T|$ obtained by deleting T from the sets of $\mathcal{S}'$. Adding T back into this sunflower gives us a sunflower in our original family of sets. So, it only remains to find sunflowers in spread families.

The main technical claim is:

Claim 4. For $r = O((1/\epsilon)\log k)$, if $\mathcal{S}$ is r-spread, and X is a random set sampled by including each element independently with probability ϵ, then X contains a set of $\mathcal{S}$ with probability at least $1/2$.

We note that the Claim holds even if $\mathcal{S}$ is a multi-set, which is useful for the application to understanding the thresholds of monotone functions. Let $X_1, \ldots, X_{2w}$ be a random partition of the universe into $2w$ sets, and set $\epsilon = 1/(2w)$, so $r = O(w \log k)$. Claim 4 implies that w of these sets will contain a set of the family in expectation, and so there must be w mutually disjoint sets: a sunflower with w petals.

To exhibit the key ideas used to prove the claim, let us settle for a weaker goal. Suppose $\mathcal{S} = \{S_1, \ldots, S_{r^k}\}$, and $X \subseteq \{1, \ldots, n\}$ is uniformly random. We will show that there must be a set $S_i \in \mathcal{S}$ such that $|X \cap S_i| \geq 0.99k$ with high probability.

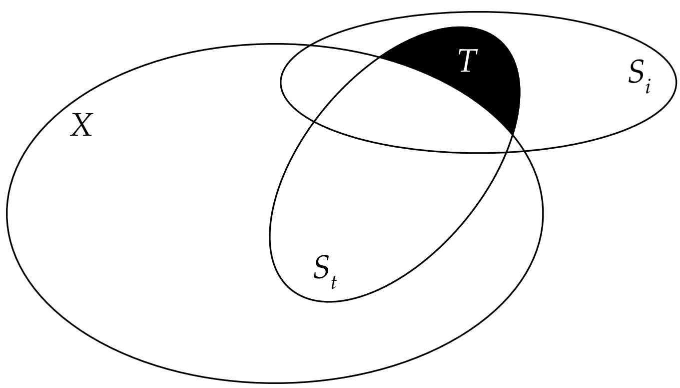

Figure 3: T must be large, and so there are few choices for S_i.

Consider the possible pairs (X, S_i) where X does *not* share $0.99k$ elements with any set of $\mathcal{S}$, and S_i is an arbitrary element of $\mathcal{S}$.

(i). There are at most 2^n choices for $X \cup S_i$.

(ii). Given $X \cup S_i$, let t be the smallest index such that $S_t \subseteq X \cup S_i$. There is exactly one choice for t.

(iii). Let $T = S_t - X$. This is a subset of S_t, so there are at most 2^k choices for T.

(*iv*). We have now identified a set T that is contained in S_i. Since X does not cover $0.99k$ elements of S_t, $|T| \geq 0.01k$. Because $\mathcal{S}$ is r-spread, there can be at most $|\mathcal{S}| \cdot r^{-|T|}$ choices for S_i.

(*v*). Finally, there are at most 2^k choices for $X \cap S_i$. Because the sets $X \cup S_i$, S_i and $X \cap S_i$ determine X, all of these choices determine the pair (X, S_i).

In this way, the number of such pairs (X, S_i) is at most $2^n \cdot 2^k \cdot |\mathcal{S}| \cdot r^{-0.01k} \cdot 2^k = |\mathcal{S}| \cdot 2^{n+2k-0.01k \log r}$. Because the number of choices for S_i is $|\mathcal{S}|$, this implies that the probability that a random choice of X fails to intersect a set in $0.99k$ elements is at most $2^{k(2-0.01 \log r)}$, which can be made very small for r a large constant. Methods from information theory along with a few other ideas can be used to prove Claim 4 in its entirety.

7 Estimating the Threshold of Monotone Functions (contd.)

Armed with our ability to reason about random sets containing a set of the spread family (Claim 4), we can now return to finish the story of the gap between the threshold and the fractional-expectation-threshold.

Suppose the fractional-expectation-threshold of a monotone function f is γ. Then recall that we have a collection of min-terms (possibly with repetitions) that correspond to sets that are r-spread, with $r = 1/(2\gamma)$. Let k be the size of the largest min-term, and let C be some large constant. Then if we set $\epsilon = C\gamma \log k$, we get that $r = C(1/(2\epsilon)) \log k$. For some large C, Claim 4 implies that a random set X, where each element is included in X with probability ϵ, will contain one of the min-terms with probability at least $1/2$. If X is viewed as a binary string, this implies that

$$\mathbb{E}[f(X)] \geq 1/2,$$

proving that the threshold of f is at most $O(\gamma \log k)$. Thus, we get that the ratio between the threshold and the fractional-expectation-threshold is at most $O(\log n)$.

8 Conclusion

Sunflowers have had an enormous impact on a surprising number of different fields. It only remains to be seen where they will be found in the future. The methods of [2] may well lead to even stronger sunflower lemmas, or find applications in places where there are no sunflowers. It is an exciting time to be playing with these concepts!

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Mathematics and the quest for vaccine-induced herd immunity threshold

Elamin H. Elbasha

Merck & Co., Inc, Kenilworth, NJ, USA

Mathematics plays a major role in providing realistic insights into the spread and control of infectious diseases. Its earliest application in this field can be dated back to 1760 when Daniel Bernoulli developed a model to assess the effectiveness of variolation of healthy people with the smallpox virus [1]. Notable applications that followed in the 20th Century include modeling measles epidemics by Hamer in 1906 and malaria by Ross in 1911 [2]. In his study of malaria transmission through mosquitos, Ross was a pioneer in thinking in terms of epidemic thresholds as exemplified by his "Mosquito Theorem" in which he challenged the prevailing wisdom that it is impossible to stop malaria transmission without eradicating mosquitos. He argued that only the density of mosquitoes in a given area must be below a critical value in order for an epidemic outbreak of malaria not to be sustained in a population [3].

Another major milestone occurred in 1927 when Kermack and McKendrick (KM) published their famous epidemic threshold result that the population density of susceptible persons to an infectious agent must exceed a critical value in order for an epidemic outbreak to occur [4]. It took several decades before Smith in 1964 [5], Smith 1970 [6], and Dietz in 1975 [7] translated the critical threshold condition using a convenient and powerful mathematical concept called the *basic reproduction number* $\mathcal{R}_0$. Dietz defines ($\mathcal{R}_0$) as "the number of secondary cases that one case can produce if introduced to a susceptible population." [8]. With this new concept, the minimum fraction of the unvaccinated susceptible population that needs to be immunized to achieve disease elimination (in a local setting) must exceed $1 - 1/\mathcal{R}_0$ [9]. This minimum fraction is now called *herd immunity* threshold (HIT) [10,11]. For example, if $\mathcal{R}_0 = 2.5$, then HIT can be achieved if at least 60% of the unvaccinated susceptible population is immunized. Because $\mathcal{R}_0$ can be readily estimated from data, there have been several applications for many vaccines and diseases. For example, crude estimates of the minimum immune fractions for herd immunity are 94% for measles, 89% for mumps, 86% for rubella, and 80% for poliomyelitis and smallpox, and 60–85% for coronavirus pandemic (COVID-19) [2,12,13,14] (Figure 1).

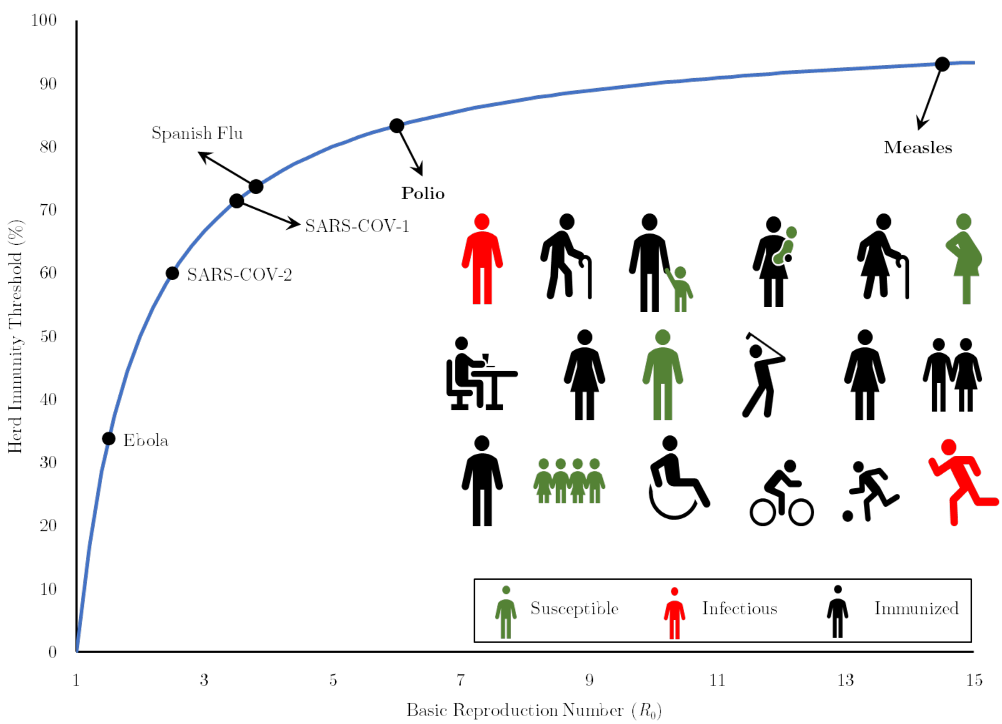


Figure 1. Relationship between vaccine-induced herd immunity threshold (HIT), $\phi^* = 1 - 1/\mathcal{R}_0$, and basic reproduction number, $\mathcal{R}_0$, in a randomly mixing homogeneous population. When the

proportion of people who are immunized (black) exceeds HIT, susceptible people are indirectly protected. As a result, introduction of new infected people (red) will not cause the disease to spread.

The above derivation of HIT is based on making several simplifying modeling assumptions regarding the properties of the vaccine, roll out of the vaccination program, and characteristics and mixing patterns of the host population receiving the vaccine. For example, it assumes that the vaccine efficacy to protect against the acquisition of infection is perfect and lasts throughout the lifetime of the vaccinated individual.

A basic epidemiologic model with a perfect vaccine

The above derivation is based on considering the following scenario for administering a hypothetical vaccine to a proportion ϕ of a cohort of newborns entering the population at rate of μ per unit time. The vaccination model follows a standard susceptible-infected-recovered (SIR) KM-type compartmental modeling formulation [4,15] with vital dynamics (i.e., births and deaths). The total population at time t is divided into three compartments of fraction unvaccinated susceptible $S(t)$, infectious $I(t)$, and recovered $R(t)$, such that $S(t) + I(t) + R(t) = 1$. The disease dynamics with compartments S, I and R is given by the flow diagram in Figure 2 and by the following deterministic system of nonlinear differential equations (ODEs) (where a dot represents differentiation with respect to time t):

$$\begin{aligned}
\dot{S} &= (1-\phi)\mu - (\mu + \beta I)S, \qquad && S(0) \geq 0, \\
\dot{I} &= \beta I S - (\mu + \gamma)I, && I(0) \geq 0, \qquad (1) \\
\dot{R} &= \phi\mu + \gamma I - \mu R, && R(0) \geq 0,
\end{aligned}$$

where β is the average number of adequate (i.e., sufficient for transmission of infection from an infective to a susceptible) contacts per person per unit time, γ is the recovery rate from infection, and μ is the all-cause mortality rate.

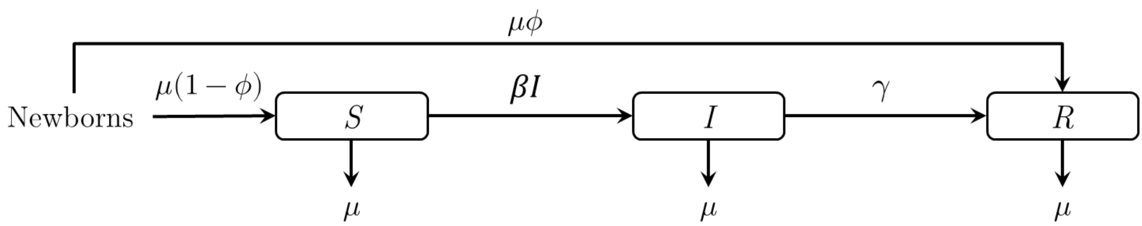


Figure 2. Flow diagram for the SIR model. The model divides the population into groups according to their susceptibilities and infectiousness. Entry and exit into the population occur at rate μ. Infection occurs at rate βI and clears at rate γ. A fraction ϕ of newborns is vaccinated.

Some of the main assumptions made in the above formulation of model (3) are:

a) Homogeneous or random mixing within the population.

b) The vaccine is perfect (i.e., it provides full, lifetime protection)

c) No disease-induced mortality and births are equal to deaths so that the total population remains constant.

d) Recovery from infection or vaccination induces permanent immunity against acquisition of future infection.

e) Exponentially distributed waiting time in each epidemiological compartment.

f) All relevant parameters (e.g., average number of contacts, transmission rate, and protective efficacy of the vaccine) are positive known quantities.

The vaccination model (1) has a disease-free equilibrium given by $(S^*, I^*, R^*) = (1-\phi, 0, \phi)$ that is globally asymptotically stable if $\mathcal{R}_v \leq 1$ and unstable if $\mathcal{R}_v > 1$. The average number of new infections generated by a typical infectious individual in a population where a fraction of the susceptible individuals is vaccinated $\mathcal{R}_v$ is known as the *control reproduction number* $\mathcal{R}_v$ [16]. In model (1), $\mathcal{R}_v$ is given by

$$\mathcal{R}_v = \mathcal{R}_0(1-\phi), \tag{2}$$

where

$$\mathcal{R}_0 = \frac{\beta}{\mu+\gamma},$$

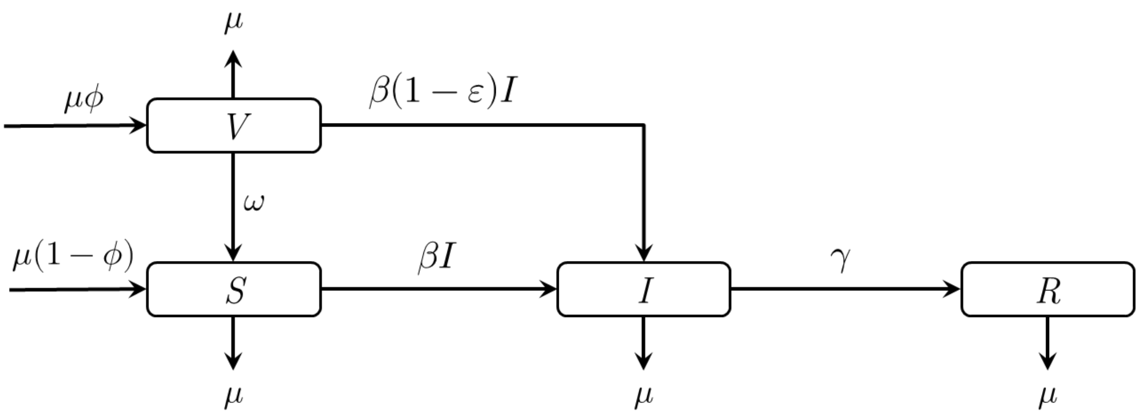
is the basic reproduction number in this model. To achieve HIT (i.e., disease-free equilibrium), the condition $\mathcal{R}_v \leq 1$ must be satisfied. HIT is obtained by setting $\mathcal{R}_v$ in equation (4) to 1 and solving for ϕ. Thus,

$$\phi^* = 1 - \frac{1}{\mathcal{R}_0}.$$

If $\boldsymbol{\mathcal{R}_0} = 2.5$, then HIT can be achieved if at least 60% of the flow of new susceptibles (e.g., newborns) are vaccinated.

A basic epidemiologic model with an imperfect vaccine.

Several compartmental mathematical models have been developed to analyze the transmission dynamics of infectious diseases and assess the impact of imperfect vaccines. McLean and Blower [17], and other researchers [18,19], derived modified HIT formulae under various assumptions for vaccine properties. Suppose the vaccine confers a degree of protection, ε, to vaccinated susceptible individuals that wanes at a rate ω per unit time in a population with average life expectation of $1/\mu$, so that the mortality-adjusted average duration of vaccine-induced protection is $1/(\mu+\omega)$. The vaccination model follows a standard susceptible-vaccinated-infected-recovered (SVIR) KM-type compartmental modeling formulation [4,15] where the total population at time t is divided into four compartments of unvaccinated susceptible $S(t)$, vaccinated susceptible $V(t)$, infectious $I(t)$, and recovered $R(t)$ (Figure 3).



The model is given by the following deterministic system of ODEs:

$$
\begin{aligned}
\dot{S} &= (1-\phi)\mu + \omega V - (\mu + \beta I)S, && S(0) \geq 0,\\
\dot{V} &= \phi\mu - [\mu + \omega + (1-\varepsilon)\beta I]V, && V(0) \geq 0,\\
\dot{I} &= \beta I[S + (1-\varepsilon)V] - (\mu+\gamma)I, && I(0) \geq 0,\\
\dot{R} &= \gamma I - \mu R, && R(0) \geq 0.
\end{aligned}
\tag{3}
$$

In addition to the main assumptions made above, model (3) assumes that the vaccine is imperfect (i.e., $0 < \varepsilon_i < 1$), and the protection offered by the vaccine wanes over time (i.e., $\omega > 0$).

Elbasha et al [15] derived $\mathcal{R}_v$ as

$$\mathcal{R}_v = \mathcal{R}_0 \left(1 - \frac{\varepsilon \phi \mu}{\mu + \omega}\right), \tag{4}$$

and proved that model (3) has disease-free equilibrium (i.e., disease is eliminated) that is globally asymptotically stable if $\mathcal{R}_v \leq 1$ and unstable if $\mathcal{R}_v > 1$. HIT is obtained by setting $\mathcal{R}_v$ in equation (4) to 1 and solving for ϕ. Thus, the critical vaccination coverage ϕ to achieve herd immunity is given by [18]

$$\phi^{**} = \left(\frac{\mu + \omega}{\mu}\right) \times \left(\frac{1}{\varepsilon}\right) \times \left(1 - \frac{1}{\mathcal{R}_0}\right) = \left(\frac{\mu + \omega}{\mu}\right) \times \left(\frac{1}{\varepsilon}\right) \phi^*,$$

where ϕ^* is as defined above. Thus, classical HIT value (ϕ) needs to be adjusted upward to reflect the imperfect efficacy of the vaccine ($0 < \varepsilon < 1$) and the fraction of the average lifetime a vaccinated individual remains protected ($1/(\mu + \omega) \div 1/\mu$). For example, with $\mathcal{R}_0$= 2.5 and a vaccine with protective efficacy of only 70% (i.e., $\varepsilon = 0.7$) that lasts 90% of a lifetime (i.e., $\frac{1}{\mu+\omega} \div \frac{1}{\mu} = 0.9$) is used in the population, at least 95.2% of the population of the unvaccinated susceptible newborns needs to be vaccinated to achieve disease elimination. This is significantly higher than the HIT value of 60% calculated above (Figure 4).

It should be noted that there are other vaccine properties that impact HIT. These include biological modes of action affecting the infectiousness and rate of recovery from or progression to more advanced stages of disease among vaccine recipients who experience breakthrough infections [20]. Lower infectiousness or faster recovery from breakthrough infections compared with infections among unvaccinated people imply lower HIT values.

It is also worth noting that recovery from some infectious diseases among unvaccinated individuals induce only temporary immunity that wanes over time. Coupled with an imperfect vaccine this may result in multiple equilibria (i.e., a stable endemic equilibrium co-existing with the disease-free equilibrium) when $\mathcal{R}_v < 1$ (a phenomenon called *backward bifurcation*). In this case, the classical epidemiological requirement $\mathcal{R}_v < 1$, although it is still necessary, does not guarantee disease elimination [15,20,21,22]. There may exist another lower sub-threshold value of $\mathcal{R}_v$ where disease elimination is possible if immunity is increased further implying a larger HIT value.

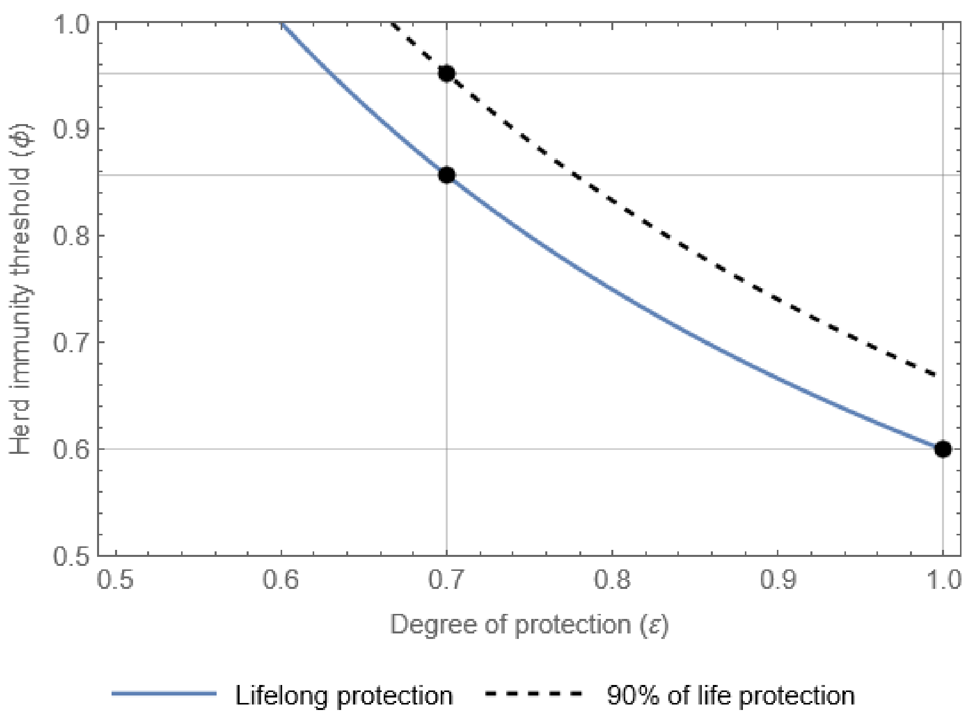


Figure 4. Herd immunity threshold (HIT) with a perfect and an imperfect vaccine. With $\mathcal{R}_0$= 2.5 and a perfect vaccine (i.e., $\varepsilon = 1$), HIT is 60%. When the degree of protection is 70% (i.e., $\varepsilon = 0.7$), HIT is 85.7% with lifelong duration of protection (i.e., $\omega = 0$), and 95.2% with 90% life protection (i.e., $\frac{1}{\mu+\omega} \div \frac{1}{\mu} = 0.9$).

Herd immunity thresholds for a two-group model with heterogeneous populations

The classical HIT formula assumes that the host population receiving the vaccine mix at random. However, the transmission of many infectious diseases occurs in a diverse heterogeneous population, and a more realistic approach is to carry out the above derivations accounting for the relevant heterogeneities [23] such as age, contact patterns, social, cultural, demographic, or geographic factors [2,24,25,26].

Consider another version of the KM-SVIR vaccination model for the transmission dynamics of an infectious disease in a heterogeneous population sub-divided into two distinct homogeneous groups. Each group is further sub-divided into four compartments such that $N_i = S_i + V_i + I_i + R_i$, with $i = 1, 2$. The resulting heterogenous two-group model is:

$$\begin{aligned}
\dot{S}_i &= (1-\phi_i)\Lambda_i + \omega_i V_i - (\mu_i + \lambda_i)S_i, \\
\dot{V}_i &= \phi\Lambda_i - [\mu_i + \omega_i + (1-\varepsilon_i)\lambda_i]V_i, \\
\dot{I}_i &= \lambda_i[S_i + (1-\varepsilon_i)V_i] - (\mu_i + \gamma_i)I_i, \\
\dot{R}_i &= \gamma_i I_i - \mu_i R_i,
\end{aligned} \tag{5}$$

where Λ_i is the group-specific per capita recruitment rate into the population and $\lambda_i(t)$ is *force of infection*:

$$\lambda_i(t) = \sum_{j=1}^{2} \beta_i a_i c_{ij} \frac{I_j(t)}{N_j(t)}, \tag{6}$$

with β_i as the transmission probability per contact for group i, a_i is the average number of contacts that an individual of group i has during a certain period of time (also called group-specific activity level), and c_{ij} is the proportions of contacts that members of group i have with other individuals of group j. Mixing should meet the following closure relation that the total number of contacts that individuals of group i have with other individuals of group j during a certain period of time should equal the total number of contacts that individuals of group j have with other individuals of group i [27].

The two-group model (5) has a unique disease-free equilibrium given by

$$(S_i^*, V_i^*, E_i^*, I_i^*, R_i^*, N_i^*) = \left(\frac{\Lambda_i}{\mu_i}\left(1 - \frac{\phi_i \mu_i}{\mu_i + \omega_i}\right), \frac{\Lambda_i \phi_i}{\mu_i + \omega_i}, 0,0,0, \frac{\Lambda_i}{\mu_i}\right), i = 1,2.$$

It is convenient to work with the fraction of the population in each group at the disease-free equilibrium. For example, the proportion of individuals in group i that are vaccinated is given by:

$$v_i^* = \frac{V_i^*}{N_i^*} = \frac{\mu_i \phi_i}{\mu_i + \omega_i}.$$

Thus, adding the fractions of all compartments of group i gives n_i, where

$$n_i^* = \frac{N_i^*}{N_1^* + N_2^*}$$

is the fraction of total number of individuals in group i relative to the total population.

The next-generation operator method [28] can be used to compute the control reproduction number as

$$\mathcal{R}_v = \frac{1}{2}\left(\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_2}\right), \tag{7}$$

where

$$\Delta_1 = (1 - v_1^* \varepsilon_1)\mathcal{R}_{11} + (1 - v_2^* \varepsilon_2)\mathcal{R}_{22},$$

$$\Delta_2 = (1 - v_1^* \varepsilon_1)(1 - v_2^* \varepsilon_2)(\mathcal{R}_{12}\mathcal{R}_{21} - \mathcal{R}_{11}\mathcal{R}_{22}).$$

In deriving equation (7), we utilized the following definition of the constituent basic reproduction numbers associated with disease transmission between individuals in group i with individuals in group j:

$$\mathcal{R}_{ij} = \frac{\beta_i a_i c_{ij}}{\gamma_j + \mu_j}. \tag{8}$$

To obtain the basic reproduction number, $\mathcal{R}_0$, associated with the two-group model, we set the vaccination coverage rates in the expression for $\mathcal{R}_v$ given by (7) to zero (i.e., $v_1^* = v_2^* = 0$). This gives,

$$\mathcal{R}_0 = \frac{1}{2}\left(\mathcal{R}_{11} + \mathcal{R}_{22} + \sqrt{\mathcal{R}_{1,1}^2 + 4\mathcal{R}_{1,2}\mathcal{R}_{2,1} - 2\mathcal{R}_{1,1}\mathcal{R}_{2,2} + \mathcal{R}_{2,2}^2}\right). \tag{9}$$

For the computation of HIT of a two-group model, the objective is to find the values of the respective HITs, v_1^* and v_2^*, such that the total vaccine coverage (i.e., the proportion of individuals in the community that is vaccinated), given by

$$\frac{V_1^* + V_2^*}{N_1^* + N_2^*} = n_1^* v_1^* + n_2^* v_2^*,$$

is at its minimum, and vaccination reproduction number, $\mathcal{R}_v$, given by (7), is less than or equal to one. Formally, the associated optimization problem can be written as choosing v_1^* and v_2^* to

$$\text{minimize } (n_1^* v_1^* + n_2^* v_2^*)$$

subject to,

$$0 \leq v_1^* \leq 1, 0 \leq v_2^* \leq 1, \mathcal{R}_v \leq 1,$$

where $\mathcal{R}_v$ is given by equation (7).

Elbasha and Gumel [29] analyzed a similar heterogeneous two-group vaccination model and found that deriving the vaccine-induced HIT in a heterogeneous population model is much more complex than deriving HIT for the corresponding model with homogeneous population. In contrast to the case of a randomly mixing homogeneous population, there may be no single HIT value in the heterogeneous two-group model. Instead, the HIT for each vaccinated group depends on the relative values of the constituent reproduction numbers (which in turn are determined by the level and duration of infectiousness of a contact for each group, contact rates for each group, as well as the type of mixing between the two groups), the relative vaccine efficacy, and the relative population sizes of the two groups. For example, under biased random mixing and when vaccinating a given group results in disproportionate prevention of higher transmission per capita, it is optimal to prioritize vaccination of that group before vaccinating the other groups. There are also situations, under biased assortative mixing assumption, where it is optimal to vaccinate (at different rates) more than one group. Importantly, population heterogeneities under various types of mixing tend to result in lower HIT values, compared with the corresponding case with homogeneous population.

It is important to highlight a common misconception regarding the concept of herd immunity. It is frequently stated that *herd immunity* occurs if enough people have disease-acquired or vaccination-acquired immunity to indirectly protect others who are susceptible. The focus here has been on vaccination-acquired herd immunity. An important distinction between disease-acquired or vaccination-acquired immunity lies in the total number of individuals who get infected and suffer from disease. In the case of disease-acquired immunity when the HIT value is reached, the incidence of disease is declining, but it is not zero. Consequently, the final proportion of the population that will be infected by the end of the epidemic is higher than the HIT value. The final proportion of the population of individuals in the community who did not acquire the infection during the course of the epidemic ($S(\infty)$) of the special case for model (3) satisfies the following equation [2]:

$$\mathcal{R}_0[S(0) + I(0) - S(\infty)] + \ln\left(\frac{S(\infty)}{S(0)}\right) = 0.$$

with the initial values approximated by $S(0) = 1$ and $I(0) = 0$ and when $\mathcal{R}_0 = 2.5$, the numerical solution of the above equation gives $S(\infty) = 11\%$. This implies that the final proportion of the population of individuals in the community who acquired the infection during the course of the epidemic is $1 - S(\infty) = 89\%$ which is significantly higher than the HIT value of 60%.

Summary

The coronavirus pandemic (COVID-19) has reignited debate in the scientific research community regarding the minimum fraction of the unvaccinated susceptible population that needs to be immunized for the pandemic to end, known as HIT. The classical HIT is frequently invoked in this debate. It is shown here that there is a need to develop a modified HIT algorithm that considers various vaccine properties, types of vaccination programs, and characteristics of the host population receiving the vaccine.

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Susan Holmes, Stanford University
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Lionel Levine, Cornell University and *Yuval Peres, Microsoft Research and University of California, Berkeley
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Amie Wilkinson, University of Chicago
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Jared S. Weinstein, Boston University
Exploring the Galois group of the rational numbers: Recent breakthroughs.

Andrea R. Nahmod, University of Massachusetts, Amherst
The nonlinear Schrödinger equation on tori: Integrating harmonic analysis, geometry, and probability.

Mina Aganagic, University of California, Berkeley
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Alex Wright, Stanford University
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Daniel Rothman, Massachusetts Institute of Technology
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Karen Vogtmann, Cornell University
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Infinitely many pairs of primes differ by no more than 70 million (and the bound's getting smaller every day)

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How many rational points does a random curve have?

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Topology of nonarchimedean analytic spaces

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Geometric group theory and 3-manifolds hand in hand: the fulfillment of Thurston's vision for three-manifolds

Lauren Williams, University of California, Berkeley
Cluster algebras

January 6, 2012 (Boston, MA)

Jeffrey Brock, Brown University
Assembling surfaces from random pants: the surface-subgroup and Ehrenpreis conjectures

Daniel Freed, University of Texas at Austin
The cobordism hypothesis: quantum field theory + homotopy invariance = higher algebra

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Dispersive equations and their role beyond PDE

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How does quantum mechanics scale?

January 6, 2011 (New Orleans, LA)

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Khot's unique games conjecture: its consequences and the evidence for and against it

Thomas Scanlon, University of California, Berkeley
Counting special points: logic, Diophantine geometry and transcendence theory

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Spaces of graphs and surfaces

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Michael Hopkins, Harvard University
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Matthew James Emerton, Northwestern University
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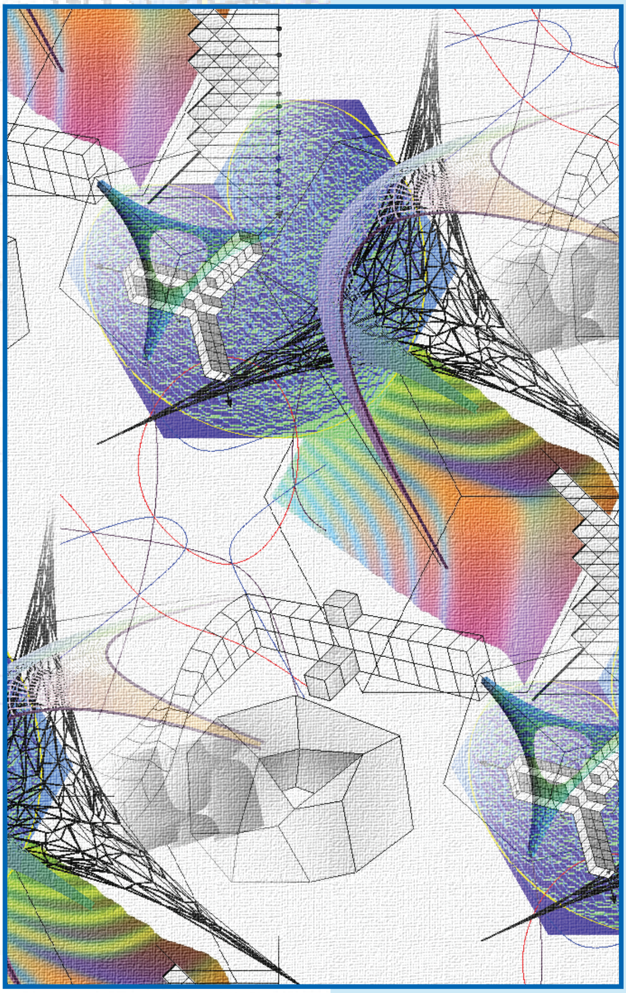

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