Deviations and Fluctuations for Mean-Field Games

Kavita Ramanan,
Brown University

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Outline

I. Interacting Diffusions: Fluctuations
II. From Interacting Diffusions to MFG: Fluctuations
III. Large Deviations for Interacting Diffusions and MFG
IV. Open Problems

I. Interacting Diffusions: Fluctuations

Consider n diffusions interacting through their empirical measure:

$$dX_t^{n,i} = b(X_t^{n,i}, \mu_t^n)dt + dW_t^i, \qquad \mu_t^n = \frac{1}{n}\sum_{k=1}^{n} \delta_{X_t^{n,k}},$$

where $\{X_0^{n,i} = \xi^i\}_{i\in\mathbb{N}}$ are iid with common law μ_0 and $(W^i)_{i=1}^n$ are iid d-dimensional Brownian motions.

Interacting Diffusions and the McKean-Vlasov limit

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Under suitable regularity conditions, $X^{n,i} \Rightarrow X$ and $\mu_t^n \to \mu_t$, where X is a non-linear Markov (or McKean-Vlasov) process:

$$dX_t = b(X_t, \mu_t)dt + dB_t, \qquad \mu_t = Law(X_t),$$

with $X_0 \sim \mu_0$ independent of B, a d-dimensional Brownian motion.

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with $X_0 \sim \mu_0$ independent of B, a d-dimensional Brownian motion. Alternatively, μ solves the (nonlinear) Fokker-Planck equation

$$\frac{d}{dt}\langle \mu_t, \varphi\rangle = \left\langle \mu_t, b(\cdot, \mu_t)\cdot \nabla\varphi + \frac{1}{2}\Delta\varphi \right\rangle, \quad \forall \varphi \in C_c^\infty.$$

Rate of Convergence to the McKean-Vlasov limit

$$dX_t^{n,i} = b(X_t^{n,i}, \mu_t^n)dt + dW_t^i, \quad X_0^{n,i} = \xi_i$$

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   - Define $\mathcal{C}^d = C([0,T] : \mathbb{R}^d)$, equipped with the uniform topology
   - Given a separable Banach space $(E, \|\cdot\|)$, let $\mathcal{P}(E)$ denote the space of Borel probability measures on E
     Note that $\mu_t^n, \mu_t \in \mathcal{P}(\mathbb{R})$ and $\mu^n, \mu \in \mathcal{P}(\mathcal{C}^d)$,

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     Note that $\mu_t^n, \mu_t \in \mathcal{P}(\mathbb{R})$ and $\mu^n, \mu \in \mathcal{P}(\mathcal{C}^d)$,
   - Given $p \in [1,\infty)$, let $\mathcal{P}^p(E) = \{\nu \in \mathcal{P} : \int_E \|x\|^p \nu(dx) < \infty\}$ equipped with the p-Wasserstein metric:

$$\mathcal{W}_{p,E}(\nu,\nu') = \inf_\pi \left(\int_{E\times E} \|x-y\|^p \pi(dx,dy)\right)^{1/p},$$

     where the infimum is over all probability measures π on $E \times E$ with first and second marginals μ and ν respectively.

$$dX_t^{n,i} = b(X_t^{n,i}, \mu_t^n)dt + dW_t^i, \qquad \mu_t^n = \frac{1}{n}\sum_{k=1}^n \delta_{X_t^{n,k}}$$

and

$$\frac{d}{dt}\langle \mu_t, \varphi \rangle = \left\langle \mu_t, b(\cdot, \mu_t) \cdot \nabla\varphi + \frac{1}{2}\Delta\varphi \right\rangle, \quad \forall \varphi \in C_c^\infty.$$

Recall $\mu = (\mu_t)_{t\geq 0}$ is the McKean-Vlasov limit.
We are interested in the limit of the signed measures capturing fluctuations:

$$S_t^n := \sqrt{n}(\mu_t^n - \mu_t), \quad t \geq 0.$$

Do you expect this sequence to converge to another signed measure?

A Simple Example

How bad can the weak limit of a signed measure be?

- Let $\nu \in \mathcal{P}(\mathbb{R})$ be a probability measure with a cdf that is not differentiable at any point:

$$F(x) := \nu((-\infty, x]), \quad x \in \mathbb{R}.$$

- For $n \in \mathbb{N}$, define ν^n to be a measure with cdf $F_n(x) := F(x + n^{-1/2})$:

$$\nu^n((-\infty, x]) = \nu((-\infty, x + n^{-1/2}]) = F(x + n^{-1/2}),$$

and define the signed measure

$$\widehat{\nu}^n := \sqrt{n}[\nu^n - \nu].$$

- Then, since F is non-differentiable,

$$\lim_{n\to\infty} \widehat{\nu}^n((-\infty, x]) = \lim_{n\to\infty} \sqrt{n}\left[F\left(x + \frac{1}{\sqrt{n}}\right) - F(x)\right] \text{ doesn't exist!}$$

A Simple Example (contd.)

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$$\langle \varphi, \nu_n \rangle = \int \varphi(x)\nu_n(dx) = \int \varphi\left(x - \frac{1}{n}\right)\nu(dx) = \left\langle \varphi\left(\cdot - \frac{1}{n}\right), \nu \right\rangle$$

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- Recall that $\hat{\nu}^n := \sqrt{n}[\nu^n - \nu]$.
  Thus, for any infinitely differentiable function $\varphi : \mathbb{R} \mapsto \mathbb{R}$ with compact support,

$$\text{So} \quad \lim_{n\to\infty} \langle \hat{\nu}^n, \varphi \rangle = \lim_{n\to\infty} \sqrt{n}[\langle \nu^n, \varphi \rangle - \langle \nu, \varphi \rangle]$$
$$= \lim_{n\to\infty} \sqrt{n} \left\langle \nu, \left[\varphi\left(\cdot - \frac{1}{\sqrt{n}}\right) - \varphi(\cdot)\right]\right\rangle$$
$$= -\langle \nu, \varphi' \rangle.$$

A Primer on the Theory of Distributions

$$\nu \qquad \leftrightarrow \qquad \varphi \mapsto \langle \nu, \varphi \rangle$$

- Let $\mathcal{D}$ be a space of test functions, e.g.,

$$\mathcal{D} = \{ f \in \mathbb{C}^\infty : \text{supp } f \text{ is compact} \}$$

equipped with a suitable topology: $\varphi_n \to \varphi$ in $\mathcal{D}$ if there exists a compact set K such that each φ_n and φ have support in K and $\partial^\alpha \varphi_n \to \partial^\alpha \varphi$ uniformly on K for all α.

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- **Definition.** A mapping $\nu : \mathcal{D} \mapsto \mathbb{R}$ is said to be a linear functional if

$$\nu(\alpha_1 \varphi_1 + \alpha_2 \varphi_2) = \alpha_1 \nu(\varphi_1) + \alpha_2 \nu(\varphi_2), \forall \alpha_i \in \mathbb{R}, \varphi_i \in \mathcal{D}, i = 1, 2.$$

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- Let $\mathcal{D}'$ denote the corresponding space of distributions, defined to be a linear functional $\nu : \mathcal{D} \mapsto \mathbb{R}$, that also satisfies the following continuity property:

$$\varphi_n \to \varphi \text{ in } \mathcal{D} \quad \Rightarrow \quad \nu(\varphi_n) \to \nu(\varphi) \text{ in } \mathbb{R}, \quad \forall \varphi \in \mathcal{D}.$$

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2. Given a σ-finite measure ν on $\mathbb{R}$, the linear functional $\varphi \mapsto \langle \nu, \varphi \rangle = \int \varphi(x)\nu(dx)$ defines a distribution
   Exercise: Prove this by verifying the continuity property
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   Note: For general $\nu \in \mathcal{D}'$, $\nu(\varphi)$ is often written as $\langle \nu, \varphi \rangle$.
3. Given a distribution $\nu \in \mathcal{D}'$, and a $\mathcal{C}^\infty$ function ψ, $\psi\nu$ denotes the distribution
   $$(\psi\nu)(\varphi) = \langle \nu, \psi\varphi \rangle, \quad \forall \varphi \in \mathcal{D}$$
   This is clearly a linear functional.
   Exercise: Verify the continuity property to show $\psi\nu$ is a distribution.

Exercise: Which of the following maps $w : \mathcal{D}(\mathbb{R}) \mapsto \mathbb{R}$ are distributions: here, $\varphi^{(k)}$ is the kth derivative of φ

1. $\langle w, \varphi \rangle := \int_{\mathbb{R}} f(x)\varphi(x)dx$ for a locally integrable function f.
2. $\langle w, \varphi \rangle := \sum_{k=0}^{\infty} \varphi(k)$.
3. $\langle w, \varphi \rangle := \sum_{k=0}^{\infty} \varphi^{(k)}(\sqrt{2})$,
4. $\langle w, \varphi \rangle := \int_{\mathbb{R}} \varphi^2(x)dx$
5. $\langle w, \varphi \rangle := \int_0^{\infty} \frac{\varphi(x)-\varphi(-x)}{x}dx$

Exercise: Show that for any $k \in \mathbb{N}$, $\langle w, \varphi \rangle := (-1)^k \langle w, \varphi^{(k)} \rangle$ is a distribution.

Convergence of Distributions

Let (ν_ℓ) be a sequence in $\mathcal{D}'$ and $\nu \in \mathcal{D}'$.

1. Then (ν_ℓ) is said to converge to ν in $\mathcal{D}'$, denoted $\nu_\ell \to \nu$ if

$$\lim_{\ell \to \infty} \langle \nu_\ell, \varphi \rangle = \langle \nu, \varphi \rangle.$$

2. Moreover, (ν_ℓ) is Cauchy in $\mathcal{D}'$ if $\forall \varphi \in \mathcal{D}$, $(\langle \nu_\ell, \varphi \rangle)$ is Cauchy in $\mathbb{R}$.

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Returning to the Example:

$$\nu^n((-\infty, x]) = \nu((-\infty, x + n^{-1/2}]), \quad \hat{\nu}^n := \sqrt{n}[\nu^n - \nu].$$

We showed that for all $\varphi \in \mathcal{D}$,

$$\lim_{n\to\infty} \langle \hat{\nu}^n, \varphi \rangle := -\langle \nu, \varphi' \rangle,$$

The linear functional $\varphi \mapsto -\langle \nu, \varphi' \rangle$ lies in $\mathcal{D}'$ (by the last exercise).
It is in fact denoted by $\partial\nu$ and is said to be the derivative of ν.
Thus, we showed that

$$\hat{\nu}^n \to \partial\nu \text{ in } \mathcal{D}'.$$

A Stochastic Example

- Let $(B^k)_{k\in\mathbb{N}}$ be a sequence of independent 1-dimensional BMs with initial distribution π_0.
- For $t > 0$, $A \in \mathcal{B}(\mathbb{R})$, define

$$N_t^n(A) := \frac{1}{n}\sum_{k=1}^{n} \mathbb{I}_{\{B_t^k \in A\}}.$$

- Exercise: Calculate $\lim_{n\to\infty} N_t^n(A)$.

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- Exercise: Calculate $\lim_{n\to\infty} N_t^n(A)$.
  Soln. Since $(\mathbb{I}_{\{B_t^k\in A\}})_{k\in\mathbb{N}}$ are iid Bernoulli random variables, the SLLN tells us that $N_t^n(A) \to \gamma_t(A)$ almost surely, where γ_t is a centered Gaussian distribution with variance t, because

$$\gamma_t(A) = \mathbb{E}\left[\mathbb{I}_{\{B_t^1\in A\}}\right] = \mathbb{P}(B_t^1 \in A) = \mathbb{P}(B_t^k \in A).$$

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  Note: In fact, one can prove convergence in $\mathcal{P}(\mathbb{R}^d)$: a.s.,

$$\lim_{n\to\infty} N_t^n(\cdot) \to \gamma_t$$

Stochastic Example (contd.)

$$N_t^n(A) = \frac{1}{n}\sum_{k=1}^{n} \mathbb{I}_{\{B_t^k \in A\}}, \quad \mathbb{E}[N_t^n(\cdot)] = \gamma_t, \quad A \in \mathcal{B}(\mathbb{R}), t \geq 0.$$

Exercise*: Find the limit of the (random) signed measure-valued proc.:

$$S_t^n(A) := \sqrt{n}\left(N_t^n(A) - \gamma_t(A)\right), \quad t \geq 0, A \in \mathcal{B}(\mathbb{R}).$$

Stochastic Example (contd.)

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Soln. View S^n as a distribution-valued process: for $t > 0, \varphi \in \mathcal{D}$,

$$S_t^n(\varphi) := \int_{\mathbb{R}} \varphi(x) S_t^n(dx) = \sqrt{n}\left[\int_{\mathbb{R}} \varphi(x) N_t^n(dx) - \int_R \varphi(x)\gamma_t(dx)\right],$$

is a random variable, in fact it is equal to

$$S_t^n(\varphi) = n^{-1/2}\sum_{k=1}^{n}\left(\varphi(B_t^k) - \langle\varphi,\gamma_t\rangle\right) = n^{-1/2}\sum_{k=1}^{n}\left(\varphi(B_t^k) - \mathbb{E}[\varphi(B_t^k)]\right)$$

- Can show $\forall \varphi \in \mathcal{D}$, $t \mapsto S_t^n(\varphi)$ is a.s. continuous and that, $t \mapsto S_t^n(\cdot)$ is a continuous $\mathcal{D}'$-valued process.

Stochastic Example (contd.)

$$S_t^n(A) := \sqrt{n}\left(N_t^n(\cdot) - \gamma_t(\cdot)\right).$$

Use the multidimensional CLT to show there exists a centered Gaussian $\mathcal{D}'$-valued process $(S_t(\varphi))_{t,\varphi}$ such that every finite-dim. distribution of $S_t^n = (S_t^n(\varphi))_{t,\varphi}$ converges to the corresponding finite-dim. distribution of the solution $S = (S_t(\varphi))_{t,\varphi}$ to the S(P)DE

$$\begin{aligned} dS_t &= (\partial \circ \sqrt{\pi_t})db_t + \frac{1}{2}\partial^2 S_t dt, \\ dS_t(\varphi) &= (\partial \circ \sqrt{\pi_t})db_t(\varphi) + \frac{1}{2}\partial^2 S_t(\varphi)dt, \quad \varphi \in \mathcal{D}, \end{aligned}$$

where

- $\{b_t\} = \{b_t(\varphi), \varphi \in \mathcal{H}_2\}$ is a standard Wiener $\mathcal{H}_2'$-valued process.
- $\pi_t = \pi_0 * \gamma_t$
- $\sqrt{\pi_t} \in C^\infty(\mathbb{R})$ is viewed as a multiplication operator in $\mathcal{D}'$
- ∂ is differentiation in $\mathcal{D}'$
- $\partial \circ \sqrt{\pi_t}$ denotes a composition of these operators.

Recap so far

We started with interacting diffusions:

$$dX_t^{n,i} = b(X_t^{n,i}, \mu_t^n)dt + dW_t^i, \qquad \mu_t^n = \frac{1}{n}\sum_{k=1}^{n} \delta_{X_t^{n,k}}$$

and recalled the McKean-Vlasov limit μ that satisfies:

$$\frac{d}{dt}\langle \mu_t, \varphi \rangle = \left\langle \mu_t, b(\cdot, \mu_t) \cdot \nabla\varphi + \frac{1}{2}\Delta\varphi \right\rangle, \quad \forall \varphi \in C_c^\infty.$$

To capture rate of convergence, we wanted to understand the limit of

$$S_t^n := \sqrt{n}(\mu_t^n - \mu_t), \quad t \geq 0.$$

Recap so far (contd.)

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To understand the form of potential limits of processes such as S_t^n, we

1. considered sequences of scaled centered deterministic signed measures of a similar form, and showed that their limits are often distributions, not (signed) measures;
2. provided a brief introduction to the theory of distributions;

Back to: Fluctuations around the McKean-Vlasov limit

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$$\frac{d}{dt}\langle \mu_t, \varphi \rangle = \left\langle \mu_t, b(\cdot, \mu_t) \cdot \nabla\varphi + \frac{1}{2}\Delta\varphi \right\rangle, \quad \forall \varphi \in C_c^\infty.$$

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3. considered the simplest stochastic case, namely to study the limit of fluctuations (or CLT – central limit theorems) for non-interacting diffusions, that is, where $b \equiv 0$, and characterized the limit as a solution to distribution-valued process, governed by an "SPDE".

$$dX_t^{n,i} = b(X_t^{n,i}, \mu_t^n)dt + dW_t^i, \qquad \mu_t^n = \frac{1}{n}\sum_{k=1}^{n} \delta_{X_t^{n,k}}$$

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$$S_t^n := \sqrt{n}(\mu_t^n - \mu_t), \quad t \geq 0.$$

To understand the form of potential limits of processes such as S_t^n, we

3. considered the simplest stochastic case, namely to study the limit of fluctuations (or CLT – central limit theorems) for non-interacting diffusions, that is, where $b \equiv 0$, and characterized the limit as a solution to distribution-valued process, governed by an "SPDE".
4. We now consider the interacting case, $b \neq 0$ and, in analogy with the non-interacting case, will view S^n as a suitable distribution-valued process.

References assuming affine dependence of drift on the empirical measure

1. H. Tanaka and M. Hitsuda. *Central limit theorem for a simple diffusion model of interacting particles*. Hiroshima Mathematical Journal **11** (1981), no. 2, 415–423.
2. A.S. Sznitman. *A fluctuation result for nonlinear diffusions.* Infinite-dimensional analysis and Stochastic Processes (1985), 145–160.
3. S. Méléard. *Asymptotic behaviour of some interacting particle systems: McKean-Vlasov and Boltzman models,* Probabilistic models for nonlinear partial differential equations, Lecture Notes in Math, vol. 1627, Springer, 1996, pp. 42–95.

Some CLT Results for Interacting Particle Systems

References that consider more general dependence on the empirical measure

1. T.G. Kurtz and J. Xiong. *A stochastic evolution equation arising from fluctuations of a class of interacting particle systems,* Communications in Mathematical Sciences **2** (2004) no. 3, 325–358.

   Comment: Only in the case where each particle takes values in $\mathbb{R}$ – one-dimensional case

2. F. Delarue, D. Lacker and K.R., "From the master equation to mean field game limit theory: a central limit theorem", Electron. J. Probab., Volume 24 (2019), paper no. 51, 54 pp.

   Comment 1: This covers more general dependence and particles taking values in $\mathbb{R}^d$ for general $d \in \mathbb{N}$.

   Comment 2: The precise space in which the limit process lies ends up depending on the dimension d.

General CLT for Interacting Particle Systems

$$dX_t^{n,i} = b(X_t^{n,i}, \mu_t^n)dt + dW_t^i, \qquad \mu_t^n = \frac{1}{n}\sum_{k=1}^{n} \delta_{X_t^{n,k}}$$

and, with μ the McKean-Vlasov limit,

$$S_t^n := \sqrt{n}(\mu_t^n - \mu_t), \quad t \geq 0.$$

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$$S_t^n := \sqrt{n}(\mu_t^n - \mu_t), \quad t \geq 0.$$

Theorem: Under suitable regularity assumptions on b, S^n converges weakly to $S = (S_t(\varphi))_{t,\varphi}$ in $\mathcal{H}_d'$, where $\mathcal{H}_d'$ is a suitable distribution space with test function space $\mathcal{H}_d$, where S solves the SPDE:

$$d\langle S_t, \varphi\rangle = \langle S_t, \mathcal{A}_{t,\mu_t}\varphi\rangle dt + dW_t(\varphi), \quad \varphi \in \mathcal{H}_d$$

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$$d\langle S_t, \varphi\rangle = \langle S_t, \mathcal{A}_{t,\mu_t}\varphi\rangle dt + dW_t(\varphi), \quad \varphi \in \mathcal{H}_d$$

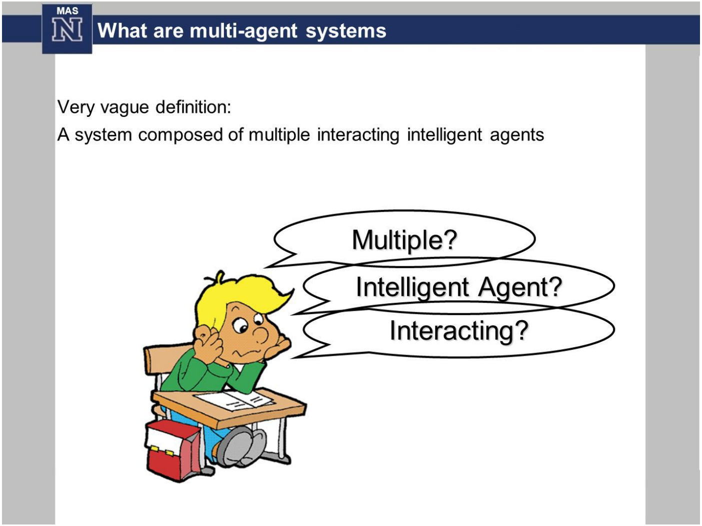
where W is a centered $\mathcal{H}_d'$-valued continuous centered Gaussian process with covariance functional

$$\mathbb{E}[W_t(\varphi_1)W_s(\varphi_2)] = \int_0^{s\wedge t} \langle \mu_r, D_x\varphi_1 \cdot D_x\varphi_2\rangle dr, \quad \varphi_1, \varphi_2 \in \mathcal{H}_d,$$

and where $\mathcal{A}_{t,\mu_t}$ is some suitable (nonlocal) operator.

2. From Interacting Diffusions to MFG: Fluctuations

Multi-agent or Many-player Systems



Symmetric n-player Differential Games

- $W^1, \ldots, W^n$ ind. d-dim BMs
- Polish action space
- drift functional $b : \mathbb{R}^d \times \mathcal{P}^p(\mathbb{R}^d) \times \mapsto \mathbb{R}^d$

State Dynamics

$$dX_t^{n,i} = b(X_t^{n,i}, \mu_t^n, \alpha^i(t, \boldsymbol{X}_t))dt + dW_t^i, \qquad \mu_t^n = \frac{1}{n}\sum_{k=1}^n \delta_{X_t^{n,k}}$$

where $\alpha^i : [0, T] \times (\mathbb{R}^d)^n \to$ is a Markovian control that is chosen to minimize the ith objective function

$$J_i^n(\alpha^1, \ldots, \alpha^n) = \mathbb{E}\left[\int_0^T f(X_t^i, \mu_t^n, \alpha^i(t, \boldsymbol{X}_t))dt + g(X_T^i, m_{\boldsymbol{X}_T}^n)\right],$$

for suitable cost functionals f and g.

Nash Equilibria

Definition A (closed-loop) Nash equilibrium is defined in the usual way as a vector of feedback functions or controls $(\alpha^1, \ldots, \alpha^n)$, where $\alpha^i : [0, T] \times (\mathbb{R}^d)^n \to$ are such that the SDE

$$dX_t^{n,i} = b(X_t^{n,i}, \mu_t^n, \alpha^i(t, \boldsymbol{X}_t))dt + dW_t^i, \qquad \mu_t^n = \frac{1}{n}\sum_{k=1}^n \delta_{X_t^{n,k}}$$

is unique in law, and

$$J^{n,i}(\alpha^1, \ldots, \alpha^n) \leq J^{n,i}(\alpha^1, \ldots, \alpha^{i-1}, \widetilde{\alpha}, \alpha^{i+1}, \ldots, \alpha^n),$$

for any alternative choice of feedback control $\widetilde{\alpha}$.

Main Point (Cardialaguet et al '15)

A verification theorem tells us that if we have a unique solution $\{v^{n,i}\}_{i=1,...,n}$ to a coupled system of n PDEs called the Nash system such that $v^{n,i}$ lies in $C^{1,2}$ for each $i = 1, \ldots, n$, then the controls

$$(0, T] \times (\mathbb{R}^d)^n \ni (t, \boldsymbol{x}) \mapsto \hat{\alpha}^{n,i}\left(\boldsymbol{x}, m^n_X, D_{x_i} v^{n,i}(t, \boldsymbol{x})\right)$$

form a closed-loop Nash equilibrium, where $m^n_x = \frac{1}{n}\sum_{i=1}^n \delta_{x_i}$

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form a closed-loop Nash equilibrium, where $m^n_x = \frac{1}{n}\sum_{i=1}^n \delta_{x_i}$
The corresponding Nash equilibrium dynamics, given by

$$dX^{n,i}_t = \hat{b}(X^{n,i}_t, \mu^n_t, D_{x_i} v^{n,i}(t, \boldsymbol{X}^n_t))dt + dW^i_t, \quad \mu^n_t = \frac{1}{n}\sum_{k=1}^n \delta_{X^{n,k}_t}$$

defines a collection of interacting diffusions,

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$$dX_t^{n,i} = \hat{b}(X_t^{n,i}, \mu_t^n, D_{x_i} v^{n,i}(t, \boldsymbol{X}_t^n))dt + dW_t^i, \quad \mu_t^n = \frac{1}{n}\sum_{k=1}^n \delta_{X_t^{n,k}}$$

defines a collection of interacting diffusions, with

$$\widehat{b}(x, m, y) = b(x, m, \widehat{\alpha}(x, m, y)),$$

being the Nash equilibrium drift, where $\widehat{\alpha}$ takes the explicit form:

$$\widehat{\alpha}(x, m, y) \in \arg\min_{a \in A} [b(x, m, a) \cdot y + f(x, m, a)] .$$

Nash Equilibrium *n*-player dynamics

Recall that the corresponding Nash equilibrium dynamics has the form

$$dX_t^{n,i} = \hat{b}(X_t^{n,i}, \mu_t^n, D_{x_i} v^{n,i}(t, \boldsymbol{X}_t^n))dt + dW_t^i, \quad \mu_t^n = \frac{1}{n}\sum_{k=1}^{n} \delta_{X_t^{n,k}}$$

In other words, it is a system of weakly interacting diffusions:

$$dX_t^i = \tilde{b}_n(t, X_t^i, \mu_t^n)dt + \sigma dB_t^i, \quad i = 1, \ldots, n,$$

where

$$\tilde{b}_n(t, x, m) = \hat{b}(x, m, D_{x_i} v^{n,i}(t, x))$$

Nash Equilibrium n-player dynamics

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$$\tilde{b}_n(t, x, m) = \hat{b}(x, m, D_{x_i} v^{n,i}(t, x))$$

But the drift is n-dependent, so this is not of the form we considered earlier. Instead, replace the n-dependent control $v^{n,i}$ by a quantity coming from the master equation.

An Approximating System

Recall: Interacting diffusions describing Nash equilibrium dynamics:

$$dX_t^{n,i} = \hat{b}(X_t^{n,i}, \mu_t^n, D_{x_i} v^{n,i}(t, \boldsymbol{X}_t^n))dt + dW_t^i, \quad \mu_t^n = \frac{1}{n}\sum_{k=1}^n \delta_{X_t^{n,k}}$$

Instead: Consider the modified system coming from the limit system:

- "**Replace**" $v^{n,i}$ by $u^{n,i}$, where

$$u^{n,i}(t, x_1, \ldots, x_n) = U(t, x_i, m_{\boldsymbol{x}}^n), \quad m_{\boldsymbol{x}}^n = \frac{1}{n}\sum_{k=1}^n \delta_{x_i}.$$

  the dependence of $u^{n,i}$ on n is **only through the empirical measure**
- That is, consider the sequence of IPS:

$$d\tilde{X}_t^{n,i} = \tilde{b}(t, \tilde{X}_t^{n,i}, m_{\tilde{\boldsymbol{X}}_t}^n) + dW_t^i, \quad i = 1, \ldots, n,$$

where

$$\tilde{b}(t, x, m) = \hat{b}(x, m, D_x U(t, x, m))$$

Interacting diffusions describing Nash equilibrium dynamics:

$$dX_t^{n,i} = \hat{b}(X_t^{n,i}, \mu_t^n, D_{x_i} v^{n,i}(t, \boldsymbol{X}_t^n))dt + dW_t^i, \quad \mu_t^n = \frac{1}{n}\sum_{k=1}^n \delta_{X_t^{n,k}}$$

Approximating diffusions in the form of an IPS

$$d\tilde{X}_t^{n,i} = \tilde{b}(t, \tilde{X}_t^{n,i}, \tilde{\mu}_t^n) + dW_t^i, \quad \tilde{\mu}_t^n = \frac{1}{n}\sum_{k=1}^n \delta_{\tilde{X}_t^{n,k}}$$

$$\tilde{b}(t,x,m) = \hat{b}(x,m,D_x U(t,x,m))$$

Overall Philosophy:Transferring LLN/CLT Results

Interacting diffusions describing Nash equilibrium dynamics:

$$dX_t^{n,i} = \hat{b}(X_t^{n,i}, \mu_t^n, D_{x_i} v^{n,i}(t, \boldsymbol{X}_t^n))dt + dW_t^i, \quad \mu_t^n = \frac{1}{n}\sum_{k=1}^n \delta_{X_t^{n,k}}$$

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$$\tilde{b}(t, x, m) = \hat{b}(x, m, D_x U(t, x, m))$$

1. Analyze master equation + Nash PDE to prove $\mathbb{E}[\mathcal{W}_{2,\mathcal{C}^d}(\mu^n, \tilde{\mu}^n)] = O(n^{-2})$
2. Invoke IPS results to deduce LLN/CLT for $\{\tilde{\mu}^n\}$.
3. Use estimate in 1. to deduce LLN/CLT for $\{\mu^n\}$.

3. Large Deviations for Interacting Particle Systems and MFG

Large Deviations Theory

- Large deviations (LD) is an asymptotic theory that characterizes the asymptotic probability of rare events

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- When a sequence of probabilities of events decay to 0, large deviations characterizes the asymptotic exponential decay rate: suppose ν^n, $n \in \mathbb{N}$ take values in (some topological space) $\mathcal{E}$, and satisfies for all "nice" $A \subset \mathcal{S}$,

$$\mathbb{P}(\nu^n \in A) \sim e^{-s_n I(A)},$$

where $I : \mathcal{E} \mapsto [0, \infty]$ is lowersemicontinuous with compact level sets, and $I(A) := \inf_{s \in A} I(s)$.

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- One says in this case that $\{\nu^n\}$ satisfies a large deviation principle (LDP) on $\mathcal{E}$ with speed $\{s_n\}$ and good rate function (GRF) I
- Thus the rate of decay of the probabilities is expressed in terms of a variational problem. Often $I(a)$ itself is also expressed in terms of a variational problem.

A Rigorous Statement of the Large Deviation Principle

- $\mathcal{E}$ - topological space
- $\{\nu^n\}$ - sequence of $\mathcal{E}$-valued random elements

Definition (Large Deviation Principle (LDP))

$\{\nu_n\}$ is said to satisfy a large deviations principle (**LDP**) with **speed** $\{s_n\}$ and **rate function** $\mathbb{I} : \mathbb{R} \mapsto [0, \infty)$ if for all measurable A,

$$-\inf_{w \in A^\circ} \mathbb{I}(w) \leq \liminf_{n\to\infty} \frac{1}{s(n)} \log \mathbb{P}(\nu^n \in A)$$
$$\leq \limsup_{n\to\infty} \frac{1}{s(n)} \log \mathbb{P}(\nu^n \in A) \leq -\inf_{w \in \bar{A}} \mathbb{I}(w),$$

where $\mathbb{I}$ is lower semicontinuous and has compact level sets.

In short, the LDP says that for all "nice" sets $A \subset \mathcal{E}$,

$$\mathbb{P}(\nu^n \in A) \approx e^{-s_n \inf_{w\in A} \mathbb{I}(w)}$$

The Contraction Principle

Theorem. Let $\mathcal{Y}$ and $\mathcal{Y}'$ be topological spaces and let $F : \mathcal{Y} \mapsto \mathcal{Y}'$ be a continuous mapping. Suppose a sequence $\{Y_n\}$ of $\mathcal{Y}$-valued random variables satisfies an LDP with rate function $I : \mathcal{Y} \mapsto [0, \infty]$. Then the sequence $\{Y_n' := F(Y_n)\}$ satisfies an LDP with rate function $J : \mathcal{Y}' \mapsto [0, \infty]$, given by

$$J(y') = \inf\{I(y) : F(y) = y' \text{ for some } y \in \mathcal{Y}\}.$$

Exercise 3: Prove the contraction principle.

Theory of Large Deviations (Sanov's Theorem)

Suppose $Y_i, i = 1, \ldots,$ are iid on some Polish space $\mathcal{Y}$ with common distribution $\theta \in \mathcal{P}(\mathcal{Y})$, and define

$$\nu_n := \frac{1}{n} \sum_{i=1}^{n} \delta_{Y_i}$$

Also, define relative entropy: given $\nu, \mu \in \mathcal{P}(\mathcal{Y})$,

$$H(\nu|\theta) := \int_{\mathcal{Y}} \ln \left(\frac{d\nu}{d\theta}(x) \right) \nu(dx).$$

if $\nu \ll \theta$ and $H(\nu|\theta) = \infty$ otherwise.

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if $\nu \ll \theta$ and $H(\nu|\theta) = \infty$ otherwise.

Sanov's Theorem

Then $\{\nu_n\}$ satisfies an LDP in $\mathcal{P}^1(\mathcal{Y})$ with good rate function $H(\cdot|\theta)$.

Exercise 4: Prove Sanov's theorem when Y_i take values in a finite state space.

Large Deviations in the non-interacting Case ($b = 0$)

$$X_t^{n,i} = X_0^{n,i} + W_t^i, \qquad \mathcal{Q}_t^n = \frac{1}{n}\sum_{k=1}^{n} \delta_{X_0^{n,k}, W^i},$$

$\{X_0^i\}_{i\in\mathbb{N}}$ iid with common distribution μ_0; $\{W^i\}_{i\in\mathbb{N}}$ iid Brownian motions.

Theorem: As an immediate consequence of Sanov's theorem we have $\{\mathcal{Q}^n\}$ satisfies an LDP on $\mathcal{P}^1(\mathbb{R}^d \times \mathbb{C}_0^d)$ with rate function

$$\mathcal{R}(\nu|\mu_0 \times \mathbb{W}),$$

where recall μ_0 is the initial distribution of X_0^i and $\mathbb{W}$ is d-dimensional Wiener measure on $\mathbb{C}_0^d$

References (LDPs for Interacting Particle Systems)

$$dX_t^{n,i} = b(t, X_t^{n,i}, \mu_t^n)dt + dW_t^i, \qquad \mu_t^n = \frac{1}{n}\sum_{k=1}^n \delta_{X_t^{n,k}},$$

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- D. Dawson and J. Gärtner, "Large deviations from the McKean-Vlasov limit for weakly interacting diffusions", *Stochastics: An International Journal of Probability and Stochastic Processes* **20** (1987), 247-308.
- A. Budhiraja, P. Dupuis, and M. Fischer, "Large deviation properties of weakly interacting processes via weak convergence methods", *Annals of Probability* (2012), 74-102.
- M.Fischer, "On the form of the large deviation rate function for the empirical measures of weakly interacting systems", *Bernoulli* **20** (2014), 1765-1801.

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$\{X_0^i\}_{i\in\mathbb{N}}$ iid with common distribution μ_0; $\{W^i\}_{i\in\mathbb{N}}$ iid Brownian motions
For the ultimate application to MFG, need to consider an extension beyond those references that includes

- random initial conditions,
- time-dependent drift
- and a weaker continuity condition on the drift b, namely, continuous as a map from $[0, T] \times \mathbb{R}^d \times \mathcal{P}^1(\mathbb{R}^d)$ to $\mathbb{R}^d$, in particular, b need not be continuous in the third variable with respect to the weak topology
- And also allows for common noise, which we do not include here.

Large Deviations in the Interacting Case

$$dX_t^{n,i} = b(t, X_t^{n,i}, \mu_t^n)dt + dW_t^i, \qquad \mu_t^n = \frac{1}{n}\sum_{k=1}^{n} \delta_{X_t^{n,k}},$$

$(X_0^{n,i})_{i\in\mathbb{N}}$ iid with common distribution μ_0.

Idea: To use the contraction principle

Need to express the law of X^n as a continuous functional of the law $\{\mathcal{Q}^n\}$ of the non-interacting particle system and a Brownian motion

Large Deviations in the Interacting Case

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Key Result: Girsanov's Theorem

Relates the law μ of the solution X to the SDE

$$dX_t = dW_t$$

with the law of μ^b of the solution X^b to the SDE with an adapted (suitably regular) drift $(r_t)_{t\geq 0}$

$$dX_t^b = r_t dt + W_t$$

Exercise 5. Prove the LDP for $\{\mu_t^n\}$ via the following steps:

1. Canonical setup: Define the mappings

$$
\begin{aligned}
e: &\quad (y \times f) \in \mathbb{R}^d \times \mathbb{C}_0^d \mapsto y \\
w_t: &\quad (y \times f) \in \mathbb{R}^d \times \mathbb{C}_0^d \mapsto f_t, \qquad t \geq 0.
\end{aligned}
$$

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2. For $\mathcal{Q} \in \mathcal{P}^1(\mathbb{R}^d \times \mathbb{C}_0^d)$, define the McKean-Vlasov equation map:

$$x_t = e + \int_0^t b(s, x_s, \mathcal{Q} \circ x_s^{-1}) ds + w_t, \tag{1}$$

where e and w_t denote the canonical maps on $\mathbb{R}^d \times \mathcal{C}_0^d$, as defined above.

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$$x_t = e + \int_0^t b(s, x_s, \mathcal{Q} \circ x_s^{-1}) ds + w_t, \tag{1}$$

   where e and w_t denote the canonical maps on $\mathbb{R}^d \times \mathcal{C}_0^d$, as defined above. Here,
   - $\mathcal{Q}$ represents the joint law of the initial condition e and driving noise w
   - $\mathcal{Q} \circ x_s^{-1} \in \mathcal{P}^1(\mathbb{R}^d)$ represents the marginal law at time s of the solution x to equation (1), under $\mathcal{Q}$.

Recall: $X_0^{n,i}$ iid, and

$$dX_t^{n,i} = b(t, X_t^{n,i}, \mu_t^n)dt + dW_t^i, \quad \mu_t^n = \frac{1}{n}\sum_{k=1}^n \delta_{X_t^{n,k}}, \quad Q^n = \frac{1}{n}\sum_{k=1}^n \delta_{(X_0^{n,k}, W^k)}$$

and for $\mathcal{Q} \in \mathcal{P}^1(\mathbb{R}^d \times \mathbb{C}_0^d)$, the McKean-Vlasov equation map is:

$$x_t = e + \int_0^t b(s, x_s, \mathcal{Q} \circ x_s^{-1})ds + w_t,$$

where e and w_t denote canonical variables on $\mathbb{R}^d \times \mathcal{C}_0^d$.

Large Deviations in the Interacting Case

Recall: $X_0^{n,i}$ iid, and

$$dX_t^{n,i} = b(t, X_t^{n,i}, \mu_t^n)dt + dW_t^i, \quad \mu_t^n = \frac{1}{n}\sum_{k=1}^n \delta_{X_t^{n,k}}, \quad Q^n = \frac{1}{n}\sum_{k=1}^n \delta_{(X_0^{n,k}, W^k)}$$

and for $\mathcal{Q} \in \mathcal{P}^1(\mathbb{R}^d \times \mathbb{C}_0^d)$, the McKean-Vlasov equation map is:

$$x_t = e + \int_0^t b(s, x_s, \mathcal{Q} \circ x_s^{-1})ds + w_t,$$

where e and w_t denote canonical variables on $\mathbb{R}^d \times \mathcal{C}_0^d$.

3. Let $\Phi : \mathcal{P}^1(\mathbb{R}^d \times \mathbb{C}_0^d) \mapsto C([0, T] : \mathcal{P}^1(\mathbb{R}^d))$ be the mapping that takes $\mathcal{Q}$ to the flow of marginal measures $(\mathcal{Q} \circ x_t^{-1})_{t \geq 0}$, and observe

$$\mu^n = \Phi(Q_n).$$

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$$\mu^n = \Phi(Q_n).$$
4. Prove $\Phi : \mathcal{P}^1(\mathbb{R}^d \times \mathcal{C}_0^d) \mapsto C([0,T] : \mathcal{P}^1(\mathbb{R}^d))$ is uniformly continuous.

Large Deviations in the Interacting Case: Summary

Given the IPS

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5. Apply the contraction principle to conclude that if b is bounded and continuous and Lipschitz continuous in the second and third arguments (uniformly in time), then $\{\mu^n\}$ satisfies an LDP with rate function

$$J(\nu) = \inf\{\mathcal{R}(\mathcal{Q}|\mu_0 \times \mathbb{W}) : \Phi(\mathcal{Q}) = \nu\}.$$

This concludes Exercise 5 and the proof of the LDP for IPS.

Large Deviations for Mean-Field Games

Recall the form of interacting diffusions describing Nash equilibrium dynamics:

$$dX_t^{n,i} = \hat{b}(X_t^{n,i}, \mu_t^n, D_{x_i} v^{n,i}(t, \boldsymbol{X}_t^n))dt + dW_t^i, \quad \mu_t^n = \frac{1}{n}\sum_{k=1}^{n} \delta_{X_t^{n,k}},$$

Aim:

To prove an LDP for the sequence $(\mu^n)_{n\in\mathbb{N}}$ of empirical measures of the sequence of Nash equilibria state processes

Principle: Transfer LDP results from IPS to MFG

Interacting diffusions describing Nash equilibrium dynamics:

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In view of the previous results on LDPs for IPS, recall the approximating diffusions we considered earlier that were in the form of an IPS:

$$d\tilde{X}_t^{n,i} = \tilde{b}(t, \tilde{X}_t^{n,i}, \tilde{\mu}_t^n) + dW_t^i, \qquad \tilde{\mu}_t^n = \frac{1}{n}\sum_{k=1}^n \delta_{\tilde{X}_t^{n,k}}$$

$$\tilde{b}(t, x, m) = \hat{b}(x, m, D_x U(t, x, m))$$

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$$d\tilde{X}_t^{n,i} = \tilde{b}(t, \tilde{X}_t^{n,i}, \tilde{\mu}_t^n) + dW_t^i, \qquad \tilde{\mu}_t^n = \frac{1}{n}\sum_{k=1}^{n} \delta_{\tilde{X}_t^{n,k}}$$

$$\tilde{b}(t, x, m) = \hat{b}(x, m, D_x U(t, x, m))$$

1. Originally, had only $\mathbb{E}[\mathcal{W}_{2,\mathcal{C}^d}(\mu^n, \tilde{\mu}^n)] = O(n^{-2})$
2. Invoke IPS results to get LDP for $\{\tilde{\mu}^n\}$.
3. Use a sharper exponential estimate in 1. to deduce LDP for $\{\mu^n\}$.

- Lacker and Ramanan, “Rare Nash equilibria and the price of anarchy in large static games,” (2019) *Mathematics of Operations Research* **44** (2019) no. 2, 400-422.
- Cardaliaguet, Delarue, Lasry and Lions, “The master equation and the convergence problem in mean-field games” (2019)
- F. Delarue, D. Lacker and K. Ramanan, “From the master equation to mean field game limit theory: Large deviations and concentration of measure," (2018) to appear in *Annals of Probability*

- Study refined convergence theorems for open-loop Nash equilibria.
- Investigate if there are cases when the MFG LDP exists, but differs from the interacting paricle system LDP obtained from the master equation.
- Establish large deviation principles for stochastic differential games in the presence of non-uniqueness (as has been done in the static case)
- Use LDPs to find interesting conditional limit laws in both the static and stochastic differential settings.
- ...