Preliminary Lecture Notes

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In this exercise session, we will derive the solution to a linear quadratic mean field game. The following preliminary notes were taken from Carmona, G., Tan [2]. In these notes, we use the probabilistic approach to MFG, and thus we can formulate the MFG equilibrium as the solution to a forward-backward SDE (FBSDE) system of McKean-Vlasov type. We can solve this system explicitly by making an ansatz which allows us to reduce the problem to ODEs. If we instead use the analytic approach (which results in a forward-backward PDE system), a similar ansatz can be made, resulting in the exact same ODEs. For further reading, see the book of Carmona and Delaure Section 3.5 [1].

1 N-Player Formulation

We consider a system of N players whose private states are denoted at time t by X_t^1, X_t^2, $\cdots$, X_t^N. To keep the presentations simple, we assume the state space is $\mathbb{R}$. We denote by μ_t^N the empirical distribution of the states, namely:

$$\mu_t^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{X_t^i}.$$

We assume that these states evolve in continuous time under the influences of controls α_t^1, α_t^2, $\cdots$, $\alpha_t^N \in \mathbb{A}$, where the set of admissible controls, $\mathbb{A}$, will be defined later. Let ν_t^N denote the empirical measure of the controls:

$$\nu_t^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{\alpha_t^i}.$$

We also assume that if and when interactions between these states and controls are present, they are of a mean field type, i.e. through μ_t^N and ν_t^N. The time evolution of the state for player i is given by the Itô dynamics:

$$dX_t^i = b(t, X_t^i, \mu_t^N, \alpha_t^i, \nu_t^N)dt + \sigma dW_t.$$

We work over the interval $[0, T]$ limited by a finite time horizon $T \in \mathbb{R}^+$. We assume the drift function $b : [0, T] \times \mathbb{R} \times \mathcal{P}(\mathbb{R}) \times \mathbb{A} \times \mathcal{P}(\mathbb{A}) \ni (t, x, \mu, \alpha, \nu) \to \mathbb{R}$ is Lipschitz in each of its inputs. For the sake of simplicity, we assume that the volatility, σ, is a positive constant.

1.1 Cost Functionals

We assume that we are given two functions $f : [0,T] \times \mathbb{R} \times \mathcal{P}(\mathbb{R}) \times \mathbb{A} \times \mathcal{P}(\mathbb{A}) \ni (t, x, \mu, \alpha, \nu) \to \mathbb{R}$ and $g : \mathbb{R} \times \mathcal{P}(\mathbb{R}) \ni (x, \mu) \to \mathbb{R}$ which we call running and terminal cost functions, respectively. We assume f and g are Lipschitz in each of their arguments. The goal of player i is to minimize their expected cost as given by:

$$J_i(\boldsymbol{\alpha}^1, \cdots, \boldsymbol{\alpha}^N) = \mathbb{E}\left[\int_0^T f(t, X_t^i, \mu_t^N, \alpha_t^i, \nu_t^N)\, dt + g(X_T^i, \mu_T^N)\right].$$

We restrict ourselves to Markovian control strategies $\boldsymbol{\alpha} = (\alpha_t)_{0 \le t \le T}$ given by feedback functions in the form $\alpha_t = \phi(t, X_t)$ and we let $\mathbb{A}$ denote the set of such controls.

2 Mean Field Game Formulation

The goal of this section is to articulate what is meant by a feedback function providing a mean field game equilibrium. To begin, we define what we call the *mean field environment*. By symmetry of the players, we suppose all of the players in the mean field game use the same feedback function, ϕ. Then the *mean field environment* specified by ϕ is characterized by $\mathcal{L}(X_t^\phi)_{0 \le t \le T}$ and $\mathcal{L}(\phi(t, X_t^\phi))_{0 \le t \le T}$ where the dynamics of $(X_t^\phi)_{0 \le t \le T}$ are given by:

$$dX_t^\phi = b(t, X_t^\phi, \mathcal{L}(X_t^\phi), \phi(t, X_t^\phi), \mathcal{L}(\phi(t, X_t^\phi))dt + \sigma dW_t. \tag{1}$$

Since we search for a Nash equilibrium, we consider a representative agent who wishes to find their best response, ϕ', to the mean field environment specified by ϕ, in which case their state is given by $\boldsymbol{X^{\phi',\phi}} = (X_t^{\phi',\phi})_{0 \le t \le T}$ solving the standard stochastic differential equation:

$$dX_t^{\phi',\phi} = b(t, X_t^{\phi',\phi}, \mathcal{L}(X_t^\phi), \phi'(t, X_t^{\phi',\phi}), \mathcal{L}(\phi(t, X_t^\phi)))dt + \sigma dW_t.$$

Consider the function:

$$J^\phi(\phi') = \left[\int_0^T < f(t, \cdot, \mathcal{L}(X_t^\phi), \phi'(t, \cdot), \mathcal{L}(\phi(t, X_t^\phi)), \mathcal{L}(X_t^{\phi',\phi})) > dt \right.$$
$$\left. + < g(\cdot, \mathcal{L}(X_T^\phi)), \mathcal{L}(X_t^{\phi',\phi}) > \right].$$

The best response for the representative agent in the mean field environment specified by ϕ is the feedback function minimizing this cost, namely $\phi^* = \arg\inf_{\phi'} J^\phi(\phi')$. Assuming the minimizer is unique (which will be the case for the models we consider), this defines a mapping $\Phi : \phi \to \phi^*$. If there is a $\hat{\phi}$ such that $\Phi(\hat{\phi}) = \hat{\phi}$, then the players are in a mean field game equilibrium.

Thus, the search for a feedback function providing a mean field game equilibrium can be summarized as the following set of two successive steps:

1. For each feedback function $\phi : [0,T] \times \mathbb{R} \ni (t,x) \to \mathbb{R}$, solve the optimal control problem
$$\phi^* = \arg\inf_{\phi'} J^{\phi}(\phi').$$
Define the mapping $\Phi(\phi) := \phi^*$.

2. Find a fixed point $\hat{\phi}$ of Φ such that $\Phi(\hat{\phi}) = \hat{\phi}$.

When these two steps can be taken successfully, we say that $\hat{\phi}$ provides a mean field game equilibrium. Note that $\boldsymbol{X}^{\hat{\phi},\hat{\phi}} = \boldsymbol{X}^{\hat{\phi}}$.

3 Linear Quadratic Extended Mean Field Games

The class of linear quadratic extended mean field games is a class of problems for which explicit solutions can be computed analytically, and thus, we can compute the price of anarchy explicitly.

We only need to specify the drift and cost functions, b, f, and g introduced in the previous sections. For linear quadratic models, we take the drift to be linear:
$$b(t,x,\mu,\alpha,\nu) = b_1(t)x + \bar{b}_1(t)\bar{\mu} + b_2(t)\alpha + \bar{b}_2(t)\bar{\nu},$$
where $\bar{\mu}$ denotes the mean of the measure μ, namely, $\bar{\mu} = \int_{\mathbb{R}} x d\mu(x)$, and similarly for $\bar{\nu}$. We take the running and terminal costs to be quadratic:
$$f(t,x,\mu,\alpha,\nu) = \frac{1}{2}\left[q(t)x^2 + \bar{q}(t)(x - s(t)\bar{\mu})^2 + r(t)\alpha^2 + \bar{r}(t)(\alpha - \bar{s}(t)\bar{\nu})^2\right],$$
$$g(x,\mu) = \frac{1}{2}\left[q_T x^2 + \bar{q}_T(x - s_T\bar{\mu})^2\right].$$

Remark 3.1. *If $\bar{b}_2(t) \equiv 0$ and $\bar{r}(t) \equiv 0$, then we have the standard mean field game.*

To solve the linear quadratic extended mean field game (LQEMFG), we begin by considering the reduced Hamiltonian for this problem:
$$H(t,x,\bar{\mu},\alpha,\bar{\nu},y) = \left[b_1(t)x + \bar{b}_1(t)\bar{\mu} + b_2(t)\alpha + \bar{b}_2(t)\bar{\nu}\right] y + \frac{1}{2}\left[q(t)x^2 + \bar{q}(t)(x - s(t)\bar{\mu})^2 + r(t)\alpha^2 + \bar{r}(t)(\alpha - \bar{s}(t)\bar{\nu})^2\right],$$
and whenever the flows $\bar{\boldsymbol{\mu}} = (\bar{\mu}_t)_{0\le t\le T}$ and $\bar{\boldsymbol{\nu}} = (\bar{\nu}_t)_{0\le t\le T}$ are fixed, we consider for each control process $\boldsymbol{\alpha} = (\alpha_t)_{0\le t\le T}$ the adjoint equation:
$$dY_t = -\partial_x H(t, X_t, \bar{\mu}_t, \alpha_t, \bar{\nu}_t, Y_t)dt + Z_t dW_t, \qquad Y_T = \partial_x g(X_T, \mathcal{L}(X_T)).$$
According to the Pontryagin stochastic maximum principle, a sufficient condition for optimality is:
$$\partial_\alpha H(t, X_t, \bar{\mu}_t, \hat{\alpha}_t, \bar{\nu}_t, Y_t) = 0.$$

Thus, we find the optimal control:

$$\hat{\alpha}_t = \frac{\bar{r}(t)\bar{s}(t)\bar{\nu} - b_2(t)Y_t}{r(t) + \bar{r}(t)}. \tag{2}$$

When solving the fixed point step, we identify $\bar{\nu}_t = \mathbb{E}(\hat{\alpha}_t)$. By taking the expectation, we find:

$$\bar{\nu}_t = \mathbb{E}(\hat{\alpha}_t) = c^{MFG}(t)\mathbb{E}(Y_t), \qquad \text{with} \quad c^{MFG}(t) = -\frac{b_2(t)}{r(t) + \bar{r}(t)(1 - \bar{s}(t))}.$$

Thus, from equation (2) we have:

$$\hat{\alpha}_t = a^{MFG}(t)Y_t + b^{MFG}(t)\mathbb{E}(Y_t), \tag{3}$$

with:

$$a^{MFG}(t) = -\frac{b_2(t)}{r(t) + \bar{r}(t)}, \quad \text{and} \quad b^{MFG}(t) = -\frac{\bar{r}(t)\bar{s}(t)b_2(t)}{(r(t) + \bar{r}(t))(r(t) + \bar{r}(t)(1 - \bar{s}(t)))}.$$

Note that $c^{MFG}(t) = a^{MFG}(t) + b^{MFG}(t)$. The solution of the mean field game equilibrium problem is given by the solution to the FBSDE system:

$$\begin{aligned} dX_t =& \Big[b_1(t)X_t + \bar{b}_1(t)\mathbb{E}X_t + a^{MFG}(t)b_2(t)Y_t \\ &+ (b^{MFG}(t)b_2(t) + c^{MFG}(t)\bar{b}_2(t))\mathbb{E}Y_t\Big]dt + \sigma dW_t \\ dY_t =& -\left[(q(t) + \bar{q}(t))X_t - \bar{q}(t)s(t)\mathbb{E}X_t + b_1(t)Y_t\right]dt + Z_t dW_t, \end{aligned} \tag{4}$$

with initial condition $X_0 = \xi$, a random variable with finite mean and variance, and terminal condition $Y_T = (q_T + \bar{q}_T)X_T - \bar{q}_T s_T \mathbb{E}X_T$.

This is a linear FBSDE of McKean-Vlasov type. Let $\bar{\eta}_t^{MFG}$, η_t^{MFG}, $\bar{x}_t^{MFG}$, and v_t^{MFG} denote the solutions for this problem as described in Appendix 4.1 so that:

$$\begin{aligned} Y_t = \eta_t^{MFG}X_t + (\bar{\eta}_t^{MFG} - \eta_t^{MFG})\bar{x}_t^{MFG}, \quad & \mathbb{E}(Y_t) = \bar{\eta}_t^{MFG}\bar{x}_t^{MFG}, \\ \mathbb{E}(X_t) = \bar{x}_t^{MFG}, \quad & Var(X_t) = v^{MFG}, \end{aligned}$$

provide a solution to the LQEMFG problem. Then from Appendix 4.1, we have:

- a scalar Riccati equation for $\bar{\eta}_t^{MFG}$:

$$\begin{aligned} \dot{\bar{\eta}}_t^{MFG} &+ \left[c^{MFG}(t)(b_2(t) + \bar{b}_2(t))\right] \cdot (\bar{\eta}_t^{MFG})^2 + \left(2b_1(t) + \bar{b}_1(t)\right) \cdot \bar{\eta}_t^{MFG} \\ &+ q(t) + \bar{q}(t)(1 - s(t)) = 0, \end{aligned} \tag{5}$$

  with terminal condition $\bar{\eta}_T^{MFG} = q_T + \bar{q}_T(1 - s_T)$,

- a linear first order ODE for $\bar{x}_t^{MFG}$:

$$\dot{\bar{x}}_t^{MFG} = \left[b_1(t) + \bar{b}_1(t) + c^{MFG}(t)(b_2(t) + \bar{b}_2(t)) \cdot \bar{\eta}_t^{MFG}\right] \cdot \bar{x}_t^{MFG}, \tag{6}$$

  with initial condition $\bar{x}_0^{MFG} = \mathbb{E}(\xi)$,

- a scalar Riccati equation for η_t^{MFG}:

  $$\dot{\eta}_t^{MFG} + a^{MFG}(t)b_2(t)\cdot(\eta_t^{MFG})^2 + 2b_1(t)\cdot\eta_t^{MFG} + q(t) + \bar{q}(t) = 0,$$

  with terminal condition $\eta_T^{MFG} = q_T + \bar{q}_T$,

and where the dot is the standard ODE notation for a derivative. And thus, we obtain explicit solutions for $\bar{x}_t^{MFG}$ and v_t^{MFG}:

$$\bar{x}_t^{MFG} = \mathbb{E}(\xi)e^{\int_0^t\left(b_1(s)+\bar{b}_1(s)+\left[c^{MFG}(s)(b_2(s)+\bar{b}_2(s))\right]\cdot\bar{\eta}_s^{MFG}\right)ds}, \tag{7}$$

$$\begin{aligned} v_t^{MFG} &= Var(\xi)e^{\int_0^t 2\left[b_1(s)+a^{MFG}(s)b_2(s)\cdot\eta_s^{MFG}\right]ds} \\ &\quad+\sigma^2\int_0^t e^{2\int_s^t\left[b_1(u)+a^{MFG}(u)b_2(u)\eta_u^{MFG}\right]du}ds. \end{aligned} \tag{8}$$

From equation (3), we have $\hat{\alpha}_t = \hat{\phi}(t, X_t)$ with:

$$\hat{\phi}(t,x) = a^{MFG}(t)\eta_t^{MFG}x + \left[a^{MFG}(t)(\bar{\eta}_t^{MFG} - \eta_t^{MFG}) + b^{MFG}(t)\bar{\eta}_t^{MFG}\right]\bar{x}_t^{MFG}.$$

Note that the Riccati equations for $\bar{\eta}_t^{MFG}$ and η_t^{MFG} can be solved explicitly under mild assumptions, or at least in the case of time-independent coefficients. See Appendix 4.2.

4 Appendix

4.1 Solving Linear FBSDEs of McKean-Vlasov Type

Consider a linear FBSDE system of McKean-Vlasov type:

$$\begin{aligned} dX_t &= \left(a_t^x X_t + a_t^{\bar{x}}\mathbb{E}X_t + a_t^y Y_t + a_t^{\bar{y}}\mathbb{E}Y_t\right)dt + \sigma dW_t, \quad X_0 = \xi, \\ dY_t &= \left(b_t^x X_t + b_t^{\bar{x}}\mathbb{E}X_t + b_t^y Y_t + b_t^{\bar{y}}\mathbb{E}Y_t\right)dt + Z_t dW_t, \quad Y_T = c^x X_T + c^{\bar{x}}\mathbb{E}X_T. \end{aligned} \tag{9}$$

For the LQEMFG model considered in Section 3, the FBSDE system in equation (4) is of the form of equation (9) if we set:

$$\begin{aligned} &a_t^x = b_1(t),\ a_t^{\bar{x}} = \bar{b}_1(t),\ a_t^y = a^{MFG}(t)b_2(t),\ a_t^{\bar{y}} = b^{MFG}(t)b_2(t) + c^{MFG}(t)\bar{b}_2(t) \\ &b_t^x = -(q(t)+\bar{q}(t)),\ b_t^{\bar{x}} = \bar{q}(t)s(t),\ b_t^y = -b_1(t),\ b_t^{\bar{y}} = 0 \\ &c_t^x = q_T + \bar{q}_T,\ c_t^{\bar{x}} = -\bar{q}_T s_T. \end{aligned}$$

Now we return to the general FBSDE system (9). By taking expectations in equation (9), and letting $\bar{x}_t$ and $\bar{y}_t$ denote $\mathbb{E}X_t$ and $\mathbb{E}Y_t$, respectively, we get:

$$\begin{aligned} \dot{\bar{x}}_t &= (a_t^x + a_t^{\bar{x}})\bar{x}_t + (a_t^y + a_t^{\bar{y}})\bar{y}_t, \quad \bar{x}_0 = \mathbb{E}(\xi), \\ \dot{\bar{y}}_t &= (b_t^x + b_t^{\bar{x}})\bar{x}_t + (b_t^y + b_t^{\bar{y}})\bar{y}_t, \quad \bar{y}_T = (c^x + c^{\bar{x}})\bar{x}_T, \end{aligned} \tag{10}$$

where the dot is the standard ODE notation for a derivative. We then make the ansatz $\bar{y}_t = \bar{\eta}_t\bar{x}_t + \bar{\chi}_t$ for deterministic functions $[0,T] \ni t \mapsto \bar{\eta}_t \in \mathbb{R}$ and

$[0,T] \ni t \mapsto \bar{\chi}_t \in \mathbb{R}$. By plugging in the ansatz, the system in equation (10) is equivalent to the ODE system:

$$\dot{\bar{\eta}}_t + (a_t^y + a_t^{\bar{y}})\bar{\eta}_t^2 + (a_t^x + a_t^{\bar{x}} - b_t^y - b_t^{\bar{y}})\bar{\eta}_t - b_t^x - b_t^{\bar{x}} = 0, \quad \bar{\eta}_T = c^x + c^{\bar{x}},$$
$$\dot{\bar{\chi}}_t + (\bar{\eta}_t(a_t^y + a_t^{\bar{y}}) - b_t^y - b_t^{\bar{y}})\bar{\chi}_t = 0, \quad \bar{\chi}_T = 0.$$

The first equation is a Riccati equation. Note that $\bar{\chi}_t$ solves a first order homogeneous linear equation. Thus $\bar{\chi}_t = 0$, $\forall t \in [0,T]$. Once the equation for $\bar{\eta}_t$ is solved, we can compute $\bar{x}_t$ by solving the linear ODE:

$$\dot{\bar{x}}_t = (a_t^x + a_t^{\bar{x}} + (a_t^y + a_t^{\bar{y}})\bar{\eta}_t)\bar{x}_t, \quad \bar{x}_0 = \mathbb{E}(\xi),$$

and thus,

$$\bar{x}_t = \mathbb{E}(\xi) e^{\int_0^t (a_u^x + a_u^{\bar{x}} + (a_u^y + a_u^{\bar{y}})\bar{\eta}_u) du}.$$

Once we have computed $(\bar{x}_t)_{0 \le t \le T}$, we can rewrite the original FBSDE system:

$$dX_t = \left(a_t^x X_t + a_t^y Y_t + a_t^0\right) dt + \sigma dW_t, \quad X_0 = \xi,$$
$$dY_t = \left(b_t^x X_t + b_t^y Y_t + b_t^0\right) dt + Z_t dW_t, \quad Y_T = c^x X_T + c^0,$$

with:

$$a_t^0 = (a_t^{\bar{x}} + a_t^{\bar{y}}\bar{\eta}_t)\bar{x}_t, \quad b_t^0 = (b_t^{\bar{x}} + b_t^{\bar{y}}\bar{\eta}_t)\bar{x}_t, \quad c^0 = c^{\bar{x}}\bar{x}_T.$$

Now we make the ansatz: $Y_t = \eta_t X_t + \chi_t$, which reduces the problem to the ODE system:

$$\dot{\eta}_t + a_t^y \eta_t^2 + (a_t^x - b_t^y)\eta_t - b_t^x = 0, \quad \eta_T = c^x,$$
$$\dot{\chi}_t + (-b_t^y + a_t^y \eta_t)\chi_t + a_t^0 \eta_t - b_t^0 = 0, \quad \chi_T = c^0,$$
$$Z_t = \sigma \eta_t.$$

Again, the first equation is a Riccati equation. Note that it is not necessary to solve for χ_t because of the relationship:

$$\bar{\eta}_t \bar{x}_t = \bar{y}_t = \mathbb{E}(Y_t) = \mathbb{E}(\eta_t X_t + \chi_t) = \eta_t \bar{x}_t + \chi_t.$$

Thus, $\chi_t = (\bar{\eta}_t - \eta_t)\bar{x}_t$.

In summary, the solution to the linear FBSDE of McKean-Vlasov type is reduced to solving linear ODEs and Riccati equations. It will also be useful to compute $Var(X_t)$, which we denote by v_t. After we have solved the above equations, we have:

$$dX_t = \left((a_t^x + a_t^y \eta_t)X_t + a_t^y \chi_t + a_t^0\right) dt + \sigma dW_t, \quad X_0 = \xi.$$

Thus,

$$v_t = Var(X_t) = Var(\xi) e^{\int_0^t 2(a_s^x + a_s^y \eta_s) ds} + \sigma^2 \int_0^t e^{2\int_s^t (a_u^x + a_u^y \eta_u) du} ds.$$

In the case where the coefficients are time-independent, the Riccati equations for $\bar{\eta}_t$ and η_t can be solved explicitly.

4.2 Scalar Riccati Equation

If the scalar Riccati equation:

$$\dot{h}_t - Bh_t^2 - 2Ah_t + C = 0,$$

with terminal condition $h_T = D$ satisfies:

$$B \neq 0,\ BD \geq 0,\ BC > 0, \tag{11}$$

then it has a unique solution:

$$h_t = \frac{C(1 - e^{-(\delta^+ - \delta^-)(T-t)}) + D(\delta^+ - \delta^- e^{-(\delta^+ - \delta^-)(T-t)})}{BD(1 - e^{-(\delta^+ - \delta^-)(T-t)}) + \delta^+ e^{-(\delta^+ - \delta^-)(T-t)} - \delta^-}, \tag{12}$$

with $\delta^{\pm} = -A \pm \sqrt{(A)^2 + BC}$.

Furthermore, if $B \to 0$ and $A \neq 0$, we can deduce that the limiting solution of the scalar Riccati equation coincides with the linear first-order differential equation:

$$\dot{h}_t - 2Ah_t + C = 0,$$

with terminal condition $h_T = D$, namely:

$$h_t = \left(D - \frac{C}{2A}\right) e^{-2A(T-t)} + \frac{C}{2A}.$$

If $B \to 0$ and $A = 0$, the limiting solution of the scalar Riccati equation coincides with the linear first-order differential equation:

$$\dot{h}_t + C = 0$$

with terminal condition $h_T = D$, namely:

$$h_t = D + C(T - t).$$

Hence, returning to the linear FBSDE (9), for $\bar{\eta}_t$, we use:

$$A = -\tfrac{1}{2}(a^x + a^{\bar{x}} - b^y - b^{\bar{y}}), \quad ,B = -(a^y + a^{\bar{y}}), \quad C = -(b^x + b^{\bar{x}}), \quad D = c^x + c^{\bar{x}}.$$

The conditions (11) are satisfied if $-(a^y + a^{\bar{y}}) > 0$, $-(b^x + b^{\bar{x}}) > 0$, and $c^x + c^{\bar{x}} \geq 0$.

For η_t, we use:

$$A = -\tfrac{1}{2}(a^x - b^y), \quad B = -a^y, \quad C = -b^x, \quad D = c^x.$$

The conditions (11) are satisfied if $-a^y > 0$, $-b^x > 0$, and $c^x \geq 0$.

References

[1] René Carmona and François Delarue. *Probabilistic Theory of Mean Field Games with Applications I*. Springer, 2018.

[2] René Carmona, Christy V Graves, and Zongjun Tan. Price of anarchy for mean field games. *ESAIM: Proceedings and Surveys*, 65:349–383, 2019.