AMS Course on Mean-Field Games

The Master Equation

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Part I. Introduction

Part I. Introduction

a. General objective

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• Purpose of mean field games is to find compromises within large populations of players in $\mathbb{R}^d$ subjected to mean field interactions

• Asymptotic equilibria are defined as trajectories $(\mu_t)_{0\leq t\leq T}$ with values in $\mathcal{P}(\mathbb{R}^d)$ solving a fixed point problem

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(1) solve the stochastic optimal control problem

$$dX_t = \alpha_t dt + dW_t$$

◦ with $\boxed{\text{cost}}$ (letting $\boldsymbol{\alpha} = (\alpha_t)_t$)

$$J(\boldsymbol{\alpha}) = \mathbb{E}\Big[g(X_T, \mu_T) + \int_0^T \Big(f(X_t, \mu_t) + \frac{1}{2}|\alpha_t|^2\Big) dt\Big]$$

◦ with W a d-dimensional Brownian motion and

$$f, g : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$$

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$$f, g : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$$

(2) let $(X_t^{\star,\boldsymbol{\mu}})_{0 \le t \le T}$ be the unique optimizer (under nice assumptions)
$\rightsquigarrow$ find $(\mu_t)_{0 \le t \le T}$ such that

$$\mu_t = \mathcal{L}(X_t^{\star,\boldsymbol{\mu}}), \quad t \in [0, T]$$

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- Characterization in terms of MFG forward-backward system

  - value function u in environment $\boldsymbol{\mu} = (\mu_t)_t$

$$\Big(\partial_t u + \frac{1}{2}\Delta_x u\Big)(t,x) - \frac{1}{2}|\partial_x u(t,x)|^2 + f(x,\mu_t) = 0$$

  coupled with Fokker Planck equation

$$\partial_t \mu_t = \mathrm{div}_x(\partial_x u(t,x)\mu_t) + \frac{1}{2}\Delta_x \mu_t$$

  - boundary conditions $\begin{cases} \mu_0 = \mu^\circ \text{ prescribed at time } 0 \\ u(T,\cdot) = g(\cdot,\mu_T) \end{cases}$

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$$u(t,\cdot) = \mathcal{U}(t,\cdot,\mu_t) \quad \rightsquigarrow \quad \text{PDE for } \mathcal{U} \quad ???$$

Part I. Introduction

b. Linear quadratic case

Linear quadratic control problem

- Interesting because the master equation reduces to a standard PDE!
- Choose $d = 1$ and dynamics of the form

$$dX_t = \alpha_t dt + dW_t$$

  - cost functional of the form

$$J(\alpha) = \mathbb{E}\Big[\tfrac{1}{2}(c_g X_T + g(\bar{\mu}_T))^2 + \int_0^T [\tfrac{1}{2}(c_f X_t + f(\bar{\mu}_t))^2 + \tfrac{1}{2}\alpha_t^2] dt\Big]$$

  - coefficients c_f, c_g may be arbitrarily chosen (say 1)
  - $\bar{\mu}_t = \int_{\mathbb{R}^d} x d\mu_t(x)$
  - noise does not matter in this toy example $\rightsquigarrow$ analysis relies on the convex structure of the problem

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- General form of the optimizer over $\boldsymbol{\alpha}$ when $\boldsymbol{\mu}$ is fixed

$$\alpha_t^{\star,\boldsymbol{\mu}} = -\eta_t X_t^{\star,\boldsymbol{\mu}} - h_t$$

  - $\boldsymbol{\eta} = (\eta_t)_t$ and $\boldsymbol{h} = (h_t)_t \rightsquigarrow$ deterministic and $\boldsymbol{\eta}$ independent of $\boldsymbol{\mu}$!
  - equation for $\boldsymbol{\eta} \rightsquigarrow$ Riccati equation

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  - Exercise: conditional on initial condition, $\boldsymbol{X}^{\star,\boldsymbol{\mu}}$ is O.-U. $\rightsquigarrow$ marginals are Gaussian with fixed variance $\rightsquigarrow$ fixed point on the mean only (dimension reduction)!

Search for equilibria

- Plug the shape of the optimal feedback into HJB $\rightsquigarrow$ equation for $\boldsymbol{h}$
  - backward linear ODE

$$\dot{h}_t = \Big(\eta_t h_t - f(\bar{\mu}_t)\Big), \quad h_T = g(\bar{\mu}_T)$$

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- Plug the shape of the optimal feedback into Fokker-Planck $\rightsquigarrow$ equation for fixed point $\boldsymbol{\mu}$

$$\partial_t \mu_t = \partial_x\Big([\eta_t x + h_t]\mu_t\Big) + \frac{1}{2}\partial_x^2 \mu_t$$

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  - take $\int_{\mathbb{R}^d} x d\mu_t(x)$

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- Compute the dynamics of the mean!
  - take $\int_{\mathbb{R}^d} x d\mu_t(x)$
- End up with forward backward ODE

$$\frac{d}{dt}\bar{\mu}_t = -\big(\eta_t \bar{\mu}_t + h_t\big)$$
$$\frac{d}{dt}h_t = \big(\eta_t h_t - f(\bar{\mu}_t)\big), \quad h_T = g(\bar{\mu}_T)$$

A word on FB systems

• Cauchy-Lipschitz theory in small time only

∘ may loose existence or uniqueness on a given time interval $\leftrightsquigarrow$ appearance of shocks in hyperbolic (system of) PDE

∘ this (system of) PDE is the candidate for being our master equation here!

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- Prototype (may easily remove the η_t term in the two equations)

$$\dot{m}_t = -h_t, \quad \dot{h}_t = -f(m_t), \quad h_T = g(m_T)$$

  ◦ characteristics system of inviscid Burgers PDE

$$\partial_t v(t,x) - \partial_x v(t,x) v(t,x) + f(x) = 0, \quad v(T,x) = g(x)$$

  ◦ if smooth solution to PDE $\rightsquigarrow h_t = v(t, m_t)$

  solve first for $\dot{m}_t = -v(t, m_t)$ and then check that $h_t = v(t, m_t)$ solves the backward equation

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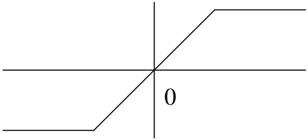
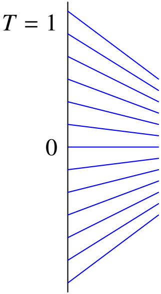
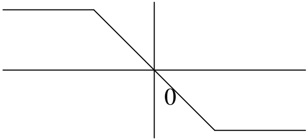
  - if smooth solution to PDE $\rightsquigarrow h_t = v(t, m_t)$
- Well-posedness to PDE if $f, g \nearrow \Rightarrow$! of characteristics
  - if not $\Rightarrow$ shocks may emerge in finite time...
  - see plot below when $f = 0$

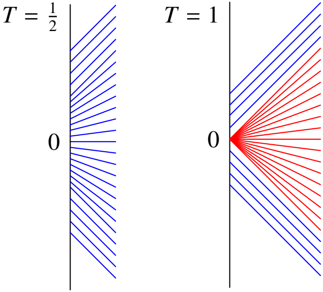
Plots of the characteristics

- If $g(x) = \pm(-1 \vee x \wedge 1)$









Part II. More on solvability

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a. Monotonicity in the general setting

Lasry Lions monotonicity condition

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  - what is monotonicity condition in the direction of the measure?

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- Lasry Lions monotonicity condition
  - monotonicity property for f and g w.r.t. μ

$$\int_{\mathbb{R}^d} (f(x,\mu) - f(x,\mu'))d(\mu - \mu')(x) \geq 0$$

$$\int_{\mathbb{R}^d} (g(x,\mu) - g(x,\mu'))d(\mu - \mu')(x) \geq 0$$

  - usually, f and g at most of quadratic growth and μ with finite second moment $\rightsquigarrow$ below, f and g bounded

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- General principle: There must be at most one equilibrium under Lasry-Lions condition
- Example:

$$h(x,\mu) = \int_{\mathbb{R}^d} L(z, \rho \star \mu(z))\rho(x-z)dz$$

  - where L is $\nearrow$ in second variable and ρ is even

Monotonicity implies uniqueness

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$$\mathbb{E}\Big[\underbrace{g(X_T^{\star,\boldsymbol{\mu}}, \mu'_T) - g(X_T^{\star,\boldsymbol{\mu}}, \mu_T)}_{\int_{\mathbb{R}^d} (g(x,\mu'_T) - g(x,\mu_T)) d\mu_T(x)} - \underbrace{\Big(g(X_T^{\star,\boldsymbol{\mu}'}, \mu'_T) - g(X_T^{\star,\boldsymbol{\mu}'}, \mu_T)\Big)}_{\int_{\mathbb{R}^d} (g(x,\mu'_T) - g(x,\mu_T)) d\mu'_T(x)} + \dots \Big] > 0$$

  - same for $f \Rightarrow$ LHS must be ≤ 0

Existence of a fixed point

- Find a fixed point $\boldsymbol{\mu} = (\mu_t)_{0\le t\le T}$ with values in $C([0,T],\mathcal{P}_2(\mathbb{R}^d))$
    - $\mathcal{P}_2(\mathbb{R}^d))$ space of probability measures μ with a finite second moment, i.e. $\int_{\mathbb{R}^d} |x|^2 d\mu(x) < \infty$, equipped with Wasserstein distance

$$\mu,\nu \in \mathcal{P}_2(\mathbb{R}^d), \quad W_2(\mu,\nu) = \left(\inf_\pi \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 d\pi(x,y)\right)^{1/2},$$

where π has μ and ν as marginals on $\mathbb{R}^d \times \mathbb{R}^d$

- X and X' two r.v.'s $\Rightarrow W_2(\mathcal{L}(X),\mathcal{L}(X')) \le \mathbb{E}[|X-X'|^2]^{1/2}$

Existence of a fixed point

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◦ $\mathcal{P}_2(\mathbb{R}^d))$ space of probability measures μ with a finite second moment, i.e. $\int_{\mathbb{R}^d} |x|^2 d\mu(x) < \infty$, equipped with Wasserstein distance

• Take f and g smooth (bounded and bounded derivatives) and Lipschitz continuous in μ

◦ for $\boldsymbol{\mu}$ given, solve HJB:

$$\Big(\partial_t u + \frac{1}{2}\Delta_x u\Big)(t,x) - \frac{1}{2}|\partial_x u(t,x)|^2 + f(x,\mu_t) = 0, \quad u(T,x) = g(x,\mu_T)$$

◦ easily solved by Cole-Hopf transform ($\exp(-u)$ solves linear equation) $\rightsquigarrow$ get a classical solution with a bounded gradient

◦ with u associate optimal path

$$dX_t^{\star,\boldsymbol{\mu}} = -\partial_x u(t, X_t^{\boldsymbol{\mu},\star})dt + dW_t$$

◦ hence create a map $\Phi : \boldsymbol{\mu} \mapsto (\mathcal{L}(X_t^{\star,\boldsymbol{\mu}}))_{0 \le t \le T}$

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• Find fixed point by Schauder like argument? Basically, suffices Φ to be continuous and have compact image!

∘ key point: sup norm of $\partial_x u$ only depends on $\|f\|_\infty, \|g\|_\infty$ and $\|\partial_x g\|_\infty$

$\Rightarrow \mathbb{E}[|X_t^{\star,\boldsymbol{\mu}}|^4] \le C$ and $\mathbb{E}[|X_t^{\star,\boldsymbol{\mu}} - X_s^{\star,\boldsymbol{\mu}}|^2] \le C|t-s|$ for C independent of $\boldsymbol{\mu}$! $\rightsquigarrow \Phi(\boldsymbol{\mu})$ lives in a compact subset

Part III. Master equation

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a. Value of the game

Generalized value function

- Initial condition of the population μ° at time t°
  - uniqueness $\rightsquigarrow$ flow $(\mu_t)_{t^\circ \le t \le T}$ describing the equilibrium
  - solution of optimal control starting from x° under $\boldsymbol{\mu} = (\mu_t)_{t^\circ \le t \le T}$

$$dX_t = -\partial_x u^{\boldsymbol{\mu}}(t, X_t)dt + dW_t \quad t \in [t^\circ, T],$$

with $X_{t^\circ} = x^\circ$ and

$$\begin{aligned} \partial_t u^{\boldsymbol{\mu}}(t,x) &= -\tfrac{1}{2}\Delta u^{\boldsymbol{\mu}}(t,x) + \tfrac{1}{2}|\partial_x u^{\boldsymbol{\mu}}(t,x)|^2 - f(x, \mu_t) \\ u^{\boldsymbol{\mu}}(T,x) &= g(x, \mu_T) \end{aligned}$$

- Generalized value function : $\mathcal{U}(t^\circ, x^\circ, \mu^\circ) = u^{\boldsymbol{\mu}: \mu_{t^\circ} = \mu^\circ}(t^\circ, x^\circ)$

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$$u^{\boldsymbol{\mu}}(T,x) = g(x,\mu_T)$$

- Generalized value function : $\mathcal{U}(t^\circ, x^\circ, \mu^\circ) = u^{\boldsymbol{\mu}:\mu_{t^\circ}=\mu^\circ}(t^\circ, x^\circ)$
  - $\mathcal{U}$ hence defined on $[0,T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \rightsquigarrow$ may follow the equilibrium $\boldsymbol{\mu}$

$$\mathcal{U}(t, x, \mu_t) = u^{\boldsymbol{\mu}:\mu_{t^\circ}=\mu^\circ}(t, x), \quad t^\circ \le t \le T$$

  - take derivative on both sides and get a PDE for $\mathcal{U}$... if $\mathcal{U}$ is indeed smooth enough!

Part III. Master equation

b. Derivation of the equation

Differential calculus on Wasserstein space

- We say that $\mathcal{V} : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is C^1 if

$$\frac{d}{d\varepsilon}_{|\varepsilon=0+} \mathcal{V}((1-\varepsilon)\mu + \varepsilon\mu') = \underbrace{\int_{\mathbb{T}^d} \frac{\delta \mathcal{V}}{\delta m}(\mu)(v) d(\mu' - \mu)(v)}_{\frac{\delta \mathcal{V}}{\delta m}(\mu)(\cdot)\cdot(\mu' - \mu)}$$

for a *continuous* map $\frac{\delta \mathcal{V}}{\delta m} : \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$ (with quadratic growth)

  - unique up to an additive constant $\rightsquigarrow$ impose zero mean under μ

Differential calculus on Wasserstein space

- We say that $\mathcal{V} : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is C^1 if

$$\frac{d}{d\varepsilon}_{|\varepsilon=0+} \mathcal{V}((1-\varepsilon)\mu + \varepsilon\mu') = \underbrace{\int_{\mathbb{T}^d} \frac{\delta \mathcal{V}}{\delta m}(\mu)(v) d(\mu' - \mu)(v)}_{\frac{\delta \mathcal{V}}{\delta m}(\mu)(\cdot)\cdot(\mu' - \mu)}$$

for a *continuous* map $\frac{\delta \mathcal{V}}{\delta m} : \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$ (with quadratic growth)

  - unique up to an additive constant $\rightsquigarrow$ impose zero mean under μ

- Connection with Wasserstein derivative

$$\partial_\mu \mathcal{V}(\mu)(v) = \partial_v \frac{\delta \mathcal{V}}{\delta m}(\mu)(v)$$

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- Connection with Wasserstein derivative

$$\partial_\mu \mathcal{V}(\mu)(v) = \partial_v \frac{\delta \mathcal{V}}{\delta m}(\mu)(v)$$

- Example: $\mathcal{U}(\mu) = \int_{\mathbb{R}^d} h(y) d\mu(y)$

  - ◦ h C^1 and ∇h at most of linear growth

$$\partial_\mu \mathcal{U}(\mu)(v) = \nabla h(v)$$

Deriving the master equation

- Write finite difference in time:

$$\frac{\mathcal{U}(t^\circ + h, x, \mu^\circ) - \mathcal{U}(t^\circ, x, \mu^\circ)}{h} = \frac{\mathcal{U}(t^\circ + h, x, \mu^\circ) - \mathcal{U}(t^\circ + h, x, \mu_{t^\circ+h})}{h} + \frac{\mathcal{U}(t^\circ + h, x, \mu_{t^\circ+h}) - \mathcal{U}(t^\circ, x, \mu^\circ)}{h}$$

  - $\mu_{t^\circ+h}$ state of the population when initialized at μ° at time t°

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  - $\mu_{t^\circ+h}$ state of the population when initialized at μ° at time t°

- Handle the second term with HJB (initial state in superscript)

$$\frac{\mathcal{U}(t^\circ + h, x, \mu_{t^\circ+h}) - \mathcal{U}(t^\circ, x, \mu^\circ)}{h} = \frac{u^{t^\circ+h,\mu_{t^\circ+h}}(t^\circ + h, x) - u^{t^\circ,\mu^\circ}(t^\circ, x)}{h} = \frac{u^{t^\circ,\mu^\circ}(t^\circ + h, x) - u^{t^\circ,\mu^\circ}(t^\circ, x)}{h}$$

  - limit as $h \searrow 0$ is

$$-\frac{1}{2} \underbrace{\Delta u^{t^\circ,\mu^\circ}(t^\circ, x)}_{\Delta_x \mathcal{U}(t^\circ, x, \mu^\circ)} - \frac{1}{2} \Big| \underbrace{\partial_x u^{t^\circ,\mu^\circ}(t^\circ, x)}_{\partial_x \mathcal{U}(t^\circ, x, \mu^\circ)} \Big|^2 - f(x, \mu^\circ)$$

Handling the first term

- Look at the term

$$\frac{\mathcal{U}(t^\circ + h, x, \mu^\circ) - \mathcal{U}(t^\circ + h, x, \mu_{t^\circ+h})}{h}$$

- Benefit from the differentiability in μ and write it as

$$-\int_0^1 \int_{\mathbb{R}^d} \frac{\delta \mathcal{U}}{\delta m}(t^\circ + h, x, \lambda\mu_{t^\circ+h} + (1-\lambda)\mu^\circ)(v) d(\mu_{t^\circ+h} - \mu^\circ)(v) d\lambda$$

  - insert Fokker-Planck $\quad \partial_t \mu_t = \mathrm{div}_x(\partial_x u(t,x)\mu_t) + \frac{1}{2}\Delta_x \mu_t$

- Limit as $h \searrow 0$

$$\int_{\mathbb{R}^d} \underbrace{\partial_v \frac{\delta \mathcal{U}}{\delta m}}_{\partial_\mu \mathcal{U}}(t^\circ, x, \mu^\circ)(v) \cdot \partial_x \mathcal{U}(t^\circ, v, \mu^\circ) d\mu^\circ(v)$$

$$-\int_{\mathbb{R}^d} \mathrm{div}_v\Big(\underbrace{\partial_v \frac{\delta \mathcal{U}}{\delta m}}_{\partial_\mu \mathcal{U}}(t^\circ, x, \mu^\circ)(v)\Big) d\mu^\circ(v)$$

  - 1st term $\rightsquigarrow$ 1st order in FP and 2nd term $\rightsquigarrow$ 2nd order in FP

Form of the master equation

- Master equation

$$
\begin{aligned}
&\partial_t \mathcal{U}(t,x,\mu) - \int_{\mathbb{R}^d} \partial_x \mathcal{U}(t,v,\mu)\cdot \partial_\mu \mathcal{U}(t,x,\mu,v)d\mu(v) \\
&\quad - \tfrac{1}{2}|\partial_x \mathcal{U}(t,x,\mu)|^2 + f(x,\mu) + \tfrac{1}{2}\mathrm{Trace}\Big(\partial_x^2 \mathcal{U}(t,x,\mu)\Big) \\
&\quad + \tfrac{1}{2}\int_{\mathbb{R}^d} \mathrm{Trace}\Big(\partial_v \partial_\mu \mathcal{U}(t,x,\mu)(v)\Big)d\mu(v) = 0
\end{aligned}
$$

  - Not a HJB! (MFG $\neq$ optimization)

Form of the master equation

- Master equation

$$\partial_t \mathcal{U}(t,x,\mu) - \int_{\mathbb{R}^d} \partial_x \mathcal{U}(t,v,\mu) \cdot \partial_\mu \mathcal{U}(t,x,\mu,v) d\mu(v)$$
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- Typical statement

  ◦ Lions, Gangbo Swiech ($T \ll 1$), Chassagneux-Crisan-D. (Carmona-D.), Cardaliaguet-D.-Lasry-Lions

  ◦ requires monotonicity and smoothness coefficients in x and μ

  ◦ then existence and uniqueness of a classical solution with $\mathcal{U}(t,\cdot,\cdot)$ having the same smoothness as f and g and continuously differentiable in time

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- Master equation

$$\partial_t \mathcal{U}(t,x,\mu) - \int_{\mathbb{R}^d} \partial_x \mathcal{U}(t,v,\mu) \cdot \partial_\mu \mathcal{U}(t,x,\mu,v) d\mu(v)$$
$$- \tfrac{1}{2}|\partial_x \mathcal{U}(t,x,\mu)|^2 + f(x,\mu) + \tfrac{1}{2}\mathrm{Trace}\Big(\partial_x^2 \mathcal{U}(t,x,\mu)\Big)$$
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- Extension Local interaction [Cardaliaguet], Long time behavior [Cardaliaguet, Porretta, (...)], Major minor [Cardaliaguet Porretta (...)]

Part III. Master Equation

c. Sketch of proof

Road map to regularity of $\mathcal{U}$

- To proceed with the $\boxed{\text{analysis} \rightsquigarrow \text{torus}}$

Road map to regularity of $\mathcal{U}$

- To proceed with the analysis $\rightsquigarrow$ torus
- Look at $\mathcal{U}$ as

$$\mathcal{U} : [0,T] \times \mathcal{P}(\mathbb{T}^d) \ni (t,\mu) \mapsto \underbrace{\left(\mathbb{T}^d \ni x \mapsto \mathcal{U}(t,x,\mu)\right)}_{\mathcal{U}(t,\cdot,\mu)}$$

  - typical example $\rightsquigarrow$ $\mathcal{U}(t,\cdot,\mu) \in C^{n+\alpha}(\mathbb{T}^d)$
  - n, α depending on the smoothness of f and g

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  - typical example $\rightsquigarrow \mathcal{U}(t,\cdot,\mu) \in C^{n+\alpha}(\mathbb{T}^d)$
  - n, α depending on the smoothness of f and g
- Objective is to understand smoothness w.r.t. μ
  - recall $\rightsquigarrow \mathcal{U}(t^\circ,\cdot,\mu) = \underbrace{u^{\mu:\mu_{t^\circ}=\mu}(t^\circ,\cdot)}_{\text{HJB with FP initialized at } (t^\circ,\mu)}$
  - differentiability w.r.t. $\mu^\circ \rightsquigarrow$ use convex perturbation

$$\frac{d}{d\varepsilon}_{|\varepsilon=0+} \mathcal{U}(t^\circ,\cdot,(1-\varepsilon)\mu+\varepsilon\mu') = \frac{d}{d\varepsilon}_{|\varepsilon=0+} u^{(1-\varepsilon)\mu+\varepsilon\mu'}(t^\circ,\cdot), \quad \mu,\mu' \in \mathcal{P}(\mathbb{T}^d)$$

Linearized MFG system

- Assume that f and g are C^1 w.r.t. m with

$$\frac{\delta f}{\delta m},\ \frac{\delta g}{\delta m} : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \times \mathbb{T}^d \ni (x, \mu, v) \mapsto \frac{\delta f}{\delta m}(x, \mu)(v),\ \frac{\delta g}{\delta m}(x, \mu)(v)$$

smooth enough in x and v

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  smooth enough in x and v

- Formal differentiation of the MFG system
  - perturbation of μ along a direction $\mu' - \mu$
  - we let $z_t = \underbrace{\frac{d}{d\varepsilon}_{|\varepsilon=0+} u^{(1-\varepsilon)\mu+\varepsilon\mu'}(t,\cdot)}_{\text{function}}, \quad m_t = \underbrace{\frac{d}{d\varepsilon}_{|\varepsilon=0+} \mu_t^{(1-\varepsilon)\mu+\varepsilon\mu'}}_{\text{distribution}}$

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  - should solve

$$\partial_t m_t - \tfrac{1}{2}\Delta m_t - \text{div}\Big(m_t \partial_x u(t,x) + \mu_t \partial_x z(t,x)\Big) = 0$$

$$\partial_t z(t,x) + \tfrac{1}{2}\Delta z(t,x) - \partial_x u(t,x) \cdot \partial_x z(t,x) + \frac{\delta f}{\delta m}(x, \mu_t)(\cdot) \cdot m_t(\cdot) = 0$$

$$z_T(x) = \frac{\delta g}{\delta m}(x, \mu_T)(\cdot) \cdot m_T(\cdot)$$

Linearized MFG system

- Assume that f and g are C^1 w.r.t. m with

$$\frac{\delta f}{\delta m}, \ \frac{\delta g}{\delta m} : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \times \mathbb{T}^d \ni (x, \mu, v) \mapsto \frac{\delta f}{\delta m}(x,\mu)(v), \ \frac{\delta g}{\delta m}(x,\mu)(v)$$

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$$\partial_t z(t,x) + \tfrac{1}{2}\Delta z(t,x) - \partial_x u(t,x)\cdot\partial_x z(t,x) + \underbrace{\frac{\delta f}{\delta m}(x,\mu_t)(\cdot)\cdot m_t(\cdot)}_{\text{balance reg in } v \text{ / singularity } m} = 0$$

Initialization of the linearized system

- $\boxed{\text{Assume}}\ \dfrac{\delta f}{\delta m}\ \dfrac{\delta g}{\delta m}\ \mathcal{C}^{n+2+\alpha}$ in (x, v), $\quad n \geq 0, \alpha \in (0, 1)$
- Fix initial condition of linearized system $m_{t^\circ}(\cdot) \in \mathcal{C}^{-(n+1+\alpha)}(\mathbb{T}^d)$
  - $\rightsquigarrow$ $\exists!$ solution to linearized system with

$$(z(t, \cdot), m_t(\cdot))_{t^\circ \leq t \leq T} \in \mathcal{C}([0, T], \mathcal{C}^{n+2+\alpha}(\mathbb{T}^d) \times \mathcal{C}^{-(n+1+\alpha)}(\mathbb{T}^d))$$

  - more than uniqueness $\rightsquigarrow$ stability

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  - more than uniqueness $\rightsquigarrow$ stability
- Example: $m_{t^\circ} = \delta_v \rightsquigarrow z(t^\circ, x) = \mathcal{V}^0(t^\circ, x, \mu^\circ)(v)$
  - if $m_{t^\circ}(\cdot)$ is finite signed measure $\rightsquigarrow$ linearity

$$z(t^\circ, x) = \int_{\mathbb{T}^d} \mathcal{V}^0(t^\circ, x, \mu^\circ)(v)dm_{t^\circ}(v)$$

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- Assume $\frac{\delta f}{\delta m}\ \frac{\delta g}{\delta m}\ C^{n+2+\alpha}$ in (x, v), $\quad n \geq 0,\ \alpha \in (0,1)$
- Fix initial condition of linearized system $m_{t^\circ}(\cdot) \in C^{-(n+1+\alpha)}(\mathbb{T}^d)$
  - $\rightsquigarrow$ ∃! solution to linearized system with

$$(z(t,\cdot), m_t(\cdot))_{t^\circ \leq t \leq T} \in C([0,T], C^{n+2+\alpha}(\mathbb{T}^d) \times C^{-(n+1+\alpha)}(\mathbb{T}^d))$$

  - more than uniqueness $\rightsquigarrow$ stability

- Example: $m_{t^\circ} = (-1)^\ell \frac{d^\ell \delta_v}{dv^\ell} \rightsquigarrow z(t^\circ, x) = \underbrace{\mathcal{V}^\ell(t^\circ, x, \mu^\circ)(v)}_{\partial_v^\ell \mathcal{V}^0(t^\circ, x, \mu^\circ)(v)}$
  - if $m_{t^\circ}(\cdot)$ is finite signed measure $\rightsquigarrow$ linearity

$$z(t^\circ, x) = \int_{\mathbb{T}^d} \mathcal{V}^0(t^\circ, x, \mu^\circ)(v) dm_{t^\circ}(v)$$

  - $\mathcal{V}^0$ is differentiable in v

Part IV. Prospects

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a. Connection with N-particle system

Revisiting the N-player game

- Controlled dynamics $\rightsquigarrow dX_t^i = \alpha_t^i dt + dW_t^i$
- Cost functionals to player i (with $\bar{\mu}_t^N = \frac{1}{N}\sum_{j=1}^N \delta_{X_t^j}$)

$$J^i(\alpha^1, \ldots, \alpha^N) = \mathbb{E}\Big[g(X_T^i, \bar{\mu}_T^N) + \int_0^T \Big(f(X_s^i, \bar{\mu}_s^N) + \frac{1}{2}|\alpha_s^i|^2\Big)ds\Big]$$

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- N player game equilibrium given by Nash system $\rightsquigarrow v^{N,i}$ value function to player i

$$\partial_t v^{N,i}(t,\boldsymbol{x}) + \frac{1}{2}\sum_j \Delta_{x_j} v^{N,i}(t,\boldsymbol{x}) - \sum_{j \neq i} \partial_{x_j} v^{N,j}(t,\boldsymbol{x}) \cdot \partial_{x_j} v^{N,i}(t,\boldsymbol{x})$$
$$- \frac{1}{2}|\partial_{x_i} v^{N,i}(t,\boldsymbol{x})|^2 + f(x_i, \bar{\mu}_{\boldsymbol{x}}^N) = 0$$

  - mean field interaction $\bar{\mu}_{\boldsymbol{x}}^N = \frac{1}{N}\sum_{j=1}^N \delta_{x_j}$

- limit should be the master equation!

Part IV. Prospects

b. Model with common noise

MFG with common noise

(1) fix a random flow of probability measures $(\mu_t)_{0 \leq t \leq T}$ (with values in $\mathcal{P}(\mathbb{R}^d)$) adapted to B (W)

MFG with common noise

(1) fix a random flow of probability measures $(\mu_t)_{0\leq t\leq T}$ (with values in $\mathcal{P}(\mathbb{R}^d)$) adapted to B (W)

(2) postulate some dynamics of a typical rational agent

$$dX_t = \alpha_t dt + dW_t + \eta dB_t$$

MFG with common noise

(1) fix a random flow of probability measures $(\mu_t)_{0\leq t\leq T}$ (with values in $\mathcal{P}(\mathbb{R}^d)$) adapted to B (W)

(2) solve the control problem in the environment $(\mu_t)_{0\leq t\leq T}$

$$dX_t = \alpha_t dt + dW_t + \eta dB_t$$

- with $\boxed{\text{cost}}$ $J(\boldsymbol{\alpha}) = \mathbb{E}\Big[g(X_T,\mu_T) + \int_0^T \Big(f(X_t,\mu_t) + \frac{1}{2}|\alpha_t|^2\Big)dt\Big]$

MFG with common noise

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(3) let $(X_t^{\star,\boldsymbol{\mu}})_{0\leq t\leq T}$ be the optimizer $\rightsquigarrow$ find $(\mu_t)_{0\leq t\leq T}$ such that

$$\mu_t = \mathcal{L}(X_t^{\star,\boldsymbol{\mu}} \,|\, B_s, s\leq t), \quad t\in[0,T]$$

MFG with common noise

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$$\mu_t = \mathcal{L}(X_t^{\star,\mu} \,|\, B_s, s \le t), \quad t \in [0, T]$$

- MFG system becomes stochastic
  - stochastic Fokker Planck equation

$$d_t \mu_t = \{\tfrac{1}{2}(1 + \eta^2)\Delta \mu_t + \mathrm{div}(\mu_t \partial_x u(t, x))\} dt - \eta\, \mathrm{div}(\mu_t dB_t)$$

  - u random value function solution of backward stochastic HJB equation

MFG with common noise

(1) fix a random flow of probability measures $(\mu_t)_{0\leq t\leq T}$ (with values in $\mathcal{P}(\mathbb{R}^d)$) adapted to B (W)

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• Master equation becomes parabolic but degenerate $\rightsquigarrow$ to get non-degeneracy, need infinite dimensional noise

◦ opens the door to uniqueness outside monotonicity under convenient common noise...

◦ ... but almost nothing is known on master equation outside monotonicity without common noise

Part V. Exercises

Part V. Exercises

a. Lions’ derivative

Exercise: Lions' derivative

- Lions' approach to differentiability of $\mathcal{U} : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$
  - Lifting of $\mathcal{U}$

$$\hat{\mathcal{U}} : L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^d) \ni X \mapsto \mathcal{U}(\mathcal{L}(X) = \mathrm{Law}(X))$$

  - $\mathcal{U}$ differentiable if $\hat{\mathcal{U}}$ Fréchet differentiable

- Exercise: Assume that $\partial_v \frac{\delta \mathcal{U}}{\delta m}(\mu)(v)$ is bounded and Lipschitz

Show that $\hat{\mathcal{U}}$ is differentiable and $D\hat{\mathcal{U}}(X) = \partial_v \frac{\delta \mathcal{U}}{\delta m}(\mathcal{L}(X))(X)$

- Remark: General property of Lions' derivative that

$$D\hat{\mathcal{U}}(X) = \partial_\mu \mathcal{U}(\mu)(X), \quad \partial_\mu \mathcal{U}(\mu) : \mathbb{R}^d \ni x \mapsto \partial_\mu \mathcal{U}(\mu)(x) \quad \mu = \mathcal{L}(X)$$

- Hint:

$$\mathcal{U}(\mathcal{L}(Y)) - \mathcal{U}(\mathcal{L}(X))$$

$$= \int_0^1 \int_0^1 \mathbb{E}\Big[\Big\langle \partial_v \frac{\delta \mathcal{U}}{\delta m}(\lambda \mathcal{L}(Y) + (1-\lambda)\mathcal{L}(X))(\lambda' Y + (1-\lambda')X), Y - X \Big\rangle\Big] d\lambda d\lambda'$$

$$
\begin{aligned}
&\mathcal{U}(\mathcal{L}(Y)) - \mathcal{U}(\mathcal{L}(X)) \\
&= \int_0^1 \int_{\mathbb{R}^d} \frac{\delta \mathcal{U}}{\delta m}\Big(\lambda \mathcal{L}(Y) + (1-\lambda)\mathcal{L}(X)\Big)(v) d\Big(\mathcal{L}(Y) - \mathcal{L}(X)\Big)(v) d\lambda \\
&= \int_0^1 \mathbb{E}\Big[\frac{\delta \mathcal{U}}{\delta m}\Big(\lambda \mathcal{L}(Y) + (1-\lambda)\mathcal{L}(X)\Big)(Y) - \frac{\delta \mathcal{U}}{\delta m}\Big(\lambda \mathcal{L}(Y) + (1-\lambda)\mathcal{L}(X)\Big)(X)\Big] d\lambda \\
&= \int_0^1 \int_0^1 \mathbb{E}\Big[\Big\langle \partial_v \frac{\delta \mathcal{U}}{\delta m}(\lambda \mathcal{L}(Y) + (1-\lambda)\mathcal{L}(X))(\lambda' Y + (1-\lambda')X), Y - X\Big\rangle\Big] d\lambda d\lambda' \\
&= \mathbb{E}\Big[\Big\langle \partial_v \frac{\delta \mathcal{U}}{\delta m}(\mathcal{L}(X))(X), Y - X\Big\rangle\Big] \\
&\quad + \int_0^1 \int_0^1 \mathbb{E}\Big[\Big\langle \partial_v \frac{\delta \mathcal{U}}{\delta m}(\lambda \mathcal{L}(Y) + (1-\lambda)\mathcal{L}(X))(\lambda' Y + (1-\lambda')X) \\
&\qquad - \Big\langle \partial_v \frac{\delta \mathcal{U}}{\delta m}(\mathcal{L}(X))(X), Y - X\Big\rangle\Big] d\lambda d\lambda'
\end{aligned}
$$

∘ $W_2(\lambda \mathcal{L}(Y) + (1-\lambda)\mathcal{L}(X), \mathcal{L}(X))^2 \leq \mathbb{E}[|\varepsilon Y + (1-\varepsilon)X - X|^2], \quad \varepsilon \sim$ $\mathrm{Ber}(\lambda) \perp\!\!\!\perp (X, Y)$

Part V. Exercises

b. Potential games

- Consider the mean field optimization problem

$$\inf_{\boldsymbol{\alpha}} \mathcal{J}(\boldsymbol{\alpha}), \quad \mathcal{J}(\boldsymbol{\alpha}) = \int_0^T \Big[\mathcal{F}(\mathcal{L}(X_t)) + \tfrac{1}{2}\mathbb{E}[|\alpha_t|^2]\Big]dt$$

  - with $\mathcal{F}(\mu) = \frac{1}{2}\int_{\mathbb{R}^d} x^2 d\mu(x) + F(\bar{\mu})$
    - $\rightsquigarrow$ where $\bar{\mu} = \int_{\mathbb{R}^d} x d\mu(x)$, $\partial_x F(x) = f(x)$,
- Show for any other process $\boldsymbol{\beta} = (\beta_t)_t$

$$\frac{d\mathcal{J}}{d\varepsilon}(\boldsymbol{\alpha} + \varepsilon\boldsymbol{\beta}) = \mathbb{E}\int_0^T [\beta_t(Y_t + \alpha_t)]dt,$$

  - where $Y_t = \mathbb{E}\Big[\int_t^T [X_s + f(\mathbb{E}(X_s))]ds \,\Big|\, \sigma(W_s, s \leq t) \vee \sigma(X_0)\Big]$
  - deduce a necessary condition for optimizers and show that the same holds true for the MFG defined in the first section provided that the coefficients therein are chosen accordingly

- Write $X_t = X_t^{\boldsymbol{\alpha}}$ and $X_t^{\boldsymbol{\alpha}+\varepsilon\boldsymbol{\beta}} = X_t + \varepsilon \int_0^t \beta_s ds$
- Then

$$
\begin{aligned}
&\frac{d\mathcal{J}}{d\varepsilon}(\boldsymbol{\alpha} + \varepsilon\boldsymbol{\beta}) \\
&= \int_0^T \Big(\mathbb{E}\Big[X_t \int_0^t \beta_s ds\Big] + f(\mathbb{E}(X_t)) \int_0^t \mathbb{E}\beta_s ds + \mathbb{E}\Big[\alpha_t \beta_t\Big]\Big) dt \\
&= \mathbb{E} \int_0^T \Big[\Big(X_t + f(\mathbb{E}(X_t))\Big) \int_0^t \beta_s ds + \alpha_t \beta_t\Big] dt \\
&= \mathbb{E} \int_0^T \Big[\int_t^T \Big(X_s + f(\mathbb{E}(X_s))\Big) ds + \alpha_t\Big] \beta_t dt \\
&= \mathbb{E} \int_0^T \Big[Y_t + \alpha_t\Big] \beta_t dt
\end{aligned}
$$