Sum-of-Squares in Theoretical Computer Science: Exercises

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Problem 1

Given a pseudo-expectation $\widetilde{\mathbb{E}}$ for MAXCUT prove that without loss of generality we can take $\widetilde{\mathbb{E}}[x_i] = 1/2$. In particular, show that if you are given a pseudo-expectation $\widetilde{\mathbb{E}}$ that satisfies

(1) $\widetilde{\mathbb{E}}$ is linear

(2) $\widetilde{\mathbb{E}}[1] = 1$

(3) $\widetilde{\mathbb{E}}[p^2] \geq 0$ for any polynomial p of degree at most $d/2$.

(4) $\widetilde{\mathbb{E}}[x_i^2 p] = \widetilde{\mathbb{E}}[x_i p]$ for any polynomial p of degree at most $d-2$.

(5) $\widetilde{\mathbb{E}}[\sum_{(i,j)\in E} (x_i - x_j)^2] = k$

show how to construct a new pseudo-expectation $\widetilde{\mathbb{E}}'$ that still satisfies conditions (1) – (5) but has $\widetilde{\mathbb{E}}'[x_i] = 1/2$ for all i.

Problem 2

Prove Hölder's inequality for pseudo-expectations. In particular, show that if $p(x_1, \cdots, x_n)$ and $q(x_1, \cdots, x_n)$ are polynomials and μ is a distribution on $x_1, \cdots, x_n$ then

$$\mathbb{E}_\mu[p(x)q(x)^3] \leq (\mathbb{E}_\mu[p(x)^4])^{1/4}(\mathbb{E}_\mu[p(x)^4])^{3/4}$$

What constraints do you need on the degree of p and q for your proof to work?

Recall the statement of the Chernoff bound: If $Z_1, \cdots, Z_m$ are i.i.d. random variables taking values in $\{0, 1\}$ and $p = \mathbb{E}[Z_i]$ then for any $\epsilon > 0$ we have

$$\mathbb{P}[\frac{1}{m}\sum_{i=1}^{m} Z_i \geq p + \epsilon] \leq e^{-\frac{m\epsilon^2}{2}}$$

Problem 3

Suppose we generate a random 3-XOR formula with n variables and m clauses where each clause chooses three variables uniformly at random and constrains their sign

$$X_i X_j X_k = C_{i,j,k}$$

where $C_{i,j,k}$ is a random ± 1 variable. Here we are thinking of each X_i as taking values in ± 1. Show that for large enough m (compared to n) that with high probability no assignment satisfies more than $\frac{1}{2} + \epsilon$ fraction of the clauses. How does m need to depend on n and ϵ for your bound to work? *Hint: Use the Chernoff bound and the union bound.*

This is the key example of an inequality that you can prove, but is difficult to prove in Sum-of-Squares. Even though a random formula is unsatisfiable with high probability, Sum-of-Squares cannot prove that the formula is indeed random (until $m \geq n^{3/2}$).

Problem 4

Suppose the MAXCUT in G cuts $\frac{1}{2}+\epsilon$ fraction of the edges. The Goemans-Williamson algorithm we analyzed would only tell you that you can cut at least 0.439 fraction of the edges. Suppose that for each edge (i, j) that $\widetilde{\mathbb{E}}[(x_i - x_j)^2] = \frac{1}{2} + \epsilon$. Show how the analysis can be improved to give an algorithm that cuts at least $\frac{1}{2} + \gamma$ fraction of the edges for some $\gamma > 0$. How does γ depend on ϵ?