Exercise Session: Linear Quadratic Mean Field Games

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Overview

1. MFG Paradigm and Approaches
2. LQMFG to FBSDEs
3. Solving Linear FBSDEs of McKean-Vlasov Type

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MFG Paradigm

- Infinite number of symmetric and infinitesimally small players.
- Only interact through some characteristic of the mean field of players (e.g. the mean).
- Since a representative player is insignificant, can freeze the mean field, and find their best response.
- Finally, address the fixed point: the mean field must match the representative player.

Two Approaches

- Analytical approach: formulate solution using coupled forward backward PDEs.
  - Backwards Hamilton Jacobi Bellman equation.
  - Forward Kolmogorov equation.
  - Nash Certainty Equivalence (NCE) equations: Huang, Malhamé, and Caines.
  - Mean field games: Lasry and Lions.
- Probabilistic approach: formulate solution using coupled forward backward SDEs.
  - Carmona and Delarue.

Goal of this session:

- Solve a linear quadratic mean field game explicitly.
- I will use the probabilistic approach (forward backward SDEs).
- The computations are very similar under the analytical approach (forward backward PDEs).
  - ...left as an exercise for the listener.

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LQMFG

Given deterministic $\bar{\boldsymbol{\mu}} = (\bar{\mu}_t)_{t\in[0,T]}$, a representative player has dynamics:

$$\begin{aligned} dX_t &= \left(b_1 X_t + \bar{b}_1 \bar{\mu}_t + b_2 \alpha_t\right) dt + \sigma dW_t, \\ X_0 &= \xi, \end{aligned} \tag{1}$$

where ξ is a random variable with finite mean and variance, and the player considers the following cost when choosing their control, $\boldsymbol{\alpha} = (\alpha_t)_{t\in[0,T]}$:

$$\begin{aligned} J(\boldsymbol{\alpha}) = \mathbb{E}\bigg[\int_0^T \frac{1}{2} \left[q X_t^2 + \bar{q}(X_t - s\bar{\mu}_t)^2 + r\alpha_t^2\right] dt \\ + \frac{1}{2} \left[q_T X_T^2 + \bar{q}_T (X_T - s_T \bar{\mu}_T)^2\right]\bigg]. \end{aligned} \tag{2}$$

Fixed point: $\bar{\mu}_t = \mathbb{E}(X_t)$, $\forall t \in [0, T]$.

Example: Flocking

$$
\begin{aligned}
dX_t &= \alpha_t dt + \sigma dW_t, \\
X_0 &= \xi,
\end{aligned} \tag{3}
$$

$$
J(\boldsymbol{\alpha}) = \mathbb{E}\Big[\int_0^T \frac{1}{2} \left[(X_t - \bar{\mu}_t)^2 + \alpha_t^2 \right] dt. \tag{4}
$$

- X_t is the velocity of a bird in the flock.
- The birds control (up to some noise), their velocity.
- Each birds tries to match the average velocity of the flock, while minimizing their efforts.

Recap of MFG Approach

1. For fixed $\bar{\boldsymbol{\mu}}$, solve optimal control problem to find best control $\boldsymbol{\alpha} = (\alpha_t)_{t\in[0,T]}$.
2. Address fixed point: $\bar{\mu}_t = \mathbb{E}(X_t)$, $\forall t \in [0, T]$.

Probabilistic Approach

Reduced Hamiltonian:

$$H(t,x,\bar{\mu},\alpha,y) = \left[b_1 x + \bar{b}_1\bar{\mu} + b_2\alpha\right] y + \frac{1}{2}\left[q x^2 + \bar{q}(x - s\bar{\mu})^2 + r\alpha^2\right]. \tag{5}$$

Adjoint equation:

$$\begin{aligned} dY_t &= -\partial_x H(t, X_t, \bar{\mu}_t, \alpha_t, Y_t)dt + Z_t dW_t, \\ Y_T &= (q_T + \bar{q}_T)X_T - \bar{q}_T s_T \bar{\mu}_T. \end{aligned} \tag{6}$$

According to the Pontryagin stochastic maximum principle, a sufficient condition for optimality is:

$$\partial_\alpha H(t, X_t, \bar{\mu}_t, \hat{\alpha}_t, Y_t) = 0.$$

Optimal control:

$$\hat{\alpha}_t = -\frac{b_2}{r} Y_t. \tag{7}$$

Replacing $\bar{\mu}_t$ by $\mathbb{E}(X_t)$, we arrive at a linear FBSDE system of McKean-Vlasov type:

$$\begin{aligned} dX_t =& \Big[b_1 X_t + \bar{b}_1 \mathbb{E} X_t - \frac{b_2^2}{r} Y_t \Big] dt + \sigma dW_t \\ X_0 =& \xi, \\ dY_t =& - \left[(q + \bar{q}) X_t - \bar{q} s \mathbb{E} X_t + b_1 Y_t \right] dt + Z_t dW_t, \\ Y_T =& (q_T + \bar{q}_T) X_T - \bar{q}_T s_T \mathbb{E} X_T. \end{aligned} \tag{8}$$

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Let:

$$
\begin{aligned}
&a^{x} := b_1,\ a^{\bar{x}} := \bar{b}_1,\ a^{y} := -\frac{b_2^2}{r},\ a^{\bar{y}} := 0 \\
&b^{x} := -(q+\bar{q}),\ b^{\bar{x}} := \bar{q}s,\ b^{y} := -b_1,\ b^{\bar{y}} := 0 \\
&c^{x} := q_T + \bar{q}_T,\ c^{\bar{x}} := -\bar{q}_T s_T.
\end{aligned}
\tag{9}
$$

General Constant Coefficient Linear FBSDE System of McKean-Vlasov Type

$$
\begin{aligned}
dX_t &= \left(a^x X_t + a^{\bar{x}} \mathbb{E} X_t + a^y Y_t + a^{\bar{y}} \mathbb{E} Y_t\right) dt + \sigma dW_t, \\
X_0 &= \xi, \\
dY_t &= \left(b^x X_t + b^{\bar{x}} \mathbb{E} X_t + b^y Y_t + b^{\bar{y}} \mathbb{E} Y_t\right) dt + Z_t dW_t, \\
Y_T &= c^x X_T + c^{\bar{x}} \mathbb{E} X_T.
\end{aligned}
\tag{10}
$$

By taking expectations in equation (10), and letting $\bar{x}_t$ and $\bar{y}_t$ denote $\mathbb{E}X_t$ and $\mathbb{E}Y_t$, respectively, we get:

$$
\begin{aligned}
\dot{\bar{x}}_t &= (a^x + a^{\bar{x}})\bar{x}_t + (a^y + a^{\bar{y}})\bar{y}_t, \\
\bar{x}_0 &= \mathbb{E}(\xi), \\
\dot{\bar{y}}_t &= (b^x + b^{\bar{x}})\bar{x}_t + (b^y + b^{\bar{y}})\bar{y}_t, \\
\bar{y}_T &= (c^x + c^{\bar{x}})\bar{x}_T.
\end{aligned}
\tag{11}
$$

Make the ansatz $\bar{y}_t = \bar{\eta}_t \bar{x}_t + \bar{\chi}_t$ for deterministic functions $[0, T] \ni t \mapsto \bar{\eta}_t \in \mathbb{R}$ and $[0, T] \ni t \mapsto \bar{\chi}_t \in \mathbb{R}$. Then we have:

$$\dot{\bar{y}}_t = (b^x + b^{\bar{x}} + (b^y + b^{\bar{y}})\bar{\eta}_t)\bar{x}_t + (b^y + b^{\bar{y}})\bar{\chi}_t, \tag{12}$$

and $\bar{y}_T = (c^x + c^{\bar{x}})\bar{x}_T$, while we also have:

$$\begin{aligned}
\dot{\bar{y}}_t &= \bar{\eta}_t \dot{\bar{x}}_t + \dot{\bar{\eta}}_t \bar{x}_t + \dot{\bar{\chi}}_t \\
&= \bar{\eta}_t \left[(a^x + a^{\bar{x}})\bar{x}_t + (a^y + a^{\bar{y}})\bar{y}_t \right] + \dot{\bar{\eta}}_t \bar{x}_t + \dot{\bar{\chi}}_t, \\
&= \bar{\eta}_t \left[(a^x + a^{\bar{x}})\bar{x}_t + (a^y + a^{\bar{y}})(\bar{\eta}_t \bar{x}_t + \bar{\chi}_t) \right] + \dot{\bar{\eta}}_t \bar{x}_t + \dot{\bar{\chi}}_t, \\
&= \left[\dot{\bar{\eta}}_t + (a^y + a^{\bar{y}})\bar{\eta}_t^2 + (a^x + a^{\bar{x}})\bar{\eta}_t \right] \bar{x}_t + \left[\dot{\bar{\chi}}_t + \bar{\eta}_t (a^y + a^{\bar{y}})\bar{\chi}_t \right],
\end{aligned} \tag{13}$$

and $\bar{y}_T = \bar{\eta}_T \bar{x}_T + \bar{\chi}_T$.

Assuming the ansatz, the system in equation (11) is equivalent to the ODE system:

$$
\begin{aligned}
\dot{\bar{\eta}}_t + (a^y + a^{\bar{y}})\bar{\eta}_t^2 + (a^x + a^{\bar{x}} - b^y - b^{\bar{y}})\bar{\eta}_t - b^x - b^{\bar{x}} &= 0, \\
\bar{\eta}_T &= c^x + c^{\bar{x}}, \\
\dot{\bar{\chi}}_t + (\bar{\eta}_t(a^y + a^{\bar{y}}) - b^y - b^{\bar{y}})\bar{\chi}_t &= 0, \\
\bar{\chi}_T &= 0.
\end{aligned}
\tag{14}
$$

Note that $\bar{\chi}_.$ solves a first order homogeneous linear equation. Thus $\bar{\chi}_t = 0$, $\forall t \in [0, T]$. $\bar{\eta}_.$ solves a Riccati equation.

Given $\bar{\eta}_\cdot$, we have:

$$\dot{\bar{x}}_t = (a^x + a^{\bar{x}} + (a^y + a^{\bar{y}})\bar{\eta}_t)\bar{x}_t, \quad \bar{x}_0 = \mathbb{E}(\xi), \tag{15}$$

and thus,

$$\bar{x}_t = \mathbb{E}(\xi)e^{\int_0^t (a^x + a^{\bar{x}} + (a^y + a^{\bar{y}})\bar{\eta}_u)du}. \tag{16}$$

Once we have computed $(\bar{x}_t)_{0\le t\le T}$, we can rewrite the original FBSDE system:

$$\begin{aligned} dX_t &= \left(a^x X_t + a^y Y_t + a_t^0\right) dt + \sigma dW_t, \quad X_0 = \xi, \\ dY_t &= \left(b^x X_t + b^y Y_t + b_t^0\right) dt + Z_t dW_t, \quad Y_T = c^x X_T + c^0, \end{aligned} \tag{17}$$

with:

$$a_t^0 = (a^{\bar{x}} + a^{\bar{y}}\bar{\eta}_t)\bar{x}_t, \quad b_t^0 = (b^{\bar{x}} + b^{\bar{y}}\bar{\eta}_t)\bar{x}_t, \quad c^0 = c^{\bar{x}}\bar{x}_T. \tag{18}$$

Now we make the ansatz: $Y_t = \eta_t X_t + \chi_t$, which reduces the problem to the ODE system:

$$
\begin{aligned}
\dot{\eta}_t + a^y \eta_t^2 + (a^x - b^y)\eta_t - b^x = 0, \quad \eta_T = c^x, \\
\dot{\chi}_t + (-b^y + a^y \eta_t)\chi_t + a_t^0 \eta_t - b_t^0 = 0, \quad \chi_T = c^0.
\end{aligned}
\tag{19}
$$

Note that it is not necessary to solve for $\chi_\cdot$ because of the relationship:

$$\bar{\eta}_t \bar{x}_t = \bar{y}_t = \mathbb{E}(Y_t) = \mathbb{E}(\eta_t X_t + \chi_t) = \eta_t \bar{x}_t + \chi_t. \tag{20}$$

Thus, $\chi_t = (\bar{\eta}_t - \eta_t)\bar{x}_t$. All that remains is to solve the Riccati equations for $\bar{\eta}_\cdot$ and $\eta_\cdot$.

Scalar Riccati Equations

If the scalar Riccati equation:

$$\dot{h}_t - Bh_t^2 - 2Ah_t + C = 0, \tag{21}$$

with terminal condition $h_T = D$ satisfies:

$$B \neq 0,\ \ BD \geq 0,\ \ BC > 0, \tag{22}$$

then it has a unique solution:

$$h_t = \frac{C(1 - e^{-(\delta^+ - \delta^-)(T-t)}) + D(\delta^+ - \delta^- e^{-(\delta^+ - \delta^-)(T-t)})}{BD(1 - e^{-(\delta^+ - \delta^-)(T-t)}) + \delta^+ e^{-(\delta^+ - \delta^-)(T-t)} - \delta^-}, \tag{23}$$

with $\delta^{\pm} = -A \pm \sqrt{(A)^2 + BC}$.

Summary

- By making an ansatz, we can reduce LQMFGs to the solution of Riccati equations and linear ODEs.

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Questions?