**ASYMPTOTIC EXPANSION OF EIGENELEMENTS
OF THE LAPLACE OPERATOR IN A DOMAIN
WITH A LARGE NUMBER OF ‘LIGHT’ CONCENTRATED MASSES
SPARSELY SITUATED ON THE BOUNDARY.
TWO-DIMENSIONAL CASE**

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ABSTRACT. This paper looks at eigenoscillations of a membrane containing a large number of concentrated masses on the boundary. The asymptotic behaviour of the frequencies of eigenoscillations is studied when a small parameter characterizing the diameter and density of the concentrated masses tends to zero. Asymptotic expansions of eigenelements of the corresponding problems are constructed and the expansions are accurately substantiated. The case where the diameter of the masses is much smaller than the distance between them is investigated under the assumption that the limit boundary condition is still a Dirichlet condition.

INTRODUCTION

The behaviour of bodies with singular density has attracted the attention of scientists for many years. They have studied the influence of concentrated masses (singular lumps) on the variation of the frequencies of eigenoscillations of such bodies. This question proved to be rather too difficult for the mathematics available at the end of the 19th century. We must give due credit to a pioneering paper from the beginning of the 20th century [1]; its author investigated the problem of the oscillation of a string loaded with concentrated masses. This paper was far ahead of its time and was forgotten for some years. It was only after asymptotic methods appeared that the interest of researchers returned to problems with concentrated masses and it became possible to investigate problems with singular density adequately.

The author of [2] considered the problem for the Laplace operator with Dirichlet boundary conditions in the three-dimensional case where the mass attached to the system is concentrated in an ε -neighbourhood of an interior point, ε being a small parameter describing the concentration and size of the mass. In that paper the methods of spectral perturbation theory were used.

Another approach was proposed in [3, 4, 5, 6, 7]. As has already frequently been pointed out, a new basic parameter for oscillatory systems with locally attached masses was introduced in these papers, namely, the ratio of the attached mass to the mass of the whole system. This approach has long been firmly established in research papers. It turns out that the introduction of this parameter made it possible not only to analyse the

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limit (effective) behaviour of such problems, but also to construct complete asymptotic series for eigenelements, accurately substantiating them.

Dimensional analysis in the problem of the spectral properties of oscillatory systems with attached masses was first carried out in [8]. The case of a single concentrated mass for a one-dimensional operator with a Dirichlet boundary condition was studied in [9]. In the case of finitely many concentrated masses, this work was done in [10].

Later, various problems were considered for arbitrary domains, for thin plates and rods, for a variety differential operators and boundary conditions, for ‘light’, ‘heavy’ and ‘critical density’ masses. The problem of a rod and of a plate with concentrated masses were studied in [11, 12] (see also [13, 14, 15]). The case of a free boundary (Neumann conditions) was analysed in [16] and [17]. In [18], the behaviour of eigenelements of the Laplace operator with a Dirichlet condition and a Fourier condition (a condition of the third kind) on the boundary was studied in the case where the density is perturbed by finitely many concentrated masses; see also [19]. The question of constructing asymptotic expansions of eigenvalues and eigenfunctions of the Dirichlet problem for the Laplace operator was investigated in [20] and [21].

The case of general position for a 3-dimensional linear stationary system in the theory of elasticity was considered in [22] (see also [23]). For the oscillation of a membrane, see [24]. Similar problems were considered in [33, 34, 35]. In [45], the problem for a linear stationary system in the theory of elasticity, in domains with concentrated masses, was considered under the assumption that the oscillation laws for the body and the masses are different (the inclusions are stiffer). We also draw the reader’s attention to the papers [36, 37, 38, 39, 40, 41, 42, 43], where the cases of second-order and fourth-order operators with various concentrated masses were considered.

The asymptotic behaviour of the oscillations of a body that has many small inclusions of high density situated periodically along the boundary was investigated in [25, 26, 27, 28, 29, 30, 31, 32]. Questions about the limit (homogenized) behaviour of eigenvalues were studied. The boundary-value problem for a stationary system in the linear theory of elasticity with nonperiodic rapidly varying boundary conditions and a large number of concentrated masses near the boundary was investigated in [44]; its asymptotic behaviour was considered, as well as the limit behaviour of the spectrum of this boundary-value problem.

The case of a boundary-value problem for a system in the theory of elasticity where the limit problem has a boundary condition of the third kind on the boundary of the domain and the masses are light was examined. Estimates were obtained for the rate of convergence of a solution of the original problem to a solution of the homogenized one, as well as for the rate of convergence of eigenelements of this type of boundary-value problem.

The paper [46] is devoted to a detailed study of the behaviour of eigenelements of the Laplace operator in a domain with nonperiodic ‘light’ concentrated masses. A multi-dimensional problem in a domain with periodic sparsely situated ‘light’ masses was considered in [47] (see also [48]). A homogenization theorem was proved and estimates were obtained for the deviation of the eigenelements of the original problem from the eigenelements of the homogenized one. Complete asymptotic expansions of eigenelements for the Laplace operator in domains with closely situated ‘light’ concentrated masses in the cases of two-dimensional and multi-dimensional domains were constructed in [49, 50, 51, 52]. The cases of simple and of multiple eigenvalues of the limit problem were considered. Results which came from investigating problems with concentrated masses in the case of ‘critical’ density (with ‘intermediate’ masses) were announced in [53].

The problem in an unbounded domain with concentrated masses on the boundary was considered in [54]. It was proved that the poles of the analytic continuation of a solution of the original problem converge to the roots of eigenvalues of the limit problem in a bounded domain.

In this paper we consider a two-dimensional problem in a domain with periodically situated ‘light’ masses. By contrast with the papers [47, 48, 49, 50, 51, 52], it is assumed that the masses are situated on the boundary rather sparsely, as was assumed in [25, 26, 27, 28, 29, 30, 31, 32], when the distance between the masses is substantially greater than their diameter. The distance between the masses is assumed to be equal to ε , the diameter of the masses equal to $a\varepsilon$, where $a = a(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$, and the density is assumed to be equal to ε^{-m} , $m < 2$. Furthermore, we assume that the limit boundary condition remains a Dirichlet condition, that is, $\varepsilon \ln a \rightarrow 0$ as $\varepsilon \rightarrow 0$ (a similar situation is investigated in [47] and [48]). In this paper, complete asymptotic expansions of eigenvalues and eigenfunctions are constructed by the method of matched asymptotic expansions [55, 56]. By contrast with the case of ‘closely’ situated masses (see [51]), it turns out that an intermediate layer of expansion appears in this problem. Then the substantiation of the asymptotics is carried out.

The paper consists of an Introduction and four sections. The problem is posed in §1, Convergence results are presented and estimates for solutions are expounded in §2. The third section is devoted to constructing the formal asymptotics of the eigenvalues and eigenfunctions. It consists of four subsections, each of which describes in detail the construction of terms for certain powers of the small parameters. In the first subsection, terms in powers of ε are constructed, in the second terms in powers of $\ln a$, in the third terms in powers of a , and in the fourth terms in powers of $\mu = \varepsilon^{2-m}a^{2-m}$. The asymptotics we have constructed are rigorously substantiated in §4. In Appendix 1 the auxiliary results needed to construct the intermediate expansion are proved. Appendix 2 is devoted to proving auxiliary results of a qualitative nature for the construction of the inner expansion.

1. THE STATEMENT OF THE PROBLEM

We begin with the construction of the domain. We use exactly the same notation and constructions as in [49, 50, 51]. We denote by Ω a domain in $\mathbb{R}^2$ contained in the upper half-plane whose boundary is piecewise smooth and consists of several parts:

$$\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4;$$

here $\Gamma_4 = \Gamma_\varepsilon \cup \gamma_\varepsilon$, where Γ_4 is the interval $(-\pi/2, \pi/2)$ on the x_1 -axis, while Γ_2 and Γ_3 belong to the straight lines $x_1 = -\pi/2$ and $x_1 = \pi/2$, respectively. Moreover, Γ_ε and γ_ε have micro-inhomogeneous structure and the small intervals that make them up alternate. Here $\varepsilon = 1/(2\mathbf{N} + 1)$ is a small parameter, $\mathbf{N}$ is a positive integer, $\mathbf{N} \gg 1$ (see Figure 1).

We now describe in more detail the fine structure of Γ_4 . Let

$$\begin{aligned} \gamma &= \{\zeta : -1 < \zeta_1 < 1, \zeta_2 = 0\}, \\ \Gamma &= \{\zeta : -\infty < \zeta_1 < -1, 1 < \zeta_1 < +\infty, \zeta_2 = 0\} \end{aligned}$$

in the variables $\zeta = x/(a\varepsilon)$, where $0 < a < \pi/2$ and $a = a(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. We set

$$\gamma_\varepsilon = \left\{ x \in \Gamma_4 : \left(\frac{x_1}{a\varepsilon} - \frac{\mathbf{n}\pi}{\varepsilon}, 0 \right) \in \gamma, \mathbf{n} \in \mathbb{Z} \right\}, \quad \Gamma_\varepsilon = \Gamma_4 \setminus \gamma_\varepsilon.$$

FIGURE 1. A membrane with a large number of concentrated masses on the boundary

We also use the following notation. Let

$$\begin{aligned} \Pi &= \left\{ \xi : -\frac{\pi}{2} < \xi_1 < \frac{\pi}{2}, \xi_2 > 0 \right\} \text{ be a half-strip in the space } \xi = \frac{x}{\varepsilon}, \\ \Pi_{\xi_2^0} &= \left\{ \xi : -\frac{\pi}{2} < \xi_1 < \frac{\pi}{2}, \xi_2 > \xi_2^0 \right\} \text{ be a 'shortened' half-strip,} \\ \check{\Sigma} &= \left\{ \xi : -\frac{\pi}{2} < \xi_1 < \frac{\pi}{2}, \xi_2 = 0 \right\}, \\ \Sigma &= \left\{ \xi : -\frac{\pi}{2} < \xi_1 < 0, 0 < \xi_1 < \frac{\pi}{2}, \xi_2 = 0 \right\}, \end{aligned}$$

and let B be the half-disc $\{\zeta : \zeta_1^2 + \zeta_2^2 < 1, \zeta_2 > 0\}$ in the space $\zeta = \xi/a = x/(a\varepsilon)$ (see Figures 2 and 3).

FIGURE 2. A periodicity cell in the intermediate expansion

We set

$$B_{\varepsilon}^{\mathbf{n}} = \left\{ x \in \Omega : \left(\frac{x_1}{a\varepsilon} - \frac{\mathbf{n}\pi}{\varepsilon}, \frac{x_2}{a\varepsilon} \right) \in B \right\}, \quad \mathbf{n} \in \mathbb{Z}, \quad B_{\varepsilon} = \bigcup B_{\varepsilon}^{\mathbf{n}},$$

and, correspondingly,

$$\gamma_\varepsilon^{\mathbf{n}} = \left\{ x \in \Gamma_4 : \left(\frac{x_1}{a\varepsilon} - \frac{\mathbf{n}\pi}{\varepsilon}, 0 \right) \in \gamma \right\}, \quad \mathbf{n} \in \mathbb{Z},$$

that is,

$$\gamma_\varepsilon = \bigcup \gamma_\varepsilon^{\mathbf{n}}.$$

FIGURE 3. A cell in the inner expansion

We assume that $a(\varepsilon)$ is a function of ε such that $a(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$ and

$$(1) \quad \lim_{\varepsilon \rightarrow 0} \varepsilon \ln a = 0.$$

Our aim is to construct the asymptotics as $\varepsilon \rightarrow 0$ of the eigenelements of the following spectral problem:

$$(2) \quad \begin{cases} -\Delta u_\varepsilon = \lambda_\varepsilon \rho_\varepsilon u_\varepsilon & \text{for } x \in \Omega, \\ u_\varepsilon = 0 & \text{for } x \in \gamma_\varepsilon, \\ \frac{\partial u_\varepsilon}{\partial \nu} = 0 & \text{for } x \in \Gamma_\varepsilon \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3, \end{cases}$$

where $\rho_\varepsilon(x)$ is a density that has the form

$$(3) \quad \rho_\varepsilon(x) = \begin{cases} 1 & \text{in } \Omega \setminus \overline{B_\varepsilon}, \\ 1 + (a\varepsilon)^{-m} & \text{in } B_\varepsilon. \end{cases}$$

Throughout what follows we take m to be a constant, $0 < m < 2$. We call the sets $B_\varepsilon^{\mathbf{n}}$ *concentrated masses*. We also consider the problem

$$(4) \quad \begin{cases} -\Delta u_0 = \lambda u_0 & \text{for } x \in \Omega, \\ u_0 = 0 & \text{for } x \in \Gamma_4, \\ \frac{\partial u_0}{\partial \nu} = 0 & \text{for } x \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_3. \end{cases}$$

In the following section we shall explain why problem (4) is the limit problem for problem (2) (in the absence of masses, see the proof in [57], as well as in [58]). Condition (1) guarantees that there is a homogeneous Dirichlet condition on Γ_4 .

2. CONVERGENCE AND ESTIMATES

We consider the boundary-value problem

$$(5) \quad \begin{cases} -\Delta U_\varepsilon = \lambda \rho_\varepsilon U_\varepsilon + f & \text{for } x \in \Omega, \\ U_\varepsilon = 0 & \text{for } x \in \gamma_\varepsilon, \\ \frac{\partial U_\varepsilon}{\partial \nu} = 0 & \text{for } x \in \Gamma_\varepsilon \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3, \end{cases}$$

where $\rho_\varepsilon(x)$ is a density that has the form (3), and the problem

$$(6) \quad \begin{cases} -\Delta U_0 = \lambda U_0 + f & \text{for } x \in \Omega, \\ U_0 = 0 & \text{for } x \in \Gamma_4, \\ \frac{\partial U_0}{\partial \nu} = 0 & \text{for } x \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_3, \end{cases}$$

which is called the *limit (homogenized) problem*. We understand a solution of (2), (4), (5), and (6) to mean a solution in the generalized sense [59] (see also [60, 61, 62]). Let $f \in L_2(\Omega)$. We denote by $\|u\|_0$ and $\|u\|_1$ the norms of the function u in the spaces $L_2(\Omega)$ and $H^1(\Omega)$, respectively. Now, we present the following three theorems.

Theorem 2.1. *Let $f \in L_2(\Omega)$ and let K be an arbitrary compact set in the complex plane $\mathbb{C}$ that does not contain eigenvalues of the limit problem (4). Then*

- 1) *there is a number $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$ and any $\lambda \in K$ there exists a unique solution of the problem (5) and the following estimate, uniform with respect to ε and λ , holds:*

$$(7) \quad \|U_\varepsilon\|_1 \leq C \|f\|_0,$$

where C is also independent of f ;

- 2) *solutions of problems (5) and (6) satisfy*

$$(8) \quad \|U_\varepsilon - U_0\|_1 \longrightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

Theorem 2.2. *Let λ_0 be an eigenvalue of the limit problem (4). Then*

- 1) *there exists an eigenvalue λ_ε of the original problem (2) converging to λ_0 as $\varepsilon \rightarrow 0$;*
- 2) *if λ_0 has multiplicity N , then there are N eigenvalues of the original problem (accounting for the total multiplicity) converging to λ_0 .*

Theorem 2.3. *Let $f \in L_2(\Omega)$. If an eigenvalue λ_0 of the limit problem (4) has multiplicity N , then*

- 1) *for any λ close to λ_0 , the solution U_ε of the boundary-value problem (5) satisfies the following uniform estimate:*

$$(9) \quad \|U_\varepsilon\|_1 \leq C \frac{\|f\|_0}{\prod_{j=1}^N |\lambda_\varepsilon^j - \lambda|},$$

where $\lambda_\varepsilon^1, \dots, \lambda_\varepsilon^N$ are eigenvalues of problem (2) converging to λ_0 .

- 2) *if the solution U_ε of problem (5) is orthogonal in $L_2(\Omega)$ to an eigenfunction u_ε^i of problem (2) corresponding to λ_ε^i , then it satisfies*

$$(10) \quad \|U_\varepsilon\|_1 \leq C \frac{\|f\|_0}{\prod_{j=1; j \neq i}^N |\lambda_\varepsilon^j - \lambda|}.$$

The proofs of these theorems are based on the scheme in [63]; see also [64, 65, 66, 67] and [58]. However, the presence of concentrated masses sparsely situated on the boundary means we have to modify this scheme. A modification was made in [51] for the case of ‘frequent’ masses. The proofs of the theorems stated above repeat the proofs in [51] almost word for word. The only difference is a change in the proof of Lemma 2.2 in [51]. For the problem with ‘sparse’ masses, the proof of Lemma 2.2 follows from results in [57]. In what follows, Theorem 2.3 will be used to estimate the discrepancy and substantiate the formal asymptotic expansion.

3. FORMAL ASYMPTOTIC ANALYSIS

To construct the asymptotics of the eigenvalues and eigenfunctions we use the method of matched asymptotic expansions (see [55, 68, 69, 70], as well as [56, 71, 72]).

3.1. The construction of the asymptotics. Step I. We consider the case of a simple eigenvalue; that is, we assume that in the spectral problem (4), λ_0 is simple, and for definiteness we assume that $u_0(x)$ is normalized in $L_2(\Omega)$. Since (4) is the limit problem for problem (2) and λ_0 and u_0 are eigenelements of problem (4), it is natural to begin by looking for an expansion of a solution to problem (2) in the form

$$(11) \quad u_\varepsilon(x) = u_0(x) + \sum_{j=1}^{+\infty} \varepsilon^j u_j(x) + \dots,$$

and an expansion of the eigenvalues in the form

$$(12) \quad \lambda_\varepsilon = \lambda_0 + \sum_{j=1}^{+\infty} \varepsilon^j \lambda_j + \dots.$$

We note here that the function $u_0(x)$ belongs to $C^\infty(\overline{\Omega})$; for the remaining coefficients $u_j(x)$ in (11) we seek solutions in $C^\infty(\overline{\Omega})$ to

$$(13) \quad \begin{cases} -\Delta u_j = \lambda_0 u_j + \lambda_1 u_{j-1} + \dots + \lambda_j u_0 & \text{for } x \in \Omega, \\ \frac{\partial u_j}{\partial \nu} = 0 & \text{for } x \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_3, \\ u_j = \alpha_j^0(x_1) & \text{for } x \in \Gamma_4, \end{cases} \quad j = 1, 2, \dots,$$

where the $\alpha_j^0(x_1)$ are, at present, unknown (arbitrary) functions whose odd derivatives vanish at $x_1 = \pm\pi/2$.

So that the problems (13) are uniquely soluble, henceforth we will choose all the solutions to be orthogonal to u_0 in $L_2(\Omega)$.

Remark 3.1. The equations in (13) and the boundary conditions outside Γ_4 are obtained by substituting the series (11) and the expansion (12) into (2) and formally equating the coefficients of terms with like powers of ε . We use the outer expansion (11) outside some small neighbourhood of Γ_4 ; therefore at present we take the boundary conditions on Γ_4 to be arbitrary and use Dirichlet boundary conditions for convenience. The requirement that the odd derivatives of the α_j^0 vanish is a necessary condition for (13) to have a solution in the class $C^\infty(\overline{\Omega})$.

In a small neighbourhood of Γ_4 , we construct other series (‘intermediate’ and ‘inner’ expansions). To do this, we expand the functions $u_j(x)$ as Taylor series in the variable x_2

as $x_2 \rightarrow 0$. Since the functions u_k are infinitely differentiable, we have

$$\begin{aligned} u_0(x) &= \alpha_0^1(x_1)x_2 + \sum_{l=2}^{+\infty} \alpha_0^l(x_1)x_2^l, \\ u_j(x) &= \alpha_j^0(x_1) + \alpha_j^1(x_1)x_2 + \sum_{l=2}^{+\infty} \alpha_j^l(x_1)x_2^l, \quad j = 1, \dots, \end{aligned}$$

where

$$\alpha_k^l = \frac{1}{(l)!} \frac{\partial^l u_k}{\partial x_2^l} \Big|_{x_2=0}, \quad k = 0, 1, \dots,$$

and using the Neumann boundary conditions on $\Gamma_2 \cup \Gamma_3$, the equation (13) and infinite differentiability, we have

$$(14) \quad \frac{d^{2j+1} \alpha_k^l}{dx_1^{2j+1}} \left(\pm \frac{\pi}{2} \right) = 0, \quad l = 1, \dots, \quad j = 0, \dots$$

We emphasize that all the $\alpha_0^l(x_1)$ are known functions (since $u_0(x)$ is the solution of (4)) and by the equation in (4) we have

$$\alpha_0^{2l}(x_1) \equiv 0, \quad l = 0, 1, \dots$$

We also note that condition (14) holds for the known functions α_0^l . For now, it is a condition that the functions α_k^0 have to satisfy. We will check that it is satisfied in what follows (then condition (14) will hold automatically for the other functions α_k^l due to the boundary-value problems (13)).

We make the change of variables $\xi_2 = x_2/\varepsilon$ (that is, we rewrite these expansions in terms of the ‘intermediate’ variables). Then as $\varepsilon\xi_2 \rightarrow 0$ we have

$$\begin{aligned} (15) \quad u_0(x_1, \varepsilon\xi_2) &= \varepsilon\alpha_0^1(x_1)\xi_2 + \sum_{l=1}^{+\infty} \varepsilon^{2l+1} \alpha_0^{2l+1}(x_1) \xi_2^{2l+1}, \\ \varepsilon^j u_j(x_1, \varepsilon\xi_2) &= \varepsilon^j \alpha_j^0(x_1) + \varepsilon^{j+1} \alpha_j^1(x_1) \xi_2 + \sum_{l=2}^{+\infty} \varepsilon^{j+l} \alpha_j^l(x_1) \xi_2^l, \quad j \in \mathbb{N}. \end{aligned}$$

Substituting (15) into (11) we see that (11) has the following form as $\varepsilon\xi_2 \rightarrow 0$:

$$(16) \quad u_\varepsilon(x) = \sum_{j=0}^{+\infty} \varepsilon^j u_j(x_1, \varepsilon\xi_2) + \dots = \sum_{j=0}^{+\infty} \varepsilon^j V_j(\xi_2; x_1) + \dots,$$

where

$$\begin{aligned} (17) \quad V_{2j+1}(\xi_2; x_1) &= \alpha_0^{2j+1}(x_1) \xi_2^{2j+1} + \alpha_1^{2j}(x_1) \xi_2^{2j} + \dots + \alpha_{2j+1}^0(x_1), \\ V_{2j}(\xi_2; x_1) &= \alpha_1^{2j-1}(x_1) \xi_2^{2j-1} + \dots + \alpha_{2j}^0(x_1), \quad j = 0, 1, \dots, \end{aligned}$$

and $\xi = x/\varepsilon$.

Following the method of matched asymptotic expansions, we conclude that the intermediate expansion must have the structure

$$(18) \quad u_\varepsilon(x) = \varepsilon v_1\left(\frac{x}{\varepsilon}; x_1\right) + \sum_{j=2}^{+\infty} \varepsilon^j v_j\left(\frac{x}{\varepsilon}; x_1\right) + \dots,$$

where

$$(19) \quad v_q(\xi; x_1) = V_q(\xi_2; x_1) + o(1) \quad \text{as } \xi_2 \rightarrow +\infty.$$

Here x_1 is a ‘slow’ variable; sometimes the dependence on x_1 will be omitted from the arguments. In terms of the variables $(\xi; x_1)$ the equation in (2) becomes

$$(20) \quad -\varepsilon^{-2} \Delta_\xi u_\varepsilon - 2\varepsilon^{-1} \frac{\partial^2 u_\varepsilon}{\partial x_1 \partial \xi_1} - \frac{\partial^2 u_\varepsilon}{\partial x_1^2} = \lambda_\varepsilon \rho_\varepsilon u_\varepsilon,$$

while the boundary conditions on Γ_2 take the form

$$(21) \quad \frac{\partial u_\varepsilon}{\partial \nu} = -\varepsilon^{-1} \frac{\partial u_\varepsilon}{\partial \xi_1} - \frac{\partial u_\varepsilon}{\partial x_1} = 0,$$

and on Γ_3 ,

$$(22) \quad \frac{\partial u_\varepsilon}{\partial \nu} = \varepsilon^{-1} \frac{\partial u_\varepsilon}{\partial \xi_1} + \frac{\partial u_\varepsilon}{\partial x_1} = 0.$$

Remark 3.2. In what follows we construct terms of the ‘intermediate’ expansion (18) in the form of functions that are π -periodic in ξ_1 (for each fixed value of x_1).

Rewriting the equation and the boundary conditions in the coordinates ξ (see (20), (21), and (22)), substituting the ‘intermediate’ expansion (18) and the series (12) into problem (2), we collect terms with like powers of ε (terms in ε^{q-1} in the equation and terms in ε^q in the boundary conditions). Passing to the limit as $a \rightarrow 0$ in the cell Π , taking account of (19), (17) and Remark 3.2, we obtain the following problems: for v_1 ,

$$(23) \quad \begin{cases} -\Delta_\xi v_1 = 0 & \text{in } \Pi, \\ \frac{\partial v_1}{\partial \xi_2} = 0 & \text{on } \Sigma, \\ \frac{\partial v_1}{\partial \xi_1} \left(\xi; \pm \frac{\pi}{2} \right) = 0 & \text{at } \xi_1 = \pm \frac{\pi}{2}, \\ v_1 \sim \alpha_0^1 \xi_2 + \alpha_1^0 & \text{as } \xi_2 \rightarrow +\infty, \end{cases}$$

and for the v_j for $j > 1$,

$$(24) \quad \begin{cases} -\Delta_\xi v_j = 2 \frac{\partial^2 v_{j-1}}{\partial x_1 \partial \xi_1} + \frac{\partial^2 v_{j-2}}{\partial x_1^2} + \sum_{k=0}^{j-2} \lambda_k v_{j-2-k} & \text{in } \Pi, \\ \frac{\partial v_j}{\partial \xi_2} = 0 & \text{on } \Sigma, \\ \frac{\partial v_j}{\partial \xi_1} \left(\xi; \pm \frac{\pi}{2} \right) = -\frac{\partial v_{j-1}}{\partial x_1} \left(\xi; \pm \frac{\pi}{2} \right) & \text{at } \xi_1 = \pm \frac{\pi}{2}, \\ v_j = V_j + o(1) & \text{as } \xi_2 \rightarrow +\infty. \end{cases}$$

From now on, we set functions with indices we have not introduced earlier to be equal to zero; that is, for example, $v_{-1} \equiv v_0 \equiv 0$.

We emphasize that from the boundary-value problems (13) and the definition of V_q , we have the following equalities:

$$(25) \quad -\Delta_\xi V_q = 2 \frac{\partial^2 V_{q-1}}{\partial x_1 \partial \xi_1} + \frac{\partial^2 V_{q-2}}{\partial x_1^2} + \lambda_0 V_{q-2} + \cdots + \lambda_{q-3} V_1,$$

$$(26) \quad \frac{\partial^{2n+1} V_q}{\partial x_1^{2n+1}} \left(\xi_2; \pm \frac{\pi}{2} \right) = 0, \quad \frac{\partial V_q}{\partial \xi_1} (\xi_2; x_1) \equiv 0.$$

We introduce into consideration the auxiliary problem

$$(27) \quad \begin{cases} -\Delta_\xi X = 0 & \text{in } \Pi, \\ \frac{\partial X}{\partial \xi_2} = 0 & \text{on } \Sigma, \\ \frac{\partial X}{\partial \xi_1} = 0 & \text{at } \xi_1 = \pm \frac{\pi}{2}, \\ X \sim \xi_2 & \text{as } \xi_2 \rightarrow +\infty. \end{cases}$$

We also consider the following function (see [71], as well as [72] and [56]):

$$(28) \quad X(\xi) = \operatorname{Re} \ln \sin z + \ln 2,$$

where $z = \xi_1 + i\xi_2$. It follows from (28) that the function

$$X(\xi) \in C^\infty\left(\overline{\mathbb{R}_+^2} \setminus \bigcup_{k \in \mathbb{Z}} \{(k\pi, 0)\}\right)$$

is π -periodic in ξ_1 ; the asymptotics of the function X at infinity as $\xi_2 \rightarrow +\infty$ have the form:

$$(29) \quad X(\xi) = \xi_2 + O(e^{-2\xi_2}),$$

while the asymptotics of this function at zero (as $\rho \equiv |\xi| \rightarrow 0$) are

$$(30) \quad X(\xi) = \ln \rho + \ln 2 + \sum_{j=1}^{+\infty} B_{2j} \rho^{2j} \cos 2j\theta,$$

where the B_{2j} are constants that can be calculated explicitly. It is easy to verify that X is a solution of (27). Then obviously

$$(31) \quad v_1(\xi; x_1) = \alpha_0^1(x_1) X(\xi) + \alpha_1^0(x_1)$$

is a solution of problem (23), where $\alpha_1^0(x_1) \in C^\infty[-\pi/2, \pi/2]$ is at present arbitrary (a boundary condition for $u_1(x)$), which satisfies (14). This function is π -periodic and even in ξ_1 . By (28) and (31), as $\xi_2 \rightarrow +\infty$, the asymptotics of the function $v_1(\xi; x_1)$ at infinity are as follows:

$$(32) \quad v_1(\xi; x_1) = \alpha_1^0(x_1) + \alpha_0^1(x_1) [\xi_2 + O(e^{-2\xi_2})].$$

In view of (30) and (31), the asymptotics of this function at zero (as $\rho \rightarrow 0$) take the form:

$$(33) \quad v_1(\xi; x_1) = \alpha_0^1(x_1) (\ln \rho + \ln 2) + \alpha_1^0(x_1) + O(\rho^2).$$

We also seek the other terms of the intermediate expansion such that

$$(34) \quad v_q = O(\ln \rho) \quad \text{as } \rho \rightarrow 0.$$

We note that according to (14) we have

$$(35) \quad \frac{\partial v_1}{\partial x_1}(\xi; x_1) = 0 \quad \text{at } x_1 = \pm \frac{\pi}{2}$$

and, consequently, by (21), (22), (23) and (35), we have

$$(36) \quad \frac{\partial v_1}{\partial \nu} \left(\frac{x}{\varepsilon}; x_1 \right) = 0 \quad \text{on } \Gamma_2 \cup \Gamma_3.$$

The function v_1 does not satisfy the Dirichlet boundary condition on γ_ε ; therefore we introduce yet another ('inner') expansion, going over to the variables

$$\zeta^{\mathbf{n}} \equiv (\zeta_1^{\mathbf{n}}, \zeta_2^{\mathbf{n}}) = \left(\frac{\xi_1}{a} - \frac{\mathbf{n}\pi}{\varepsilon}, \frac{\xi_2}{a} \right), \quad \mathbf{n} \in \mathbb{Z}.$$

We rewrite the asymptotics of $\varepsilon v_1(\xi; x_1)$ as $\rho \rightarrow 0$ in terms of the variables $\zeta^{\mathbf{n}}$ (more precisely, in the polar coordinates $(\tau^{\mathbf{n}}, \theta^{\mathbf{n}})$ for $\zeta^{\mathbf{n}}$), taking (33) into account. As $a\tau^{\mathbf{n}} \rightarrow 0$, we have

$$(37) \quad \varepsilon v_1(a\zeta^{\mathbf{n}}; x_1) = \varepsilon \alpha_0^1(x_1) [\ln a + (\ln \tau^{\mathbf{n}} + \ln 2)] + \varepsilon \alpha_1^0(x_1) + O(\varepsilon a^2(\tau^{\mathbf{n}})^2).$$

Rewriting the asymptotics of the series (18) as $\rho \rightarrow 0$ in terms of $\zeta^{\mathbf{n}}$, taking account of (37) and (34), and following the method of matched asymptotic expansions, we conclude that the expansion of the function u_ε in the new variables $\zeta^{\mathbf{n}}$ must have the form

$$(38) \quad u_\varepsilon(x) = \varepsilon \ln a w_{1,1,0}^{\mathbf{n}}(\zeta^{\mathbf{n}}; x_1) + \varepsilon w_{1,0,0}^{\mathbf{n}}(\zeta^{\mathbf{n}}; x_1) + \cdots,$$

where, as $\tau^{\mathbf{n}} \rightarrow +\infty$, the term $w_{1,0,0}^{\mathbf{n}}$ has the asymptotic form

$$(39) \quad w_{1,0,0}^{\mathbf{n}} \sim \alpha_0^1(\ln \tau^{\mathbf{n}} + \ln 2) + \alpha_1^0.$$

For brevity we omit the index $\mathbf{n}$ from the functions w and variables ζ in what follows. Then in the variables $(\zeta; x_1)$ the equation in problem (2) takes the form

$$(40) \quad -\varepsilon^{-2} a^{-2} \Delta_\zeta u_\varepsilon - 2\varepsilon^{-1} a^{-1} \frac{\partial^2 u_\varepsilon}{\partial x_1 \partial \zeta_1} - \frac{\partial^2 u_\varepsilon}{\partial x_1^2} = \lambda_\varepsilon \rho_\varepsilon u_\varepsilon.$$

Taking the form of equation (40) in the inner variables ζ into account, we substitute (38) and the series (12) into (2) and collect terms with like powers of ε , a , and $\ln a$ (in the equation with $\varepsilon^{-1} a^{-2}$, and in the boundary conditions with a^{-1} , respectively). In view of (39) we obtain a problem for the function $w_{1,0,0}$ of the form

$$(41) \quad \begin{cases} \Delta_\zeta w_{1,0,0} = 0 & \text{in } \mathbb{R}_+^2, \\ \frac{\partial w_{1,0,0}}{\partial \zeta_2} = 0 & \text{on } \Gamma, \\ w_{1,0,0} = 0 & \text{on } \gamma, \\ w_{1,0,0} \sim \alpha_0^1(\ln \tau + \ln 2) + \alpha_1^0 & \text{as } \tau \rightarrow +\infty. \end{cases}$$

Recall that

$$\gamma = \{\zeta : -1 < \zeta_1 < 1, \zeta_2 = 0\} \quad \text{and} \quad \Gamma = \{\zeta : -\infty < \zeta_1 < -1, 1 < \zeta_1 < +\infty, \zeta_2 = 0\}.$$

Consider the following function Y (see [73]):

$$(42) \quad Y(\zeta) = \operatorname{Re} \ln(y + \sqrt{y^2 - 1}),$$

where $y = \zeta_1 + i\zeta_2$. It is easy to verify that $Y \in H_{\text{loc}}^1(\mathbb{R}_+^2)$ is a solution of the problem

$$(43) \quad \begin{cases} \Delta_\zeta Y = 0 & \text{in } \mathbb{R}_+^2, \\ \frac{\partial Y}{\partial \zeta_2} = 0 & \text{on } \Gamma, \\ Y = 0 & \text{on } \gamma. \end{cases}$$

It follows from the explicit form of the function Y that, as $\tau \rightarrow +\infty$, it has the asymptotic form

$$(44) \quad Y(\zeta) = (\ln \tau + \ln 2) + O(\tau^{-1}).$$

Taking (44) and problem (43) into account, we conclude that the function

$$(45) \quad w_{1,0,0}(\zeta; x_1) = \alpha_0^1(x_1) Y(\zeta)$$

is a solution of problem (41) for

$$(46) \quad \alpha_1^0(x_1) \equiv 0.$$

Therefore we take $w_{1,0,0}$ and α_1^0 to be given by (45) and (46). Then by (31) the function v_1 is determined by the equality

$$(47) \quad v_1(\xi; x_1) = \alpha_0^1(x_1) X(\xi),$$

and (13), the problem for u_1 , takes the form

$$(48) \quad \begin{cases} -\Delta u_1 = \lambda_0 u_1 + \lambda_1 u_0 & \text{for } x \in \Omega, \\ u_1 = 0 & \text{for } x \in \Gamma_4, \\ \frac{\partial u_1}{\partial \nu} = 0 & \text{for } x \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_3. \end{cases}$$

Obviously, the pair

$$(49) \quad u_1 \equiv 0, \quad \lambda_1 = 0$$

satisfies (48). Thus, it is clear that

$$(50) \quad \alpha_1^l \equiv 0.$$

Remark 3.3. Note that the arbitrary $\alpha_1^0(x_1)$ is fixed for the function $v_1(\xi; x_1)$ by the matching condition for the functions $v_1(\xi; x_1)$ and $w_{1,0,0}(\zeta; x_1)$, and this is how $v_1(\xi; x_1)$ is finally determined. We also emphasize that in what follows the arbitrary summands independent of ξ , appearing in the terms

$$\varepsilon^{j+p(m-2)} a^{k+p(m-2)} \ln^l a v_{j,k,l,p}(\xi; x_1)$$

in the intermediate expansion, are determined from the matching condition with the terms of the inner expansion

$$\varepsilon^{j+p(m-2)} a^{k+p(m-2)} \ln^l a w_{j,k,l,p}(\zeta; x_1)$$

(for the same exponents j, k, l).

We continue our investigation of problem (24). We write out the problem for $j = 2$, taking (17), (35) and (50) into account:

$$(51) \quad \begin{cases} -\Delta_\xi v_2 = 2 \frac{\partial^2 v_1}{\partial x_1 \partial \xi_1} & \text{in } \Pi, \\ \frac{\partial v_2}{\partial \xi_2} = 0 & \text{on } \Sigma, \\ \frac{\partial v_2}{\partial \xi_1} \left(\xi; \pm \frac{\pi}{2} \right) = 0 & \text{at } \xi_1 = \pm \frac{\pi}{2}, \\ v_2 \sim \alpha_2^0 & \text{as } \xi_2 \rightarrow +\infty, \end{cases}$$

where, up to now, the function $\alpha_2^0(x_1) \in C^\infty[-\pi/2, \pi/2]$ is an arbitrary summand (a boundary condition for $u_2(x)$) that satisfies (14).

We consider the auxiliary problem

$$(52) \quad \begin{cases} -\Delta_\xi X_2 = 2 \frac{\partial X}{\partial \xi_1} & \text{in } \Pi, \\ \frac{\partial X_2}{\partial \xi_2} = 0 & \text{on } \Sigma, \\ X_2 = 0 & \text{at } \xi_1 = \pm \frac{\pi}{2}, \\ X_2 = o(1) & \text{as } \xi_2 \rightarrow +\infty. \end{cases}$$

We introduce an auxiliary smooth cutoff function $0 < \chi(\rho) < 1$ such that

$$(53) \quad \chi(\rho) = \begin{cases} 1, & \rho > \frac{\pi}{4}, \\ 0, & \rho < \frac{\pi}{8}. \end{cases}$$

Lemma 3.1. *There exists a solution of (52) that is π -periodic and odd in ξ_1 and is such that*

$$X_2 \in C^\infty\left(\overline{\mathbb{R}_+^2} \setminus \bigcup_{k \in \mathbb{Z}} \{(k\pi, 0)\}\right), \quad e^{\varpi \xi_2} \chi(\rho) X_2 \in H^1(\Pi), \quad 0 < \varpi < 2,$$

and the asymptotic behaviour

$$\begin{aligned} X_2(\xi) &= Q\rho \ln \rho \cos \theta + \sum_{j=0}^{\infty} \beta_{2j+1} \rho^{2j+3} \cos(2j+1)\theta \\ &\quad + \sum_{j=0}^{\infty} \tilde{\beta}_{2j+1} \rho^{2j+1} \cos(2j+1)\theta \end{aligned}$$

holds as $\rho \rightarrow 0$, where Q , β_k , and $\tilde{\beta}_k$ are some constants.

The proof of the lemma is given in Appendix 1 (§4).

By Lemma 3.1, (31), (52) and (14), the function

$$(54) \quad v_2(\xi; x_1) = (\alpha_0^1)'(x_1) X_2(\xi) + \alpha_2^0(x_1)$$

is a solution of (51) with the asymptotic behaviour

$$(55) \quad v_2(\xi; x_1) = \alpha_2^0(x_1) + O(\rho \ln \rho) \quad \text{as } \rho \rightarrow 0$$

and

$$(56) \quad v_2 = \alpha_2^0(x_1) + \hat{v}_2(\xi; x_1), \quad \|\hat{v}_2\|_{H^1(\Pi_{\xi_2})} \leq C e^{-\varpi \xi_2} \quad \text{as } \xi_2 \rightarrow +\infty,$$

at infinity, where the constant C is independent of x_1 . Recall that

$$\Pi_b = \left\{ \xi : -\frac{\pi}{2} < \xi_1 < \frac{\pi}{2}, \xi_2 > b \right\}.$$

Note that because of (14) and the boundary conditions in (52) we have

$$(57) \quad \frac{\partial v_2}{\partial x_1}(\xi; x_1) = 0 \quad \text{at } x_1 = \pm \frac{\pi}{2}$$

and, consequently, by (21), (22), (51) and (57),

$$(58) \quad \frac{\partial v_2}{\partial \nu} \left(\frac{x}{\varepsilon}; x_1 \right) = 0 \quad \text{on } \Gamma_2 \cup \Gamma_3.$$

Rewriting (55) in terms of ζ :

$$(59) \quad v_2(\xi; x_1) = \alpha_2^0(x_1) + O(a\tau \ln(a\tau)) \quad \text{as } a\tau \rightarrow 0,$$

and then rewriting the asymptotics of the series (18) as $\rho \rightarrow 0$ in terms of the variables ζ , taking (59) and (34) into account, and following the method of matched asymptotic expansions, we see that a new term must be introduced into the expansion (38):

$$(60) \quad u_\varepsilon(x) = \varepsilon \ln a \, w_{1,1,0}(\zeta; x_1) + \varepsilon w_{1,0,0}(\zeta; x_1) + \varepsilon^2 w_{2,0,0}(\zeta; x_1) + \cdots,$$

where

$$(61) \quad w_{2,0,0} \sim \alpha_2^0, \quad \tau \rightarrow +\infty.$$

We now give the boundary-value problem for $w_{2,0,0}$. We substitute the expansion (60) into problem (2), rewritten in terms of the variables ζ , and collect terms with the powers

a^{-2} in the equation and with the powers εa^{-1} in the boundary condition. In view of (61) we obtain the problem

$$(62) \quad \begin{cases} \Delta_{\zeta} w_{2,0,0} = 0 & \text{in } \mathbb{R}_+^2, \\ \frac{\partial w_{2,0,0}}{\partial \zeta_2} = 0 & \text{on } \Gamma, \\ w_{2,0,0} = 0 & \text{on } \gamma, \\ w_{2,0,0} \sim \alpha_2^0 & \text{as } \tau \rightarrow +\infty. \end{cases}$$

Obviously, (62) is solvable only for

$$(63) \quad \alpha_2^0(x_1) \equiv 0,$$

and the function

$$(64) \quad w_{2,0,0}(\zeta; x_1) \equiv 0$$

is a solution. Therefore we define α_2^0 and $w_{2,0,0}$ in accordance with (63) and (64). Then it follows from (54) that v_2 has the final form

$$(65) \quad v_2(\xi; x_1) = (\alpha_0^1)'(x_1) X_2(\xi).$$

In view of (49) and (63), the boundary-value problem (13) for $j = 2$ takes the form

$$(66) \quad \begin{cases} -\Delta u_2 = \lambda_0 u_2 + \lambda_2 u_0 & \text{for } x \in \Omega, \\ u_2 = 0 & \text{for } x \in \Gamma_4, \\ \frac{\partial u_2}{\partial \nu} = 0 & \text{for } x \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_3. \end{cases}$$

It is also obvious that the pair

$$(67) \quad u_2(x) = 0, \quad \lambda_2 = 0$$

satisfies (66). Hence, $\alpha_2^l \equiv 0$ for $dl = 1, 2, \dots$.

We now rewrite problem (24) for $j = 3$ taking (57) into account:

$$(68) \quad \begin{cases} -\Delta_{\xi} v_3 = 2 \frac{\partial^2 v_2}{\partial x_1 \partial \xi_1} + \frac{\partial^2 v_1}{\partial x_1^2} + \lambda_0 v_1 & \text{in } \Pi, \\ \frac{\partial v_3}{\partial \xi_2} = 0 & \text{on } \Sigma, \\ \frac{\partial v_3}{\partial \xi_1}(\xi; \pm \frac{\pi}{2}) = 0 & \text{at } \xi_1 = \pm \frac{\pi}{2}, \\ v_3 \sim V_3(\xi_2; x_1) & \text{as } \xi_2 \rightarrow +\infty. \end{cases}$$

We observe that by (54) and (63), as well as by (31) and (46), the right-hand side of the equation in problem (68) takes the following form:

$$(69) \quad F = 2(\alpha_0^1)'' \frac{\partial X_2}{\partial \xi_1} + (\alpha_0^1)'' X + \lambda_0 \alpha_0^1 X.$$

For our convenience when we analyse problems with a right-hand side of the form (69), we introduce the following classes of functions.

- We let $A_{\text{even}}^{(2k,q)}$ denote the set of functions

$$f_{\text{even}} \in C^{\infty} \left(\overline{\mathbb{R}_+^2} \setminus \bigcup_{k \in \mathbb{Z}} \{(k\pi, 0)\} \right)$$

that are π -periodic and even in ξ_1 , have asymptotic behaviour at zero of the form

$$(70) \quad \begin{aligned} f_{\text{even}}(\xi) = & \delta_0^q \sum_{n=1}^{k+1} \beta_{\text{even}}^{(n)} \rho^{2n-2} \cos 2n\theta + \sum_{n=0}^k \sum_{j=0}^n \widehat{\beta}_{\text{even}}^{(n,j)} \rho^{2n} \cos 2(n-j)\theta \ln \rho \\ & + \sum_{n=0}^{\infty} \sum_{j=0}^n \widetilde{\beta}_{\text{even}}^{(n,j)} \rho^{2n} \cos 2(n-j)\theta, \quad \rho \rightarrow 0, \end{aligned}$$

where δ_0^q is the Kronecker delta, and have the property

$$e^{\varpi \xi_2} \chi(\rho) f_{\text{even}} \in H^1(\Pi), \quad 0 < \varpi < 2.$$

It follows from the definition of $X(\xi)$ (see (28)) and $X_2(\xi)$ (see Lemma 3.1) that

$$(71) \quad \frac{\partial X_2}{\partial \xi_1} \in A_{\text{even}}^{(0,0)}, \quad X = \xi_2 + \widetilde{X}, \quad \widetilde{X} \in A_{\text{even}}^{(0,1)}.$$

We also note that

$$A_{\text{even}}^{(2k,1)} \subset A_{\text{even}}^{(2k+2s,1)}, \quad A_{\text{even}}^{(2k,0)} \subset A_{\text{even}}^{(2k+2s,0)}, \quad A_{\text{even}}^{(2k,1)} \subset A_{\text{even}}^{(2k+2s,0)}, \quad s = 0, 1, \dots$$

We set

$$\begin{aligned} \mathcal{A}_{\text{even}}^{(2k,q)} = & \left\{ V(\xi; x_1) : V(\xi; x_1) = \sum_{j=1}^J \sigma_j(x_1) Y_j(\xi), \quad Y_j \in A_{\text{even}}^{(2k,q)}, \right. \\ & \left. \sigma_j \in C^\infty \left[-\frac{\pi}{2}, \frac{\pi}{2} \right], \quad (\sigma_j)^{(2n+1)} \left(\pm \frac{\pi}{2} \right) = 0, \quad n = 0, 1, \dots \right\}, \end{aligned}$$

where J is arbitrary (not fixed but finite). Note that

$$\frac{\partial V}{\partial x_1} \left(\xi; \pm \frac{\pi}{2} \right) = 0 \quad \text{for all } V \in \mathcal{A}_{\text{even}}^{(2k,q)}.$$

Thus, it follows from (14) and (71) that the right-hand side of the equation in problem (68) (see (69)) can be represented in the form

$$(72) \quad F = \mathcal{P}_1(\xi_2; x_1) + f,$$

where

$$(73) \quad \mathcal{P}_1(\xi_2; x_1) = ((\alpha_0^1)'' + \lambda_0 \alpha_0^1) \xi_2, \quad -\Delta_\xi V_3 = \mathcal{P}_1, \quad f \in \mathcal{A}_{\text{even}}^{(0,0)}.$$

Here, and in what follows, $\mathcal{P}_t(\xi_2; x_1)$ denotes a polynomial of order t in the variable ξ_2 whose coefficients are functions of x_1 that are infinitely differentiable on $[-\pi/2, \pi/2]$ and have odd derivatives vanishing at $x_1 = \pm\pi/2$.

Lemma 3.2. a) Let F be represented in the form $F = \mathcal{P}_t + f$, where $f \in \mathcal{A}_{\text{even}}^{(2k,0)}$ and $\mathcal{P}_{t+2}$ is a polynomial such that $-\mathcal{P}_{t+2}'' = \mathcal{P}_t$. Then there exists a solution of the boundary-value problem

$$(74) \quad \begin{cases} -\Delta_\xi W = F & \text{in } \Pi, \\ \frac{\partial W}{\partial \xi_2} = 0 & \text{on } \Sigma, \\ \frac{\partial W}{\partial \xi_1} = 0 & \text{at } \xi_1 = \pm \frac{\pi}{2} \end{cases}$$

that can be represented in the form $W = \mathcal{P}_{t+2} + W_1$, where $W_1 \in \mathcal{A}_{\text{even}}^{(2k+2,1)}$.

b) Furthermore,

$$\frac{\partial W}{\partial x_1} \left(\xi; \pm \frac{\pi}{2} \right) = 0, \quad \frac{\partial W}{\partial x_1} \left(\frac{x}{\varepsilon}; x_1 \right) = 0$$

at $x_1 = \pm\pi/2$.

This lemma is proved in Appendix 1 (§4).

By (17) the polynomial $V_3(\xi_2; x_1)$ can be represented in the form

$$(75) \quad V_3 = \tilde{V}_3 + \alpha_3^0,$$

where $\tilde{V}_3 = \mathcal{P}_3(\xi_2; x_1)$ is a given polynomial, independent of α_3^0 (it was defined before this step). By Lemma 3.2 and (73) there exists a solution of the boundary-value problem (68) which can be represented in the form

$$(76) \quad v_3 = V_3 + \tilde{v}_3,$$

where $\tilde{v}_3 \in \mathcal{A}_{\text{even}}^{(2,1)}$ is a completely defined function that is independent of α_3^0 . Note that

$$\frac{\partial v_3}{\partial x_1}(\xi; x_1) = 0 \quad \text{at } x_1 = \pm \frac{\pi}{2}, \quad \frac{\partial v_3}{\partial \nu} \left(\frac{x}{\varepsilon}; x_1 \right) = 0 \quad \text{on } \Gamma_2 \cup \Gamma_3.$$

It follows from (76) that v_3 has asymptotics at zero of the form

$$(77) \quad v_3(\xi; x_1) = \mathcal{M}_3(x_1) \ln \rho + \widehat{\mathcal{M}}_3(x_1) + \alpha_3^0(x_1) + O(\rho^2 \ln \rho) \quad \text{as } \rho \rightarrow 0,$$

where $\mathcal{M}_3(x_1)$ and $\widehat{\mathcal{M}}_3(x_1)$ are completely defined functions such that

$$(78) \quad \frac{d^{2j+1} \mathcal{M}_3}{dx_1^{2j+1}} \left(\pm \frac{\pi}{2} \right) = 0, \quad \frac{d^{2j+1} \widehat{\mathcal{M}}_3}{dx_1^{2j+1}} \left(\pm \frac{\pi}{2} \right) = 0, \quad j = 0, \dots$$

Recall also that $\alpha_3^0(x_1)$ is some, so far undefined, function (completely analogous to $\alpha_1^0(x_1)$ in the definition (31) of the function v_1).

We rewrite (77), which gives the asymptotics of the function v_3 at zero, in terms of ζ . As $a\tau \rightarrow 0$, we have

$$(79) \quad \begin{aligned} \varepsilon^3 v_3(a\zeta; x_1) &= \varepsilon^3 \mathcal{M}_3(x_1) [\ln a + \ln \tau] + \varepsilon^3 \widehat{\mathcal{M}}_3(x_1) \\ &\quad + \varepsilon^3 \alpha_3^0(x_1) + O(\varepsilon^3 (a\tau)^2 \ln(a\tau)). \end{aligned}$$

Rewriting the asymptotics of the series (18) as $\rho \rightarrow 0$ in terms of ζ , taking (79) and (34) into account and following the method of matched asymptotic expansions, we conclude that the expansion of the function u_ε in the new variables ζ must have the form

$$(80) \quad \begin{aligned} u_\varepsilon &= \varepsilon \ln a \, w_{1,1,0}(\zeta; x_1) + \varepsilon w_{1,0,0}(\zeta; x_1) + \varepsilon^2 a \ln a \, w_{2,1,1}(\zeta; x_1) \\ &\quad + \varepsilon^2 a w_{2,0,1} + \varepsilon^3 \ln a \, w_{3,1,0}(\zeta; x_1) + \varepsilon^3 w_{3,0,0}(\zeta; x_1) + \dots, \end{aligned}$$

where, as $\tau \rightarrow +\infty$, the term $w_{3,0,0}$ has asymptotic behaviour

$$(81) \quad w_{3,0,0} \sim \mathcal{M}_3 \ln \tau + \widehat{\mathcal{M}}_3 + \alpha_3^0.$$

We rewrite the operator in problem (2) in terms of the inner variables ζ (see (40)), substitute the expansion (80) and the series (12) into (2) and, in the equation, collect terms in powers of εa^{-2} , and in the boundary conditions, in powers of $\varepsilon^2 a^{-1}$, respectively. In view of (81) we obtain a problem for the function $w_{3,0,0}$ of the form

$$(82) \quad \begin{cases} \Delta_\zeta w_{3,0,0} = 0 & \text{in } \mathbb{R}_+^2, \\ \frac{\partial w_{3,0,0}}{\partial \zeta_2} = 0 & \text{on } \Gamma, \\ w_{3,0,0} = 0 & \text{on } \gamma, \\ w_{3,0,0} \sim \mathcal{M}_3 \ln \tau + \widehat{\mathcal{M}}_3 + \alpha_3^0 & \text{as } \tau \rightarrow +\infty. \end{cases}$$

Taking into account the asymptotic behaviour in (44) and problem (43), we conclude that the function

$$(83) \quad w_{3,0,0}(\zeta; x_1) = \mathcal{M}_3(x_1) Y(\zeta)$$

is a solution of problem (82) for

$$(84) \quad \alpha_3^0(x_1) = \mathcal{M}_3 \ln 2 - \widehat{\mathcal{M}}_3,$$

where $Y(\zeta)$ is the function defined in (42). Therefore we fix $w_{3,0,0}$ and α_3^0 in accordance with (83) and (84). We observe that by (78) condition (14) holds. We also point out that this completes the construction of the function v_3 .

Taking account of (49) and (67), we obtain the boundary-value problem (13) for $j = 3$:

$$(85) \quad \begin{cases} -\Delta u_3 = \lambda_0 u_3 + \lambda_3 u_0 & \text{for } x \in \Omega, \\ u_3 = \alpha_3^0 & \text{for } x \in \Gamma_4, \\ \frac{\partial u_3}{\partial \nu} = 0 & \text{for } x \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_3, \end{cases}$$

where α_3^0 is the function we have just defined in (84). From the condition that this problem be solvable, we determine λ_3 and, correspondingly, find u_3 .

Thus, because we have completely determined the function $v_3(\xi; x_1)$ compatible with $w_{3,0,0}$, we have found the boundary condition for u_3 . In other words, we have constructed v_3 , $w_{3,0,0}$, u_3 , and λ_3 in this step.

In what follows, we will construct λ_j , u_j , v_j , and $w_{j,0,0}$ in a similar way.

We will be looking at (24) with special right-hand sides, and for convenience we introduce the following classes of functions.

- We denote by $A_{\text{odd}}^{(2k-1,q)}$ the set of functions $f_{\text{odd}} \in C^\infty(\overline{\mathbb{R}_+^2} \setminus \bigcup_{k \in \mathbb{Z}} \{(k\pi, 0)\})$ that are π -periodic and odd in ξ_1 , have asymptotic behaviour at zero of the form

$$\begin{aligned} f_{\text{odd}}(\xi) = & \delta_0^q \sum_{n=0}^k \beta_{\text{odd}}^{(n)} \rho^{2n-1} \cos(2n+1)\theta \\ & + \sum_{n=0}^{k-1} \sum_{j=0}^n \widehat{\beta}_{\text{odd}}^{(n,j)} \rho^{2n+1} \cos(2n-2j+1)\theta \ln \rho \\ & + \sum_{n=0}^{\infty} \sum_{j=0}^n \widetilde{\beta}_{\text{odd}}^{(n,j)} \rho^{2n+1} \cos(2n-2j+1)\theta, \quad \rho \rightarrow 0, \end{aligned}$$

where δ_0^q is the Kronecker delta, and have the property

$$e^{\varpi \xi_2} \chi(\rho) f_{\text{odd}} \in H^1(\Pi), \quad 0 < \varpi < 2.$$

We note here that the symbol $A_{\text{odd}}^{(-1,q)}$ (that is, when $k = 0$) denotes the class of functions whose asymptotics as $\rho \rightarrow 0$ do not have summands involving $\ln \rho$.

We observe that

$$(86) \quad \begin{aligned} A_{\text{odd}}^{(2k-1,1)} &\subset A_{\text{odd}}^{(2k+2s-1,1)}, & A_{\text{odd}}^{(2k-1,0)} &\subset A_{\text{odd}}^{(2k+2s-1,0)}, \\ A_{\text{odd}}^{(2k-1,1)} &\subset A_{\text{odd}}^{(2k+2s-1,0)}, & s = 0, 1, \dots, \\ \frac{\partial X}{\partial \xi_1} &\in A_{\text{odd}}^{(-1,0)}, & X_2 &\in A_{\text{odd}}^{(1,1)}. \end{aligned}$$

We also set

$$\begin{aligned} A_{\text{odd}}^{(2k-1,q)} = & \left\{ V(\xi; x_1) : V(\xi; x_1) = \sum_{j=1}^J \sigma_j(x_1) Y_j(\xi), Y_j \in A_{\text{odd}}^{(2k-1,q)}, \right. \\ & \left. \sigma_j \in C^\infty \left[-\frac{\pi}{2}, \frac{\pi}{2} \right], (\sigma_j)^{(2n)} \left(\pm \frac{\pi}{2} \right) = 0, n = 0, 1, \dots \right\}, \end{aligned}$$

where J is arbitrary (not fixed and finite).

It follows immediately from the definition of these classes, as well as from (14), that the following hold.

$$(87) \quad \begin{aligned} 1. \quad & \text{If } V \in \mathcal{A}_{\text{odd}}^{(2k-1,1)}, \quad \text{then} \quad \frac{\partial^2 V}{\partial \xi_1 \partial x_1} \in \mathcal{A}_{\text{even}}^{(2k-2,0)} \quad \text{and} \quad \frac{\partial^2 V}{\partial x_1^2} \in \mathcal{A}_{\text{odd}}^{(2k-1,1)}. \\ 2. \quad & \text{If } V \in \mathcal{A}_{\text{even}}^{(2k,1)}, \quad \text{then} \quad \frac{\partial^2 V}{\partial \xi_1 \partial x_1} \in \mathcal{A}_{\text{odd}}^{(2k-1,0)} \quad \text{and} \quad \frac{\partial^2 V}{\partial x_1^2} \in \mathcal{A}_{\text{even}}^{(2k,1)}. \end{aligned}$$

We introduce the following classes of functions:

$$\begin{aligned} \mathbb{A}^{(2k+1,q)} &= \{V(\xi; x_1) : V(\xi; x_1) = \widehat{Z}(\xi; x_1) + \check{Z}(\xi; x_1), \widehat{Z} \in \mathcal{A}_{\text{odd}}^{(2k+1,q)}, \check{Z} \in \mathcal{A}_{\text{even}}^{(2k,q)}\}, \\ \mathbb{A}^{(2k,q)} &= \{V(\xi; x_1) : V(\xi; x_1) = \widehat{Z}(\xi; x_1) + \check{Z}(\xi; x_1), \widehat{Z} \in \mathcal{A}_{\text{odd}}^{(2k-1,q)}, \check{Z} \in \mathcal{A}_{\text{even}}^{(2k,q)}\}, \end{aligned}$$

where J is arbitrary (finite but not fixed).

These classes satisfy:

$$(88) \quad \mathbb{A}^{(k,1)} \subset \mathbb{A}^{(k+2s,1)}, \quad \mathbb{A}^{(k,0)} \subset \mathbb{A}^{(k+2s,0)}, \quad \mathbb{A}^{(k,1)} \subset \mathbb{A}^{(k+2s,0)}, \quad s = 0, 1, \dots$$

Theorem 3.1. a) Let F be represented in the form $F = \mathcal{P}_t + f$, where $f \in \mathbb{A}^{(k,0)}$ and $\mathcal{P}_{t+2}$ is a polynomial such that $-\mathcal{P}_{t+2}'' = \mathcal{P}_t$. Then there exists a solution of the boundary-value problem

$$(89) \quad \begin{cases} -\Delta_\xi W = F & \text{in } \Pi, \\ \frac{\partial W}{\partial \xi_2} = 0 & \text{on } \Sigma, \\ \frac{\partial W}{\partial \xi_1} = 0 & \text{at } \xi_1 = \pm \frac{\pi}{2}, \quad x_1 = \pm \frac{\pi}{2}, \end{cases}$$

that can be represented in the form $W = \mathcal{P}_{t+2} + W_1$, where $W_1 \in \mathbb{A}^{(k+2,1)}$.

b) The function W satisfies the equalities

$$\begin{aligned} \frac{\partial W}{\partial x_1} \left(\xi; \pm \frac{\pi}{2} \right) &= 0 \quad \text{at} \quad \xi_1 = \pm \frac{\pi}{2}, \\ \frac{\partial W}{\partial x_1} \left(\frac{x}{\varepsilon}; x_1 \right) &= 0 \quad \text{at} \quad x_1 = \pm \frac{\pi}{2}. \end{aligned}$$

The proof of the theorem will be given below in Appendix 1 (§4).

Lemma 3.3. a) Suppose that

$$V(\xi; x_1) = \sum_{k=1}^{\infty} \varepsilon^k v_k(\xi; x_1),$$

where $v_k = \mathcal{P}_k + \check{v}_k$ for $\check{v}_k \in \mathbb{A}^{(k,1)}$ with asymptotics of the form $\check{v}_k = \ln \rho \mathcal{M}_k + \check{\mathcal{M}}_k$ as $\rho \rightarrow 0$. Then the asymptotics of the series V as $\rho \rightarrow 0$ rewritten in terms of ζ can be represented in the form

$$(90) \quad \begin{aligned} V(\xi; x_1) &= \ln a \sum_{k=1}^{\infty} \varepsilon^k \mathcal{M}_k(x_1) + \sum_{k=1}^{\infty} \varepsilon^k (\mathcal{M}_k(x_1) \ln \tau + \check{\mathcal{M}}_k(x_1)) \\ &+ O(a\tau \ln(a\tau)) \quad \text{as} \quad a\tau \rightarrow 0. \end{aligned}$$

Furthermore,

$$(91) \quad \frac{d^{2j+1} \mathcal{M}_k}{dx_1^{2j+1}} \left(\pm \frac{\pi}{2} \right) = 0, \quad \frac{d^{2j+1} \check{\mathcal{M}}_k}{dx_1^{2j+1}} \left(\pm \frac{\pi}{2} \right) = 0, \quad j = 0, \dots$$

b) If in addition the v_j are solutions of the problems

$$(92) \quad \begin{cases} -\Delta_\xi v_j = 2 \frac{\partial^2 v_{j-1}}{\partial x_1 \partial \xi_1} + \frac{\partial^2 v_{j-2}}{\partial x_1^2} + \sum_{k=0}^{j-2} \lambda_k v_{j-2-k} & \text{in } \Pi, \\ \frac{\partial v_j}{\partial \xi_2} = 0 & \text{on } \Sigma, \\ \frac{\partial v_j}{\partial \xi_1}(\xi; \pm \frac{\pi}{2}) = 0 & \text{at } \xi_1 = \pm \frac{\pi}{2}, \\ v_j = V_j + o(1) & \text{as } \xi_2 \rightarrow +\infty, \end{cases}$$

then the v_j are solutions of the problems (24).

Proof. The assertion of part a) follows immediately from the definition of the class $\mathbb{A}^{(k,1)}$. We will prove part b). Let the v_j be solutions of the homogeneous boundary-value problems (92). It follows from part b) of Theorem 3.1 that

$$\frac{\partial v_j}{\partial x_1} \left(\xi; \pm \frac{\pi}{2} \right) = 0$$

for all $j = 1, 2, \dots$ and, consequently, the v_j are solutions of the problems (24). $\square$

We seek the terms of the intermediate expansion v_k that can be represented in the form of the sum of the polynomial V_k and a function in the class $\mathbb{A}^{(k,1)}$ and are solutions of (92) and, consequently, by part b) of Lemma 3.3, also solutions of (24). Notice that the functions v_1, v_2 , and v_3 constructed earlier possess these properties.

Following the method of matched asymptotic expansions, by part a) of Lemma 3.3 we deduce from the representation (90) that the inner expansion must be sought in the form

$$(93) \quad u_\varepsilon = \sum_{k=1}^{\infty} \varepsilon^k \ln a w_{k,1,0} + \sum_{k=1}^{\infty} \varepsilon^k w_{k,0,0} + \dots,$$

where the term $w_{k,0,0}$ has an asymptotic as $\tau \rightarrow +\infty$ of the form

$$(94) \quad w_{k,0,0} \sim \mathcal{M}_k \ln \tau + \check{\mathcal{M}}_k,$$

while the term $w_{1,1,0}$ has an asymptotic as $\tau \rightarrow +\infty$ of the form

$$(95) \quad w_{1,1,0} \sim \mathcal{M}_1.$$

We rewrite the operator of the problem (2) in terms of the inner variables ζ , substitute the expansion (93) and the series (12) into the problem (2), and collect terms with like powers of ε , a , and $\ln a$ (in the equation with $\varepsilon^{k-2}a^{-2}$, in the boundary conditions with $\varepsilon^{k-1}a^{-1}$). Taking the asymptotic behaviour in (94) into account, the functions $w_{k,0,0}$ must satisfy problems of the form

$$(96) \quad \begin{cases} \Delta_\zeta w_{k,0,0} = 0 & \text{in } \mathbb{R}_+^2, \\ \frac{\partial w_{k,0,0}}{\partial \zeta_2} = 0 & \text{on } \Gamma, \\ w_{k,0,0} = 0 & \text{on } \gamma, \\ w_{k,0,0} \sim \mathcal{M}_k \ln \tau + \check{\mathcal{M}}_k & \text{as } \tau \rightarrow +\infty. \end{cases}$$

Obviously, if

$$(97) \quad \check{\mathcal{M}}_k = \mathcal{M}_k \ln 2,$$

then the problem turns into

$$(98) \quad \begin{cases} \Delta_\zeta w_{k,0,0} = 0 & \text{in } \mathbb{R}_+^2, \\ \frac{\partial w_{k,0,0}}{\partial \zeta_2} = 0 & \text{on } \Gamma, \\ w_{k,0,0} = 0 & \text{on } \gamma, \\ w_{k,0,0} \sim \mathcal{M}_k (\ln \tau + \ln 2) & \text{as } \tau \rightarrow +\infty; \end{cases}$$

and the functions

$$(99) \quad w_{k,0,0} = \mathcal{M}_k Y$$

are solutions of (98).

We now show that we can satisfy (97) by finding all the α_k^0 in turn. Moreover, we have already done this for $k = 1, 2, 3$ by choosing α_1^0 , α_2^0 , and α_3^0 in accordance with (46), (63), and (84).

We will construct the rest by induction.

Suppose that α_k^0 , v_k , u_k , and λ_k ($k \leq n$) are defined such that

- ($\phi 1$) the α_k^0 satisfy (14);
- ($\phi 2$) the pairs $u_k \in C^\infty(\overline{\Omega})$ and λ_k satisfy (13);
- ($\phi 3$) $v_k = V_k + \tilde{v}_k$, $\tilde{v}_k \in \mathbb{A}^{(k,1)}$, and the v_k are solutions of (92) (and also of (24) by part b) of Lemma 3.3);
- ($\phi 4$) the v_k have the asymptotic behaviour

$$(100) \quad v_k(\xi; x_1) = \mathcal{M}_k(x_1) (\ln \rho + \ln 2) + O(\rho \ln \rho) \quad \text{as } \rho \rightarrow 0,$$

where the $\mathcal{M}_k$ satisfy (91).

Remark 3.4. We emphasize that we have already determined α_1^0 , α_2^0 , and α_3^0 , u_1 , u_2 , and u_3 , λ_1 , λ_2^0 , and λ_3^0 , as well as v_1 , v_2 , and v_3 such that assertions ($\phi 1$)–($\phi 4$) hold. Note, in particular, that

$$(101) \quad \mathcal{M}_1 = \alpha_0^1.$$

Consider the $(n+1)$ th step. By (17), the polynomial V_{n+1} can be represented in the form

$$(102) \quad V_{n+1} = \tilde{V}_{n+1} + \alpha_{n+1}^0,$$

where $\tilde{V}_{n+1} = \mathcal{P}_{n+1}$ is a given polynomial (defined earlier) that is independent of α_{n+1}^0 . By part a) of Theorem 3.1 and (25) there exists a solution of the boundary-value problem (92) (for $j = n+1$) that can be represented in the form

$$(103) \quad v_{n+1} = V_{n+1} + \tilde{v}_{n+1},$$

where $\tilde{v}_{n+1} \in \mathbb{A}^{(n+1,1)}$ is a completely defined function that is independent of α_{n+1}^0 . Thus, part ($\phi 3$) holds for $k = n+1$.

By (103), the definition of the class $\mathbb{A}^{(n,1)}$ implies that at zero the function v_{n+1} has the asymptotic behaviour

$$(104) \quad v_{n+1}(\xi; x_1) = \mathcal{M}_{n+1}(x_1) \ln \rho + \widehat{\mathcal{M}}_{n+1}(x_1) + \alpha_{n+1}^0(x_1) + O(\rho \ln \rho) \quad \text{as } \rho \rightarrow 0,$$

where

$$(105) \quad \frac{d^{2j+1} \mathcal{M}_{n+1}}{dx_1^{2j+1}} \left(\pm \frac{\pi}{2} \right) = 0, \quad \frac{d^{2j+1} \widehat{\mathcal{M}}_{n+1}}{dx_1^{2j+1}} \left(\pm \frac{\pi}{2} \right) = 0, \quad j = 0, 1, \dots$$

We set

$$(106) \quad \alpha_{n+1}^0(x_1) = \mathcal{M}_{n+1} \ln 2 - \widehat{\mathcal{M}}_{n+1}.$$

Then in view of (105), parts $(\phi 1)$ and $(\phi 4)$ hold for $k = n + 1$.

Taking (49) and (67) into account, we rewrite the boundary-value problem (13) for $j = n + 1$:

$$(107) \quad \begin{cases} -\Delta u_{n+1} = \lambda_0 u_{n+1} + \lambda_3 u_{n-2} + \cdots + \lambda_{n+1} u_0 & \text{for } x \in \Omega, \\ u_{n+1} = \alpha_{n+1}^0 & \text{for } x \in \Gamma_4, \\ \frac{\partial u_{n+1}}{\partial \nu} = 0 & \text{for } x \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_3, \end{cases}$$

where α_{n+1}^0 is the function already defined by equality (106). As this problem is solvable, we determine λ_{n+1} and, correspondingly, find $u_{n+1} \in C^\infty(\overline{\Omega})$. Thus, part $(\phi 2)$ holds for $k = n + 1$. Recall that w_{n+1} is chosen in accordance with (99).

As a result we have proved the following theorem.

Theorem 3.2. *There exist series (11), (12), (18), and (93) such that*

- a) *the pairs $u_j \in C^\infty(\overline{\Omega})$ and λ_j are solutions of the problems (13);*
- b) *the coefficients satisfy $\lambda_1 = \lambda_2 = 0$ and $u_1 \equiv 0$, $u_2 \equiv 0$;*
- c) *the functions v_j can be represented as the sums $v_j = V_j + \tilde{v}_j$, $\tilde{v}_j \in \mathbb{A}^{(j,1)}$, and are solutions of the problems (24);*
- d) *the series (18) has asymptotic behaviour as $\rho \rightarrow 0$, rewritten in the variables ζ , that has the form*

$$(108) \quad \begin{aligned} u_\varepsilon = \ln a \sum_{k=1}^{\infty} \varepsilon^k \mathcal{M}_k(x_1) + \sum_{k=1}^{\infty} \varepsilon^k \mathcal{M}_k(x_1) (\ln \tau + \ln 2) \\ + O(a\tau \ln(a\tau)) \quad \text{as } a\tau \rightarrow 0, \end{aligned}$$

where

$$(109) \quad \frac{d^{2j+1} \mathcal{M}_k}{dx_1^{2j+1}} \left(\pm \frac{\pi}{2} \right) = 0, \quad j = 0, 1, \dots;$$

- e) *the coefficients of the series (93), $w_{k,0,0} = \mathcal{M}_k Y$, are solutions of problems (98).*

Remark 3.5. In the asymptotic expansion (108) the last expression $O(a\tau \ln(a\tau))$ denotes an infinite sum of terms each of which has this order.

This finishes the construction of the terms in powers of ε .

3.2. The construction of the asymptotics. Second step. In this subsection we assume that the coefficients v_k , u_k , λ_k , and $w_{k,0,0}$ have all been constructed according to the scheme in the preceding subsection; that is, they all satisfy the statement of Theorem 3.2.

We now return to the terms $w_{k,1,0}$ in the expansion (93). We give the problem for the function $w_{1,1,0}$. We rewrite the operator of problem (2) in the inner variables ζ , substitute the expansion (93) and the series (12) into (2), and collect terms with like powers of ε , a , and $\ln a$ (in the equation with $\varepsilon^{-1} a^{-2} \ln a$, in the boundary conditions with $a^{-1} \ln a$). In view of (95) and (101) the problem has the form

$$(110) \quad \begin{cases} \Delta_\zeta w_{1,1,0} = 0 & \text{in } \mathbb{R}_+^2, \\ \frac{\partial w_{1,1,0}}{\partial \zeta_2} = 0 & \text{on } \Gamma, \\ w_{1,1,0} = 0 & \text{on } \gamma, \\ w_{1,1,0} \sim \alpha_0^1 & \text{as } \tau \rightarrow +\infty. \end{cases}$$

This problem has no solution; therefore the term $w_{1,1,0}$ cannot be present in the expansion. So that this can happen, in the intermediate expansion we introduce the additional term $\varepsilon \ln a v_{1,1}$ with asymptotic behaviour

$$(111) \quad v_{1,1} \sim -\alpha_0^1 \quad \text{as } \rho \rightarrow 0;$$

that is, the expansion takes the form

$$(112) \quad u_\varepsilon(x) = \sum_{j=1}^{+\infty} \varepsilon^j v_j\left(\frac{x}{\varepsilon}; x_1\right) + \varepsilon \ln a v_{1,1}\left(\frac{x}{\varepsilon}; x_1\right) + \dots$$

Rewriting the asymptotics of the series (112) as $\rho \rightarrow 0$ in terms of the inner variables ζ , we see that, by (111) and (108), the asymptotics takes the form

$$(113) \quad \begin{aligned} u_\varepsilon = \ln a \sum_{k=2}^{\infty} \varepsilon^k \mathcal{M}_k(x_1) + \sum_{k=1}^{\infty} \varepsilon^k \mathcal{M}_k(x_1) (\ln \tau + \ln 2) \\ + O(a\tau \ln(a\tau)) \quad \text{as } a\tau \rightarrow 0, \end{aligned}$$

where the $\mathcal{M}_k$ are the same as before. Next, by following the matching method, we conclude that the inner expansion has the form

$$(114) \quad u_\varepsilon = \sum_{k=2}^{\infty} \varepsilon^k \ln a w_{k,1,0} + \sum_{k=1}^{\infty} \varepsilon^k w_{k,0,0} + \dots;$$

that is, in the expansion (93) there is no term $w_{1,1,0}$, while the asymptotics as $\tau \rightarrow +\infty$ for the terms $w_{k,0,0}$ have not changed and therefore they remain to be chosen in accordance with (99).

Substituting the series (112) and the expansion (12) into the problem (2) (first having rewritten it in terms of the coordinates ξ ; see (20)) and collecting terms in powers $\varepsilon^{-1} \ln a$ in the equation and in $\ln a$ in the boundary conditions, by using (111) we conclude that the problem for $v_{1,1}$ has the form

$$(115) \quad \begin{cases} \Delta_\xi v_{1,1} = 0 & \text{in } \Pi, \\ \frac{\partial v_{1,1}}{\partial \xi_2} = 0 & \text{on } \Sigma, \\ \frac{\partial v_{1,1}}{\partial \xi_1}(\xi; \pm \frac{\pi}{2}) = 0 & \text{at } \xi_1 = \pm \frac{\pi}{2}, \\ v_{1,1} \sim -\alpha_0^1 & \text{as } \rho \rightarrow 0. \end{cases}$$

Obviously, the function

$$(116) \quad v_{1,1}(\xi; x_1) \equiv -\alpha_0^1(x_1)$$

is a solution of (115).

We rewrite the asymptotics of the series (112) as $\xi_2 \rightarrow +\infty$, taking the asymptotics of terms (32), (56), (103), (116) which we have already constructed and the equality $\alpha_1^0(x_1) \equiv 0$ into account; following the matching method we conclude that the outer expansion must be sought in the form

$$(117) \quad u_\varepsilon(x) = u_0(x) + \sum_{j=1}^{+\infty} \varepsilon^j \ln a u_{j,1} + \sum_{j=1}^{+\infty} \varepsilon^j u_j(x) + \dots,$$

where $u_{1,1} \sim -\alpha_0^1$ as $x_2 \rightarrow 0$. We seek the functions $u_{j,1}$ in $C^\infty(\overline{\Omega})$; therefore this asymptotic behaviour is equivalent to the boundary conditions

$$(118) \quad u_{1,1}(x_1, 0) = -\alpha_0^1(x_1), \quad x_1 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Corresponding to (117), we seek an asymptotic expansion of the eigenvalue λ_ε in the form

$$(119) \quad \lambda_\varepsilon = \lambda_0 + \sum_{j=1}^{+\infty} \varepsilon^j \ln a \lambda_{j,1} + \sum_{j=1}^{+\infty} \varepsilon^j \lambda_j + \dots$$

By substituting the expansion (117) for u_ε in terms of the variables x and the expansion (119) into the problem and collecting terms in the powers $\varepsilon \ln a$, we obtain the following problem for $u_{1,1}$:

$$(120) \quad \begin{cases} -\Delta u_{1,1} = \lambda_0 u_{1,1} + \lambda_{1,1} u_0 & \text{for } x \in \Omega, \\ u_{1,1} = -\alpha_0^1 & \text{for } x \in \Gamma_4, \\ \frac{\partial u_{1,1}}{\partial \nu} = 0 & \text{for } x \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_3. \end{cases}$$

Correspondingly, collecting terms for the corresponding powers $\varepsilon^j \ln a$ ($j > 1$) in the equation and boundary conditions, based on the same considerations as in Remark 3.1, we obtain the following problem for $u_{j,1}$:

$$(121) \quad \begin{cases} -\Delta u_{j,1} = \lambda_0 u_{j,1} + \dots + \lambda_{j-1} u_{1,1} + \lambda_{1,1} u_{j-1} + \dots + \lambda_{j,1} u_0 & \text{for } x \in \Omega, \\ u_{j,1} = \alpha_{j,1}^0 & \text{for } x \in \Gamma_4, \\ \frac{\partial u_{j,1}}{\partial \nu} = 0 & \text{for } x \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_3, \end{cases}$$

where the $\alpha_{j,1}^0(x_1)$ are, for the moment, unknown (arbitrary) functions, whose odd derivatives vanish at $x_1 = \pm\pi/2$.

We now obtain an explicit formula for $\lambda_{1,1}$. We multiply the equation by u_0 , integrate over the domain Ω taking the normalization of u_0 into account, and apply Green's formula twice to the left-hand side of the equality. Finally we obtain

$$(122) \quad \lambda_{1,1} = \int_{-\pi/2}^{\pi/2} \alpha_0^{12}(x_1) dx_1 > 0 \quad \left(\text{or } \lambda_{1,1} = \int_{\Gamma_4} \left(\frac{\partial u_0}{\partial \nu} \right)^2 ds > 0 \right).$$

So that problems (120) and (121) are uniquely solvable, we choose all solutions to be orthogonal to u_0 in $L_2(\Omega)$.

We expand the functions $u_{j,1}(x)$ in a Taylor series in the variable x_2 as $x_2 \rightarrow 0$. We make the change of variables $\xi_2 = x_2/\varepsilon$ (that is, we rewrite these expansions in the 'intermediate' variables). Then the asymptotic expression of the series (117) as $\varepsilon \xi_2 \rightarrow 0$ has the form

$$(123) \quad V_{k,1}(\xi_2; x_1) + \sum_{k=1}^{\infty} \varepsilon^k V_k(\xi_2; x_1) + \dots,$$

where the $V_k(\xi_2; x_1)$ are already known, while the $V_{k,1}(\xi_2; x_1)$ are polynomials of order $k-1$ in the variable ξ_2 , with coefficients that are infinitely differentiable functions of x_1 on $[-\pi/2, \pi/2]$, whose odd derivatives vanish at $x_1 = \pm\pi/2$. We have the representation

$$(124) \quad V_{k,1}(\xi_2; x_1) = \tilde{V}_{k,1}(\xi_2; x_1) + \alpha_{k,1}^0(x_1)$$

and the $\tilde{V}_{k,1}$ are independent of $\alpha_{j,1}^0$, $j \geq k$. In this notation, $\alpha_{1,1}^0 \equiv -\alpha_0^1$ and $\tilde{V}_{1,1} \equiv 0$. By the boundary-value problems (120), (121) and the definition of the $V_{q,1}$, we have the

following equalities:

$$(125) \quad -\Delta_\xi V_{q,1} = 2 \frac{\partial^2 V_{q-1,1}}{\partial x_1 \partial \xi_1} + \frac{\partial^2 V_{q-2,1}}{\partial x_1^2} + \lambda_0 V_{q-2,1} + \dots \\ + \lambda_{q-3} V_{1,1} + \lambda_{1,1} V_{q-3} + \dots + \lambda_{q-3,1} V_1,$$

$$(126) \quad \frac{\partial V_{q,1}}{\partial x_1}(\xi_2; \pm \frac{\pi}{2}) = 0, \quad \frac{\partial V_{q,1}}{\partial \xi_1}(\xi_2; x_1) \equiv 0.$$

We point out that, as before, the condition

$$(127) \quad \frac{d^{2j+1} \alpha_{k,1}^0}{dx_1^{2j+1}} \left(\pm \frac{\pi}{2} \right) = 0, \quad j = 0, 1, \dots, \quad k = 2, 3, \dots$$

is important and we shall keep an eye on its validity.

Taking (123) and (117) into account and following the method of matched asymptotic expansions, we see that the intermediate expansion must have the structure

$$(128) \quad u_\varepsilon(x) = \sum_{k=1}^{+\infty} \varepsilon^k \ln a v_{k,1} \left(\frac{x}{\varepsilon}; x_1 \right) + \sum_{j=1}^{+\infty} \varepsilon^j v_j \left(\frac{x}{\varepsilon}; x_1 \right) + \dots,$$

where

$$(129) \quad v_{q,1}(\xi; x_1) = V_{q,1}(\xi_2; x_1) + o(1) \quad \text{as } \xi_2 \rightarrow +\infty.$$

In what follows we construct the coefficients $v_{k,1}$ in the form $v_{k,1} = V_{k,1} + \tilde{v}_{k,1}$, where $\tilde{v}_{k,1} \in \mathbb{A}^{(k-1,1)}$. Substituting the series (128) and the expansion (122) into the problem (2) (first having rewritten it in terms of ξ ; see (20)) and collecting terms in $\varepsilon^{k-2} \ln a$ in the equation, and in $\varepsilon^{k-1} \ln a$ in the boundary conditions, using (129) we conclude that the $v_{k,1}$ satisfy problems of the form

$$(130) \quad \begin{cases} \Delta_\xi v_{k,1} = 2 \frac{\partial^2 v_{k-1,1}}{\partial x_1 \partial \xi_1} + \frac{\partial^2 v_{k-2,1}}{\partial x_1^2} + \lambda_0 v_{k-2,1} + \dots + \lambda_{k-3} v_{1,1} \\ \quad + \lambda_{1,1} v_{k-3} + \dots + \lambda_{k-3,1} v_1 \quad \text{in } \Pi, \\ \frac{\partial v_{k,1}}{\partial \xi_2} = 0 \quad \text{on } \Sigma, \\ \frac{\partial v_{k,1}}{\partial \xi_1} \left(\xi; \pm \frac{\pi}{2} \right) = - \frac{\partial v_{k-1,1}}{\partial x_1} \left(\xi; \pm \frac{\pi}{2} \right) \quad \text{at } \xi_1 = \pm \frac{\pi}{2}, \\ v_{k,1} = V_{k,1} + o(1) \quad \text{as } \xi_2 \rightarrow +\infty. \end{cases}$$

Rewriting the problem for $k = 2$, taking (116) and (127) into account, we obtain

$$(131) \quad \begin{cases} -\Delta_\xi v_{2,1} = 0 \quad \text{in } \Pi, \\ \frac{\partial v_{2,1}}{\partial \xi_2} = 0 \quad \text{on } \Sigma, \\ \frac{\partial v_{2,1}}{\partial \xi_1} \left(\xi; \pm \frac{\pi}{2} \right) = 0 \quad \text{at } \xi_1 = \pm \frac{\pi}{2}, \\ v_{2,1} = \alpha_{1,1}^1 \xi_2 + \alpha_{2,1}^0 \quad \text{as } \xi_2 \rightarrow +\infty. \end{cases}$$

By Theorem 3.1,

$$(132) \quad v_{2,1} = V_{2,1} + \tilde{v}_{2,1}$$

is a solution of problem (131), where $\tilde{v}_{2,1} \in \mathbb{A}^{(1,1)}$ is a completely defined function that is independent of $\alpha_{2,1}^0$.

In view of (132), the asymptotics of this function at zero (as $\rho \rightarrow 0$) look like:

$$(133) \quad v_{2,1}(\xi; x_1) = \mathcal{M}_{2,1}(x_1) \ln \rho + \widehat{\mathcal{M}}_{2,1}(x_1) + \alpha_{2,1}^0(x_1) + O(\rho).$$

We rewrite the asymptotics of $\varepsilon^2 \ln a v_{2,1}(\xi; x_1)$ as $\rho \rightarrow 0$ in terms of ζ , taking (133) into account. As $a\tau \rightarrow 0$,

$$(134) \quad \begin{aligned} \varepsilon^2 \ln a v_{2,1}(a\zeta; x_1) &= \varepsilon^2 \ln a \mathcal{M}_{2,1}(x_1) [\ln a + \ln \tau] + \varepsilon^2 \ln a \widehat{\mathcal{M}}_{2,1}(x_1) \\ &\quad + \varepsilon^2 \ln a \alpha_{2,1}^0(x_1) + O(\varepsilon^2 \ln a a^2 \tau^2). \end{aligned}$$

Rewriting the asymptotics of the series (128) as $\rho \rightarrow 0$ in terms of ζ , taking (134) into account, and following the method of matched asymptotic expansions, we conclude that in (114) as $\tau \rightarrow +\infty$ the term $w_{2,1,0}$ has asymptotics of the form

$$(135) \quad w_{2,1,0} \sim \mathcal{M}_{2,1} \ln \tau + \widehat{\mathcal{M}}_{2,1} + \mathcal{M}_2 + \alpha_{2,1}^0.$$

Taking the form of equation (40) into account in the inner variables ζ , we substitute the expansion (114) and the series (119) into the problem (2) and collect terms with like powers of ε , a and $\ln a$ (in the equation with $a^{-2} \ln a$, and in the boundary conditions, with $\varepsilon a^{-1} \ln a$, respectively). In view of (135) we obtain a problem for the function $w_{2,1,0}$ of the form

$$(136) \quad \begin{cases} \Delta_\zeta w_{2,1,0} = 0 & \text{in } \mathbb{R}_+^2, \\ \frac{\partial w_{2,1,0}}{\partial \zeta_2} = 0 & \text{on } \Gamma, \\ w_{2,1,0} = 0 & \text{on } \gamma, \\ w_{2,1,0} \sim \mathcal{M}_{2,1} \ln \tau + \widehat{\mathcal{M}}_{2,1} + \mathcal{M}_2 + \alpha_{2,1}^0 & \text{as } \tau \rightarrow +\infty. \end{cases}$$

Taking into account the form of the asymptotics (44) and problems (43), we conclude that the function

$$(137) \quad w_{2,1,0}(\zeta; x_1) = \mathcal{M}_{2,1}(x_1) Y(\zeta)$$

is a solution of problem (136) for

$$(138) \quad \alpha_{2,1}^0(x_1) \equiv \mathcal{M}_{2,1} \ln 2 - \widehat{\mathcal{M}}_{2,1} - \mathcal{M}_2.$$

Therefore we fix $w_{2,1,0}$ and $\alpha_{2,1}^0$ in accordance with (137) and (138). Then the function $v_{2,1}$ is uniquely determined.

From (121), solved for $u_{2,1}$, we find $\lambda_{2,1}$ and determine $u_{2,1}$.

It is easy to show that the following assertion is true, similarly to Lemma 3.3.

Lemma 3.4. a) *Let*

$$V(\xi; x_1) = \sum_{k=1}^{\infty} \varepsilon^k v_k(\xi; x_1) + \sum_{k=1}^{\infty} \varepsilon^k \ln a v_{k,1}(\xi; x_1),$$

where $v_{k,1} = \mathcal{P}_{k-1} + \widetilde{v}_{k,1}$, $\widetilde{v}_{k,1} \in \mathbb{A}^{(k-1,1)}$, $k \geq 2$, $v_{1,1} = \check{\mathcal{M}}_{1,1}$, and the v_k satisfy the assertions of Theorem 3.2. Then the asymptotics of the series V as $\rho \rightarrow 0$, rewritten in

terms of ζ , can be represented in the form

$$\begin{aligned}
 V(\xi; x_1) = & \ln^2 a \sum_{k=2}^{\infty} \varepsilon^k \mathcal{M}_{k,1}(x_1) + \ln a \varepsilon (\check{\mathcal{M}}_{1,1}(x_1) + \mathcal{M}_1(x_1)) \\
 (139) \quad & + \ln a \sum_{k=2}^{\infty} \varepsilon^k (\mathcal{M}_{k,1}(x_1) \ln \tau + \check{\mathcal{M}}_{k,1}(x_1) + \mathcal{M}_k(x_1)) \\
 & + \sum_{k=1}^{\infty} \varepsilon^k \mathcal{M}_k(x_1) (\ln \tau + \ln 2) + O(a\tau \ln(a\tau)) \quad \text{as } a\tau \rightarrow 0,
 \end{aligned}$$

where the coefficients $\mathcal{M}_k(x_1)$ and $\check{\mathcal{M}}_{k,1}(x_1)$, which satisfy

$$(140) \quad \frac{d^{2j+1} \mathcal{M}_{k,1}}{dx_1^{2j+1}} \left(\pm \frac{\pi}{2} \right) = 0, \quad \frac{d^{2j+1} \check{\mathcal{M}}_{k,1}}{dx_1^{2j+1}} \left(\pm \frac{\pi}{2} \right) = 0, \quad j = 0, \dots,$$

are determined from the asymptotic behaviour of $\tilde{v}_{k,1} = \mathcal{M}_{k,1} \ln \rho + \check{\mathcal{M}}_{k,1}$, $k \geq 2$, as $\rho \rightarrow 0$.

b) If in addition the $v_{j,1}$ are solutions of the problems

$$(141) \quad \begin{cases} \Delta_{\xi} v_{k,1} = 2 \frac{\partial^2 v_{k-1,1}}{\partial x_1 \partial \xi_1} + \frac{\partial^2 v_{k-2,1}}{\partial x_1^2} + \lambda_0 v_{k-2,1} + \dots + \lambda_{k-3} v_{1,1} \\ \quad + \lambda_{1,1} v_{k-3} + \dots + \lambda_{k-3,1} v_1 \quad \text{in } \Pi, \\ \frac{\partial v_{k,1}}{\partial \xi_2} = 0 \quad \text{on } \Sigma, \\ \frac{\partial v_{k,1}}{\partial \xi_1} \left(\xi; \pm \frac{\pi}{2} \right) = 0 \quad \text{at } \xi_1 = \pm \frac{\pi}{2}, \\ v_{k,1} = V_{k,1} + o(1) \quad \text{as } \xi_2 \rightarrow +\infty, \end{cases}$$

then the $v_{j,1}$ are solutions of the problems (130).

We again seek the terms of the intermediate expansion $v_{k,1}$ that are solutions of (141) and, consequently, by part b) of Lemma 3.4, also solutions of (130). Notice that the functions $v_{1,1}$ and $v_{2,1}$ constructed earlier possess these properties.

Following the method of matched asymptotic expansions, by part a) of Lemma 3.4, from the representation (139) we conclude that the inner expansion must be sought in the form

$$(142) \quad u_{\varepsilon} = \sum_{k=2}^{\infty} \varepsilon^k \ln^2 a w_{k,2,0} + \sum_{k=2}^{\infty} \varepsilon^k \ln a w_{k,1,0} + \sum_{k=1}^{\infty} \varepsilon^k w_{k,0,0} + \dots,$$

where, as $\tau \rightarrow +\infty$, the terms $w_{k,1,0}$ have asymptotics of the form

$$(143) \quad w_{k,1,0} \sim \mathcal{M}_{k,1} \ln \tau + \check{\mathcal{M}}_{k,1} + \mathcal{M}_k,$$

and the term $w_{2,2,0}$ has asymptotics of the form

$$(144) \quad w_{2,2,0} \sim \mathcal{M}_{2,1}.$$

Completely similarly to Theorem 3.2, taking Lemma 3.4 (instead of Lemma 3.3) into account, one can prove the following assertion.

Theorem 3.3. *There exist series (117), (119), (128) and (142) such that*

- a) *the coefficients λ_k , u_k , v_k and $w_{k,0,0}$ satisfy the assertions of Theorem 3.2;*
- b) *the pairs $u_{j,1} \in C^{\infty}(\bar{\Omega})$ and $\lambda_{j,1}$ are solutions of problems (121);*
- c) *the coefficient $\lambda_{1,1}$ is determined from formula (122);*

- d) the functions $v_{j,1}$ can be represented as the sums $v_{j,1} = V_{j,1} + \tilde{v}_{j,1}$, $\tilde{v}_{j,1} \in \mathbb{A}^{(j-1,1)}$, $v_{1,1} = -\mathcal{M}_1$, and are solutions of problems (130);
e) in terms of the variables ζ , as $\rho \rightarrow 0$ the series (128) has asymptotics of the form

$$\begin{aligned}
(145) \quad u_\varepsilon = & \ln^2 a \sum_{k=2}^{\infty} \varepsilon^k \mathcal{M}_{k,1}(x_1) \\
& + \ln a \sum_{k=2}^{\infty} \varepsilon^k \mathcal{M}_{k,1}(x_1) (\ln \tau + \ln 2) \\
& + \sum_{k=1}^{\infty} \varepsilon^k \mathcal{M}_k(x_1) (\ln \tau + \ln 2) \\
& + O(a\tau \ln(a\tau)) \quad \text{as } a\tau \rightarrow 0,
\end{aligned}$$

where

$$(146) \quad \frac{d^{2j+1} \mathcal{M}_{k,1}}{dx_1^{2j+1}} \left(\pm \frac{\pi}{2} \right) = 0, \quad j = 0, 1, \dots;$$

- f) the summands $w_{k,1,0} = \mathcal{M}_{k,1} Y$ are solutions of the problems

$$(147) \quad \begin{cases} \Delta_\zeta w_{k,1,0} = 0 & \text{in } \mathbb{R}_+^2, \\ \frac{\partial w_{k,1,0}}{\partial \zeta_2} = 0 & \text{on } \Gamma, \\ w_{k,1,0} = 0 & \text{on } \gamma, \\ w_{k,1,0} \sim \mathcal{M}_{k,1} (\ln \tau + \ln 2) & \text{as } \tau \rightarrow +\infty. \end{cases}$$

The succeeding terms in the expansions in terms of powers of $\ln a$ are also constructed completely similarly. Moreover, the algorithm to construct the terms $u_{k,2}$, $\lambda_{k,2}$, $v_{k,2}$, and $w_{k,2,0}$ of the corresponding series and to eliminate $w_{2,2,0}$ repeats the algorithm used to construct the terms $u_{k,1}$, $\lambda_{k,1}$, $v_{k,1}$, and $w_{k,1,0}$ of the series (117), (119), (128), and (142), respectively, and to eliminate $w_{1,1,0}$. In a similar fashion, the term $w_{3,3,0}$ appears, which plays the same role as the terms $w_{1,1,0}$ and $w_{2,2,0}$. To go further we repeat our arguments.

Indeed, by formally rewriting the problem for the term $w_{2,2,0}$ with the asymptotics (144) we verify that this problem has no solutions. As before, we introduce in the intermediate expansion the term $v_{2,2} \equiv -\mathcal{M}_{2,1}(x_1)$, which is independent of ξ and plays the same role as $v_{1,1}$ given in (116). Then we find the terms $\lambda_{2,2}$ and $u_{2,2}$, which play exactly the same role as $\lambda_{1,1}$ and $u_{1,1}$. Next, by rewriting the asymptotics of the intermediate expansion as $\rho \rightarrow 0$, including these new terms, in terms of the variables ζ , we conclude that the term $w_{3,3,0}$ appears in the inner expansion, which plays the same role as $w_{1,1,0}$ and $w_{2,2,0}$. The rest of the procedure to construct the terms and eliminate $w_{3,3,0}$ is completely similar.

Thus, by continuing the process of constructing the terms of the expansions, we show that we have the next result. We set $\lambda_{k,0} := \lambda_k$, $u_{k,0} := u_k$, $v_{k,0} := v_k$ ($k \geq 1$) and $\lambda_{0,0} := \lambda_0$, $u_{0,0} := u_0$, $v_{0,0} \equiv 0$. We have the following theorem.

Theorem 3.4. *There exist series*

$$(148) \quad u_\varepsilon(x) = \sum_{k=0}^{+\infty} \sum_{j=k}^{+\infty} \varepsilon^j \ln^k a u_{j,k}(x),$$

$$(149) \quad \lambda_\varepsilon = \sum_{k=0}^{+\infty} \sum_{j=k}^{+\infty} \varepsilon^j \ln^k a \lambda_{j,k},$$

$$(150) \quad u_\varepsilon(x) = \sum_{k=0}^{+\infty} \sum_{j=k}^{+\infty} \varepsilon^j \ln^k a v_{j,k}(\xi; x_1), \quad v_{0,0} \equiv 0,$$

$$(151) \quad u_\varepsilon(x) = \sum_{k=0}^{+\infty} \sum_{j=k+1}^{+\infty} \varepsilon^j \ln^k a w_{j,k,0}(\zeta; x_1),$$

such that

a) the pairs $u_{j,k} \in C^\infty(\overline{\Omega})$ and $\lambda_{j,k}$ are solutions of the problems

$$(152) \quad \begin{cases} -\Delta u_{j,k} = \sum_{p=0}^j \sum_{q=p}^{k+j-p} \lambda_{p,q} u_{j-p,k-q} & \text{in } \Omega, \\ \frac{\partial u_{j,k}}{\partial \nu} = 0 & \text{on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_3, \\ u_{j,k} = \alpha_{j,k}^0 & \text{on } \Gamma_4; \end{cases}$$

b) the series (148) has the following asymptotics as $x_2 \rightarrow 0$, rewritten in terms of the variables ξ :

$$(153) \quad u_\varepsilon(x) = \sum_{k=0}^{+\infty} \sum_{j=k}^{+\infty} \varepsilon^j \ln^k a V_{j,k}(\xi_2; x_1) + \dots,$$

where the $V_{j,k}(\xi_2; x_1)$ are polynomials of order $j - k$ in the variable ξ_2 whose coefficients are infinitely differentiable functions of x_1 on $[-\pi/2, \pi/2]$ and whose odd derivatives vanish at $x_1 = \pm\pi/2$;

c) the polynomials $V_{j,k}$ satisfy the equalities

$$(154) \quad \begin{aligned} -\Delta_\xi V_{j,k} &= 2 \frac{\partial^2 V_{j-1,k}}{\partial x_1 \partial \xi_1} + \frac{\partial^2 V_{j-2,k}}{\partial x_1^2} + \sum_{p=0}^{j-2} \sum_{q=p}^{k+j-p} \lambda_{p,q} V_{j-2-p,k-q}, \\ \frac{\partial V_{j,k}}{\partial x_1}(\xi_2; \pm \frac{\pi}{2}) &= 0, \quad \frac{\partial V_{j,k}}{\partial \xi_1}(\xi_2; x_1) \equiv 0; \end{aligned}$$

d) the coefficient $\lambda_{1,1}$ is determined from formula (122);

e) the functions $v_{j,k}$ can be represented as the sums $v_{j,k} = V_{j,k} + \tilde{v}_{j,k}$, $\tilde{v}_{j,k} \in \mathbb{A}^{(j-k,1)}$, $v_{j,j} = \mathcal{M}_{j,j}$, and are solutions of the problems

$$(155) \quad \begin{cases} -\Delta_\xi v_{j,k} = 2 \frac{\partial^2 v_{j-1,k}}{\partial x_1 \partial \xi_1} + \frac{\partial^2 v_{j-2,k}}{\partial x_1^2} \\ \quad + \sum_{p=0}^{j-2} \sum_{q=p}^{k+j-p} \lambda_{p,q} v_{j-2-p,k-q} & \text{in } \Pi, \\ \frac{\partial v_{j,k}}{\partial \xi_2} = 0 & \text{on } \Sigma, \\ \frac{\partial v_{j,k}}{\partial \xi_1}(\xi; \pm \frac{\pi}{2}) = 0 & \text{at } \xi_1 = \pm \frac{\pi}{2}; \end{cases}$$

f) as $\rho \rightarrow 0$ the series (150), rewritten in terms of ζ , has asymptotics of the form

$$(156) \quad u_\varepsilon = \sum_{k=0}^{+\infty} \sum_{j=k}^{+\infty} \varepsilon^j \ln^k a \mathcal{M}_{j,k}(x_1) (\ln \tau + \ln 2) + O(a\tau \ln(a\tau)), \quad \text{as } a\tau \rightarrow 0,$$

where

$$(157) \quad \frac{d^{2l+1} \mathcal{M}_{k,j}}{dx_1^{2l+1}} \left(\pm \frac{\pi}{2} \right) = 0, \quad l = 0, 1, \dots;$$

g) the summands $w_{k,j,0} = \mathcal{M}_{k,j} Y$ are solutions of the problems

$$(158) \quad \begin{cases} \Delta_\zeta w_{k,j,0} = 0 & \text{in } \mathbb{R}_+^2, \\ \frac{\partial w_{k,j,0}}{\partial \zeta_2} = 0 & \text{on } \Gamma, \\ w_{k,j,0} = 0 & \text{on } \gamma, \\ w_{k,j,0} \sim \mathcal{M}_{k,j}(\ln \tau + \ln 2) & \text{as } \tau \rightarrow +\infty. \end{cases}$$

We point out that if there are negative limits of summation in problems (152) and (158) or later in the paper, this means that these sums do not appear; similarly, terms with ‘smaller’ indices, that have not so far appeared, are assumed to be identically zero.

3.3. The construction of the asymptotics. Third step. We now determine the asymptotics (33) more precisely, using (30), the asymptotics (55), the asymptotic behaviour of the function X_2 in Lemma 3.1, and the consequent asymptotics of the terms occurring in the intermediate expansion.

Due to parts e) and f) of Theorem 3.4, rewriting the asymptotics of the series (150) as $\rho \rightarrow 0$ in terms of ζ we obtain

$$(159) \quad \begin{aligned} u_\varepsilon &= \sum_{k=0}^{+\infty} \sum_{j=k}^{+\infty} \varepsilon^j \ln^k a W_{j,k,0}(\zeta; x_1) + \sum_{k=0}^{+\infty} \sum_{j=k}^{+\infty} a \varepsilon^j \ln^k a W_{j,k,1}(\zeta; x_1) \\ &\quad + O(a^2 \tau^2 \ln(a\tau)) \quad \text{as } a\tau \rightarrow 0, \\ W_{j,k,0}(\zeta; x_1) &= \mathcal{M}_{j,k}(x_1) (\ln \tau + \ln 2), \quad j \geq k, \\ W_{j,k,1}(\zeta; x_1) &= (N_{j,k,1}(x_1) \tau \ln \tau \cos \theta + \hat{N}_{j,k,1}(x_1) \tau \cos \theta), \quad j \geq k+1, \\ W_{k,k,1}(\zeta; x_1) &= \hat{N}_{k,k,1}(x_1) \tau \cos \theta, \\ W_{0,0,1} &\equiv 0, \quad W_{1,0,1} \equiv 0, \quad W_{1,1,1} \equiv 0, \quad N_{2,1,1} \equiv 0, \quad W_{2,2,1} \equiv 0. \end{aligned}$$

We observe that using the boundary-value problems (155) and the definition of the $W_{j,k,l}$, the following equality holds:

$$(160) \quad -\Delta_\zeta W_{j,k,1} = 2 \frac{\partial^2 W_{j-1,k,0}}{\partial x_1 \partial \zeta_1},$$

and by the definition of the class $\mathbb{A}^{(k,q)}$ we have

$$(161) \quad \frac{\partial^{2i} N_{j,k,1}}{\partial x_1^{2i}} \left(\xi; \pm \frac{\pi}{2} \right) = 0, \quad \frac{\partial^{2i} \hat{N}_{j,k,1}}{\partial x_1^{2i}} \left(\xi; \pm \frac{\pi}{2} \right) = 0, \quad i = 0, 1, \dots$$

and, consequently,

$$(162) \quad \frac{\partial^{2i} W_{j,k,1}}{\partial x_1^{2i}} \left(\xi; \pm \frac{\pi}{2} \right) = 0, \quad i = 0, 1, \dots$$

Following the method of matched asymptotic expansions, by (159) we conclude that the inner expansion must be sought in the form

$$(163) \quad u_\varepsilon(x) = \sum_{k=0}^{+\infty} \sum_{j=k+1}^{+\infty} \varepsilon^j \ln^k a w_{j,k,0}(\zeta; x_1) + \sum_{k=0}^{+\infty} \sum_{j=k+1}^{+\infty} a \varepsilon^j \ln^k a w_{j,k,1}(\zeta; x_1) + \dots,$$

where $w_{0,0,1} \equiv 0$, $w_{1,0,1} \equiv 0$, $w_{1,1,1} \equiv 0$, $w_{2,2,1} \equiv 0$.

We rewrite the operator in problem (2) in terms of the inner variables ζ , substitute the expansion (163) and the series (149) into (2), and collect the terms with the powers $a^{-1} \varepsilon^{j-2} \ln^k a$ in the equation, and $\varepsilon^{j-1} \ln^k a$ in the boundary conditions. We obtain a problem for the functions $w_{j,k,1}$ of the form

$$(164) \quad \begin{cases} -\Delta_\zeta w_{j,k,1} = 2 \frac{\partial^2 w_{j-1,k,0}}{\partial x_1 \partial \zeta_1} & \text{in } \mathbb{R}_+^2, \\ \frac{\partial w_{j,k,1}}{\partial \zeta_2} = 0 & \text{on } \Gamma, \\ w_{j,k,1} = 0 & \text{on } \gamma, \\ w_{j,k,1} \sim W_{j,k,1} & \text{as } \tau \rightarrow +\infty. \end{cases}$$

We introduce into consideration the auxiliary problem

$$(165) \quad \begin{cases} -\Delta_\zeta Y_1 = 2 \frac{\partial Y}{\partial \zeta_1} & \text{in } \mathbb{R}_+^2, \\ \frac{\partial Y_1}{\partial \zeta_2} = 0 & \text{on } \Gamma, \\ Y_1 = 0 & \text{on } \gamma, \\ Y_1 \sim \tau \ln \tau \cos \theta & \text{as } \tau \rightarrow +\infty. \end{cases}$$

We observe that a solution $Y_1 \in H_{\text{loc}}^1(\mathbb{R}_+^2)$ of (165) can be represented in the form

$$(166) \quad Y_1(\zeta) = \zeta_1 Y(\zeta).$$

We consider the following function Y_2 (see [73]):

$$(167) \quad Y_2(\zeta) = \operatorname{Re} \sqrt{y^2 - 1},$$

where $y = \zeta_1 + i\zeta_2$. It is easy to verify that $Y_2 \in H_{\text{loc}}^1(\mathbb{R}_+^2)$ is a solution of the problem

$$(168) \quad \begin{cases} -\Delta_\zeta Y_2 = 0 & \text{in } \mathbb{R}_+^2, \\ \frac{\partial Y_2}{\partial \zeta_2} = 0 & \text{on } \Gamma, \\ Y_2 = 0 & \text{on } \gamma, \\ Y_2 \sim \tau \cos \theta & \text{as } \tau \rightarrow +\infty. \end{cases}$$

It follows from the explicit form of the functions Y_1 and Y_2 that

$$(169) \quad \begin{aligned} Y_1(\zeta) &= \tau(\ln \tau + \ln 2) \cos \theta + \sum_{j=0}^{\infty} C_{2j+1} \tau^{-2j-1} (\cos((2j+1)\theta) \\ &\quad + \cos((2j+3)\theta)), \\ Y_2(\zeta) &= \tau \cos \theta + \sum_{j=0}^{+\infty} c_{2j+1} \tau^{-2j-1} \cos((2j+1)\theta) \quad \text{as } \tau \rightarrow +\infty. \end{aligned}$$

Now, equation (160) implies that

$$(170) \quad -N_{j,k,1}(x_1) = \mathcal{M}'_{j,k}(x_1).$$

Taking into account the asymptotics (169) and problems (165), (168), from equality (170) we conclude that the function

$$(171) \quad w_{j,k,1}(\zeta; x_1) = N_{j,k,1}(x_1)Y_1(\zeta) + (\widehat{N}_{j,k,1}(x_1) - N_{j,k,1}(x_1) \ln 2)Y_2(\zeta)$$

is a solution of problem (164).

Thus, we have proved the following theorem.

Theorem 3.5. *There exist series (148), (149), (150), and (163) such that*

- a) *the coefficients $\lambda_{j,k,0} = \lambda_{j,k}$, $u_{j,k,0} \equiv u_{j,k}$, $v_{j,k,0} \equiv v_{j,k}$, $w_{j,k,0}$ satisfy the assertions of Theorem 3.4;*
- b) *the asymptotics of the series (150) as $\rho \rightarrow 0$, rewritten in terms of the variables ζ , have the form (159), where the $W_{j,k,1}$ satisfy (162);*
- c) *the terms of the series $w_{j,k,0}$ satisfy the equalities*

$$(172) \quad w_{j,k,0} = W_{j,k,0} + O(\tau^{-2}) \quad \text{as } \tau \rightarrow +\infty;$$

- d) *the functions $w_{j,k,1}$ are solutions of (164) and have the asymptotics*

$$(173) \quad w_{j,k,1} = W_{j,k,1} + O(\tau^{-1}) \quad \text{as } \tau \rightarrow +\infty;$$

- e) *the asymptotics of the series (163) as $\tau \rightarrow +\infty$, rewritten in terms of ξ , have the form*

$$(174) \quad u_\varepsilon = \sum_{k=0}^{+\infty} \sum_{j=k}^{+\infty} \varepsilon^j \ln^k a \mathcal{V}_{j,k,0}(\xi; x_1) + O(a^2 \rho^{-1}) \quad \text{as } \rho a^{-1} \rightarrow +\infty;$$

- f) *the terms of the series $v_{j,k,0}$ satisfy the equalities*

$$(175) \quad v_{j,k,0} = \mathcal{V}_{j,k,0} + O(\rho^2 \ln \rho) \quad \text{as } \rho \rightarrow 0.$$

It follows from part e) of Theorem 3.5 that we do not need to introduce terms of order a in the intermediate expansion (150) and, consequently, we do not need them either in the outer expansion (148), or in the expansion of the eigenvalues (149).

We will determine the asymptotic behaviour (159) again, more precisely. By parts e) and f) of Theorem 3.4 we obtain

$$(176) \quad \begin{aligned} u_\varepsilon &= \sum_{k=0}^{+\infty} \sum_{j=k}^{+\infty} \varepsilon^j \ln^k a W_{j,k,0}(\zeta; x_1) + \sum_{k=0}^{+\infty} \sum_{j=k}^{+\infty} a \varepsilon^j \ln^k a W_{j,k,1}(\zeta; x_1) \\ &\quad + \sum_{k=0}^{+\infty} \sum_{j=k+1}^{+\infty} a^2 \varepsilon^j \ln^k a \widetilde{W}_{j,k,2}(\zeta; x_1) + O(a^3 \tau^3 \ln(a\tau)) \quad \text{as } a\tau \rightarrow 0, \\ \widetilde{W}_{j,k,2}(\zeta; x_1) &= (\tau^2 \ln \tau (N_{j,k,2}^1(x_1) + N_{j,k,2}^2(x_1) \cos 2\theta)) \\ &\quad + \tau^2 (\widehat{N}_{j,k,2}^1(x_1) + \widehat{N}_{j,k,2}^2(x_1) \cos 2\theta), \quad j \geq k+2, \\ \widetilde{W}_{k+1,k,2}(\zeta; x_1) &= \tau^2 (\widehat{N}_{k+1,k,2}^1(x_1) + \widehat{N}_{k+1,k,2}^2(x_1) \cos 2\theta), \\ \widetilde{W}_{1,0,2}(\zeta; x_1) &= \widehat{N}_{1,0,2}^2(x_1) \tau^2 \cos 2\theta, \end{aligned}$$

where the $W_{j,k,0}$ and $W_{j,k,1}$ satisfy (159). We observe that by the boundary-value problems (155) and the definition of the $W_{j,k,0}$, $W_{j,k,1}$, $\widetilde{W}_{j,k,2}$ we have the following equality:

$$(177) \quad -\Delta_\zeta \widetilde{W}_{j,k,2} = 2 \frac{\partial^2 W_{j-1,k,1}}{\partial x_1 \partial \zeta_1} + \frac{\partial^2 W_{j-2,k,0}}{\partial x_1^2} + \sum_{p=0}^{j-2} \sum_{q=p}^{k+j-p} \lambda_{p,q,0} W_{j-2-p,k-q,0},$$

and by the definition of the class $\mathbb{A}^{(k,q)}$ we have

$$(178) \quad \begin{aligned} \frac{\partial^{2i+1} N_{j,k,2}^1}{\partial x_1^{2i+1}} \left(\xi; \pm \frac{\pi}{2} \right) &= 0, & \frac{\partial^{2i+1} \widehat{N}_{j,k,2}^2}{\partial x_1^{2i+1}} \left(\xi; \pm \frac{\pi}{2} \right) &= 0, \\ \frac{\partial^{2i+1} N_{j,k,2}^1}{\partial x_1^{2i+1}} \left(\xi; \pm \frac{\pi}{2} \right) &= 0, & \frac{\partial^{2i+1} \widehat{N}_{j,k,2}^2}{\partial x_1^{2i+1}} \left(\xi; \pm \frac{\pi}{2} \right) &= 0, \quad i = 0, 1, \dots \end{aligned}$$

and, consequently,

$$(179) \quad \frac{\partial^{2i+1} \widetilde{W}_{j,k,2}}{\partial x_1^{2i+1}} \left(\xi; \pm \frac{\pi}{2} \right) = 0, \quad i = 0, 1, \dots$$

Following the method of matched asymptotic expansions, by (176) we conclude that the inner expansion must be sought in the form

$$(180) \quad \begin{aligned} u_\varepsilon(x) &= \sum_{k=0}^{+\infty} \sum_{j=k}^{+\infty} \varepsilon^j \ln^k a w_{j,k,0}(\zeta; x_1) + \sum_{k=0}^{+\infty} \sum_{j=k}^{+\infty} a \varepsilon^j \ln^k a w_{j,k,1}(\zeta; x_1) \\ &+ \sum_{k=0}^{+\infty} \sum_{j=k+1}^{+\infty} a^2 \varepsilon^j \ln^k a w_{j,k,2}(\zeta; x_1) + \dots, \end{aligned}$$

where $w_{0,0,1} \equiv 0$, $w_{1,0,1} \equiv 0$, $w_{1,1,1} \equiv 0$, $w_{2,2,1} \equiv 0$.

We rewrite the operator in problem (2) in terms of the inner variables ζ (see (40)), substitute the expansion (180) and the series (149) into (2), and collect terms in the powers $\varepsilon^{j-2} \ln^k a$ in the equation, and $a \varepsilon^{j-1} \ln^k a$ in the boundary conditions. We obtain problems for the functions $w_{j,k,2}$ of the form

$$(181) \quad \left\{ \begin{aligned} -\Delta_\zeta w_{j,k,2} &= 2 \frac{\partial^2 w_{j-1,k,1}}{\partial x_1 \partial \zeta_1} + \frac{\partial^2 w_{j-2,k,0}}{\partial x_1^2} \\ &+ \sum_{p=0}^{j-2} \sum_{q=p}^{k+j-p} \lambda_{p,q} w_{j-2-p,k-q,0} \quad \text{in } \mathbb{R}_+^2, \\ \frac{\partial w_{j,k,2}}{\partial \zeta_2} &= 0 \quad \text{on } \Gamma, \\ w_{j,k,2} &= 0 \quad \text{on } \gamma, \\ w_{j,k,2} &\sim \widetilde{W}_{j,k,2} \quad \text{as } \tau \rightarrow +\infty. \end{aligned} \right.$$

In particular, the problem for $w_{1,0,2}$ takes the form

$$(182) \quad \left\{ \begin{aligned} -\Delta_\zeta w_{1,0,2} &= 0 \quad \text{in } \mathbb{R}_+^2, \\ \frac{\partial w_{1,0,2}}{\partial \zeta_2} &= 0 \quad \text{on } \Gamma, \\ w_{1,0,2} &= 0 \quad \text{on } \gamma, \\ w_{1,0,2} &\sim \widehat{N}_{1,0,2}^2 \tau^2 \cos \theta \quad \text{as } \tau \rightarrow +\infty. \end{aligned} \right.$$

From the explicit form of the function X and its asymptotics (30) we see that the coefficient satisfies $\widehat{N}_{1,0,2}^1 \equiv 0$. Consider the following function Y_3 (see [73]):

$$(183) \quad Y_3(\zeta) = \operatorname{Re} y \sqrt{y^2 - 1},$$

where $y = \zeta_1 + i\zeta_2$. It is easy to verify that $Y_2 \in H_{\text{loc}}^1(\mathbb{R}_+^2)$ is a solution of

$$(184) \quad \begin{cases} \Delta_\zeta Y_3 = 0 & \text{in } \mathbb{R}_+^2, \\ \frac{\partial Y_3}{\partial \zeta_2} = 0 & \text{on } \Gamma, \\ Y_3 = 0 & \text{on } \gamma, \\ Y_3 \sim \tau^2 \cos 2\theta & \text{as } \tau \rightarrow +\infty. \end{cases}$$

It follows from the explicit form of the function Y_3 that

$$(185) \quad Y_3(\zeta) = \tau^2 \cos 2\theta - \frac{1}{2} + \sum_{j=1}^{+\infty} \mathcal{C}_{2j} \tau^{-2j} \cos(2j\theta) \quad \text{as } \tau \rightarrow +\infty.$$

Taking into account the asymptotic behaviour (185) and problem (184) we conclude that the function

$$(186) \quad w_{1,0,2}(\zeta; x_1) = \widehat{N}_{1,0,2}^2(x_1) Y_3(\zeta)$$

is a solution of (182). Rewriting the asymptotics of $w_{1,0,2}$ as $\tau \rightarrow +\infty$ in terms of ξ , we obtain

$$(187) \quad \varepsilon a^2 w_{1,0,2} = \varepsilon \widehat{N}_{1,0,2}^2 \left(\rho^2 \cos 2\theta - \frac{a^2}{2} + \sum_{j=1}^{+\infty} a^{2j+2} \mathcal{C}_{2j} \rho^{-2j} \cos 2j\theta \right)$$

as $\rho a^{-1} \rightarrow +\infty$.

Rewriting the asymptotics of $w_{1,0,2}$ as $\tau \rightarrow +\infty$ in terms of ξ , from the explicit form of the function (42) we obtain

$$(188) \quad \varepsilon w_{1,0,2} = \varepsilon \alpha_0^1(x_1) \left(\ln \rho + \ln 2 - \ln a + \sum_{j=2}^{\infty} a^{2j} A_{2j} \rho^{-2j} \cos 2j\theta \right)$$

as $\rho a^{-1} \rightarrow +\infty$.

Suppose that all the $w_{j,k,2}$ are solutions of the problems (181). Then, by rewriting the asymptotics of the series (180) as $\tau \rightarrow +\infty$ in terms of ξ , taking (187) and (188) into account, and following the method of matched asymptotic expansions, we conclude that an additional term appears in the intermediate expansion (150):

$$(189) \quad u_\varepsilon(x) = \sum_{k=0}^{+\infty} \sum_{j=k}^{+\infty} \varepsilon^j \ln^k a v_{j,k,0}(\xi; x_1) + \varepsilon a^2 v_{1,0,2} + \dots$$

with the asymptotics at zero

$$(190) \quad v_{1,0,2}(\xi; x_1) \sim A_2 \alpha_0^1(x_1) \rho^{-2} \cos 2\theta - \frac{1}{2} \widehat{N}_{1,0,2}^2 \quad \text{as } \rho \rightarrow 0.$$

After rewriting the equation and the boundary conditions in terms of the coordinates ξ (see (20), (21), and (22)) and substituting the ‘intermediate’ expansion (189) and the series (149) into the problem (2), we collect terms with like powers of ε and a . In view of (190) we obtain the following problem for $v_{1,0,2}$:

$$(191) \quad \begin{cases} \Delta_\xi v_{1,0,2} = 0 & \text{in } \Pi, \\ \frac{\partial v_{1,0,2}}{\partial \xi_2} = 0 & \text{on } \Sigma, \\ \frac{\partial v_{1,0,2}}{\partial \xi_1}(\xi; \pm \frac{\pi}{2}) = 0 & \text{at } \xi_1 = \pm \frac{\pi}{2}, \\ v_{1,0,2} \sim A_2 \alpha_0^1(x_1) \rho^{-2} \cos 2\theta - \frac{1}{2} \widehat{N}_{1,0,2}^2 & \text{as } \rho \rightarrow 0. \end{cases}$$

We introduce into consideration the auxiliary problem

$$(192) \quad \begin{cases} -\Delta_\xi X_3 = 0 & \text{in } \Pi, \\ \frac{\partial X_3}{\partial \xi_2} = 0 & \text{on } \Sigma, \\ \frac{\partial X_3}{\partial \xi_1} = 0 & \text{at } \xi_1 = \pm \frac{\pi}{2}, \\ X_3 \sim \rho^{-2} \cos 2\theta & \text{as } \rho \rightarrow 0. \end{cases}$$

By (28), giving the definition of the function $X(\xi)$ and the boundary-value problem (27), it follows that the function

$$(193) \quad X_3(\xi) = \frac{\partial^2 X}{\partial \xi_2^2}(\xi)$$

is a solution of (192) with asymptotic behaviour as $\xi_2 \rightarrow +\infty$,

$$(194) \quad X_3(\xi) = O(e^{-2\xi_2}),$$

and the asymptotics of this function at zero (as $\rho \rightarrow 0$) is as follows:

$$(195) \quad X_3(\xi) = \rho^{-2} \cos 2\theta + \check{B}_0 + \sum_{j=1}^{+\infty} \check{B}_{2j} \rho^{2j} \cos 2j\theta,$$

where $\check{B}_{2j}$, $j = 0, 1, \dots$, are constants that can be explicitly calculated. By the boundary-value problem (192) for $X_3(\xi)$ it follows that the function

$$(196) \quad v_{1,0,2}(\xi; x_1) = A_2 \alpha_0^1(x_1) X_3(\xi) + \mathcal{K}_{1,0,2}(x_1)$$

is a solution of (191) that is π -periodic and even in ξ_1 , where

$$\mathcal{K}_{1,0,2}(x_1) = -\frac{1}{2} \hat{N}_{1,0,2}^2(x_1) - \check{B}_0 A_2 \alpha_0^1(x_1)$$

and

$$(197) \quad \frac{\partial^{2j+1} \mathcal{K}_{1,0,2}}{\partial x_1^{2j+1}} \left(\pm \frac{\pi}{2} \right) = 0, \quad j = 0, 1, \dots,$$

by (47) and (14). It follows from (196) and (194) that the asymptotic expression for the function $v_{1,0,2}$ as $\xi_2 \rightarrow +\infty$ has the form

$$(198) \quad v_{1,0,2}(\xi; x_1) = \mathcal{K}_{1,0,2}(x_1) + O(e^{-2\xi_2}).$$

Note that due to (14) and (197) we have

$$(199) \quad \frac{\partial v_{1,0,2}}{\partial x_1}(\xi; x_1) = 0 \quad \text{at } x_1 = \pm \frac{\pi}{2}$$

and so, using (21), (22), (23) and (35), we have

$$(200) \quad \frac{\partial v_{1,0,2}}{\partial \nu} \left(\frac{x}{\varepsilon}; x_1 \right) = 0 \quad \text{on } \Gamma_2 \cup \Gamma_3.$$

The more precise asymptotics of the series (189) as $\xi_2 \rightarrow +\infty$, in view of the asymptotic behaviour (198), makes it necessary for us to introduce new terms in the outer expansion. We seek an outer expansion in the form

$$(201) \quad u_\varepsilon(x) = \sum_{k=0}^{+\infty} \sum_{j=k}^{+\infty} \varepsilon^j \ln^k a u_{j,k,0}(x) + \sum_{k=0}^{+\infty} \sum_{j=k}^{+\infty} a^2 \varepsilon^j \ln^k a u_{j,k,2}(x) + \dots,$$

where

$$(202) \quad u_{1,0,2} \sim \mathcal{K}_{1,0,2} \quad \text{as } x_2 \rightarrow 0.$$

Corresponding to (201), we seek an asymptotic expansion of the eigenvalue λ_ε in the form

$$(203) \quad \lambda_\varepsilon = \sum_{k=0}^{+\infty} \sum_{j=k}^{+\infty} \varepsilon^j \ln^k a \lambda_{j,k,0} + \sum_{k=0}^{+\infty} \sum_{j=k}^{+\infty} a^2 \varepsilon^j \ln^k a \lambda_{j,k,2} + \cdots.$$

We seek functions $u_{j,k,2}$ in $C^\infty(\overline{\Omega})$; therefore, in particular, the asymptotic behaviour (202) is equivalent to the boundary conditions

$$(204) \quad u_{1,0,2}(x_1, 0) = \alpha_{1,0,2}^0 \equiv \mathcal{K}_{1,0,2}, \quad x_1 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Substituting the expansion (201) of the function u_ε in terms of x and the expansion (203) into the problem and collecting terms with like powers of $a, \varepsilon, \ln a$, we obtain the following problem for $u_{1,0,2}$:

$$(205) \quad \begin{cases} -\Delta u_{1,0,2} = \lambda_0 u_{1,0,2} + \lambda_{1,0,2} u_0 & \text{for } x \in \Omega, \\ u_{1,0,2} = \alpha_{1,0,2}^0 & \text{for } x \in \Gamma_4, \\ \frac{\partial u_{1,0,2}}{\partial \nu} = 0 & \text{for } x \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \end{cases}$$

and, correspondingly, from the same considerations as in Remark 3.1, we obtain the following problems for the $u_{j,k,2}$:

$$(206) \quad \begin{cases} -\Delta u_{j,k,2} = \sum_{p=0}^j \sum_{q=p}^{k+j-p} \lambda_{p,q,0} u_{j-p,k-q,2} \\ \quad + \sum_{p=0}^j \sum_{q=p}^{k+j-p} \lambda_{p,q,2} u_{j-p,k-q,0} & \text{in } \Omega, \\ u_{j,k,2} = \alpha_{j,k,2}^0 & \text{for } x \in \Gamma_4, \\ \frac{\partial u_{j,k,2}}{\partial \nu} = 0 & \text{for } x \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_3, \end{cases}$$

where $\alpha_{1,0,2}^0 \equiv \mathcal{K}_{1,0,2}$, while the other $\alpha_{j,k,2}^0(x_1)$ are so far unknown (arbitrary) functions whose odd derivatives vanish at $x_1 = \pm\pi/2$.

Next, from the condition that (205) is solvable, we find $\lambda_{1,0,2}$ and determine $u_{1,0,2}$.

In order that problems (205) and (206) be uniquely solvable, we choose all solutions to be orthogonal to u_0 in $L_2(\Omega)$.

We expand the functions $u_{j,k,2}(x)$ into a Taylor series in terms of x_2 as $x_2 \rightarrow 0$. We make the change of variables $\xi_2 = x_2/\varepsilon$ (that is, we rewrite these expansions in the ‘intermediate’ variables). Then the asymptotics of the series (201) as $\varepsilon \xi_2 \rightarrow 0$ has the form

$$(207) \quad u_\varepsilon(x) = \sum_{k=0}^{+\infty} \sum_{j=k}^{+\infty} \varepsilon^j \ln^k a V_{j,k,0}(\xi_2; x_1) + \sum_{k=0}^{+\infty} \sum_{j=k}^{+\infty} a^2 \varepsilon^j \ln^k a V_{j,k,2}(\xi_2; x_1) + \cdots,$$

where the $V_{j,k,0}(\xi_2; x_1)$ are already known, while the $V_{j,k,2}(\xi_2; x_1)$ are polynomials of order $j - k$ in the variable ξ_2 whose coefficients are functions of x_1 that are infinitely differentiable on $[-\pi/2, \pi/2]$ and whose odd derivatives vanish at $x_1 = \pm\pi/2$. We have the representation

$$(208) \quad V_{j,k,2}(\xi_2; x_1) = \tilde{V}_{j,k,2}(\xi_2; x_1) + \alpha_{j,k,2}^0(x_1), \quad \tilde{V}_{1,0,2} \equiv 0,$$

where the $\tilde{V}_{j,k,2}$ are independent of the $\alpha_{s,k,2}^0$, $s \geq j$. By the boundary-value problems (205), (206) and the definition of the $V_{j,k,2}$ we have the following equalities:

$$(209) \quad \begin{aligned} -\Delta_\xi V_{j,k,2} &= 2 \frac{\partial^2 V_{j-1,k,2}}{\partial x_1 \partial \xi_1} + \frac{\partial^2 V_{j-2,k,2}}{\partial x_1^2} \\ &+ \sum_{p=0}^{j-2} \sum_{q=p}^{k+j-p} \lambda_{p,q,0} V_{j-2-p,k-q,2} + \sum_{p=0}^{j-2} \sum_{q=p}^{k+j-p} \lambda_{p,q,2} V_{j-2-p,k-q,0}, \\ \frac{\partial V_{j,k,2}}{\partial x_1} \left(\xi_2; \pm \frac{\pi}{2} \right) &= 0, \quad \frac{\partial V_{j,k,2}}{\partial \xi_1} (\xi_2; x_1) \equiv 0. \end{aligned}$$

We note as before that the condition

$$(210) \quad \frac{d^{2l+1} \alpha_{j,k,2}^0}{dx_1^{2l+1}} \left(\pm \frac{\pi}{2} \right) = 0, \quad l = 0, 1, \dots, \quad j = 2, 3, \dots, \quad k = 0, 1, \dots,$$

is important and we shall check its validity.

By taking (207) and (201) into account and following the method of matched asymptotic expansions, we see that the intermediate expansion must have the following structure:

$$(211) \quad u_\varepsilon(x) = \sum_{k=0}^{+\infty} \sum_{j=k}^{+\infty} \varepsilon^j \ln^k av_{j,k,0}(\xi; x_1) + \sum_{k=0}^{+\infty} \sum_{j=k}^{+\infty} a^2 \varepsilon^j \ln^k av_{j,k,2}(\xi; x_1) + \dots,$$

where

$$(212) \quad v_{q,p,2}(\xi; x_1) = V_{q,p,2}(\xi_2; x_1) + o(1) \quad \text{as } \xi_2 \rightarrow +\infty.$$

We substitute the series (211) and the expansion (203) into (2) (first having rewritten it in terms of ξ) and collect terms with like powers of a , ε , $\ln a$ in the equation and in the boundary conditions; using (212) we conclude that the problems for the $v_{j,k,2}$ have the form

$$(213) \quad \left\{ \begin{aligned} -\Delta_\xi v_{j,k,2} &= 2 \frac{\partial^2 v_{j-1,k,2}}{\partial x_1 \partial \xi_1} + \frac{\partial^2 v_{j-2,k,2}}{\partial x_1^2} \\ &+ \sum_{p=0}^{j-2} \sum_{q=p}^{k+j-p} \lambda_{p,q,0} v_{j-2-p,k-q,2} \\ &+ \sum_{p=0}^{j-2} \sum_{q=p}^{k+j-p} \lambda_{p,q,2} v_{j-2-p,k-q,0} \quad \text{in } \Pi, \\ \frac{\partial v_{j,k,2}}{\partial \xi_2} &= 0 \quad \text{on } \Sigma, \\ \frac{\partial v_{j,k,2}}{\partial \xi_1} \left(\xi; \pm \frac{\pi}{2} \right) &= - \frac{\partial v_{j-1,k,2}}{\partial x_1} \left(\xi; \pm \frac{\pi}{2} \right) \quad \text{at } \xi_1 = \pm \frac{\pi}{2}, \\ v_{j,k,2} &\sim V_{j,k,2} \quad \text{as } \xi_2 \rightarrow +\infty. \end{aligned} \right.$$

The behaviour of the functions $v_{j,k,2}$ at zero will be determined more precisely later. We return to analysing problem (181). To do this we introduce the following classes of functions.

• In order to define new classes of functions, we introduce the following notation: $\tilde{B}$ is the half-disc $\{\zeta : \tau < 1, \zeta_2 \geq 0\}$, $\mathcal{O}_{\varsigma,\pm}$ are ς -neighbourhoods of the points $(\pm 1, 0)$. We denote by $B_{\text{even}}^{(2k, 2n, q)}$ the set of functions

$$g_{\text{even}} \in H_{\text{loc}}^1(\mathbb{R}_+^2) \cap C^\infty(\tilde{B}) \cap H_{\text{loc}}^2(\mathbb{R}_+^2 \setminus \overline{(\mathcal{O}_{\varsigma,+} \cup \mathcal{O}_{\varsigma,-})}) \quad \text{for any } \varsigma$$

that are even in ζ_1 and have an asymptotic expansion at infinity of the form

$$\begin{aligned}
 g_{\text{even}}(\zeta) = & \delta_0^q \sum_{i=1}^{k+1} \eta_{\text{even}}^{(i)} \tau^{2i-2} \cos 2i\theta \\
 & + \sum_{i=0}^k \sum_{j=0}^i \hat{\eta}_{\text{even}}^{(i,j)} \tau^{2i} \cos 2(i-j)\theta \ln \tau \\
 & + \sum_{i=0}^{\infty} \sum_{j=0}^i \tilde{\eta}_{\text{even}}^{(i,j)} \tau^{-2i+2j} \cos 2i\theta \\
 & + \sum_{i=1}^n \sum_{j=0}^i \check{\eta}_{\text{even}}^{(i,j)} \tau^{2i-2j} \cos 2i\theta, \quad \tau \rightarrow +\infty,
 \end{aligned} \tag{214}$$

where δ_0^q is the Kronecker delta. We also point out that

$$\begin{aligned}
 B_{\text{even}}^{(2k,2n,1)} & \subset B_{\text{even}}^{(2k+2s,2n+2t,1)}, & B_{\text{even}}^{(2k,2n,0)} & \subset B_{\text{even}}^{(2k+2s,2n+2t,0)}, \\
 B_{\text{even}}^{(2k,2n,1)} & \subset B_{\text{even}}^{(2k+2s,2n+2t,0)}, & s, t & = 0, 1, \dots
 \end{aligned}$$

We set

$$\begin{aligned}
 \mathcal{B}_{\text{even}}^{(2k,2n,q)} = & \left\{ V(\zeta; x_1) : V(\zeta; x_1) = \sum_{j=1}^J \sigma_j(x_1) Y_j(\zeta), Y_j \in B_{\text{even}}^{(2k,2n,q)}, \right. \\
 & \left. \sigma_j \in C^\infty \left[-\frac{\pi}{2}, \frac{\pi}{2} \right], (\sigma_j)^{(2n+1)} \left(\pm \frac{\pi}{2} \right) = 0, n = 0, 1, \dots \right\},
 \end{aligned}$$

where J is arbitrary (not fixed, but finite). We point out that

$$\frac{\partial V}{\partial x_1}(\zeta; \pm \pi/2) = 0 \quad \text{for all } V \in \mathcal{B}_{\text{even}}^{(2k,2n,q)}.$$

- We denote by $B_{\text{odd}}^{(2k-1,2n-1,q)}$ the set of functions

$$g_{\text{odd}} \in H_{\text{loc}}^1(\mathbb{R}_+^2) \cap C^\infty(\tilde{B}) \cap H_{\text{loc}}^2(\mathbb{R}_+^2 \setminus \overline{(\mathcal{O}_{\varsigma,+} \cup \mathcal{O}_{\varsigma,-})}) \quad \text{for any } \varsigma$$

that are odd in ζ_1 and have asymptotic behaviour at infinity of the form

$$\begin{aligned}
 f_{\text{odd}}(\zeta) = & \delta_0^q \sum_{i=0}^k \eta_{\text{odd}}^{(i)} \tau^{2i-1} \cos(2i+1)\theta \\
 & + \sum_{i=0}^{k-1} \sum_{j=0}^i \hat{\eta}_{\text{odd}}^{(i,j)} \tau^{2i+1} \cos(2i-2j+1)\theta \ln \tau \\
 & + \sum_{i=0}^{\infty} \sum_{j=0}^i \tilde{\eta}_{\text{odd}}^{(i,j)} \tau^{-2i+2j-1} \cos(2i+1)\theta \\
 & + \sum_{i=0}^{n-1} \sum_{j=0}^i \check{\eta}_{\text{odd}}^{(i,j)} \tau^{2i-2j+1} \cos(2i+1)\theta, \quad \tau \rightarrow +\infty,
 \end{aligned}$$

where δ_0^q is the Kronecker delta. We note immediately that the symbol $B_{\text{odd}}^{(2k-1,2n-1,q)}$ for $k = 0$ denotes the class of functions whose asymptotic expansion as $\tau \rightarrow +\infty$ has no terms with $\ln \tau$; correspondingly, for $n = 0$, it denotes the class of functions whose asymptotic expansion as $\tau \rightarrow +\infty$ has no terms with negative powers of τ .

We observe that

$$B_{\text{odd}}^{(2k-1, 2n-1, 1)} \subset B_{\text{odd}}^{(2k+2s-1, 2n+2t-1, 1)}, \quad B_{\text{odd}}^{(2k-1, 2n-1, 0)} \subset B_{\text{odd}}^{(2k+2s-1, 2n+2t-1, 0)},$$

$$B_{\text{odd}}^{(2k-1, 2n-1, 1)} \subset B_{\text{odd}}^{(2k+2s-1, 2n+2t-1, 0)}, \quad s = 0, 1, \dots,$$

and we denote

$$\mathcal{B}_{\text{odd}}^{(2k-1, 2n-1, q)} = \left\{ V(\zeta; x_1) : V(\zeta; x_1) = \sum_{j=1}^J \sigma_j(x_1) Y_j(\zeta), \quad Y_j \in B_{\text{odd}}^{(2k-1, 2n-1, q)}, \right.$$

$$\left. \sigma_j \in C^\infty \left[-\frac{\pi}{2}, \frac{\pi}{2} \right], \quad (\sigma_j)^{(2n)} \left(\pm \frac{\pi}{2} \right) = 0, \quad n = 0, 1, \dots \right\},$$

where J is arbitrary (not fixed, but finite).

It follows immediately from the definition of the classes that the following hold.

$$(215) \quad \begin{aligned} 1) \quad & \text{If } V \in \mathcal{B}_{\text{odd}}^{(2k-1, 2n-1, 1)}, \text{ then} \\ & \frac{\partial^2 V}{\partial \zeta_1 \partial x_1} \in \mathcal{B}_{\text{even}}^{(2k-2, 2n-2, 0)} \quad \text{and} \quad \frac{\partial^2 V}{\partial x_1^2} \in \mathcal{B}_{\text{odd}}^{(2k-1, 2n-1, 1)}. \\ 2) \quad & \text{If } V \in \mathcal{B}_{\text{even}}^{(2k, 2n, 1)}, \text{ then} \\ & \frac{\partial^2 V}{\partial \zeta_1 \partial x_1} \in \mathcal{B}_{\text{odd}}^{(2k-1, 2n-1, 0)} \quad \text{and} \quad \frac{\partial^2 V}{\partial x_1^2} \in \mathcal{B}_{\text{even}}^{(2k, 2n, 1)}. \end{aligned}$$

We introduce the following class of functions:

$$\mathbb{B}^{(2k+1, 2n+1, q)} = \{ V(\zeta; x_1) : V(\zeta; x_1) = \widehat{Z}(\zeta; x_1) + \check{Z}(\zeta; x_1), \\ \widehat{Z} \in \mathcal{B}_{\text{odd}}^{(2k+1, 2n+1, q)}, \quad \check{Z} \in \mathcal{B}_{\text{even}}^{(2k, 2n, q)} \},$$

$$\mathbb{B}^{(2k, 2n, q)} = \{ V(\zeta; x_1) : V(\zeta; x_1) = \widehat{Z}(\zeta; x_1) + \check{Z}(\zeta; x_1), \\ \widehat{Z} \in \mathcal{B}_{\text{odd}}^{(2k-1, 2n-1, q)}, \quad \check{Z} \in \mathcal{B}_{\text{even}}^{(2k, 2n, q)} \},$$

where J is arbitrary (finite, but not fixed).

We point out the following properties of these classes:

$$(216) \quad \begin{aligned} \mathbb{B}^{(k, n, 1)} &\subset \mathbb{B}^{(k+2s, n+2t, 1)}, \quad \mathbb{B}^{(k, n, 0)} \subset \mathbb{B}^{(k+2s, n+2t, 0)}, \\ \mathbb{B}^{(k, n, 1)} &\subset \mathbb{B}^{(k+2s, n+2t, 0)}, \quad s, t = 0, 1, \dots \end{aligned}$$

- We denote by $H_{\text{odd}}^{(2k-1)}$, $k = 1, \dots$, the set of polynomials in τ of the form

$$\sum_{i=0}^{k-1} \sum_{j=0}^i \kappa_{\text{odd}}^{(i, j)}(x_1) \tau^{2i+1} \cos(2i - 2j + 1)\theta,$$

with infinitely differentiable coefficients on the segment $[-\pi/2, \pi/2]$, whose even derivatives with respect to x_1 vanish at $x_1 = \pm\pi/2$.

We denote by $H_{\text{even}}^{(1, 2k)}$, $k = 1, \dots$, the set of polynomials in τ of the form

$$\sum_{i=1}^k \sum_{j=0}^i \kappa_{\text{even}}^{(i, j)}(x_1) \tau^{2i} \cos 2(i - j)\theta,$$

with infinitely differentiable coefficients on the segment $[-\pi/2, \pi/2]$, whose odd derivatives with respect to x_1 vanish at $x_1 = \pm\pi/2$.

We denote by $H_{\text{even}}^{(0,2k)}$, $k = 0, 1, \dots$, the set of polynomials in τ of the form

$$\sum_{i=0}^k \sum_{j=0}^i \kappa_{\text{even}}^{(i,j)}(x_1) \tau^{2i} \cos 2(i-j)\theta,$$

with infinitely differentiable coefficients on the segment $[-\pi/2, \pi/2]$, whose odd derivatives with respect to x_1 vanish at $x_1 = \pm\pi/2$.

We also set

$$\begin{aligned} F^{(1,k)} &= \left\{ V(\zeta; x_1) : V(\zeta; x_1) = \sum_{j=1}^k \ln \tau h_j(\zeta; x_1) + \sum_{j=1}^k \tilde{h}_j(\zeta; x_1), \right. \\ &\quad \left. h_{2j-1} \in H_{\text{odd}}^{(2j-1)}, \tilde{h}_{2j-1} \in H_{\text{odd}}^{(2j-1)}, h_{2j} \in H_{\text{even}}^{(1,2j)}, \tilde{h}_{2j} \in H_{\text{even}}^{(1,2j)} \right\}, \\ F^{(0,k)} &= \left\{ V(\zeta; x_1) : V(\zeta; x_1) = \sum_{j=0}^k \ln \tau h_j(\zeta; x_1) + \sum_{j=0}^k \tilde{h}_j(\zeta; x_1), \right. \\ &\quad \left. h_{2j+1} \in H_{\text{odd}}^{(2j+1)}, \tilde{h}_{2j+1} \in H_{\text{odd}}^{(2j+1)}, h_{2j} \in H_{\text{even}}^{(0,2j)}, \tilde{h}_{2j} \in H_{\text{even}}^{(0,2j)} \right\}. \end{aligned}$$

We observe that the definition of the class $F^{(q,k)}$ implies that $\widetilde{W}_{j,k,2} \in F^{(1,2)}$.

Theorem 3.6. *Let $F \in \mathbb{B}^{(k,n,0)}$ and suppose that a function $Q \in F^{(1,n+2)}$ satisfies the equality $F - \Delta Q = O(\tau^{-2})$ as $\tau \rightarrow +\infty$. Then there exists a solution $W \in \mathbb{B}^{(k+2,n+2,1)}$ of the boundary-value problem*

$$(217) \quad \begin{cases} \Delta_{\zeta} W = F & \text{in } \mathbb{R}_+^2, \\ \frac{\partial W}{\partial \zeta_2} = 0 & \text{on } \Gamma, \\ W = 0 & \text{on } \gamma, \\ W \sim Q & \text{as } \tau \rightarrow +\infty. \end{cases}$$

Theorem 3.6 will be proved in Appendix 2 (§4).

By Theorem 3.6 there exists a solution $\widehat{w}_{j,k,2} \in \mathbb{B}^{j-k,2,0}$ of problem (181) such that

$$(218) \quad \widehat{w}_{j,k,2} = \widehat{W}_{j,k,2} + O(\tau^{-1}) \quad \text{as } \tau \rightarrow +\infty,$$

where

$$(219) \quad \widehat{W}_{j,k,2} \in F^{(0,2)}, \quad \widehat{W}_{j,k,2} - \widetilde{W}_{j,k,2} = O(1) \quad \text{as } \tau \rightarrow +\infty.$$

Consider a function of the form

$$(220) \quad w_{j,k,2} = \widehat{w}_{j,k,2} + a_{j,k,2} Y,$$

where the function Y was defined in (42), while the coefficients $a_{j,k,2}$ so far remain arbitrary, but satisfy the property

$$(221) \quad \frac{d^{2j+1} a_{2,0,2}}{dx_1^{2j+1}} \left(\pm \frac{\pi}{2} \right) = 0, \quad j = 0, \dots$$

The function (220) is a solution of problem (181) such that

$$(222) \quad w_{j,k,2} = \widehat{W}_{j,k,2} + a_{j,k,2} \ln \tau + a_{j,k,2} \ln 2 + O(\tau^{-1}) \quad \text{as } \tau \rightarrow +\infty.$$

- We denote by $\mathcal{H}_{\text{odd}}^{2k-1}(\xi; x_1)$ the set of polynomials of the form

$$\sum_{i=0}^{k-1} \sum_{j=0}^i \kappa_{\text{odd}}^{(i,j)}(x_1) \rho^{-2i-1} \cos(2i+2j+1)\theta,$$

with infinitely differentiable coefficients on the segment $[-\pi/2, \pi/2]$, whose even derivatives with respect to x_1 vanish at $x_1 = \pm\pi/2$.

We denote by $\mathcal{H}_{\text{even}}^{2k}(\xi; x_1)$ the set of polynomials of the form

$$\sum_{i=0}^k \sum_{j=0}^i \kappa_{\text{even}}^{(i,j)}(x_1) \rho^{-2i} \cos 2(i+j)\theta,$$

with infinitely differentiable coefficients on the segment $[-\pi/2, \pi/2]$, whose odd derivatives with respect to x_1 vanish at $x_1 = \pm\pi/2$.

We also denote

$$\mathcal{F}^{(k)} = \left\{ V(\xi; x_1) : V(\xi; x_1) = h_0(x_1) \ln \rho + \sum_{j=1}^k h_j(\xi; x_1), \right. \\ \left. h_{2j+1} \in \mathcal{H}_{\text{odd}}^{2j+1}, \quad h_{2j} \in \mathcal{H}_{\text{even}}^{2j}, \quad j = 0, \dots, \left\lfloor \frac{k}{2} \right\rfloor \right\}.$$

We rewrite the asymptotics of the series (180) as $\tau \rightarrow +\infty$ in terms of the variables ξ , taking (64), (171) and (169) into account:

$$(223) \quad u_\varepsilon = \sum_{k=0}^{+\infty} \sum_{j=k}^{+\infty} \varepsilon^j \ln^k a \mathcal{V}_{j,k,0}(\xi; x_1) + \sum_{k=0}^{+\infty} \sum_{j=k}^{+\infty} a^2 \varepsilon^j \ln^k a \mathcal{V}_{j,k,2}(\xi; x_1) \\ + O(a^3 \rho^{-3}) \quad \text{as } \rho a^{-1} \rightarrow +\infty, \\ \mathcal{V}_{j,k,0} = \mathcal{M}_{j,k}(x_1) (\ln \tau + \ln 2), \quad j \geq k, \\ \mathcal{V}_{j,k,2} = \tilde{\mathcal{V}}_{j,k,2} + a_{j,k,2} \ln \rho + a_{j,k,2} \ln 2,$$

where $\tilde{\mathcal{V}}_{j,k,2} \in \mathcal{F}^{(2)}$ is independent of $a_{p,q,2}$, $p \geq j$, $q \geq k$.

Taking the reexpression of the asymptotics in (223) into account, we conclude that solutions of the problem (213) must have asymptotic behaviour

$$(224) \quad v_{j,k,2}(\xi; x_1) = \tilde{\mathcal{V}}_{j,k,2}(\xi; x_1) + a_{j,k,2}(x_1) \ln \rho + a_{j,k,2}(x_1) \ln 2 \\ + O(\rho \ln \rho) \quad \text{as } \rho \rightarrow 0.$$

To analyse problems (213), (224), we introduce the following classes of functions.

• We denote by $A_{\text{even}}^{(2k, 2n, q)}$ the set of functions $f_{\text{even}} \in C^\infty(\overline{\mathbb{R}_+^2} \setminus \bigcup_{k \in \mathbb{Z}} \{(k\pi, 0)\})$ that are π -periodic and even in ξ_1 , have an asymptotic expansion as $\rho \rightarrow 0$ at zero of the form

$$(225) \quad f_{\text{even}}(\xi) = \delta_0^q \sum_{i=1}^{k+1} \beta_{\text{even}}^{(i)} \rho^{2i-2} \cos 2i\theta + \sum_{i=0}^k \sum_{j=0}^i \hat{\beta}_{\text{even}}^{(i,j)} \rho^{2i} \cos 2(i-j)\theta \ln \rho \\ + \sum_{i=0}^{\infty} \sum_{j=0}^i \tilde{\beta}_{\text{even}}^{(i,j)} \rho^{2i} \cos 2(i-j)\theta + \sum_{i=1}^n \sum_{j=0}^i \check{\beta}_{\text{even}}^{(i,j)} \rho^{-2i+2j} \cos 2i\theta, \\ \rho \rightarrow 0,$$

where δ_0^q is the Kronecker delta, and have the property

$$e^{\varpi \xi_2} \chi(\rho) f_{\text{even}} \in H^1(\Pi), \quad 0 < \varpi < 2.$$

We also point out that

$$A_{\text{even}}^{(2k, 2n, 1)} \subset A_{\text{even}}^{(2k+2s, 2n+2t, 1)}, \quad A_{\text{even}}^{(2k, 2n, 0)} \subset A_{\text{even}}^{(2k+2s, 2n+2t, 0)}, \\ A_{\text{even}}^{(2k, 2n, 1)} \subset A_{\text{even}}^{(2k+2s, 2n+2t, 0)}, \quad s, t = 0, 1, \dots$$

We set

$$\mathcal{A}_{\text{even}}^{(2k,2n,q)} = \left\{ V(\xi; x_1) : V(\xi; x_1) = \sum_{j=1}^J \sigma_j(x_1) Y_j(\xi), Y_j \in A_{\text{even}}^{(2k,2n,q)}, \right. \\ \left. \sigma_j \in C^\infty \left[-\frac{\pi}{2}, \frac{\pi}{2} \right], (\sigma_j)^{(2n+1)} \left(\pm \frac{\pi}{2} \right) = 0, n = 0, 1, \dots \right\},$$

where J is arbitrary (not fixed, but finite). We note that

$$\frac{\partial V}{\partial x_1} \left(\xi; \pm \frac{\pi}{2} \right) = 0 \quad \text{for all } V \in \mathcal{A}_{\text{even}}^{(2k,2n,q)}.$$

- We denote by $A_{\text{odd}}^{(2k-1,2n-1,q)}$ the set of functions

$$f_{\text{odd}} \in C^\infty(\overline{\mathbb{R}_+^2} \setminus \bigcup_{k \in \mathbb{Z}} \{(k\pi, 0)\})$$

that are π -periodic and odd in ξ_1 , have an asymptotic expansion as $\rho \rightarrow 0$ at zero of the form

$$f_{\text{odd}}(\xi) = \delta_0^q \sum_{i=0}^k \beta_{\text{odd}}^{(i)} \rho^{2i-1} \cos(2i+1)\theta + \sum_{i=0}^{k-1} \sum_{j=0}^i \widehat{\beta}_{\text{odd}}^{(i,j)} \rho^{2i+1} \cos(2i-2j+1)\theta \ln \rho \\ + \sum_{i=0}^{\infty} \sum_{j=0}^i \widetilde{\beta}_{\text{odd}}^{(i,j)} \rho^{2i+1} \cos(2i-2j+1)\theta + \sum_{i=0}^{n-1} \sum_{j=0}^i \check{\beta}_{\text{odd}}^{(i,j)} \rho^{-2i+2j-1} \cos(2i+1)\theta,$$

where δ_0^q is the Kronecker delta, and have the property

$$e^{\varpi \xi_2} \chi(\rho) f_{\text{odd}} \in H^1(\Pi), \quad 0 < \varpi < 2.$$

We point out at once that the symbol $A_{\text{odd}}^{(2k-1,2n-1,q)}$ for $k=0$ denotes the class of functions whose asymptotic expansion as $\rho \rightarrow 0$ has no terms involving $\ln \rho$; correspondingly, for $n=0$ it denotes the class of functions whose asymptotics as $\rho \rightarrow 0$ has no terms with negative powers of ρ .

We observe that

$$A_{\text{odd}}^{(2k-1,2n-1,1)} \subset A_{\text{odd}}^{(2k+2s-1,2n+2t-1,1)}, \quad A_{\text{odd}}^{(2k-1,2n-1,0)} \subset A_{\text{odd}}^{(2k+2s-1,2n+2t-1,0)}, \\ A_{\text{odd}}^{(2k-1,2n-1,1)} \subset A_{\text{odd}}^{(2k+2s-1,2n+2t-1,0)}, \quad s = 0, 1, \dots$$

We also denote

$$\mathcal{A}_{\text{odd}}^{(2k-1,2n-1,q)} = \left\{ V(\xi; x_1) : V(\xi; x_1) = \sum_{j=1}^J \sigma_j(x_1) Y_j(\xi), Y_j \in A_{\text{odd}}^{(2k-1,2n-1,q)}, \right. \\ \left. \sigma_j \in C^\infty \left[-\frac{\pi}{2}, \frac{\pi}{2} \right], (\sigma_j)^{(2n)} \left(\pm \frac{\pi}{2} \right) = 0, n = 0, 1, \dots \right\},$$

where J is arbitrary (not fixed and finite).

It follows immediately from the definition that the following hold.

- 1) If $V \in \mathcal{A}_{\text{odd}}^{(2k-1,2n-1,1)}$, then

$$\frac{\partial^2 V}{\partial \xi_1 \partial x_1} \in \mathcal{A}_{\text{even}}^{(2k-2,2n-2,0)} \quad \text{and} \quad \frac{\partial^2 V}{\partial x_1^2} \in \mathcal{A}_{\text{odd}}^{(2k-1,2n-1,1)}.$$
- 2) If $V \in \mathcal{A}_{\text{even}}^{(2k,2n,1)}$, then

$$\frac{\partial^2 V}{\partial \xi_1 \partial x_1} \in \mathcal{A}_{\text{odd}}^{(2k-1,2n-1,0)} \quad \text{and} \quad \frac{\partial^2 V}{\partial x_1^2} \in \mathcal{A}_{\text{even}}^{(2k,2n,1)}.$$

(226)

We introduce the following class of functions:

$$\begin{aligned} \mathbb{A}^{(2k+1, 2n+1, q)} &= \left\{ V(\xi; x_1) : V(\xi; x_1) = \widehat{Z}(\xi; x_1) + \check{Z}(\xi; x_1), \right. \\ &\quad \left. \widehat{Z} \in \mathcal{A}_{\text{odd}}^{(2k+1, 2n+1, q)}, \check{Z} \in \mathcal{A}_{\text{even}}^{(2k, 2n, q)} \right\}, \\ \mathbb{A}^{(2k, 2n, q)} &= \left\{ V(\xi; x_1) : V(\xi; x_1) = \widehat{Z}(\xi; x_1) + \check{Z}(\xi; x_1), \right. \\ &\quad \left. \widehat{Z} \in \mathcal{A}_{\text{odd}}^{(2k-1, 2n-1, q)}, \check{Z} \in \mathcal{A}_{\text{even}}^{(2k, 2n, q)} \right\}, \end{aligned}$$

where J is arbitrary (finite but not fixed).

These classes have the following properties:

$$(227) \quad \begin{aligned} \mathbb{A}^{(k, n, 1)} &\subset \mathbb{A}^{(k+2s, n+2t, 1)}, & \mathbb{A}^{(k, n, 0)} &\subset \mathbb{A}^{(k+2s, n+2t, 0)}, \\ \mathbb{A}^{(k, n, 1)} &\subset \mathbb{A}^{(k+2s, n+2t, 0)}, & s, t &= 0, 1, \dots \end{aligned}$$

The following theorem is proved similarly to Theorem 3.1.

Theorem 3.7. a) Let F be represented in the form $F = \mathcal{P}_t + f$, where $f \in \mathbb{A}^{(k, n, 0)}$ and $\mathcal{P}_{t+2}$ is a polynomial such that $-\mathcal{P}_{t+2}'' = \mathcal{P}_t$. Suppose that a function $\mathcal{Q} \in \mathcal{F}^{(n+2)}$ satisfies the equality $F - \Delta \mathcal{Q} = O(\rho^{-1})$ as $\rho \rightarrow 0$. Then there exists a solution of the boundary-value problem

$$(228) \quad \begin{cases} \Delta_\xi W = F & \text{in } \Pi, \\ \frac{\partial W}{\partial \xi_2} = 0 & \text{on } \Sigma, \\ \frac{\partial W}{\partial \xi_1} = 0 & \text{at } \xi_1 = \pm \frac{\pi}{2}, \quad x_1 = \pm \frac{\pi}{2}, \\ W = \mathcal{Q} + a \ln \rho + \tilde{a} + O(\rho^{-1}) & \text{as } \rho \rightarrow 0 \end{cases}$$

that can be represented in the form $W = \mathcal{P}_{t+2} + W_1$, where $W_1 \in \mathbb{A}^{(k+2, n+2, 1)}$.

b) The function W satisfies the equalities

$$\frac{\partial W}{\partial x_1} \left(\xi; \pm \frac{\pi}{2} \right) = 0 \quad \text{at } \xi_1 = \pm \frac{\pi}{2}, \quad \frac{\partial W}{\partial x_1} \left(\frac{x}{\varepsilon}; x_1 \right) = 0 \quad \text{at } x_1 = \pm \frac{\pi}{2}.$$

The definition of the classes ensures that the following statement is true.

Lemma 3.5. Suppose that the $v_{j, k, 2} = V_{j, k, 2} + \tilde{v}_{j, k, 2}$, where $\tilde{v}_{j, k, 2} \in \mathbb{A}^{(j-k, 2, 1)}$, are solutions of the problems

$$(229) \quad \begin{cases} -\Delta_\xi v_{j, k, 2} = 2 \frac{\partial^2 v_{j-1, k, 2}}{\partial x_1 \partial \xi_1} + \frac{\partial^2 v_{j-2, k, 2}}{\partial x_1^2} + \sum_{p=0}^{j-2} \sum_{q=p}^{k+j-p} \lambda_{p, q, 0} v_{j-2-p, k-q, 2} \\ \quad + \sum_{p=0}^{j-2} \sum_{q=p}^{k+j-p} \lambda_{p, q, 2} v_{j-2-p, k-q, 0} & \text{in } \Pi, \\ \frac{\partial v_{j, k, 2}}{\partial \xi_2} = 0 & \text{on } \Sigma, \\ \frac{\partial v_{j, k, 2}}{\partial \xi_1} \left(\xi; \pm \frac{\pi}{2} \right) = 0 & \text{at } \xi_1 = \pm \frac{\pi}{2}, \\ v_{j, k, 2} = V_{j, k, 2} + o(1) & \text{as } \xi_2 \rightarrow +\infty. \end{cases}$$

Then the $v_{j, k, 2}$ are solutions of problems (213).

Remark 3.6. As in the preceding sections, we will seek solutions of the problems (213), (224) by turns. Having found the term $v_{1,0,2} = V_{1,0,2} + \tilde{v}_{1,0,2}$ with $\tilde{v}_{1,0,2} \in \mathbb{A}^{(1,2,1)}$, we then construct the terms $v_{p,0,2} = V_{p,0,2} + \tilde{v}_{p,0,2}$ with $\tilde{v}_{p,0,2} \in \mathbb{A}^{(p,2,1)}$ consecutively, for $p = 2, 3, \dots$. At the second step we construct $v_{p,1,2} = V_{p,1,2} + \tilde{v}_{p,1,2}$ with $\tilde{v}_{p,1,2} \in \mathbb{A}^{(p-1,2,1)}$ for $p = 1, 2, \dots$, and so on. Therefore for fixed indices, in problem (213), (224), on the right-hand side of the equation, which is denoted by $F_{j,k,2}$, there will be functions that by now are known, and consequently we shall have the equality

$$(230) \quad -\Delta \mathcal{V}_{j,k,2} = F_{j,k,2} + O(\rho^{-1}) \quad \text{as } \rho \rightarrow 0.$$

By Theorem 3.7 there exists a solution of the problem (213), (224) that can be represented in the form $v_{j,k,2}(\xi; x_1) = V_{j,k,2}(\xi; x_1) + \tilde{v}_{j,k,2}(\xi; x_1)$, where $\tilde{v}_{j,k,2} \in \mathbb{A}^{(j-k,2,1)}$, and has the following asymptotics at zero:

$$(231) \quad \begin{aligned} v_{j,k,2}(\xi; x_1) &= \tilde{V}_{j,k,2}(\xi; x_1) + \widehat{\mathcal{M}}_{j,k,2}(x_1) \ln \rho \\ &\quad + \widetilde{\mathcal{M}}_{j,k,2}(x_1) + \alpha_{j,k,2}^0(x_1) + O(\rho \ln \rho) \quad \text{as } \rho \rightarrow 0, \end{aligned}$$

where $\widehat{\mathcal{M}}_{j,k,2}(x_1)$ and $\widetilde{\mathcal{M}}_{j,k,2}(x_1)$ are completely defined functions that are infinitely differentiable on the segment $[-\pi/2, \pi/2]$ and whose odd derivatives vanish at $x_1 = \pm\pi/2$, while $\alpha_{j,k,2}^0(x_1)$ is arbitrary at present.

On the other hand, the asymptotics must have the form (224). We now show that this can be achieved by choosing the $a_{j,k,2}$ and $\alpha_{j,k,2}^0$, which so far have been arbitrary. By choosing

$$(232) \quad a_{j,k,2} = \widehat{\mathcal{M}}_{j,k,2}, \quad \alpha_{j,k,2} = \widehat{\mathcal{M}}_{j,k,2} \ln 2 - \widetilde{\mathcal{M}}_{j,k,2},$$

we verify that the function $v_{j,k,2}$ now has the asymptotics (224); that is, we finally determine the function $v_{j,k,2}$, and so we finally fix the function $w_{j,k,2}$. Thus we determine the boundary condition for $u_{j,k,2}$ in the problem (206); then, from the condition that the problem is solvable, we find $\lambda_{j,k,2}$ and determine $u_{j,k,2}$.

Finally, at this step we completely determine the terms $w_{j,k,2}$, $v_{j,k,2}$, $\lambda_{j,k,2}$, and $u_{j,k,2}$.

We conduct the subsequent steps (for $a^3, a^4, \dots$) along the same lines as we used for the terms for a^2 . As a result we see that the following assertion holds.

Theorem 3.8. *There exist series*

$$(233) \quad \lambda_\varepsilon = \lambda_0 + \sum_{j=1}^{\infty} \sum_{l=2}^{\infty} \sum_{k=0}^{j-l+1} \varepsilon^j \ln^k a \, a^l \, \lambda_{j,k,l} + \sum_{j=1}^{\infty} \sum_{k=0}^j \varepsilon^j \ln^k a \, \lambda_{j,k,0},$$

$$(234) \quad u_\varepsilon(x) = u_0(x) + \sum_{j=1}^{\infty} \sum_{l=2}^{\infty} \sum_{k=0}^{j-l+1} \varepsilon^j \ln^k a \, a^l \, u_{j,k,l}(x) + \sum_{j=1}^{\infty} \sum_{k=0}^j \varepsilon^j \ln^k a \, u_{j,k,0}(x),$$

$$(235) \quad u_\varepsilon(x) = \sum_{j=1}^{\infty} \sum_{l=2}^{\infty} \sum_{k=0}^{j-l+1} \varepsilon^j \ln^k a \, a^l \, v_{j,k,l}(\xi; x_1) + \sum_{j=1}^{\infty} \sum_{k=0}^j \varepsilon^j \ln^k a \, v_{j,k,0}(\xi; x_1),$$

$$(236) \quad u_\varepsilon(x) = \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} \sum_{k=0}^{j-l} \varepsilon^j \ln^k a \, a^l \, w_{j,k,l}(\zeta; x_1) + \sum_{j=1}^{\infty} \sum_{k=0}^{j-1} \varepsilon^j \ln^k a \, w_{j,k,0}(\zeta; x_1),$$

such that

a) the pairs $u_{j,k,l} \in C^\infty(\overline{\Omega})$ and $\lambda_{j,k,l}$ are solutions of the problems

$$(237) \quad \left\{ \begin{array}{l} -\Delta u_{j,k,l} = \sum_{s=2}^{l-2} \sum_{p=0}^j \sum_{q=p}^{k+j-p} \lambda_{p,q,s} u_{j-p,k-q,l-s} \\ \quad + \sum_{p=0}^j \sum_{q=p}^{k+j-p} \lambda_{p,q,0} u_{j-p,k-q,l} + \sum_{p=0}^j \sum_{q=p}^{k+j-p} \lambda_{p,q,l} u_{j-p,k-q,0} \quad \text{in } \Omega, \\ \frac{\partial u_{j,k}}{\partial \nu} = 0 \quad \text{on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_3, \\ u_{j,k} = \alpha_{j,k}^0 \quad \text{on } \Gamma_4, \end{array} \right.$$

where $u_{0,0,0} = u_0$ and $\lambda_{0,0,0} = \lambda_0$;

b) the series (234) has an asymptotic expansion as $x_2 \rightarrow 0$, rewritten in terms of the variables ξ , of the form

$$(238) \quad \begin{aligned} u_\varepsilon(x) &= \sum_{j=1}^{\infty} \sum_{l=2}^{\infty} \sum_{k=0}^{j-l+1} \varepsilon^j \ln^k a \, a^l V_{j,k,l}(\xi_2; x_1) \\ &+ \sum_{j=1}^{\infty} \sum_{k=0}^j \varepsilon^j \ln^k a \, V_{j,k,0}(\xi_2; x_1) \quad \text{as } \varepsilon \xi_2 \rightarrow 0, \end{aligned}$$

where the $V_{j,k,l}(\xi_2; x_1)$ are polynomials of order $j - k$ in the variable ξ_2 whose coefficients are functions of x_1 that are infinitely differentiable on $[-\pi/2, \pi/2]$ and whose odd derivatives vanish at $x_1 = \pm\pi/2$;

c) the coefficient $\lambda_{1,1,0}$ is determined from formula (122);

d) the functions $v_{j,k,l}$ can be represented as the sums $v_{j,k,l} = V_{j,k,l} + \tilde{v}_{j,k,l}$, where $\tilde{v}_{j,k,l} \in \mathbb{A}^{(j-k,l,1)}$ and $v_{j,j,0} = \mathcal{M}_{j,j}$, and are solutions of the problem

$$(239) \quad \left\{ \begin{array}{l} -\Delta_\xi v_{j,k,l} = 2 \frac{\partial^2 v_{j-1,k,l}}{\partial x_1 \partial \xi_1} + \frac{\partial^2 v_{j-2,k,l}}{\partial x_1^2} + \sum_{s=2}^{l-2} \sum_{p=0}^{j-2} \sum_{q=p}^{k+j-p} \lambda_{p,q,s} v_{j-2-p,k-q,l-s} \\ \quad + \sum_{p=0}^{j-2} \sum_{q=p}^{k+j-p} \lambda_{p,q,0} v_{j-2-p,k-q,l} + \sum_{p=0}^{j-2} \sum_{q=p}^{k+j-p} \lambda_{p,q,l} v_{j-2-p,k-q,0} \quad \text{in } \Pi, \\ \frac{\partial v_{j,k,l}}{\partial \xi_2} = 0 \quad \text{on } \Sigma, \\ \frac{\partial v_{j,k,l}}{\partial \xi_1} \left(\xi; \pm \frac{\pi}{2} \right) = 0 \quad \text{at } \xi_1 = \pm \frac{\pi}{2}; \end{array} \right.$$

furthermore,

$$\frac{\partial v_{j,k,l}}{\partial x_1} \left(\xi; \pm \frac{\pi}{2} \right) \equiv 0 \quad \text{and} \quad \frac{d^{2l+1} \mathcal{M}_{j,j}}{dx_1^{2l+1}} \left(\pm \frac{\pi}{2} \right) = 0, \quad l = 0, 1, \dots;$$

e) the functions $v_{j,k,l}$ have the asymptotic behaviour

$$(240) \quad v_{j,k,l} = \mathcal{V}_{j,k,l} \quad \text{as } \rho \rightarrow 0,$$

where $\mathcal{V}_{j,k,l} \in \mathcal{F}^{(l-j+k)}$;

f) the functions $w_{k,j,l} \in \mathbb{B}^{j-k,l,1}$ are solutions of the problem

$$(241) \quad \begin{cases} -\Delta_{\zeta} w_{j,k,l} = 2 \frac{\partial^2 w_{j-1,k,l-1}}{\partial x_1 \partial \zeta_1} + \frac{\partial^2 w_{j-2,k,l-2}}{\partial x_1^2} \\ \quad + \sum_{s=0}^{l-2} \sum_{p=0}^{j-2} \sum_{q=p}^{k+j-p} \lambda_{p,q,s} w_{j-2-p,k-q,l-2-s} \quad \text{in } \mathbb{R}_+^2, \\ \frac{\partial w_{j,k,l}}{\partial \zeta_2} = 0 \quad \text{on } \Gamma, \\ w_{j,k,l} = 0 \quad \text{on } \gamma; \end{cases}$$

g) the series (236) has an asymptotic expansion as $\tau \rightarrow +\infty$, rewritten in terms of the variables ξ , of the form

$$(242) \quad \begin{aligned} u_{\varepsilon}(x) &= \sum_{j=1}^{\infty} \sum_{l=2}^{\infty} \sum_{k=0}^{j-l+1} \varepsilon^j \ln^k a \, a^l \, \mathcal{V}_{j,k,l}(\xi; x_1) \\ &\quad + \sum_{j=1}^{\infty} \sum_{k=0}^j \varepsilon^j \ln^k a \, \mathcal{V}_{j,k,0}(\xi; x_1) \quad \text{as } \rho a^{-1} \rightarrow +\infty. \end{aligned}$$

3.4. The construction of the asymptotics. Fourth step. To construct the terms that depend on the concentrated masses, we introduce the following notation. We set $\mu := \varepsilon^{2-m} a^{2-m}$; then, in the new notation, the density $\rho_{\varepsilon}(x)$ defined in (3) has the form

$$(243) \quad \rho_{\varepsilon}(x) = \begin{cases} 1 & \text{in } \Omega \setminus \overline{B}_{\varepsilon}, \\ 1 + \mu(a\varepsilon)^{-2} & \text{in } B_{\varepsilon}. \end{cases}$$

We assume the small parameter μ to be independent.

We denote the series (236) by $\tilde{w}_{\varepsilon}$, the series (233) by $\tilde{\lambda}_{\varepsilon}$, the series (234) by $\tilde{u}_{\varepsilon}$ and the series (235) by $\tilde{v}_{\varepsilon}$. Substituting $\tilde{w}_{\varepsilon}$ and $\tilde{\lambda}_{\varepsilon}$ into the equation in problem (2), we obtain

$$(244) \quad -\Delta \tilde{w}_{\varepsilon} = \rho_{\varepsilon} \tilde{\lambda}_{\varepsilon} \tilde{w}_{\varepsilon} = \tilde{\lambda}_{\varepsilon} \tilde{w}_{\varepsilon} + \mu(a\varepsilon)^{-2} \tilde{\lambda}_{\varepsilon} \tilde{w}_{\varepsilon}.$$

Hence it immediately follows that we need to introduce new terms in the inner expansion. We seek an inner expansion in the form

$$(245) \quad u_{\varepsilon} = \tilde{w}_{\varepsilon} + \mu \hat{w}_{\varepsilon} + \dots$$

and, correspondingly,

$$(246) \quad \lambda_{\varepsilon} = \tilde{\lambda}_{\varepsilon} + \mu \hat{\lambda}_{\varepsilon} + \dots$$

Substituting the series (245) and (246) into (2) and collecting terms for the corresponding minimal powers of a , ε , $\ln a$, and μ , we conclude that

$$(247) \quad \hat{w}_{\varepsilon} = \varepsilon w_{1,0,0,1} + \dots$$

and the problem for the function $w_{1,0,0,1}$ has the form

$$(248) \quad \begin{cases} -\Delta_{\zeta} w_{1,0,0,1} = \begin{cases} 0 & \text{in } \overline{\mathbb{R}_+^2} \setminus B, \\ \lambda_0 w_{1,0,0,0} & \text{in } B, \end{cases} \\ w_{1,0,0,1} = 0 \quad \text{on } \gamma, \\ \frac{\partial w_{1,0,0,1}}{\partial \zeta_2} = 0 \quad \text{on } \Gamma, \end{cases}$$

where $w_{1,0,0,0} \equiv w_{1,0,0}$. We consider the auxiliary problem

$$(249) \quad \begin{cases} -\Delta_\zeta Y_4 = \begin{cases} 0 & \text{in } \overline{\mathbb{R}_+^2} \setminus B, \\ \lambda_0 Y & \text{in } B, \end{cases} \\ Y_4 = 0 & \text{on } \gamma, \\ \frac{\partial Y_4}{\partial \zeta_2} = 0 & \text{on } \Gamma. \end{cases}$$

Lemma 3.6. *Suppose that $F \in L_2(\mathbb{R}_+^2) \cap C^\infty(\tilde{B})$ has the asymptotic behaviour*

$$F = O(\tau^{-N-1}) \quad \text{as } \tau \rightarrow +\infty.$$

Then there exists a bounded function

$$W \in H_{\text{loc}}^1(\mathbb{R}_+^2) \cap C^\infty(\tilde{B}) \cap H_{\text{loc}}^2(\mathbb{R}_+^2 \setminus (\overline{\mathcal{O}_{\varsigma,+}} \cup \overline{\mathcal{O}_{\varsigma,-}})),$$

for arbitrary ς , that is a solution of the boundary-value problem

$$(250) \quad \begin{cases} \Delta_\zeta W = F & \text{in } \mathbb{R}_+^2, \\ \frac{\partial W}{\partial \zeta_2} = 0 & \text{on } \Gamma, \\ W = 0 & \text{on } \gamma \end{cases}$$

with the asymptotic expansion

$$(251) \quad W = \sum_{j=0}^{N-2} \tilde{c}_j \tau^{-j} \cos(j\theta) + O(\tau^{-N+1}) \quad \text{as } \tau \rightarrow \infty.$$

If in addition F is even, then $\tilde{c}_{2j+1} = 0$; if F is odd, then $\tilde{c}_{2j} = 0$.

Lemma 3.6 will be proved in Appendix 2 (§4). It follows from Lemma 3.6 and the arbitrariness in the choice of N that there exists a bounded solution of (249) having asymptotics

$$(252) \quad Y_4 \sim M_4 + A_{4,2} \tau^{-2} \cos 2\theta + \sum_{j=2}^{\infty} A_{4,2j} \tau^{-2j} \cos 2j\theta \quad \text{as } \tau \rightarrow +\infty.$$

It is easy to compute the constant M_4 . First multiply the equation in problem (249) by the function Y defined in (42) and integrate over the half-disc $B_R = \{\zeta \in \mathbb{R}_+^2, |\zeta| < R\}$. Then apply Green's formula twice and pass to the limit as $R \rightarrow +\infty$ to obtain

$$(253) \quad \pi M_4 = \lambda_0 \int_B Y^2(\zeta) d\zeta > 0.$$

Obviously, the function

$$(254) \quad w_{1,0,0,1}(\zeta, x_1) = \alpha_0^1(x_1) Y_4(\zeta)$$

satisfies (248) and has the following asymptotics (as $\tau \rightarrow +\infty$):

$$(255) \quad w_{1,0,0,1} \sim \alpha_0^1 \left(M_4 + A_{4,2} \tau^{-2} \cos 2\theta + \sum_{j=2}^{\infty} A_{4,2j} \tau^{-2j} \cos 2j\theta \right).$$

The appearance of the function $M_4 \alpha_0^1(x_1)$ multiplied by $\varepsilon \mu$ in the inner expansion gives rise to a discrepancy in the intermediate expansion. Rewriting the asymptotics of the series (245) as $\tau \rightarrow +\infty$ in terms of the variables ξ and following the matching method, we see that in the intermediate expansion there must be a term $\varepsilon \mu v_{1,0,0,1}$ with the asymptotics

$$(256) \quad v_{1,0,0,1}(\xi; x_1) \sim M_4 \alpha_0^1(x_1) \quad \text{as } \rho \rightarrow 0;$$

that is, the intermediate expansion has the form

$$(257) \quad u_\varepsilon(x) = \tilde{v}_\varepsilon + \varepsilon\mu v_{1,0,0,1} + \cdots,$$

where $\tilde{v}_\varepsilon$ is the series (235). After substituting the series (257) and (246) into the problem (2) we collect like terms for the powers of ε , a , and $\ln a$. In view of (256) we obtain the following problem for $v_{1,0,0,1}$:

$$(258) \quad \begin{cases} \Delta_\xi v_{1,0,0,1} = 0 & \text{in } \Pi, \\ \frac{\partial v_{1,0,0,1}}{\partial \xi_2} = 0 & \text{on } \Sigma, \\ \frac{\partial v_{1,0,0,1}}{\partial \xi_1} \left(\xi; \pm \frac{\pi}{2} \right) = 0 & \text{at } \xi_1 = \pm \frac{\pi}{2}, \\ v_{1,0,0,1}(\xi; x_1) \sim M_4 \alpha_0^1(x_1) & \text{as } \rho \rightarrow 0. \end{cases}$$

Obviously, the function $v_{1,0,0,1} \equiv M_4 \alpha_0^1(x_1)$ is a solution of (258) with asymptotics at infinity given by

$$(259) \quad v_{1,0,0,1} = M_4 \alpha_0^1 \quad \text{as } \xi_2 \rightarrow +\infty.$$

Matching the asymptotic expansions, we introduce the following term into the outer expansion:

$$(260) \quad u_\varepsilon(x) = \tilde{u}_\varepsilon + \varepsilon\mu u_{1,0,0,1} + \cdots,$$

where $\tilde{u}_\varepsilon$ is the series (234), while $u_{1,0,0,1} \sim M_4 \alpha_0^1(x_1)$ as $x_2 \rightarrow 0$. We seek the function $u_{1,0,0,1} \in C^\infty(\bar{\Omega})$; therefore such an asymptotic is equivalent to the boundary conditions

$$u_{1,0,0,1}(x_1, 0) = M_4 \alpha_0^1(x_1), \quad x_1 \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right).$$

Substituting the expansion (260) of the function u_ε in terms of x and the expansion (246) into the problem and collecting terms in $\varepsilon\mu$, we conclude that

$$(261) \quad \hat{\lambda}_\varepsilon = \varepsilon \lambda_{1,0,0,1} + \cdots$$

and the problem for the function $u_{1,0,0,1}$ has the form

$$(262) \quad \begin{cases} -\Delta u_{1,0,0,1} = \lambda_0 u_{1,0,0,1} + \lambda_{1,0,0,1} u_0 & \text{for } x \in \Omega, \\ u_{1,0,0,1} = M_4 \alpha_0^1(x_1) & \text{for } x \in \Gamma_4, \\ \frac{\partial u_{1,0,0,1}}{\partial \nu} = 0 & \text{for } x \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_3. \end{cases}$$

From the condition that this problem is solvable we obtain a formula for $\lambda_{1,0,0,1}$. We multiply the equation by u_0 , integrate over the domain Ω , recalling the normalization of u_0 in $L_2(\Omega)$, and apply Green's formula twice to the left-hand side of the equality. We obtain

$$\lambda_{1,0,0,1} = -M_4 \int_{-\pi/2}^{\pi/2} \alpha_0^{12}(x_1) dx_1.$$

We now recall the form of the constant M_4 (see (253)) and substitute it into this expression. Finally we obtain

$$(263) \quad \lambda_{1,0,0,1} = -\frac{\lambda_0}{\pi} \int_{\Gamma_4} \left(\frac{\partial u_0}{\partial \nu} \right)^2 ds \int_B Y^2(\zeta) d\zeta.$$

In order that (262) be uniquely solvable we will assume that

$$\int_{\Omega} u_0(x) u_{1,0,0,1}(x) dx = 0.$$

Next, in just the same way as we proceeded before, we construct the terms $v_{j,k,l,1}$, $w_{j,k,l,1}$, $u_{j,k,l,1}$, and $\lambda_{j,k,l,1}$ in three steps. Their appearance immediately implies that the terms $v_{j,k,l,2}$, $w_{j,k,l,2}$, $u_{j,k,l,2}$, and $\lambda_{j,k,l,2}$ appear. The way we construct them also repeats the way we constructed the terms for μ^0 in the preceding sections. Then we construct the terms for μ^3 , μ^4 , and so on. As a result we obtain that the following assertion holds.

We set

$$\begin{aligned}\lambda_{j,k,l,0} &:= \lambda_{j,k,l}, & u_{j,k,l,0} &:= u_{j,k,l}, \\ v_{j,k,l,0} &:= v_{j,k,l}, & \text{and} & & w_{j,k,l,0} &:= w_{j,k,l}.\end{aligned}$$

Theorem 3.9. *There exist series*

$$\begin{aligned}(264) \quad \lambda_\varepsilon &= \lambda_0 + \sum_{p=0}^{\infty} \sum_{j=1}^{\infty} \sum_{l=2}^{\infty} \sum_{k=0}^{j-l+1} \varepsilon^j \ln^k a \, a^l \mu^p \, \lambda_{j,k,l,p} \\ &+ \sum_{p=0}^{\infty} \sum_{j=1}^{\infty} \sum_{k=0}^j \varepsilon^j \ln^k a \mu^p \, \lambda_{j,k,0,p},\end{aligned}$$

$$\begin{aligned}(265) \quad u_\varepsilon(x) &= u_0(x) + \sum_{p=0}^{\infty} \sum_{j=1}^{\infty} \sum_{l=2}^{\infty} \sum_{k=0}^{j-l+1} \varepsilon^j \ln^k a \, a^l \mu^p \, u_{j,k,l,p}(x) \\ &+ \sum_{p=0}^{\infty} \sum_{j=1}^{\infty} \sum_{k=0}^j \varepsilon^j \ln^k a \mu^p \, u_{j,k,0,p}(x),\end{aligned}$$

$$\begin{aligned}(266) \quad u_\varepsilon(x) &= \sum_{p=0}^{\infty} \sum_{j=1}^{\infty} \sum_{l=2}^{\infty} \sum_{k=0}^{j-l+1} \varepsilon^j \ln^k a \, a^l \mu^p \, v_{j,k,l,p}(\xi; x_1) \\ &+ \sum_{p=0}^{\infty} \sum_{j=1}^{\infty} \sum_{k=0}^j \varepsilon^j \ln^k a \mu^p \, v_{j,k,0,p}(\xi; x_1),\end{aligned}$$

$$\begin{aligned}(267) \quad u_\varepsilon(x) &= \sum_{p=0}^{\infty} \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} \sum_{k=0}^{j-l} \varepsilon^j \ln^k a \, a^l \mu^p \, w_{j,k,l,p}(\zeta; x_1) \\ &+ \sum_{p=0}^{\infty} \sum_{j=1}^{\infty} \sum_{k=0}^{j-1} \varepsilon^j \ln^k a \mu^p \, w_{j,k,0,p}(\zeta; x_1),\end{aligned}$$

such that

a) The pairs $u_{j,k,l,p} \in C^\infty(\bar{\Omega})$ and $\lambda_{j,k,l,p}$ are solutions of the problems

$$(268) \quad \begin{cases} -\Delta u_{j,k,l,p} = \sum_{j_1, k_1, l_1, p_1} \lambda_{j_1, k_1, l_1, p_1} u_{j-j_1, k-k_1, l-l_1, p-p_1} & \text{in } \Omega, \\ \frac{\partial u_{j,k,l,p}}{\partial \nu} = 0 & \text{on } \Gamma_1 \cup \Gamma_2 \cup \Gamma_3. \end{cases}$$

Henceforth we assume that the indices j_1 , k_1 , l_1 , and p_1 vary only within the limits indicated in the corresponding sums (264), (265), (266), and (267); here $u_{0,0,0,0} = u_0$ and $\lambda_{0,0,0,0} = \lambda_0$.

- b) The series (265) has an asymptotic expansion as $x_2 \rightarrow 0$, rewritten in terms of the variable ξ , of the form

$$(269) \quad \begin{aligned} u_\varepsilon(x) = & \sum_{p=0}^{\infty} \sum_{j=1}^{\infty} \sum_{l=2}^{\infty} \sum_{k=0}^{j-l+1} \varepsilon^j \ln^k a \, a^l \mu^p V_{j,k,l,p}(\xi_2; x_1) \\ & + \sum_{p=0}^{\infty} \sum_{j=1}^{\infty} \sum_{k=0}^j \varepsilon^j \ln^k a \mu^p V_{j,k,0,p}(\xi_2; x_1) \quad \text{as } \varepsilon \xi_2 \rightarrow 0, \end{aligned}$$

where the $V_{j,k,l,p}(\xi_2; x_1)$ are polynomials of order $j - k$ in the variable ξ_2 whose coefficients are functions of x_1 that are infinitely differentiable on $[-\pi/2, \pi/2]$ and whose odd derivatives vanish at $x_1 = \pm\pi/2$.

- c) The coefficient $\lambda_{1,1,0,0}$ is determined by the formula

$$\lambda_{1,1,0,0} = \int_{\Gamma_4} \left(\frac{\partial u_0}{\partial \nu} \right)^2 ds > 0.$$

- d) The coefficient $\lambda_{1,0,0,1}$ is given by the formula

$$\lambda_{1,0,0,1} = -\frac{\lambda_0}{\pi} \int_{\Gamma_4} \left(\frac{\partial u_0}{\partial \nu} \right)^2 ds \int_B Y^2(\zeta) d\zeta,$$

where Y is the function defined in (42).

- e) The functions $v_{j,k,l,p}$ can be represented as the sums

$$v_{j,k,l,p} = V_{j,k,l,p} + \tilde{v}_{j,k,l,p}, \quad \tilde{v}_{j,k,l,p} \in \mathbb{A}^{(j-k,l,1)}, \quad v_{j,j,0,0} = \mathcal{M}_{j,j},$$

and are solutions of the problem

$$(270) \quad \left\{ \begin{aligned} -\Delta_\xi v_{j,k,l,p} &= 2 \frac{\partial^2 v_{j-1,k,l,p}}{\partial x_1 \partial \xi_1} + \frac{\partial^2 v_{j-2,k,l,p}}{\partial x_1^2} \\ &+ \sum_{j_1, k_1, l_1, p_1} \lambda_{j_1, k_1, l_1, p_1} v_{j-j_1-2, k-k_1, l-l_1, p-p_1} \quad \text{in } \Pi, \\ \frac{\partial v_{j,k,l,p}}{\partial \xi_2} &= 0 \quad \text{on } \Sigma, \\ \frac{\partial v_{j,k,l,p}}{\partial \xi_1} \left(\xi; \pm \frac{\pi}{2} \right) &= 0 \quad \text{at } \xi_1 = \pm \frac{\pi}{2}; \end{aligned} \right.$$

furthermore,

$$\frac{\partial v_{j,k,l,p}}{\partial x_1} \left(\xi; \pm \frac{\pi}{2} \right) \equiv 0 \quad \text{and} \quad \frac{d^{2l+1} \mathcal{M}_{j,j}}{dx_1^{2l+1}} \left(\pm \frac{\pi}{2} \right) = 0, \quad l = 0, 1, \dots$$

- f) The functions $v_{j,k,l,p}$ have the asymptotic behaviour

$$(271) \quad v_{j,k,l,p} = \mathcal{V}_{j,k,l,p} \quad \text{as } \rho \rightarrow 0,$$

where $\mathcal{V}_{j,k,l,p} \in \mathcal{F}^{(l-j+k)}$.

g) The functions $w_{k,j,l,p} \in \mathbb{B}^{j-k,l,1}$ are solutions of the problems

$$(272) \quad \left\{ \begin{array}{l} -\Delta_{\zeta} w_{j,k,l,p} = 2 \frac{\partial^2 w_{j-1,k,l-1,p}}{\partial x_1 \partial \zeta_1} + \frac{\partial^2 w_{j-2,k,l-2,p}}{\partial x_1^2} \\ \quad + \left\{ \begin{array}{l} \sum_{j_1,k_1,l_1,p_1} \lambda_{j_1,k_1,l_1,p_1} w_{j-j_1-2,k-k_1,l-l_1-2,p-p_1} \quad \text{in } \overline{\mathbb{R}_+^2} \setminus B, \\ \sum_{j_1,k_1,l_1,p_1} \lambda_{j_1,k_1,l_1,p_1} w_{j-j_1,k-k_1,l-l_1,p-p_1-1} \quad \text{in } B, \end{array} \right. \\ w_{j,k,l,p} = 0 \quad \text{on } \gamma, \\ \frac{\partial w_{j,k,l,p}}{\partial \zeta_2} = 0 \quad \text{on } \Gamma. \end{array} \right.$$

h) The series (267) has an asymptotic expansion as $\tau \rightarrow +\infty$, rewritten in terms of the variables ξ , of the form

$$(273) \quad \begin{aligned} u_{\varepsilon}(x) &= \sum_{p=0}^{\infty} \sum_{j=1}^{\infty} \sum_{l=2}^{\infty} \sum_{k=0}^{j-l+1} \varepsilon^j \ln^k a \, a^l \mu^p \, \mathcal{V}_{j,k,l,p}(\xi; x_1) \\ &+ \sum_{p=0}^{\infty} \sum_{j=1}^{\infty} \sum_{k=0}^j \varepsilon^j \ln^k a \mu^p \, \mathcal{V}_{j,k,0,p}(\xi; x_1) \quad \text{as } \rho a^{-1} \rightarrow +\infty. \end{aligned}$$

4. SUBSTANTIATION OF THE ASYMPTOTICS THAT HAVE BEEN CONSTRUCTED

Let $\mathcal{N} = (\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3)$ be the vector of indices, $\mathcal{N}_s \in \mathbb{N}$. We denote partial sums of the series (264), (265), (266), and (267), respectively, by

$$(274) \quad \begin{aligned} \tilde{\lambda}_{\varepsilon}^{\mathcal{N}} &= \lambda_0 + \sum_{p=0}^{\mathcal{N}_3} \sum_{j=1}^{\mathcal{N}_1} \sum_{l=2}^{\mathcal{N}_2} \sum_{k=0}^{j-l+1} \varepsilon^j \ln^k a \, a^l \mu^p \, \lambda_{j,k,l,p} \\ &+ \sum_{p=0}^{\mathcal{N}_3} \sum_{j=1}^{\mathcal{N}_1} \sum_{k=0}^j \varepsilon^j \ln^k a \mu^p \, \lambda_{j,k,0,p}; \end{aligned}$$

$$(275) \quad \begin{aligned} u_{\varepsilon}^{\mathcal{N}}(x) &= u_0(x) + \sum_{p=0}^{\mathcal{N}_3} \sum_{j=1}^{\mathcal{N}_1} \sum_{l=2}^{\mathcal{N}_2} \sum_{k=0}^{j-l+1} \varepsilon^j \ln^k a \, a^l \mu^p \, u_{j,k,l,p}(x) \\ &+ \sum_{p=0}^{\mathcal{N}_3} \sum_{j=1}^{\mathcal{N}_1} \sum_{k=0}^j \varepsilon^j \ln^k a \mu^p \, u_{j,k,0,p}(x); \end{aligned}$$

$$(276) \quad \begin{aligned} v_{\varepsilon}^{\mathcal{N}}(\xi; x_1) &= \sum_{p=0}^{\mathcal{N}_3} \sum_{j=1}^{\mathcal{N}_1} \sum_{l=2}^{\mathcal{N}_2} \sum_{k=0}^{j-l+1} \varepsilon^j \ln^k a \, a^l \mu^p \, v_{j,k,l,p}(\xi; x_1) \\ &+ \sum_{p=0}^{\mathcal{N}_3} \sum_{j=1}^{\mathcal{N}_1} \sum_{k=0}^j \varepsilon^j \ln^k a \mu^p \, v_{j,k,0,p}(\xi; x_1); \end{aligned}$$

$$(277) \quad \begin{aligned} w_{\varepsilon}^{\mathcal{N}}(\zeta; x_1) &= \sum_{p=0}^{\mathcal{N}_3} \sum_{j=1}^{\mathcal{N}_1} \sum_{l=1}^{\mathcal{N}_2} \sum_{k=0}^{j-l} \varepsilon^j \ln^k a \, a^l \mu^p \, w_{j,k,l,p}(\zeta; x_1) \\ &+ \sum_{p=0}^{\mathcal{N}_3} \sum_{j=1}^{\mathcal{N}_1} \sum_{k=0}^{j-1} \varepsilon^j \ln^k a \mu^p \, w_{j,k,0,p}(\zeta; x_1). \end{aligned}$$

Let $0 < \beta < 1$ and $\Upsilon > 0$ be arbitrary numbers, and $\chi(t)$ a cutoff function, $0 < \chi < 1$, that is equal to 0 for $t < 1$ and to 1 for $t > 2$. We define

$$\chi_{\beta, \Upsilon}(x) = \chi\left(\frac{x_2}{\Upsilon \varepsilon^\beta}\right), \quad \chi_{\beta, \Upsilon}^{\mathbf{n}}(x) = \chi\left(\frac{\sqrt{(x_1 - \mathbf{n}\pi a)^2 + x_2^2}}{\Upsilon \varepsilon^\beta a^\beta}\right),$$

$$w_\varepsilon^{\mathbf{n}, \mathcal{N}}\left(\frac{x}{a\varepsilon}; x_1\right) = w_\varepsilon^{\mathcal{N}}\left(\frac{x_1}{a\varepsilon} - \frac{\mathbf{n}\pi}{\varepsilon}, \frac{x_2}{a\varepsilon}; x_1\right), \quad \mathbf{n} \in \mathbb{Z}.$$

Lemma 4.1. *The function*

$$(278) \quad \begin{aligned} \tilde{u}_\varepsilon^{\mathcal{N}}(x) = & u_\varepsilon^{\mathcal{N}} \chi_{\beta, \Upsilon}(x) + v_\varepsilon^{\mathcal{N}}\left(\frac{x}{\varepsilon}; x_1\right) (1 - \chi_{\beta, \Upsilon}(x)) \prod_{\mathbf{n}=-\mathbf{N}}^{\mathbf{N}} \chi_{\beta, \Upsilon}^{\mathbf{n}}(x) \\ & + \sum_{\mathbf{n}=-\mathbf{N}}^{\mathbf{N}} w_\varepsilon^{\mathbf{n}, \mathcal{N}}\left(\frac{x}{a\varepsilon}; x_1\right) (1 - \chi_{\beta, \Upsilon}^{\mathbf{n}}(x)) \end{aligned}$$

is an $H^1(\Omega)$ solution of the boundary-value problem

$$(279) \quad \begin{cases} -\Delta \tilde{u}_\varepsilon^{\mathcal{N}} = \tilde{\lambda}_\varepsilon^{\mathcal{N}} \rho_\varepsilon \tilde{u}_\varepsilon^{\mathcal{N}} + f_\varepsilon^{\mathcal{N}} & \text{in } \Omega, \\ \tilde{u}_\varepsilon^{\mathcal{N}} = 0 & \text{on } \gamma_\varepsilon, \\ \frac{\partial \tilde{u}_\varepsilon^{\mathcal{N}}}{\partial \nu} = 0 & \text{on } \Gamma_\varepsilon \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3, \end{cases}$$

where ρ_ε is given by formula (3),

$$(280) \quad \|f_\varepsilon^{\mathcal{N}}\|_0 = O(\varepsilon^{\mathcal{N}_4} a^{\mathcal{N}_5} \mu^{\mathcal{N}_6}),$$

and the constants $\mathcal{N}_4$, $\mathcal{N}_5$, and $\mathcal{N}_6$ can be explicitly calculated and satisfy

$$(281) \quad \mathcal{N}_s \rightarrow +\infty, \quad s = 4, 5, 6 \quad \text{as} \quad \mathcal{N}_q \rightarrow +\infty, \quad q = 1, 2, 3.$$

Furthermore,

$$(282) \quad \tilde{u}_\varepsilon^{\mathcal{N}} \longrightarrow u_0 \quad \text{strongly in } L_2(\Omega) \quad \text{as} \quad a, \varepsilon, \mu \rightarrow 0.$$

Proof. Using the boundary conditions for the problems (268), (270), and (272), the function $\tilde{u}_\varepsilon^{\mathcal{N}}$ satisfies the boundary conditions for (279). Acting on the function $\tilde{u}_\varepsilon^{\mathcal{N}}$ by the operator $(-\Delta - \tilde{\lambda}_\varepsilon^{\mathcal{N}} \rho_\varepsilon)$ we find that

$$(283) \quad f_\varepsilon^{\mathcal{N}} = I_1 + I_2 + I_3 + I_4 + I_5,$$

where

$$\begin{aligned}
(284) \quad I_1 &= \chi_{\beta, \Upsilon} (-\Delta u_\varepsilon^\mathcal{N} - \tilde{\chi}_\varepsilon^\mathcal{N} u_\varepsilon^\mathcal{N}), \\
I_2 &= (1 - \chi_{\beta, \Upsilon}) \prod_{\mathbf{n}=-\mathbf{N}}^{\mathbf{N}} \chi_{\beta, \Upsilon}^\mathbf{n} (-\Delta v_\varepsilon^\mathcal{N} - \tilde{\chi}_\varepsilon^\mathcal{N} \rho_\varepsilon v_\varepsilon^\mathcal{N}), \\
I_3 &= \sum_{\mathbf{n}=-\mathbf{N}}^{\mathbf{N}} (-\Delta w_\varepsilon^{\mathbf{n}, \mathcal{N}} - \tilde{\chi}_\varepsilon^\mathcal{N} w_\varepsilon^{\mathbf{n}, \mathcal{N}}) (1 - \chi_{\beta, \Upsilon}^\mathbf{n}), \\
I_4 &= 2 \frac{1}{\Upsilon \varepsilon^\beta} \frac{d\chi}{dt} \left(\frac{\partial v_\varepsilon^\mathcal{N}}{\partial x_2} - \frac{\partial u_\varepsilon^\mathcal{N}}{\partial x_2} \right) \\
&\quad + \frac{1}{\Upsilon^2 \varepsilon^{2\beta}} \frac{d^2 \chi}{dt^2} (v_\varepsilon^\mathcal{N} - u_\varepsilon^\mathcal{N}), \\
I_5 &= -2 \nabla v_\varepsilon^\mathcal{N} (1 - \chi_{\beta, \Upsilon}) \nabla \left(\prod_{\mathbf{n}=-\mathbf{N}}^{\mathbf{N}} \chi_{\beta, \Upsilon}^\mathbf{n} \right) \\
&\quad - v_\varepsilon^\mathcal{N} (1 - \chi_{\beta, \Upsilon}) \Delta \left(\prod_{\mathbf{n}=-\mathbf{N}}^{\mathbf{N}} \chi_{\beta, \Upsilon}^\mathbf{n} \right) \\
&\quad + 2 \sum_{\mathbf{n}=-\mathbf{N}}^{\mathbf{N}} \nabla w_\varepsilon^{\mathbf{n}, \mathcal{N}} \nabla \chi_{\beta, \Upsilon}^\mathbf{n} + \sum_{\mathbf{n}=-2\mathbf{N}}^{2\mathbf{N}} w_\varepsilon^{\mathbf{n}, \mathcal{N}} \Delta \chi_{\beta, \Upsilon}^\mathbf{n}.
\end{aligned}$$

It follows from parts a), e), and g) of Theorem 3.9 that

$$\begin{aligned}
(285) \quad \|I_s\| &= O(\varepsilon^{\mathcal{N}_4} a^{\mathcal{N}_5} \mu^{\mathcal{N}_6}), \\
\mathcal{N}_p &\rightarrow +\infty, \quad p = 4, 5, 6 \quad \text{as} \quad \mathcal{N}_q \rightarrow +\infty, \quad q = 1, 2, 3,
\end{aligned}$$

for $s = 1, 2, 3$, respectively. We observe that the support of I_4 is contained in the layer $\Upsilon \varepsilon^\beta < x_2 < 2\Upsilon \varepsilon^\beta$, while the support of I_5 is contained in the union of the half-rings

$$\begin{aligned}
\Upsilon a^\beta \varepsilon^\beta &< \sqrt{(x_1 - \mathbf{n}\pi a)^2 + x_2^2} < \Upsilon a^\beta \varepsilon^\beta, \\
\mathbf{n} &\in \mathbb{Z}, \quad -\mathbf{N} \leq \mathbf{n} \leq \mathbf{N};
\end{aligned}$$

there are of the order of ε^{-1} of these. This fact and parts b), e) and f), h) of Theorem 3.9 imply that the estimate (285) holds for $s = 4$ and 5 , respectively. The estimate (280) follows from (283) and (285) for $s = 1, 2, 3, 4, 5$. The limit (282) follows from the definition of $\tilde{u}_\varepsilon^\mathcal{N}$. The lemma is proved. $\square$

We have thus completed the construction of the formal asymptotics of the eigenvalues and eigenfunctions. We now pass to a rigorous substantiation of the asymptotics we have constructed. The following statement holds.

Theorem 4.1. *The asymptotic of the eigenvalues and the corresponding eigenfunctions has the form*

$$(286) \quad \begin{aligned} \lambda_\varepsilon = \lambda_0 &+ \sum_{p=0}^{\infty} \sum_{j=1}^{\infty} \sum_{l=2}^{\infty} \sum_{k=0}^{j-l+1} \varepsilon^j \ln^k a \, a^l \mu^p \, \lambda_{j,k,l,p} \\ &+ \sum_{p=0}^{\infty} \sum_{j=1}^{\infty} \sum_{k=0}^j \varepsilon^j \ln^k a \mu^p \, \lambda_{j,k,0,p}, \end{aligned}$$

$$(287) \quad \begin{aligned} \mathcal{U}_\varepsilon(x) = u_0(x) &+ \sum_{p=0}^{\infty} \sum_{j=1}^{\infty} \sum_{l=2}^{\infty} \sum_{k=0}^{j-l+1} \varepsilon^j \ln^k a \, a^l \mu^p \, u_{j,k,l,p}(x) \\ &+ \sum_{p=0}^{\infty} \sum_{j=1}^{\infty} \sum_{k=0}^j \varepsilon^j \ln^k a \mu^p \, u_{j,k,0,p}(x) \quad \text{for } x_2 \geq \varepsilon^\beta, \end{aligned}$$

$$(288) \quad \begin{aligned} \mathcal{U}_\varepsilon(x) = &\sum_{p=0}^{\infty} \sum_{j=1}^{\infty} \sum_{l=2}^{\infty} \sum_{k=0}^{j-l+1} \varepsilon^j \ln^k a \, a^l \mu^p \, v_{j,k,l,p}\left(\frac{x}{\varepsilon}; x_1\right) \\ &+ \sum_{p=0}^{\infty} \sum_{j=1}^{\infty} \sum_{k=0}^j \varepsilon^j \ln^k a \mu^p \, v_{j,k,0,p}\left(\frac{x}{\varepsilon}; x_1\right) \\ &\text{for } x_2 < 2\varepsilon^\beta, \, (x_1 - a\mathbf{n}\pi)^2 + x_2^2 \geq \varepsilon^{2\beta} a^{2\beta}, \end{aligned}$$

$$(289) \quad \begin{aligned} \mathcal{U}_\varepsilon(x) = &\sum_{p=0}^{\infty} \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} \sum_{k=0}^{j-l} \varepsilon^j \ln^k a \, a^l \mu^p \, w_{j,k,l,p}\left(\frac{x_1}{a\varepsilon} - \frac{\mathbf{n}\pi}{\varepsilon}, \frac{x_2}{a\varepsilon}; x_1\right) \\ &+ \sum_{p=0}^{\infty} \sum_{j=1}^{\infty} \sum_{k=0}^{j-1} \varepsilon^j \ln^k a \mu^p \, w_{j,k,0,p}\left(\frac{x_1}{a\varepsilon} - \frac{\mathbf{n}\pi}{\varepsilon}, \frac{x_2}{a\varepsilon}; x_1\right) \\ &\text{for } (x_1 - a\mathbf{n}\pi)^2 + x_2^2 < 2\varepsilon^{2\beta} a^{2\beta}, \quad \mathbf{n} \in \mathbb{Z}, \quad -\mathbf{N} \leq \mathbf{n} \leq \mathbf{N}, \end{aligned}$$

in the norm of the Sobolev space $H^1(\Omega)$. Here the $u_{j,k,l,p}$, $\lambda_{j,k,l,p}$, $v_{j,k,l,p}$, and $w_{j,k,l,p}$ satisfy the conditions of Theorem 3.9 and, in particular,

$$\lambda_{1,1,0,0} = \int_{-\pi/2}^{\pi/2} (\alpha_0^1)^2(x_1) dx_1 > 0, \quad \lambda_{1,0,0,1} = -\frac{\lambda_0}{\pi} \int_{-\pi/2}^{\pi/2} (\alpha_0^1)^2(x_1) dx_1 \int_B Y^2(\zeta) d\zeta,$$

where

$$Y(\zeta) = \operatorname{Re} \ln(y + \sqrt{y^2 - 1}),$$

$y = \zeta_1 + i\zeta_2$, $w_{1,0,0,0}(\zeta; x_1) = \alpha_0^1(x_1)Y(\zeta)$, $v_{1,0,0,0}(\xi; x_1) = \alpha_0^1(x_1)X(\xi)$, where $X(\xi) = \operatorname{Re} \ln \sin z + \ln 2$, $z = \xi_1 + i\xi_2$, $\alpha_0^1(x_1) = \frac{\partial u_0}{\partial x_2} \Big|_{x_2=0}$.

Remark 4.1. We point out that in the theorem an asymptotic of a non-normalized eigenfunction is constructed, that is, of $\mathcal{U}_\varepsilon(x) = T_\varepsilon u_\varepsilon(x)$, with the constant satisfying $T_\varepsilon = 1 + o(1)$ as $\varepsilon \rightarrow 0$, where u_ε is an eigenfunction of the problem (2) normalized in $L_2(\Omega)$.

Proof. Consider the function $\tilde{u}_\varepsilon^\mathcal{N}$ defined in (279). Taking the fact that $\|\tilde{u}_\varepsilon^\mathcal{N}\|_0 \rightarrow 1$ as $\varepsilon \rightarrow 0$ into account and applying Theorem 2.3 (part 1) with $N = 1$ for $\lambda = \tilde{\lambda}_\varepsilon^\mathcal{N}$, $U_\varepsilon = \tilde{u}_\varepsilon^\mathcal{N}$, and $f = f_\varepsilon^\mathcal{N}$, we conclude that

$$(290) \quad |\tilde{\lambda}_\varepsilon^\mathcal{N} - \lambda_\varepsilon| = O(\varepsilon^{\mathcal{N}_4} a^{\mathcal{N}_5} \mu^{\mathcal{N}_6}).$$

As the choice of $\mathcal{N}$ is arbitrary, using condition (281), the asymptotics of λ_ε have the form (286). We pass to substantiating the asymptotics constructed for the eigenfunction.

Recall that by the theorem on convergence, $u_\varepsilon \rightarrow u_0$ strongly in $L_2(\Omega)$ as $\varepsilon \rightarrow 0$. Consequently, by (282) we have

$$(291) \quad \|\tilde{u}_\varepsilon^\mathcal{N} - u_\varepsilon\|_0 \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0,$$

where u_ε is a solution of the problem (2). The representation $\tilde{u}_\varepsilon^\mathcal{N} = u_\varepsilon \int_\Omega u_\varepsilon \tilde{u}_\varepsilon^\mathcal{N} dx + u_\varepsilon^\perp$ with $u_\varepsilon^\perp \perp u_\varepsilon$ and (291) imply that

$$(292) \quad \tilde{u}_\varepsilon^\mathcal{N} = T_\varepsilon^\mathcal{N} u_\varepsilon + u_\varepsilon^\perp, \quad T_\varepsilon^\mathcal{N} = 1 + o(1).$$

It follows from (290), (292), and Lemma 4.1 that the function $u_\varepsilon^\perp$ is a solution of the boundary-value problem

$$(293) \quad \begin{cases} -\Delta u_\varepsilon^\perp = \tilde{\lambda}_\varepsilon^\mathcal{N} \rho_\varepsilon u_\varepsilon^\perp + F_\varepsilon^\mathcal{N} & \text{for } x \in \Omega, \\ u_\varepsilon^\perp = 0 & \text{for } x \in \gamma_\varepsilon, \\ \frac{\partial u_\varepsilon^\perp}{\partial \nu} = 0 & \text{for } x \in \Gamma_\varepsilon \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3, \end{cases}$$

where

$$(294) \quad F_\varepsilon^\mathcal{N} = f_\varepsilon^\mathcal{N} + \rho_\varepsilon T_\varepsilon u_\varepsilon (\tilde{\lambda}_\varepsilon^\mathcal{N} - \lambda_\varepsilon) = O(\varepsilon^{\mathcal{N}_4-m} a^{\mathcal{N}_5-m} \mu^{\mathcal{N}_6}).$$

It follows from (293), (294), and part 2) of Theorem 2.3 with $N = 1$ for $\lambda = \tilde{\lambda}_\varepsilon^\mathcal{N}$, $U_\varepsilon = \tilde{u}_\varepsilon^\mathcal{N}$, and $f = f_\varepsilon^\mathcal{N}$ that

$$(295) \quad \|u_\varepsilon^\perp\|_0 = O(\varepsilon^{\mathcal{N}_4-m} a^{\mathcal{N}_5-m} \mu^{\mathcal{N}_6}).$$

Finally, it follows from (292), (295), from the fact that the choice of $\mathcal{N}$, Υ is arbitrary, and from (281) that the asymptotic behaviour of the eigenfunction has the form (287), (288), (289). Thus, the theorem is proved. $\square$

Remark 4.2. When $a = \varepsilon^h$ and in addition $(2h+1)/(h+1) < m < 2$, the correction (263) comes second in the asymptotic expansion of the eigenvalue, that is,

$$\lambda_\varepsilon = \lambda_0 + \varepsilon \ln a \lambda_{1,1,0,0} + \varepsilon \mu \lambda_{1,0,0,1} + \dots$$

and the other terms have lower order.

APPENDIX 1. AUXILIARY RESULTS FOR THE CONSTRUCTION OF THE ‘INTERMEDIATE’ EXPANSION

Proof of Lemma 3.1. It follows from the definition (28) and from (30) that

$$(296) \quad \frac{\partial X}{\partial \xi_1} = A \rho^{-1} \cos \theta + \sum_{j=0}^{\infty} \tilde{B}_{2j} \rho^{2j+1} \cos(2j+1)\theta \quad \text{as } \rho \rightarrow 0.$$

Let $Z_j(\xi)$ be an arbitrary harmonic polynomial of order j . It is known (see, for example, [69]) that the equations

$$(297) \quad \Delta u = Z_j \rho^\alpha, \quad \alpha \neq -2j-2, \quad \alpha \neq -2, \quad \Delta u = Z_j \rho^{-2}$$

have solutions of the form $u = a_1 Z_j \rho^{\alpha+2}$ and $u = a_2 Z_j \ln \rho$, respectively. The constants a_1 and a_2 can be calculated explicitly. It follows from this fact and from (296) that there exists a series

$$\hat{X}_2(\xi) = Q \rho \ln \rho \cos \theta + \sum_{j=0}^{\infty} \beta_{2j+1} \rho^{2j+3} \cos(2j+1)\theta$$

(where Q, β_k are constants that can be calculated explicitly) that is a formal asymptotic solution of the equation

$$(298) \quad \Delta \hat{X}_2 = -2 \frac{\partial X}{\partial \xi_1} \quad \text{as } \rho \rightarrow 0.$$

We set

$$(299) \quad \widehat{X}_2^N(\xi) = Q\rho \ln \rho \cos \theta + \sum_{j=0}^N \beta_{2j+1} \rho^{2j+3} \cos(2j+1)\theta$$

in Π and extend $(1 - \chi(\rho))\widehat{X}_2^N(\xi)$ periodically in ξ_1 to $\mathbb{R}_+^2$. We seek $X_2(\xi)$ in the form

$$(300) \quad X_2(\xi) = (1 - \chi(\rho))\widehat{X}_2^N(\xi) + \check{X}_2(\xi)$$

in Π , also extending $\check{X}_2(\xi)$ periodically in ξ_1 to $\mathbb{R}_+^2$. By (52), for the function $\check{X}_2$ we obtain the boundary-value problem

$$(301) \quad \begin{cases} -\Delta_\xi \check{X}_2 = F & \text{in } \Pi, \\ \frac{\partial \check{X}_2}{\partial \xi_2} = 0 & \text{on } \check{\Sigma}, \\ \check{X}_2 = 0 & \text{at } \xi_1 = \pm \frac{\pi}{2}, \end{cases}$$

where

$$(302) \quad e^{\varpi \xi_2} F \in L_2(\Pi),$$

$$(303) \quad F = O(\rho^{2N+3}) \quad \text{as } \rho \rightarrow 0.$$

Using (28), (299), and (300) we see that the function F is odd and π -periodic in ξ_1 and

$$(304) \quad F \in C^\infty\left(\overline{\mathbb{R}_+^2} \setminus \bigcup_{k \in \mathbb{Z}} \{(k\pi, 0)\}\right) \cap C^{2N+3}(\overline{\mathbb{R}_+^2}).$$

By (302) there exists a unique solution of the problem (301) such that

$$(305) \quad e^{\varpi \xi_2} \check{X}_2 \in H^1(\Pi)$$

(see [74, Ch. 2,5]). Furthermore, using (304), the problem (301), and the fact that the function F is odd and π -periodic in ξ_1 , we conclude that

$$\check{X}_2 \in C^\infty\left(\overline{\mathbb{R}_+^2} \setminus \bigcup_{k \in \mathbb{Z}} \{(k\pi, 0)\}\right) \cap C^{2N+3}(\overline{\mathbb{R}_+^2}),$$

$\check{X}_2$ is an odd function in ξ_1 (by the uniqueness of the solution of the problem (301)), and by (303) it has an asymptotic expansion of the form

$$(306) \quad \check{X}_2 = \sum_{j=0}^{N+1} \varrho_{2j+1} \rho^{2j+1} \cos(2j+1)\theta + O(\rho^{2N+5}) \quad \text{as } \rho \rightarrow 0.$$

Hence, all the assertions of the lemma follow when we apply (299), (300), the fact that N is an arbitrary number, and that the problem (301) is uniquely solvable in the class of functions under consideration. $\square$

It is known (see, for example, [69]) that the equation

$$(307) \quad \Delta u = Z_j \rho^\alpha \ln \rho, \quad \alpha \geq 0,$$

has a solution of the form $u = a_3 Z_j \rho^{\alpha+2} + a_4 Z_j \rho^{\alpha+2} \ln \rho$. The constants a_3 and a_4 can be calculated explicitly. In view of this fact, similarly to Lemma 3.1, we can prove the following assertion.

Lemma 4.2. *Let $F \in A_{\text{odd}}^{(2k-1,0)}$; then the problem*

$$(308) \quad \begin{cases} -\Delta_{\xi} W = F & \text{in } \Pi, \\ W = 0 & \text{at } \xi_1 = \pm \frac{\pi}{2}, \\ \frac{\partial W}{\partial \xi_2} = 0 & \text{on } \Sigma \end{cases}$$

has a solution $W \in A_{\text{odd}}^{(2k+1,1)}$.

The definition of the class $\mathcal{A}_{\text{odd}}^{(2j+1,q)}$ and Lemma 4.2 imply the following lemma.

Lemma 4.3. *Let $F \in \mathcal{A}_{\text{odd}}^{(2k-1,0)}$; then the problem*

$$(309) \quad \begin{cases} -\Delta_{\xi} W = F & \text{in } \Pi, \\ \frac{\partial W}{\partial \xi_2} = 0 & \text{on } \Sigma, \\ W = 0 & \text{at } \xi_1 = \pm \frac{\pi}{2} \end{cases}$$

has a solution $W \in \mathcal{A}_{\text{odd}}^{(2k+1,1)}$.

Lemma 4.4. *Let $e^{\varpi \xi_2} F \in L_2(\Pi)$; then the problem*

$$(310) \quad \begin{cases} -\Delta_{\xi} W = F & \text{in } \Pi, \\ \frac{\partial W}{\partial \xi_1} = 0 & \text{at } \xi_1 = \pm \frac{\pi}{2}, \\ \frac{\partial W}{\partial \xi_2} = 0 & \text{on } \Sigma \end{cases}$$

has a solution W that can be represented in the form

$$(311) \quad W = c\xi_2 + \widetilde{W},$$

where $e^{\varpi \xi_2} \widetilde{W} \in H^1(\Pi)$.

Proof. We represent the function W in the form of a sum

$$W = W_1 + W_2,$$

where W_1 is a solution of the problem

$$(312) \quad \begin{cases} -\Delta_{\xi} W_1 = F & \text{in } \Pi, \\ \frac{\partial W_1}{\partial \xi_1} = 0 & \text{at } \xi_1 = \pm \frac{\pi}{2}, \\ W_1 = 0 & \text{on } \Sigma, \end{cases}$$

and W_2 solves the problem

$$(313) \quad \begin{cases} -\Delta_{\xi} W_2 = 0 & \text{in } \Pi, \\ \frac{\partial W_2}{\partial \xi_1} = 0 & \text{at } \xi_1 = \pm \frac{\pi}{2}, \\ \frac{\partial W_2}{\partial \xi_2} = -\frac{\partial W_1}{\partial \xi_2} & \text{on } \Sigma. \end{cases}$$

By the hypothesis of the lemma, the problem (312) is solvable in the class

$$(314) \quad e^{\varpi \xi_2} W_1 \in H^1(\Pi)$$

(see [74, Ch. 2,5]). It follows from the theorem on the increase of smoothness that

$$\frac{\partial W_1}{\partial \xi_2} \in L_2(\check{\Sigma}).$$

Next, solving the problem (313) by the standard method of separation of variables, we find that a solution of the form $W_2 = c\xi_2 + \widetilde{W}_2$ exists, where $e^{\varpi \xi_2} \widetilde{W}_2 \in H^1(\Pi)$. This fact and (314) prove the lemma. $\square$

Taking account of Lemma 4.4, similarly to Lemma 3.1, we can prove that the following assertion holds.

Lemma 4.5. *Let $F \in A_{\text{even}}^{(2k,0)}$; then the problem (310) has a solution W that can be represented in the form*

$$(315) \quad W = c\xi_2 + \widetilde{W},$$

where $\widetilde{W} \in A_{\text{even}}^{(2k+2,1)}$, $\widetilde{W} = O(1)$ as $\rho \rightarrow 0$, and c is some constant.

By subtracting the function $cX(\xi)$ from (315) we obtain the following.

Corollary. *Suppose that F satisfies the hypotheses of Lemma 4.5; then the problem (310) has a solution $W \in A_{\text{even}}^{(2k+2,1)}$.*

Throughout what follows, $P_j(\xi_2)$ will denote a polynomial of order j . The following assertion follows immediately from Lemma 4.5.

Lemma 4.6. *Suppose that F can be represented in the form $F = P_t + f$, where $f \in A_{\text{even}}^{(2k,0)}$ and P_{t+2} is a polynomial such that $-P_{t+2}'' = P_t$. Then there exists a solution of the boundary-value problem (310) that can be represented in the form $W = P_{t+2} + W_1$, where $W_1 \in A_{\text{even}}^{(2k+2,1)}$.*

Proof of Lemma 3.2. The definitions of $A_{\text{even}}^{(2k,q)}$ and $\mathcal{P}_t(\xi_2; x_1)$ and Lemma 4.6 imply the proof of part a) of Lemma 3.2. The first equality of part b) follows from the definitions of $A_{\text{even}}^{(2k,q)}$ and $\mathcal{P}_t(\xi_2; x_1)$. Since

$$\frac{\partial W}{\partial \xi_1} = 0 \quad \text{at } \xi_1 = \pm \frac{\pi}{2}$$

(by part a)), the first equality implies the second equality of part b). $\square$

Proof of Theorem 3.1. The proof of part a) of the theorem follows immediately from Lemmas 4.3 and 4.6. The first equality in part b) follows from the definition of the class $\mathbb{A}^{(k+2,1)}$ and from Lemma 4.3. The first equality and the fact that $\partial W / \partial \xi_1(\xi; \pm \pi/2) = 0$ at $\xi_1 = \pm \pi/2$ (by part a)) imply that the second equality in part b) holds. $\square$

APPENDIX 2. AUXILIARY RESULTS FOR THE CONSTRUCTION OF THE 'INNER' EXPANSION

The explicit form of the Green's function for the Neumann problem for the half-plane and the theorem on the increase of smoothness yield the following result.

Lemma 4.7. *Suppose that $F \in L_2(\mathbb{R}_+^2) \cap C^\infty(\widetilde{B})$ has asymptotic behaviour*

$$F = O(\tau^{-N-1}) \quad \text{as } \tau \rightarrow +\infty$$

and that $\varphi \in C_0^\infty(\gamma)$ is such that $\int_{\mathbb{R}} \varphi d\zeta_1 = 1$. Then there exists a function

$$W_1 \in H_{\text{loc}}^2(\mathbb{R}_+^2) \cap C^\infty(\widetilde{B})$$

that is a solution of the boundary-value problem

$$(316) \quad \begin{cases} \Delta_{\zeta} W_1 = F & \text{in } \mathbb{R}_+^2, \\ \frac{\partial W_1}{\partial \zeta_2} = \left(\int_{\mathbb{R}_+^2} F d\zeta \right) \varphi & \text{at } \zeta_2 = 0 \end{cases}$$

with the asymptotics

$$(317) \quad W_1 = \sum_{j=0}^{N-2} \tilde{a}_j \tau^{-j} \cos(j\theta) + O(\tau^{-N+1}), \quad \tau \rightarrow \infty.$$

Proof of Lemma 3.6. We seek a solution of the problem (250) in the form of a sum $W = W_1 + W_2$, where W_1 satisfies the assertion of Lemma 4.7. Then the function W_2 must satisfy the following boundary-value problem:

$$(318) \quad \begin{cases} \Delta_{\zeta} W_2 = 0 & \text{in } \mathbb{R}_+^2, \\ \frac{\partial W_2}{\partial \zeta_2} = 0 & \text{on } \Gamma, \\ W_2 = -W_1 & \text{on } \gamma. \end{cases}$$

It follows from Lemma 4.7 that $W_1 \in H^{3/2}(\gamma) \cap C^{\infty}(\gamma)$ (and, consequently, satisfies a Hölder condition at each point of γ). Therefore the Keldysh–Sedov formula (see, for example, [75, Ch. III, § 3]) and the theorem on the increase of smoothness imply that there exists a solution

$$W_2 \in H_{\text{loc}}^1(\mathbb{R}_+^2) \cap C^{\infty}(\mathbb{R}_+^2 \cup \Gamma \cup \gamma)$$

of the problem (318) with the asymptotics

$$(319) \quad W_2 = \sum_{j=0}^{+\infty} \tilde{b}_j \tau^{-j} \cos(j\theta), \quad \tau \rightarrow \infty.$$

This fact and the statement of Lemma 4.7 imply that the first statement in Lemma 3.6 holds. The second part of the lemma follows from the fact that the solution of problem (318) which is bounded at infinity is unique. $\square$

- We denote by $\overline{H}_{\text{odd}}^{(2k-1)}$, $k = 1, \dots$, the set of polynomials in τ of the form

$$\sum_{i=0}^{k-1} \sum_{j=0}^i \mathcal{K}_{\text{odd}}^{(i,j)} \tau^{2i+1} \cos(2i - 2j + 1)\theta$$

with constant coefficients.

- We denote by $\overline{H}_{\text{even}}^{(1,2k)}$, $k = 1, \dots$, the set of polynomials in τ of the form

$$\sum_{i=1}^k \sum_{j=0}^i \mathcal{K}_{\text{even}}^{(i,j)} \tau^{2i} \cos 2(i - j)\theta$$

with constant coefficients.

- We denote by $\overline{H}_{\text{even}}^{(0,2k)}$, $k = 0, 1, \dots$, the set of polynomials in τ of the form

$$\sum_{i=0}^k \sum_{j=0}^i \mathcal{K}_{\text{even}}^{(i,j)}(x_1) \tau^{2i} \cos 2(i - j)\theta$$

with constant coefficients.

We also set

$$\begin{aligned}\overline{F}_{\text{odd}}^{(2k-1)} &= \left\{ V(\zeta) : V(\zeta) = \sum_{j=1}^k \ln \tau h_{2j-1}(\zeta) + \sum_{j=1}^k \tilde{h}_{2j-1}(\zeta), \quad h_{2j-1}, \tilde{h}_{2j-1} \in \overline{H}_{\text{odd}}^{(2j-1)} \right\}, \\ \overline{F}_{\text{even}}^{(1,2k)} &= \left\{ V(\zeta) : V(\zeta) = \sum_{j=1}^k \ln \tau h_{2j}(\zeta) + \sum_{j=1}^k \tilde{h}_{2j}(\zeta), \quad h_{2j}, \tilde{h}_{2j} \in \overline{H}_{\text{even}}^{(1,2j)} \right\}, \\ \overline{F}_{\text{even}}^{(0,2k)} &= \left\{ V(\zeta) : V(\zeta) = \sum_{j=0}^k \ln \tau h_{2j}(\zeta) + \sum_{j=0}^k \tilde{h}_{2j}(\zeta), \quad h_{2j}, \tilde{h}_{2j} \in \overline{H}_{\text{even}}^{(0,2j)} \right\}.\end{aligned}$$

Lemma 4.8. *Let $F \in B_{\text{odd}}^{(2k-1, 2n-1, 0)}$ and let Q be a function in $\overline{F}_{\text{odd}}^{(2n+1)}$ such that*

$$(320) \quad F - \Delta Q = O(\tau^{-2}) \quad \text{as } \tau \rightarrow +\infty.$$

Then there exists a solution $W \in B_{\text{odd}}^{(2k+1, 2n+1, 1)}$ of the boundary-value problem

$$(321) \quad \begin{cases} \Delta_{\zeta} W = F & \text{in } \mathbb{R}_+^2, \\ \frac{\partial W}{\partial \zeta_2} = 0 & \text{on } \Gamma, \\ W = 0 & \text{on } \gamma, \\ W \sim Q & \text{as } \tau \rightarrow +\infty. \end{cases}$$

Proof. The form of the solutions of equations (297) and (307) implies that there exists a series

$$(322) \quad \tilde{Q} = \tilde{Q} + \sum_{i=0}^{+\infty} \sum_{j=0}^i \tilde{d}_{i,j}^{\text{odd}} \tau^{-2i+2j-1} \cos(2i+1)\theta,$$

where $\tilde{Q} \in \overline{F}_{\text{odd}}^{(2n+1)}$, that is a formal asymptotic solution as $\tau \rightarrow +\infty$ of the problem

$$(323) \quad \begin{cases} \Delta_{\zeta} Q = F & \text{in } \mathbb{R}_+^2, \\ \frac{\partial Q}{\partial \zeta_2} = 0 & \text{at } \zeta_2 = 0. \end{cases}$$

Taking account of (320) we conclude that $\Delta Q - \Delta \tilde{Q} = O(\tau^{-2})$. Consequently, the series

$$(324) \quad Q = Q + \sum_{i=0}^{+\infty} \sum_{j=0}^i d_{i,j}^{\text{odd}} \tau^{-2i+2j-1} \cos(2i+1)\theta$$

is also a formal asymptotic solution as $\tau \rightarrow +\infty$ of (323). We seek a solution of the boundary-value problem (321) in the form

$$(325) \quad W(\zeta) = W_N(\zeta) = (1 - \chi(\tau R)) Q_{2N+1}(\zeta) + W_N^{(0)}(\zeta),$$

where Q_{2N+1} is a partial sum of the series (324), and χ is the cutoff function introduced in §4. Then for $W_N^{(0)}$ we obtain the boundary-value problem

$$(326) \quad \begin{cases} \Delta_{\zeta} W_N^{(0)} = F_N & \text{in } \mathbb{R}_+^2, \\ \frac{\partial W_N^{(0)}}{\partial \zeta_2} = 0 & \text{on } \Gamma, \\ W_N^{(0)} = 0 & \text{on } \gamma, \end{cases}$$

where the function $F_N \in L_2(\mathbb{R}_+^2) \cap C^\infty(\tilde{B})$ has the asymptotic behaviour $F_N = O(\tau^{-2N-3})$ as $\tau \rightarrow \infty$ and is odd in ζ_1 . By Lemma 3.6 there exists a solution

$$W_N^{(0)} \in H_{\text{loc}}^1(\mathbb{R}_+^2) \cap C^\infty(\tilde{B}) \cap H_{\text{loc}}^2(\mathbb{R}_+^2 \setminus \overline{(\mathcal{O}_{\zeta,+} \cup \mathcal{O}_{\zeta,-})})$$

of the problem (326) that is odd in ζ_1 and has the asymptotic expansion

$$(327) \quad W_N^{(0)} = \sum_{j=0}^{N-1} \tilde{C}_{2j+1} \tau^{-2j-1} \cos(2j+1)\theta + O(\tau^{-2N-1}) \quad \text{as } \tau \rightarrow \infty.$$

Consequently, by (325) the function W_N is odd. We claim that the function W_N is independent of N . Let $N_1 \geq N \geq 4$ and let $W^{(N,N_1)} = W_N - W_{N_1}$. Then for $W^{(N,N_1)}$ we obtain the boundary-value problem

$$\begin{cases} \Delta_\xi W^{(N,N_1)} = 0 & \text{in } \mathbb{R}_+^2, \\ \frac{\partial W^{(N,N_1)}}{\partial \zeta_2} = 0 & \text{on } \Gamma, \\ W^{(N,N_1)} = 0 & \text{on } \gamma, \end{cases}$$

and the function $W^{(N,N_1)}$ decreases as $\tau \rightarrow \infty$. Then by the uniqueness of the solution of this boundary-value problem, the function satisfies $W^{(N,N_1)} \equiv 0$ for $\zeta_2 \geq 0$ (see, for example, [75]) and, consequently, W_N is independent of N . As N is arbitrary, by (325), (327), and the theorem on the increase of smoothness, it follows that $W \in B_{\text{odd}}^{(2k+1, 2n+1, 1)}$. The lemma is proved. $\square$

The following result is proved completely analogously.

Lemma 4.9. *Let $F \in B_{\text{even}}^{(2k, 2n, 0)}$ and let Q be a function in $\overline{F}_{\text{even}}^{(0, 2n)}$ such that*

$$(328) \quad F - \Delta Q = O(\tau^{-2}) \quad \text{as } \tau \rightarrow +\infty.$$

Then there exists a solution $W \in B_{\text{even}}^{(2k, 2n, 1)}$ of the boundary-value problem

$$(329) \quad \begin{cases} \Delta_\zeta W = F & \text{in } \mathbb{R}_+^2, \\ \frac{\partial W}{\partial \zeta_2} = 0 & \text{on } \Gamma, \\ W = 0 & \text{on } \gamma, \\ W \sim Q & \text{as } \tau \rightarrow +\infty. \end{cases}$$

Proof of Theorem 3.6. The proof of the theorem follows from the definition of the spaces $\mathbb{B}^{(k, n, 0)}$, $F^{(1, k)}$ and the statements of Lemmas 4.8 and 4.9. $\square$

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