ON THE POLES OF PICARD POTENTIALS

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ABSTRACT. We study the existence of a global meromorphic fundamental system of solutions for a system of two differential equations $E_x = (az + q(x))E$, where a is a constant diagonal matrix, and $q(x)$ is an off-diagonal meromorphic function, for each $z \in \mathbb{C}$. Following Gesztesy and Weikard (1998), who investigated this property of functions $q(x)$ and its connection to finite-gap solutions of soliton equations, we call such $q(x)$ Picard potentials. We obtain conditions for the Picard property of various potentials $q(x)$.

1. INTRODUCTION AND STATEMENTS OF MAIN RESULTS

Suppose that a_{11} and a_{22} are distinct complex numbers. Following ([4, Definition 4.7]), we say that a pair $u(x), v(x)$ of meromorphic functions on $\mathbb{C}$ defines a *Picard potential* if, for each $z \in \mathbb{C}$, the system of two linear differential equations

$$(1) \quad E_x = \begin{pmatrix} a_{11}z & u(x) \\ v(x) & a_{22}z \end{pmatrix} E$$

has a fundamental system of solutions which is meromorphic on the entire complex plane $\mathbb{C}$ of the variable x . The main result of [4] shows that a pair of elliptic functions with a common period lattice defines a Picard potential if and only if it is an initial condition of a finite-gap solution for some class of integrable nonlinear equations (for example, the Korteweg–de Vries equation with $v(x) \equiv 1$ and $a_{11} = 1/2, a_{22} = -1/2$ or the nonlinear Schrödinger equation with $v(x) \equiv \pm u(\bar{x})$ and $a_{11} = i/2, a_{22} = -i/2$, see [6, Chapter 1, §8]). Moreover, it follows from [1] that if $w(x, t)$ is a locally holomorphic solution of one of the above equations, then for each fixed t_0 the function $u(x) = w(x, t_0)$ analytically continues to a function meromorphic in $\mathbb{C}$, and, together with the corresponding function $v(x)$, defines a Picard potential. All that provides motivation for a more detailed study of Picard potentials with a goal of a more explicit and complete description of them. In [3], such a description was obtained in the case when $v(x) \equiv 1$, and $u(x)$ is either elliptic or periodic and bounded near the ends of the period strip or rational and holomorphic at ∞ : in that case the pair $u(x), 1$ is Picard if and only if $u(x)$ satisfies a suitable variant of the Calogero–Moser conditions. However, these examples are practically all the known results up to date about an explicit description of Picard potentials.

In Theorem 1 of this paper, we obtain a necessary condition for an arbitrary pair $u(x), v(x)$ of meromorphic functions to be Picard, provided that at least one of them has a pole $x_0 \in \mathbb{C}$: the function $u(x)v(x)$ must have, at x_0 , a pole of order two with Laurent coefficient of special form. This necessary condition is rather restrictive, but it is still far from being sufficient, which is illustrated by Theorem 2, giving a necessary and sufficient condition for the Picard property of monomial functions $u(x), v(x)$. Finally, in

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Theorem 3 we consider pairs with $v(x) = u(x)$ for $a_{11} = 1/2$ and $a_{22} = -1/2$ and obtain an analog (Corollary 2) of the Calogero–Moser conditions for that case (corresponding, in the terminology of [6, Chapter 1, §8], to the modified Korteweg–de Vries equation).

Theorem 1. *Let a_{11}, a_{22} be distinct complex numbers, and $u(x), v(x)$ meromorphic functions in a neighborhood of $x_0 \in \mathbb{C}$ such that at least one of them has a pole of order n at x_0 . Suppose system (1) has, for each $z \in \mathbb{C}$, a fundamental system of solutions which is meromorphic in a neighborhood of x_0 . Then $u(x)v(x)$ has a pole of order two at x_0 and*

$$u(x)v(x) = \frac{N^2 - (n-1)^2}{4}(x-x_0)^{-2} + O((x-x_0)^{-1})$$

for some $N \in \mathbb{N}$ such that n and N are of distinct parities (and $N \neq n-1$).

Theorem 2. *Let a_{11} and a_{22} be distinct complex numbers, $m, l \in \mathbb{Z}$, $A, B \in \mathbb{C} \setminus \{0\}$, and $x_0 \in \mathbb{C}$. Then the system*

$$(2) \quad E_x = \begin{pmatrix} a_{11}z & A(x-x_0)^m \\ B(x-x_0)^l & a_{22}z \end{pmatrix} E$$

has, for each $z \in \mathbb{C}$, a meromorphic in a neighborhood of x_0 fundamental system of solutions if and only if either $m, l \geq 0$ or, if $m < 0$ ($l < 0$), then $m + l = -2$ and $AB = \frac{N^2 - (-m-1)^2}{4}$, where $N = -m + 1, -m + 3, -m + 5, \dots$ (respectively, replacing m by l).

Theorem 3. *Suppose $u(x)$ is a meromorphic function in a neighborhood of $x_0 \in \mathbb{C}$, with a pole at that point. Suppose the system*

$$(3) \quad E_x = \begin{pmatrix} z/2 & u(x) \\ u(x) & -z/2 \end{pmatrix} E$$

has, for each $z \in \mathbb{C}$, a meromorphic in a neighborhood of x_0 fundamental system of solutions. Then $u(x)$ has a pole of order one at x_0 and, in a neighborhood of x_0 ,

$$(4) \quad u(x) = \frac{\pm N}{x-x_0} + \sum_{k=0}^{\infty} u_k(x-x_0)^k,$$

where $N \in \mathbb{N}$ and $u_{2k} = 0$ for $k = 0, \dots, N-1$.

The main tool used in the proofs of all three theorems is the Frobenius method. For our case, we describe it in Section 2, the general case and the proof can be found in [5, Chapter XVI]. The proof of Theorem 1 is given in Section 3, and of Theorem 2, in Section 4. In Section 5, one can find the proof of Theorem 3 and some of its consequences related to Picard potentials for $v(x) = \pm u(x)$.

2. THE FROBENIUS METHOD

The Frobenius method allows us to find a series representation for a fundamental system of solutions (FSS) of a differential equation in a neighborhood of a singular point. To apply it to system (1) we use the following result.

Proposition 1. *A linear 2×2 system*

$$(5) \quad \begin{pmatrix} y_1'(x) \\ y_2'(x) \end{pmatrix} = \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix} \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$$

with meromorphic coefficients, where $a_{12}(x) \not\equiv 0$, has a meromorphic FSS in a neighborhood of x_0 if and only if the equation

$$(6) \quad y'' = \left(a_{22} + a_{11} + \frac{a_{12}'}{a_{12}} \right) y' + \left(a_{21}a_{12} + a_{11}' - a_{22}a_{11} - a_{11}\frac{a_{12}'}{a_{12}} \right) y$$

has a meromorphic FSS in a neighborhood of x_0 .

Proof. To obtain (6), we differentiate the second equation of (5) and substitute in the resulting equation the values for y'_2 and y_2 obtained, respectively, from the second and the first equation of the system. Therefore, if $(\frac{y_1}{y_2}, \frac{\tilde{y}_1}{\tilde{y}_2})$ is an FSS of (5) in a neighborhood of x_0 , then $y_1, \tilde{y}_1$ are solutions of (6). Moreover, since $y'_1 = a_{11}y_1 + a_{12}y_2$ (this is the second equation of (5)), the Wronskians

$$\det \begin{pmatrix} y_1 & \tilde{y}_1 \\ y_2 & \tilde{y}_2 \end{pmatrix} \quad \text{and} \quad \det \begin{pmatrix} y_1 & y'_1 \\ \tilde{y}_1 & \tilde{y}'_1 \end{pmatrix}$$

vanish simultaneously. Hence, $y_1, \tilde{y}_1$ is an FSS of (6). Therefore, if (5) has a meromorphic FSS in a neighborhood of x_0 , then so does (6).

Conversely, suppose (6) has a meromorphic FSS in a neighborhood of x_0 . By what has just been proved, if $(\frac{y_1}{y_2})$ is a solution of (5), then y_1 is meromorphic in a neighborhood of x_0 . Hence, because of the second equation of (5), y_2 is meromorphic in the same neighborhood. $\square$

For this reason, in the proofs of Theorems 1 and 2, we shall reduce system (1) to the corresponding second-order differential equation and apply the Frobenius method, which we now describe.

Suppose we have a differential equation

$$(7) \quad y'' + \frac{g_1(x)}{x}y' + \frac{g_0(x)}{x^2}y = 0,$$

where the functions g_1, g_0 are holomorphic in some neighborhood of zero $\mathcal{O}_\varepsilon := \{x \in \mathbb{C} : |x| < \varepsilon\}$. Then, in that neighborhood, we can write an FSS $(y_1(x), y_2(x))$ of equation (7) as a convergent series.

More precisely, define a function

$$f(x, r) = r(r-1) + rg_1(x) + g_0(x),$$

where $x \in \mathcal{O}_\varepsilon, r \in \mathbb{C}$. For each r , it is holomorphic in $\mathcal{O}_\varepsilon$. Hence, in that neighborhood, we can expand it into a series

$$f(x, r) = \sum_{k=0}^{\infty} f_k(r)x^k.$$

The equation

$$(8) \quad f_0(r) := r(r-1) + rg_1(0) + g_0(0) = 0$$

is called the characteristic equation. Suppose that r_1, r_2 are its roots ($\operatorname{Re} r_1 \geq \operatorname{Re} r_2$).

1. If $r_1 - r_2 \notin \mathbb{Z}$, then set $c_0(r) = 1$ and recurrently define, in small neighborhoods of r_1 and r_2 , functions $c_k(r)$ ($k \in \mathbb{N}$):

$$(9) \quad c_k(r) = -\frac{1}{f_0(r+k)} \sum_{j=0}^{k-1} c_{k-j}(r) f_j(r+k-j).$$

For $i = 1, 2$ we set

$$y_i(x) = x^{r_i} \sum_{k=0}^{\infty} c_k(r_i)x^k.$$

2. If $r_1 - r_2 \in \mathbb{Z}$, then set $c_0(r) = \prod_{j=1}^{r_1-r_2} f_0(r+j)$ (if $r_1 = r_2$, then $c_0(r) = 1$). Functions $c_k(r)$ are again defined recurrently by formula (9) ($c_{r_1-r_2}(r)$ is extended to r_2 by continuity). Then

$$y_1(x) = x^{r_1} \sum_{k=0}^{\infty} c_k(r_1) x^k,$$

$$y_2(x) = x^{r_2} \left(\sum_{k=0}^{\infty} c'_k(r_2) x^k + \ln x \sum_{k=0}^{\infty} c_k(r_2) x^k \right).$$

Claim 1 (The Frobenius method). The functions $y_1(x), y_2(x)$ form an FSS of equation (7) in $\mathcal{O}_\varepsilon$.

Proof. See [5, Chapter XVI]. □

Corollary 1. *The differential equation (7) has a meromorphic FSS in a neighborhood of $x_0 = 0$ if and only if:*

- 1) *the characteristic equation (8) has distinct integer roots r_1, r_2 ($r_1 > r_2$);*
- 2) *$\gamma_{r_1-r_2}(r_2) = 0$, where $\gamma_0 = 1$, and $\gamma_k(r)$, $k \in \mathbb{N}$ is defined recurrently as*

$$(10) \quad \gamma_k(r) = - \sum_{j=1}^k \gamma_{k-j}(r) f_j(r+k-j) \prod_{i=1}^{j-1} f_0(r+k-i).$$

Proof. If equation (7) has a meromorphic FSS in $\mathcal{O}_\varepsilon$, then any solution of (7) is meromorphic in $\mathcal{O}_\varepsilon$. If condition 1) fails (i.e., either one of r_1, r_2 is not an integer or r_1, r_2 are equal integers), then, among the solutions $y_1(x), y_2(x)$, there would be at least one which is not meromorphic in $\mathcal{O}_\varepsilon$ (has a branch point at $x = 0$). If condition 1) is satisfied, then $y_1(x)$ is meromorphic, and $y_2(x)$ is meromorphic if and only if $c_k(r_2) = 0$ for any $k = 0, 1, \dots$, where $c_k(r)$ are defined by (9). For that (as can be gleaned from the formula itself) it suffices to have $c_{r_1-r_2}(r_2) = 0$, since $f_0(r_2+k) \neq 0$ for $k \in \mathbb{N}$ and $k \neq r_1 - r_2$, and $c_0(r_2) = 0$ by definition. To finish the proof, we show that $c_{r_1-r_2}(r) = \gamma_{r_1-r_2}(r)$. This is a consequence of the following formula expressing $\gamma_k(r)$ in terms of $c_k(r)$ for $k = 0, 1, \dots, r_1 - r_2$:

$$c_k(r) = \frac{c_0(r)}{\prod_{j=1}^k f_0(r+j)} \gamma_k(r). \quad \square$$

3. PROOF OF THEOREM 1

Without loss of generality, we may assume that $x_0 = 0$ and that $u(x)$ has a pole of order n at zero.

To show that the conditions imposed on $u(x)$ and $v(x)$ are necessary for the existence, in a neighborhood of zero, of a meromorphic FSS of (1) for each $z \in \mathbb{C}$, we apply Proposition 1 and transform system (1) into an equation of type (6):

$$(11) \quad y'' - \left((a_{11} + a_{22})z + \frac{u'}{u} \right) y' - \left(uv - a_{11}a_{22}z^2 - a_{11}z \frac{u'}{u} \right) y = 0.$$

This equation must have a meromorphic FSS in a neighborhood of zero for each $z \in \mathbb{C}$. Since the function $u'(x)/u(x)$ has a pole of order one at zero, it is easy to see that if (11) has a meromorphic solution in a neighborhood of zero, then $u(x)v(x)$ has pole of order at most two at zero. Hence, equation (11) is of the form (7).

Now apply Corollary 1. In accordance with the Frobenius method, we find

$$f(x, r, z) = r(r-1) - \left((a_{11} + a_{22})z + \frac{u'}{u} \right) xr - \left(uv - a_{11}a_{22}z^2 - a_{11}z \frac{u'}{u} \right) x^2.$$

Here, we write $f(x, r, z)$ to emphasize the dependence on z . To use series expansions of $u'(x)/u(x)$ and $u(x)v(x)$, set

$$\frac{u'(x)}{u(x)} = -\frac{n}{x} + \sum_{k=0}^{\infty} \sigma_k x^k, \quad u(x)v(x) = \sum_{k=-2}^{\infty} d_k x^k.$$

Now

$$\begin{aligned} f_0(r, z) &= r^2 + (n-1)r - d_{-2}; \\ f_1(r, z) &= -((a_{11} + a_{22})z + \sigma_0)r - d_{-1} - a_{11}nz; \\ f_2(r, z) &= -\sigma_1 r - d_0 + a_{11}a_{22}z^2 + a_{11}\sigma_0 z; \\ f_k(r, z) &= -\sigma_{k-1}r - d_{k-2} + a_{11}\sigma_{k-2}z \quad \text{for } k \geq 3. \end{aligned}$$

Therefore, the characteristic equation is $r^2 + (n-1)r - d_{-2} = 0$. Since its roots r_1 and r_2 are to be distinct integers and since $r_1 + r_2 = 1 - n$, we have $r_1 - r_2 = N \in \mathbb{N}$ and n and N are of distinct parities. Hence, $d_{-2} = \frac{N^2 - (n-1)^2}{4}$.

To finish the proof, it remains to show that $d_{-2} \neq 0$. Assume the opposite, $d_{-2} = 0$. Then the roots of the characteristic equation $r^2 + (n-1)r = 0$ are $r_1 = 0$, $r_2 = 1 - n$. By Corollary 1, they are distinct integers (i.e., $n \geq 2$) and $\gamma_{n-1}(1-n, z) \equiv 0$ in z . We shall show that $\gamma_{n-1}(1-n, z) \not\equiv 0$ in z for $n \geq 2$.

The recurrence relation (10) and the expressions for the $f_j(r)$ show that $\gamma_k(1-n, z)$ is a polynomial in z of degree k . Suppose

$$\gamma_k(1-n, z) = \alpha_k z^k + O(z^{k-1}).$$

We shall show that $\alpha_{n-1} \neq 0$. Using (10) and the expressions for the f_j we find a recurrence relation for the α_k ,

$$\alpha_k = (ka_{11} + (k-n)a_{22})\alpha_{k-1} - a_{11}a_{22}(k-n)(k-1)\alpha_{k-2},$$

together with initial conditions $\alpha_0 = 1$, $\alpha_1 = a_{11} + (1-n)a_{22}$. In general, such recurrence relations with nonconstant coefficients do not admit an explicit solution, but this relation has one (for proof, induct on k):

$$\alpha_k = \sum_{l=0}^k \frac{k!}{l!} (1-n)(2-n) \cdots (l-n) a_{11}^{k-l} a_{22}^l.$$

Therefore,

$$\alpha_{n-1} = (n-1)! \sum_{l=0}^{n-1} (-1)^l \binom{n-1}{l} a_{11}^{n-1-l} a_{22}^l = (n-1)! (a_{11} - a_{22})^{n-1}.$$

Since $a_{11} \neq a_{22}$, we have $\alpha_{n-1} \neq 0$. Hence, $\gamma_{n-1}(1-n, z) \not\equiv 0$ in z , which was claimed.

4. PROOF OF THEOREM 2

If $m, l \geq 0$, then the assertion of the theorem is obvious since (2) has no singular points. Thus, without loss of generality, we may assume that $n := -m \in \mathbb{N}$. The necessity of the condition $l - n = -2$ follows from Theorem 1; hence $l = n - 2$ in what follows. Moreover, we assume that $x_0 = 0$.

To find a necessary and sufficient condition for the meromorphicity of an FSS of (2) in a neighborhood of zero for each $z \in \mathbb{C}$, apply Proposition 1 and reduce our system to a type (6) equation

$$y'' + \left(-(a_{11} + a_{22})z + \frac{n}{x} \right) y' + \left(a_{11}a_{22}z^2 - a_{11}\frac{n}{x}z - \frac{AB}{x^2} \right) y = 0.$$

Now apply Corollary 1. In accordance with the Frobenius method, we find

$$\begin{aligned} f(x, r, z) &= r(r-1) + (-(a_{11} + a_{22})xz + n)r + a_{11}a_{22}x^2z^2 - a_{11}nxz - AB; \\ f_0(r, z) &= r^2 + (n-1)r - AB; \\ f_1(r, z) &= -(a_{11} + a_{22})zr - a_{11}nz; \\ f_2(r, z) &= a_{11}a_{22}z^2; \\ f_k(r, z) &= 0 \quad \text{for } k \geq 3. \end{aligned}$$

Therefore, the characteristic equation has the form $r^2 + (n-1)r - AB = 0$. Since we again need its roots r_1 and r_2 ($r_1 > r_2$) to be distinct integers, we have (as in Theorem 1) $r_1 = \frac{1-n+N}{2}$, $r_2 = \frac{1-n-N}{2}$, and $AB = \frac{N^2 - (n-1)^2}{4}$, where $N := r_1 - r_2 \in \mathbb{N}$, and n and N are of distinct parities.

By Corollary 1, it remains to find the values of n and N such that $\gamma_{r_1-r_2}(r_2, z) \equiv 0$ in z , i.e., $\gamma_N(\frac{1-n-N}{2}, z) \equiv 0$ in z . Recurrence relations (10) for $\gamma_k(r, z)$ and the expressions for the $f_j(r, z)$ show that $\gamma_k(r, z)$ is a monomial in z of degree k , and, if we introduce α_k via

$$\gamma_k\left(\frac{1-n-N}{2}, z\right) = \frac{\alpha_k z^k}{2^k},$$

then they would satisfy the recurrence relation

$\alpha_k = ((2k-1-N+n)a_{11} + (2k-1-N-n)a_{22})\alpha_{k-1} + 4(k-1)(N-k+1)a_{11}a_{22}\alpha_{k-2}$ with the initial condition $\alpha_0 = 1$, $\alpha_1 = (1-N+n)a_{11} + (1-N-n)a_{22}$. This recurrence relation has the following solution (to see that, induct on k):

$$\alpha_k = \sum_{l=0}^k \prod_{j=1}^l (2j-1-N+n) \prod_{j=1}^{k-l} (2j-1-N-n) \binom{k}{l} a_{11}^l a_{22}^{k-l}.$$

Therefore,

$$\alpha_N = \prod_{j=1}^N (2j-1-N+n) \sum_{l=0}^N (-1)^{N-l} \binom{N}{l} a_{11}^l a_{22}^{N-l} = \prod_{j=1}^N (2j-1-N+n) (a_{11} - a_{22})^N.$$

Hence (as $a_{11} \neq a_{22}$), $\alpha_N = 0$ if and only if $n = N-1, N-3, \dots, 1-N$. Since $\gamma_N(\frac{1-n-N}{2}, z) = \frac{\alpha_N z^N}{2^N}$ and n and N are natural numbers of distinct parities, we have that $\gamma_N(\frac{1-n-N}{2}, z) \equiv 0$ in z if and only if $N = n+1, n+3, \dots$.

5. PROOF OF THEOREM 3

Without loss of generality, we may assume that $x_0 = 0$. Applying the necessary condition from Theorem 1, we have that $u(x)$ has a pole of order one at zero with residue $\pm N$, where $N \in \mathbb{N}$.

To obtain the more refined necessary condition, stated in Theorem 3, for the meromorphicity of an FSS for (3) in a neighborhood of zero for each $z \in \mathbb{C}$, we apply Proposition 1 and reduce the system to a type (6) equation

$$(12) \quad y''(x) = \frac{u'(x)}{u(x)} y'(x) + \left(\frac{z^2}{4} - \frac{z}{2} \frac{u'(x)}{u(x)} + u^2(x) \right) y(x).$$

As (12) does not change when $u(x)$ is replaced by $-u(x)$, we shall assume that

$$u(x) = \frac{N}{x-x_0} + \sum_{k=0}^{\infty} u_k(x-x_0)^k, \quad \text{where } N \in \mathbb{N}.$$

Now apply Corollary 1 to (12). In accordance with the Frobenius method, we find

$$f(x, r, z) = r(r-1) - x \frac{u'(x)}{u(x)} r - \frac{z^2}{4} x^2 + \frac{z}{2} \frac{u'(x)}{u(x)} x^2 - u^2(x) x^2.$$

To use series expansions of $u'(x)/u(x)$ and $u^2(x)$, set

$$\begin{aligned} \frac{u'(x)}{u(x)} &= -\frac{1}{x} + \sum_{k=0}^{\infty} \sigma_k x^k, \\ u^2(x) &= \frac{N^2}{x^2} + \sum_{k=-1}^{\infty} \tau_k x^k. \end{aligned}$$

Then

$$\begin{aligned} f_0(r, z) &= r^2 - N^2; \\ f_1(r, z) &= -\sigma_0 r - \tau_{-1} - \frac{z}{2}; \\ f_2(r, z) &= -\sigma_1 r - \tau_0 + \sigma_0 \frac{z}{2} - \frac{z^2}{4}; \\ f_k(r, z) &= -\sigma_{k-1} r - \tau_{k-2} + \sigma_{k-2} \frac{z}{2} \quad \text{for } k \geq 3. \end{aligned}$$

Since $f_0(r, z)$ does not depend on z , we shall simply write $f_0(r)$.

The roots $r_1 = N$ and $r_2 = -N$ of the characteristic equation $r^2 - N^2 = 0$ are distinct integers; hence the first condition from Corollary 1 is satisfied.

We shall prove that $u_{2k} = 0$ for $k = 0, 1, \dots, N-1$. Suppose this is not the case and let l be the smallest of the numbers k such that $u_{2k} \neq 0$ and $l < N$. We claim that in this case $\gamma_{2N}(-N, z) \neq 0$ in z , and therefore the second condition from Corollary 1 is not satisfied. Recurrence relations (10) for $\gamma_k(r, z)$ and the formulas for the $f_k(r, z)$ show that $\gamma_k(r, z)$ is a polynomial in z of degree at most k . We shall show that the coefficient of $z^{2N-1-2l}$ in the polynomial $\gamma_{2N}(-N, z)$ is not 0.

Since $u_{2k} = 0$ for $k = 0, 1, \dots, l-1$,

$$\begin{aligned} \sigma_0 &= \sigma_2 = \dots = \sigma_{2l-2} = 0, & \sigma_{2l} &= \frac{2l+1}{N} u_{2l}, \\ \tau_{-1} &= \tau_1 = \dots = \tau_{2l-3} = 0, & \tau_{2l-1} &= 2N u_{2l}. \end{aligned}$$

Therefore, in particular, $f_1(r, z), f_3(r, z), \dots, f_{2l-1}(r, z)$ contain only the first power of z , whereas $f_2(r, z), f_4(r, z), \dots, f_{2l}(r, z)$ contain only even powers of z ($f_2(r, z)$ contains degrees zero and two; all others contain only degree zero). It is now easy to see (using induction on n) that the coefficients of $z^{n-1}, z^{n-3}, \dots, z^{n+1-2l}$ in the polynomial $\gamma_n(r, z)$ are zero.

Furthermore,

$$\begin{aligned} f_{2l+1}(r, z) &= \left(-\frac{2l+1}{N} r - 2N \right) u_{2l}; \\ f_{2l+2}(r, z) &= \left(\frac{2l+1}{N} \right) u_{2l} \frac{z}{2} - \sigma_{2l+1} r - \tau_{2l}. \end{aligned}$$

Let α_n be the coefficient of $(\frac{z}{2})^n$ and β_n the coefficient of $(\frac{z}{2})^{n-1-2l}$ in the polynomial $\gamma_n(-N, z)$. We claim that β_{2N} is the product of u_{2l} and a nonzero number. Using the recurrence relation (10) for $\gamma_n(r, z)$, the obtained formulas for the $f_k(r, z)$, and the

absence of the degrees $(n-1), (n-3), \dots, (n+1-2l)$ in z in the polynomial $\gamma_n(r, z)$, we have recurrence relations for β_n :

$$(13) \quad \begin{aligned} \beta_n &= \beta_{n-1} + f_0(-N+n-1)\beta_{n-2} \\ &+ \left(\frac{2l+1}{N}(-N+n-2l-1) + 2N \right) u_{2l} \prod_{j=1}^{2l} f_0(-N+n-j)\alpha_{n-2l-1} \\ &- \frac{2l+1}{N} u_{2l} \prod_{j=1}^{2l+1} f_0(-N+n-j)\alpha_{n-2l-2} \end{aligned}$$

with initial condition

$$(14) \quad \beta_{2l} = 0, \quad \beta_{2l+1} = -((2l+1) - 2N)u_{2l} \prod_{j=1}^{2l} f_0(-N+j).$$

It is clear that to find β_n we need to find α_n . The α_n satisfy the following recurrence relation (it follows from the recurrence relations (10) for $\gamma_n(r, z)$ and the formulas for the corresponding $f_k(r, z)$):

$$(15) \quad \alpha_n = \alpha_{n-1} + f_0(-N+n-1)\alpha_{n-2} = \alpha_{n-1} + (n-1)(n-1-2N)\alpha_{n-2}$$

with initial condition $\alpha_0 = \alpha_1 = 1$. By induction, we easily have

$$(16) \quad \alpha_{2m} = 2 \cdot 6 \cdots (4m-2) \prod_{j=1}^m (j-N), \quad \alpha_{2m+1} = (2m+1)\alpha_{2m}.$$

To solve recurrence relation (13), we first find a fundamental system of solutions of the corresponding homogeneous recurrence relation, and then apply the variation of constants method.

The homogeneous recurrence relation corresponding to (13) coincides with recurrence relation (15) for α_n . We know one of its solutions: the coefficients α_n (with initial conditions $\alpha_0 = \alpha_1 = 1$). We shall find another solution of (15) in the form $h_n = a_n \alpha_n$ with initial condition $h_0 = 0, h_1 = 1$ (therefore, $a_0 = 0, a_1 = 1$). Since the h_n satisfy the recurrence relation for the α_n , we have

$$a_n \alpha_n = a_{n-1} \alpha_{n-1} + (n-1)(n-1-2N)a_{n-2} \alpha_{n-2}.$$

Substituting in this formula the expressions for α_n from (15) and letting b_k denote $a_k - a_{k-1}$, we have a recurrence relation for the b_n ,

$$b_n = -(n-1)(n-1-2N) \frac{\alpha_{n-2}}{\alpha_n} b_{n-1},$$

with initial condition $b_1 = a_1 - a_0 = 1$. Substituting the explicit formulas for α_n from (16), we have

$$(17) \quad b_{2m} = -\frac{2m-1-2N}{2(m-N)} b_{2m-1}, \quad b_{2m+1} = -\frac{2m}{2m+1} b_{2m}.$$

One can see that the expression for b_{2N} has a zero (of order one) in the denominator, but $h_{2N} = \alpha_{2N} \sum_{j=1}^{2N} b_j$, and the expression for α_{2N} has a zero of order one; hence $h_{2N} = \alpha_{2N} b_{2N}$ is well defined and is not zero. (For a more rigorous argument, one can introduce a formal parameter $\tilde{N}$ in a neighborhood of N and then make it approach N .)

Let B_n be the constant term in the recurrence relation for β_n . Then we can rewrite (13) as

$$\beta_n = \beta_{n-1} + f_0(-N+n-1)\beta_{n-2} + B_n.$$

We have two independent solutions of the corresponding homogeneous relations: α_n and h_n . In accordance with the variation of constants method ([2, Chapter 5, §3]), the solution of the inhomogeneous recurrence relation on β_n has the form

$$\beta_n = p_n \alpha_n + q_n h_n,$$

where

$$\begin{aligned} p_n - p_{n-1} &= -B_{n+1} \frac{h_n}{\alpha_n h_{n+1} - \alpha_{n+1} h_n}, \\ q_n - q_{n-1} &= B_{n+1} \frac{\alpha_n}{\alpha_n h_{n+1} - \alpha_{n+1} h_n}, \end{aligned}$$

and the initial conditions p_{2l} and q_{2l} are determined by the relations

$$(18) \quad \begin{cases} p_{2l} \alpha_{2l} + q_{2l} h_{2l} = \beta_{2l}, \\ (p_{2l} + (p_{2l+1} - p_{2l})) \alpha_{2l+1} + (q_{2l} + (q_{2l+1} - q_{2l})) h_{2l+1} = \beta_{2l+1}. \end{cases}$$

Substituting $h_n = a_n \alpha_n$, we have

$$(19) \quad p_n - p_{n-1} = -\frac{a_n B_{n+1}}{b_{n+1} \alpha_{n+1}}, \quad q_n - q_{n-1} = \frac{B_{n+1}}{b_{n+1} \alpha_{n+1}}.$$

Since $\beta_{2l} = 0$, the solution of system (18) has the form

$$p_{2l} = -\frac{a_{2l} \beta_{2l+1}}{b_{2l+1} \alpha_{2l+1}}, \quad q_{2l} = \frac{\beta_{2l+1}}{b_{2l+1} \alpha_{2l+1}},$$

where β_{2l+1} are given by (14).

Taking account of the explicit formulas (16) for α_n , we have

$$\begin{aligned} B_{2m} &= u_{2l}(2N + 2l + 1) \alpha_{2m-2l-1} \prod_{j=1}^{2l} f_0(-N + 2m - j), \\ B_{2m+1} &= u_{2l}(2N - 2l - 1) \alpha_{2m-2l} \prod_{j=1}^{2l} f_0(-N + 2m + 1 - j). \end{aligned}$$

Since p_{2l} and $p_n - p_{n-1}$ have no singularities for $n < 2N$ (by (19)) and $\alpha_{2N} = 0$,

$$p_{2N} \alpha_{2N} = (p_{2N} - p_{2N-1}) \alpha_{2N} = -\frac{a_{2N} B_{2N+1} \alpha_{2N}}{b_{2N+1} \alpha_{2N+1}}.$$

Therefore,

$$\begin{aligned} \beta_{2N} &= p_{2N} \alpha_{2N} + q_{2N} h_{2N} = -\frac{a_{2N} B_{2N+1} \alpha_{2N}}{b_{2N+1} \alpha_{2N+1}} + \left(q_{2l} + \sum_{n=2l+1}^{2N} \frac{B_{n+1}}{b_{n+1} \alpha_{n+1}} \right) a_{2N} \alpha_{2N} \\ &= h_{2N} \left(q_{2l} + \sum_{n=2l+1}^{2N-1} \frac{B_{n+1}}{b_{n+1} \alpha_{n+1}} \right). \end{aligned}$$

Formula (15) shows that, for $m = 0, \dots, N-1$, the signs of α_{2m} and α_{2m+1} equal $(-1)^m$. Formula (17) shows that, for $n = 1, \dots, 2N-1$, the sign of b_n equals $(-1)^{n+1}$. Moreover, since $f_0(r) = r^2 - N^2$, the numbers $f_0(-N + j)$ are negative for $j = 1, \dots, 2N-1$. Therefore, in view of the explicit formulas for B_n , we have that q_{2l} and $\frac{B_{n+1}}{b_{n+1} \alpha_{n+1}}$, for $n = 2l+1, \dots, 2N-2$, equal the product of u_{2l} and a number of sign $(-1)^l$. Furthermore, as we mentioned above, $b_{2N} \alpha_{2N}$ is a well-defined nonzero number of sign $(-1)^{N+1}$ (as can be seen from (15) and (17)). Therefore, $\frac{B_{n+1}}{b_{n+1} \alpha_{n+1}}$ is the product of u_{2l} and a number of sign $(-1)^l$. We have thus proved that β_{2N} is the product of u_{2l} and a nonzero number, and therefore $\gamma_{2N}(-N, z) \not\equiv 0$ in z , contrary to the second condition from Corollary 1.

Remark 1. We have only established the necessity of the conditions from Theorem 3 for the existence, for each $z \in \mathbb{C}$, of a meromorphic FSS for system (3) in a neighborhood of a pole. But we believe that those conditions are also sufficient. If this conjecture is true, then the analog (21) of the Calogero–Moser conditions, mentioned below, would be necessary and sufficient for the Picard property of potentials for the modified Korteweg–de Vries equation.

Corollary 2. *Suppose that $u(x)$ is a rational function, holomorphic at ∞ , with poles at $x_1, x_2, \dots, x_M \in \mathbb{C}$. If the pair $u(x), u(x)$ defines a Picard potential with $a_{11} = 1/2$, $a_{22} = -1/2$ (i.e., system (3) has a global meromorphic FSS), then*

$$(20) \quad u(x) = c + \sum_{\mu=1}^M \frac{\pm N_{\mu}}{x - x_{\mu}},$$

where $N_{\mu} \in \mathbb{N}$, $c \in \mathbb{C}$, and, for any $\mu = 1, \dots, M$, we have the following Calogero–Moser type conditions:

$$(21) \quad c + \sum_{\nu \neq \mu} \frac{\pm N_{\nu}}{x_{\mu} - x_{\nu}} = 0, \quad \sum_{\nu \neq \mu} \frac{\pm N_{\nu}}{(x_{\mu} - x_{\nu})^{2k+1}} = 0$$

for $k = 1, \dots, N_{\mu} - 1$ (in these relations the sign for N_{ν} is the same as for N_{ν} in (20)).

Proof. Since $u(x)$ is a rational function, holomorphic at ∞ , $u(x)$ has finitely many poles. By Theorem 3, the principal part at each pole x_{μ} equals $\frac{\pm N_{\mu}}{x - x_{\mu}}$, where $N_{\mu} \in \mathbb{N}$. Therefore, $u(x)$ has the form (20), and relations (21) are conditions (4). $\square$

Remark 2. It is easy to obtain analogs of Theorem 3 and Corollary 2 for Picard potentials defined by the pairs $u(x), -u(x)$ for $a_{11} = 1/2, a_{22} = -1/2$. In the statements of those results, one should just replace N by iN . To see that, notice that the equation similar to (12) should be of the form

$$(22) \quad y''(x) = \frac{u'(x)}{u(x)} y'(x) + \left(\frac{z^2}{4} - \frac{z}{2} \frac{u'(x)}{u(x)} - u^2(x) \right) y(x).$$

Equations (12) and (22) are transformed into one another when $u(x)$ and $iu(x)$ are interchanged. Hence, the pair $u(x), -u(x)$ is Picard if and only if the pair $iu(x), iu(x)$ is.

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