THE FOKKER–PLANCK–KOLMOGOROV EQUATIONS WITH A POTENTIAL AND A NON-UNIFORMLY ELLIPTIC DIFFUSION MATRIX

S. V. SHAPOSHNIKOV

ABSTRACT. We study solutions of the Fokker–Planck–Kolmogorov equation with unbounded coefficients and a non-uniformly elliptic diffusion matrix. Upper bounds for solutions are obtained. In addition, new estimates with a Lyapunov function are obtained.

§ 1. INTRODUCTION

The purpose of this paper is to obtain upper bounds for solutions of the Fokker–Planck–Kolmogorov equation

$$(1.1) \quad \partial_t \mu = \partial_{x_i} \partial_{x_j} (a^{ij} \mu) - \partial_{x_i} (b^i \mu) + c \mu.$$

Below we use the summation convention, so there is summation over repeated indices. Given a number $T > 0$, we say that a locally finite Borel measure μ on a set $\mathbb{R}^d \times (0, T)$ is defined by a flow of Borel measures $(\mu_t)_{t \in (0, T)}$. We use the notation $\mu = \mu_t dt$ if for every Borel set $B \subset \mathbb{R}^d$ the map $t \mapsto \mu_t(B)$ is measurable and the equation

$$\int_{\mathbb{R}^d \times (0, T)} u(x, t) \mu(dx dt) = \int_0^T \int_{\mathbb{R}^d} u(x, t) \mu_t(dx) dt$$

holds for every function $u \in C_0^\infty(\mathbb{R}^d \times (0, T))$. A typical example is $\mu(B) = P(x_t \in B) dt$, where x_t is a random process. We set

$$Lu = a^{ij} \partial_{x_i} \partial_{x_j} u + b^i \partial_{x_i} u + cu.$$

We say that a measure $\mu = (\mu_t)_{t \in (0, T)}$ satisfies equation (1.1) if the functions a^{ij} , b^i , and c are locally integrable with respect to the total variation $|\mu|$ of the measure μ and

$$\int_0^T \int_{\mathbb{R}^d} [\partial_t u(x, t) + Lu(x, t)] \mu_t(dx) dt = 0$$

for every function $u \in C_0^\infty(\mathbb{R}^d \times (0, T))$. The measure μ satisfies the initial condition $\mu|_{t=0} = \nu$, where ν is a locally finite Borel measure on $\mathbb{R}^d$, if

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^d} \zeta(x) \mu_t(dx) = \int_{\mathbb{R}^d} \zeta(x) \nu(dx)$$

for every function $\zeta \in C_0^\infty(\mathbb{R}^d)$.

2010 *Mathematics Subject Classification.* Primary 35R15; Secondary 35K10, 60J60.

Key words and phrases. Parabolic equations for measures, Fokker–Planck–Kolmogorov equation, diffusion processes.

This research was supported by the Russian Foundation for Basic Research (grant nos. 13-01-00332-a, 11-01-00348-a, 12-01-33009, 11-01-12018-of-m-2011, 13-01-92100 JF) and Programme SFB 701 of Bielefeld University, Germany.

We consider three classes of solutions: non-negative solutions, that is, locally finite Borel measures μ defined by a flow of non-negative measures μ_t ; sub-probability solutions, that is, finite Borel measures μ defined by a flow of non-negative measures μ_t such that $\mu_t(\mathbb{R}^d) \leq 1$ for almost all $t \in (0, T)$; and, finally, in the case $c \leq 0$, solutions μ defined by a flow of non-negative measures μ_t such that $|c| \in L^1(\mu)$ and

$$(1.2) \quad \mu_t(\mathbb{R}^d) \leq \nu(\mathbb{R}^d) + \int_0^t \int_{\mathbb{R}^d} c(x, s) \mu_s(dx) ds$$

for almost all $t \in (0, T)$.

We consider the class of solutions satisfying condition (1.2) separately. When the coefficients are globally bounded every solution μ defined by a family of sub-probability measures $(\mu_t)_{t \in (0, T)}$ satisfies (1.2). Furthermore, if the coefficient c is a continuous function and μ_t is the weak limit of a sequence of measures μ_t^n satisfying (1.2), then μ_t also satisfies (1.2). Thus, this condition holds for every solution constructed as a weak limit of solutions of equations with bounded coefficients.

The main results in our paper consist of obtaining local and global L^p and L^∞ estimates for the densities of non-negative solutions of equation (1.1) with unbounded and non-uniformly elliptic coefficients, deriving a priori estimates with a Lyapunov function for solutions satisfying condition (1.2), and studying the behaviour of the densities of solutions in this class as $|x| \rightarrow +\infty$.

In the case when the matrix A is uniformly elliptic, and the coefficients a^{ij} , b^i and c are bounded or have linear growth, estimates of Gaussian type are well known (see, for example, [5] and [8]). Global boundedness and upper bounds for the densities were obtained in [2] for solutions of Cauchy's problem for equation (1.1), in the case when the matrix A is Lipschitz and uniformly elliptic but with no restrictions on the growth of the coefficients b and c . This was done under the assumption that the initial condition is defined by the density and has finite entropy. In [9], [13], [14] and [16] the transition kernels of the semigroup $\{T_t\}$ were studied. For every continuous non-negative bounded function f these define the minimal non-negative solution $T_t f$ of the Cauchy problem $\partial_t u = Lu$, $u|_{t=0} = f$. In the above papers it was assumed that the coefficients were independent of t and locally Hölder continuous, that the diffusion matrix A was uniformly elliptic and continuously differentiable, and that $c \leq 0$. In addition, the elements of the matrix A and the coefficients b , c had arbitrary growth as $|x| \rightarrow +\infty$. The main results in these papers establish upper estimates for the transition densities of the semigroup $\{T_t\}$ and the continuity of this semigroup in various function classes. Conditions were stated in terms of a Lyapunov function, for which a priori estimates independent of the initial condition were obtained. Note that the kernels of the semigroup $\{T_t\}$ satisfy equation (1.1), but the initial condition is Dirac's measure, and so the results in [2] cannot be applied.

In [6] and [7], estimates for the densities were obtained in the case of an arbitrary initial condition and the coefficient b was only assumed to be locally integrable. However, the diffusion matrix was assumed to be uniformly bounded, uniformly elliptic and uniformly Lipschitz, and it was assumed that $c = 0$. By contrast with the preceding papers, which used Moser's iteration technique to derive estimates over the whole of $\mathbb{R}^d$ directly, in [6] and [7] a new approach was proposed consisting of deriving global estimates from the local ones obtained in [11].

In all the above papers the assumption is made that $A(x, t) \geq \lambda I$ for all $(x, t) \in \mathbb{R}^d \times (0, T)$ and some number $\lambda > 0$. This assumption is essential in deriving global estimates. For example, in [2], [9], [13], [14], and [16] a priori estimates were derived using Moser's technique, and uniform ellipticity is of fundamental importance in this. We note that uniform ellipticity is not required for the existence of a solution, since the

existence of the diffusion process corresponding to the operator L only requires local regularity of the coefficients, and the transition probabilities of this diffusion process satisfy the Fokker–Planck–Kolmogorov equation (1.1). Therefore our main aim in this paper is to consider the case where the diffusion matrix can be both unbounded and non-uniformly elliptic. Using the ideas in [6] and [7] we derive global estimates from local ones, first making the dependence of the constants in the local estimates on the matrix A more precise. Furthermore, the local estimates we obtain here are of interest in their own right and generalize the results in [11]. Note that the conditions are stated in terms of the integrability of A , b and c with respect to the solution μ itself, rather than with respect to Lebesgue measure, which makes it possible to consider unbounded coefficients. A priori integrability of the coefficients with respect to a solution is verified using a Lyapunov function. Recall that a function $V \in C^{1,2}(\mathbb{R}^d \times (0, T)) \cap C(\mathbb{R}^d \times [0, T])$ is called a Lyapunov function if the equation

$$\lim_{|x| \rightarrow +\infty} \min_{t \in [a, b]} V(x, t) = +\infty$$

holds for every closed interval $[a, b] \subset (0, T)$. Below we generalize the estimates using a Lyapunov function for the transition densities of a semigroup obtained in [14] and [16].

We look at an example which illustrates the estimates obtained here. Suppose that $d = d_1 + d_2 \geq 2$; we write $x = (x', x'')$, where $x' \in \mathbb{R}^{d_1}$, $x'' \in \mathbb{R}^{d_2}$. Let $r > 2$, $k > r$, and $\delta \in (0, 1)$. We set

$$A(x, t) = e^{|x'|^{r-\delta} - |x''|^{r-\delta}} I, \quad b(x, t) = -x|x|^{r-2} e^{|x'|^{r-\delta} - |x''|^{r-\delta}}, \quad c(x, t) = -|x|^k.$$

Then, if ϱ is the density of a non-negative solution satisfying condition (1.2), the estimate

$$\varrho(x, t) \leq c_1 \exp(-c_2 |x|^r) \exp(c_3 t^{-r/(k-r)})$$

holds for all $(x, t) \in \mathbb{R}^d \times (0, T)$ and some positive numbers c_1 , c_2 and c_3 . Note that the estimate is independent of the initial condition ν . In Example 3.9 at the end of the paper we consider a much more general case.

Lower bounds for solutions were obtained in [3]. The existence and uniqueness of a solution to the Cauchy problem for the Fokker–Planck–Kolmogorov equation were studied in [10] and [12]. A detailed survey of research into elliptic and parabolic equations for measures was given in [1].

In the next section we consider a priori estimates with a Lyapunov function, and the last section of the paper is devoted to local and global estimates of solutions.

§ 2. ESTIMATES WITH A LYAPUNOV FUNCTION

Here we derive a priori estimates with a Lyapunov function, that is, we obtain sufficient conditions under which a Lyapunov function V is integrable with respect to a solution μ . We also obtain estimates for the integral

$$\int_{\mathbb{R}^d} V(x, t) \mu_t(dx).$$

At the end of the section we give several examples with exponential and polynomial Lyapunov functions.

Recall that, given a diffusion process x_t with generator L and Lyapunov function V for which the inequality $V_t + LV \leq CV$ holds for some number C , Itô's formula and Gronwall's inequality make it possible to obtain the following inequality:

$$\mathbb{E}V(x_t, t) \leq \mathbb{E}V(x_0, 0)e^{Ct}.$$

We set $\mu_t(B) := P(x_t \in B)$. Then

$$\mathbb{E}V(x_t, t) = \int_{\mathbb{R}^d} V(x, t) \mu_t(dx).$$

Below we derive similar estimates for solutions of the Fokker–Planck–Kolmogorov equation, but we do not use probabilistic methods, just the definition of a solution. In this case, restrictions on the coefficients are minimal. We assume that $c \leq 0$ and a solution μ is defined by a family of non-negative measures μ_t which satisfy (1.2), that is, $|c| \in L^1(\mu)$ and

$$\mu_t(\mathbb{R}^d) \leq \nu(\mathbb{R}^d) + \int_0^t \int_{\mathbb{R}^d} c(x, s) \mu_s(dx) ds.$$

In particular, the μ_t are sub-probability measures on $\mathbb{R}^d$. We do not assume any other conditions on the coefficients a^{ij} , b^i , and c . Note that the kernels considered in [13] satisfy (1.2).

The following auxiliary assertion was proved in [10], [12].

Lemma 2.1. *Suppose that $\mu = (\mu_t)_{t \in (0, T)}$ is a solution of equation (1.1) and that a function u in the class $C^{1,2}(\mathbb{R}^d \times (0, T))$ is such that $u(x, t) = 0$ for $x \notin U$ for some ball $U \subset \mathbb{R}^d$. Then there exists a set $J_u \subset (0, T)$ of full Lebesgue measure in $(0, T)$ such that*

$$\int_{\mathbb{R}^d} u(x, t) \mu_t(dx) = \int_{\mathbb{R}^d} u(x, s) \mu_s(dx) + \int_s^t \int_{\mathbb{R}^d} [\partial_\tau u(x, \tau) + Lu(x, \tau)] \mu_\tau(dx) d\tau$$

for all $s, t \in J_u$. Moreover, if $u \in C(\mathbb{R}^d \times [0, T])$, the measure $\mu = (\mu_t)_{0 < t < T}$ satisfies the initial condition $\mu|_{t=0} = \nu$, and $a^{ij}, b^i, c \in L^1(U \times [0, T], \mu)$, then

$$\int_{\mathbb{R}^d} u(x, t) \mu_t(dx) = \int_{\mathbb{R}^d} u(x, 0) \nu(dx) + \int_0^t \int_{\mathbb{R}^d} [\partial_\tau u(x, \tau) + Lu(x, \tau)] \mu_\tau(dx) d\tau$$

for all $t \in J_u$.

In the following theorem we derive an a priori estimate with a Lyapunov function.

Theorem 2.2. *Suppose that $\mu = (\mu_t)_{0 < t < T}$ is a solution of the Cauchy problem $\partial_t \mu = L^* \mu$, $\mu|_{t=0} = \nu$, where $c \leq 0$ and μ_t, ν are sub-probability measures on $\mathbb{R}^d$ satisfying (1.2). Suppose that there exists a Lyapunov function V such that, for some functions $K, H \in L^1((0, T))$ with $H \geq 0$ and for all $(x, t) \in \mathbb{R}^d \times (0, T)$, we have the inequality*

$$\partial_t V(x, t) + LV(x, t) \leq K(t) + H(t)V(x, t).$$

Suppose also that $V(\cdot, 0) \in L^1(\nu)$. Then for almost all $t \in (0, T)$ we have

$$\begin{aligned} \mu_t(\mathbb{R}^d) &= \nu(\mathbb{R}^d) + \int_0^t \int_{\mathbb{R}^d} c(x, s) \mu_s(dx) ds, \\ \int_{\mathbb{R}^d} V(x, t) \mu_t(dx) &\leq Q(t) + R(t) \int_{\mathbb{R}^d} V(x, 0) \nu(dx), \end{aligned}$$

where

$$R(t) = \exp\left(\int_0^t H(s) ds\right), \quad Q(t) = R(t) \int_0^t \frac{K(s)}{R(s)} ds.$$

Proof. Let the function $\zeta_N \in C^2([0, +\infty))$ be such that $0 \leq \zeta'_N \leq 1$, $\zeta''_N \leq 0$, $\zeta_N(s) = s$ for $s \leq N - 1$, and $\zeta_N(s) = N$ for $s > N + 1$. Substituting the function $u = \zeta_N(V) - N$

into the equation in Lemma 2.1 we obtain

$$\begin{aligned} \int_{\mathbb{R}^d} \zeta_N(V(x, t)) \mu_t(dx) &= \int_{\mathbb{R}^d} \zeta_N(V(x, s)) \mu_s(dx) \\ &\quad + \left(\mu_t(\mathbb{R}^d) - \nu(\mathbb{R}^d) - \int_s^t \int_{\mathbb{R}^d} c(x, \tau) \mu_\tau(dx) d\tau \right) N \\ &\quad + \int_s^t \int_{\mathbb{R}^d} \left(\zeta'_N(V)(\partial_t V + LV) + \zeta''_N(V) |\sqrt{A} \nabla V|^2 \right) \mu_\tau(dx) d\tau \\ &\quad + \int_s^t \int_{\mathbb{R}^d} c(\zeta_N(V) - \zeta'_N(V)V) \mu_\tau(dx) d\tau. \end{aligned}$$

Note that $z\zeta'_N(z) \leq \zeta_N(z)$. Therefore,

$$\begin{aligned} \int_{\mathbb{R}^d} \zeta_N(V(x, t)) \mu_t(dx) &\leq \int_{\mathbb{R}^d} \zeta_N(V(x, s)) \mu_s(dx) \\ &\quad + \left(\mu_t(\mathbb{R}^d) - \nu(\mathbb{R}^d) - \int_s^t \int_{\mathbb{R}^d} c(x, \tau) \mu_\tau(dx) d\tau \right) N \\ &\quad + \int_s^t \left(K(\tau) + H(\tau) \int_{\mathbb{R}^d} \zeta_N(V(x, \tau)) \mu_\tau(dx) \right) d\tau. \end{aligned}$$

Letting $s \rightarrow 0$ we arrive at the inequality

$$\begin{aligned} (2.1) \quad \int_{\mathbb{R}^d} \zeta_N(V(x, t)) d\mu_t &\leq \int_{\mathbb{R}^d} \zeta_N(V(x, 0)) d\nu \\ &\quad + \left(\mu_t(\mathbb{R}^d) - \nu(\mathbb{R}^d) - \int_0^t \int_{\mathbb{R}^d} c(x, \tau) \mu_\tau(dx) d\tau \right) N \\ &\quad + \int_0^t \left(K(\tau) + H(\tau) \int_{\mathbb{R}^d} \zeta_N(V(x, \tau)) \mu_\tau(dx) \right) d\tau. \end{aligned}$$

Since

$$\mu_t(\mathbb{R}^d) \leq \nu(\mathbb{R}^d) + \int_0^t \int_{\mathbb{R}^d} c(x, \tau) \mu_\tau(dx) d\tau,$$

the last inequality can be rewritten in the following form:

$$\begin{aligned} \int_{\mathbb{R}^d} \zeta_N(V(x, t)) \mu_t(dx) &\leq \int_{\mathbb{R}^d} \zeta_N(V(x, 0)) \nu(dx) \\ &\quad + \int_0^t \left(K(\tau) + H(\tau) \int_{\mathbb{R}^d} \zeta_N(V(x, \tau)) \mu_\tau(dx) \right) d\tau. \end{aligned}$$

Applying Gronwall's inequality we obtain

$$\int_{\mathbb{R}^d} \zeta_N(V(x, t)) \mu_t(dx) \leq Q(t) + R(t) \int_{\mathbb{R}^d} \zeta_N(V(x, 0)) \nu(dx).$$

Letting $N \rightarrow \infty$ we obtain the required estimate. Note that if

$$\mu_t(\mathbb{R}^d) < \nu(\mathbb{R}^d) + \int_0^t \int_{\mathbb{R}^d} c(x, \tau) \mu_\tau(dx) d\tau,$$

then

$$\begin{aligned} \int_{\mathbb{R}^d} V(x, t) \mu_t(dx) - \int_{\mathbb{R}^d} V(x, 0) \nu(dx) \\ - \int_0^t \left(K(\tau) + H(\tau) \int_{\mathbb{R}^d} V(x, \tau) \mu_\tau(dx) \right) d\tau = -\infty, \end{aligned}$$

which is impossible. Thus, we have the equation

$$\mu_t(\mathbb{R}^d) = \nu(\mathbb{R}^d) + \int_0^t \int_{\mathbb{R}^d} c(x, \tau) \mu_\tau(dx) d\tau.$$

The theorem is proved. $\square$

Corollary 2.3. *Suppose that $\mu = (\mu_t)_{0 \leq t \leq T}$ is a solution of the Cauchy problem $\partial_t \mu = L^* \mu$, $\mu|_{t=0} = \nu$, where $c \leq 0$ and μ_t, ν are sub-probability measures on $\mathbb{R}^d$ satisfying condition (1.2). Suppose that the positive function $W \in C^2(\mathbb{R}^d)$ is such that*

$$\lim_{|x| \rightarrow +\infty} W(x) = +\infty.$$

(i) *If for some number $C > 0$ and almost all $(x, t) \in \mathbb{R}^d \times (0, T)$ the inequality*

$$LW(x, t) \leq C + CW(x)$$

holds, then the estimate

$$\int_{\mathbb{R}^d} W(x) \mu_t(dx) \leq \exp(Ct) + \exp(Ct) \int_{\mathbb{R}^d} W(x) \nu(dx)$$

holds for almost all $t \in (0, T)$.

(ii) *Suppose that G is a positive continuous increasing function on $[0, +\infty)$ such that*

$$\int_1^{+\infty} \frac{ds}{sG(s)} < +\infty.$$

Let η be the continuous function on $[0, T)$ defined by the equation

$$t = \int_0^{\eta(t)} \frac{ds}{sG(s-\delta)}, \quad \delta \in (0, 1).$$

If the inequality

$$LW(x, t) \leq C - W(x)G(W(x))$$

holds for some number $C > 0$ and all $(x, t) \in \mathbb{R}^d \times (0, T)$, then

$$\int_{\mathbb{R}^d} W(x) \mu_t(dx) \leq \frac{1}{(1-\delta)\eta^\delta(t)} + \frac{C}{\eta(t)} \int_0^t \eta(s) ds$$

for almost all $t \in (0, T)$.

(iii) *Let G and η be the functions defined in (ii). Suppose that the inequality*

$$LW(x, t) + \eta(t) |\sqrt{A(x, t)} \nabla W(x)|^2 \leq C - W(x)G(W(x))$$

holds for some number $C > 0$ and all $(x, t) \in \mathbb{R}^d \times (0, T)$. Then

$$\int_{\mathbb{R}^d} \exp(\eta(t)W(x)) \mu_t(dx) \leq \exp\left((1-\delta)^{-1}\eta^{1-\delta}(t) + C \int_0^t \eta(s) ds\right)$$

for almost all $t \in (0, T)$.

Proof. To prove (i) it suffices to apply Theorem 2.2 with the functions $H(t) = K(t) = C$ and $V(x, t) = W(x)$.

We now prove (ii). We set $V(x, t) = \eta(t)W(x)$. Then

$$\partial_t V(x, t) + LV(x, t) \leq \eta'(t)W(x) - \eta(t)W(x)G(W(x)) + C\eta(t).$$

We observe that the inequality

$$\alpha\beta \leq \alpha G^{-1}(\alpha) + \beta G(\beta)$$

holds for all non-negative numbers α and β , where G^{-1} is the inverse function of G . Using this inequality with $\alpha = \eta'/\eta$ and $\beta = W$ we obtain

$$\partial_t V(x, t) + LV(x, t) \leq \eta'(t)G^{-1}\left(\frac{\eta'(t)}{\eta(t)}\right) + C\eta(t) = \frac{\eta'(t)}{\eta^\delta(t)} + C\eta(t),$$

since $\eta'(t) = \eta(t)G(\eta^{-\delta}(t))$ by hypothesis.

Applying Theorem 2.2 with $H(t) = 0$ and $K(t) = \eta'(t)/\eta^\delta(t) + C\eta(t)$, we obtain the required inequality.

We now prove (iii). Let $V(x, t) = \exp(\eta(t)W(x))$. Then

$$\partial_t V(x, t) + LV(x, t) \leq [\eta'(t)W(x) - \eta(t)W(x)G(W(x)) + C\eta(t)] \exp(\eta(t)W(x)).$$

Consequently,

$$\partial_t V(x, t) + LV(x, t) \leq \left[\frac{\eta'(t)}{\eta^\delta(t)} + C\eta(t) \right] \exp(\eta(t)W(x)).$$

Applying Theorem 2.2 with $K(t) = 0$ and

$$H(t) = \frac{\eta'(t)}{\eta^\delta(t)} + C\eta(t),$$

we arrive at the required assertion. $\square$

Now we look at some examples.

Example 2.4. Let $V(x, t) = |x|^r$, where $r \geq 2$. Then

$$\begin{aligned} LV(x, t) &= r|x|^{r-2} \operatorname{tr} A(x, t) + r(r-2)|x|^{r-4}(A(x, t)x, x) \\ &\quad + r|x|^{r-2}(b(x, t), x) + |x|^r c(x, t). \end{aligned}$$

Suppose that the inequality

$$r \operatorname{tr} A(x, t) + r(r-2)|x|^{-2}(A(x, t)x, x) + r(b(x, t), x) + |x|^2 c(x, t) \leq C_1 + C_2|x|^2$$

holds for some numbers $C_1 > 0$, $C_2 > 0$ and all $(x, t) \in \mathbb{R}^d \times [0, T]$. Let $|x|^r \in L^1(\nu)$. Then

$$\int_{\mathbb{R}^d} |x|^r \mu_t(dx) \leq e^{C_3 t} + e^{C_3 t} \int_{\mathbb{R}^d} |x|^r \nu(dx)$$

for almost all $t \in (0, T)$ and some number $C_3 > 0$.

Example 2.5. Let $V(x, t) = \exp(\alpha|x|^r)$, where $r \geq 2$. Then

$$\begin{aligned} LV(x, t) &= \exp(\alpha|x|^r) \left[\alpha r|x|^{r-2} \operatorname{tr} A(x, t) + \alpha r(r-2)|x|^{r-4}(A(x, t)x, x) \right. \\ &\quad \left. + \alpha^2 r^2 |x|^{2r-4}(A(x, t)x, x) + \alpha r|x|^{r-2}(b(x, t), x) + c(x, t) \right]. \end{aligned}$$

Suppose that the inequality

$$\begin{aligned} \alpha r|x|^{r-2} \operatorname{tr} A(x, t) + \alpha r(r-2)|x|^{r-4}(A(x, t)x, x) \\ + \alpha^2 r^2 |x|^{2r-4}(A(x, t)x, x) + \alpha r|x|^{r-2}(b(x, t), x) + c(x, t) \leq C_1 \end{aligned}$$

holds for some number C_1 and for all $(x, t) \in \mathbb{R}^d \times [0, T]$.

If $\exp(|x|^r) \in L^1(\nu)$, then

$$\int_{\mathbb{R}^d} \exp(\alpha|x|^r) \mu_t(dx) \leq e^{C_2 t} + e^{C_2 t} \int_{\mathbb{R}^d} \exp(\alpha|x|^r) \nu(dx)$$

for almost all $t \in (0, T)$ and some positive number C_2 .

Example 2.6. Let $k > 2$ and $r \geq 2$. Suppose that for all $(x, t) \in \mathbb{R}^d \times (0, T)$ and some numbers $C_1 > 0$ and $C_2 > 0$,

$$r \operatorname{tr} A(x, t) + r(r-2)|x|^{-2}(A(x, t)x, x) + r(b(x, t), x) + |x|^2 c(x, t) \leq C_1 - C_2|x|^k.$$

Then

$$L|x|^r \leq C_3 - C_3|x|^{r+k-2}$$

for some number $C_3 > 0$. Set $W(x) = |x|^r$ and $G(z) = C_3 z^\sigma$, where $\sigma = \frac{k-2}{r} > 0$. Then

$$LW(x, t) \leq C_3 - WG(W(x))$$

and $\eta(t) = C_4 t^{1/(\delta\sigma)}$, where C_4 depends only on C_3 , δ , and σ . Applying Corollary 2.3 we obtain the estimate

$$\int_{\mathbb{R}^d} |x|^r \mu_t(dx) \leq \frac{\gamma}{t^{r/(k-2)}},$$

where γ depends on C_1 , C_2 , δ , σ .

Example 2.7. Let $r > 2$ and $k > r$. Suppose that the inequality

$$\begin{aligned} \alpha r|x|^{r-2} \operatorname{tr} A(x, t) + \alpha r(r-2)|x|^{r-4}(A(x, t)x, x) + \alpha^2 r^2 |x|^{2r-4}(A(x, t)x, x) \\ + \alpha r|x|^{r-2}(b(x, t), x) + c(x, t) \leq C_1 - C_2|x|^k \end{aligned}$$

holds for all $(x, t) \in \mathbb{R}^d \times (0, T)$ and some numbers $C_1 > 0$ and $C_2 > 0$. Then

$$L \exp(\alpha|x|^r) \leq C_3 - C_3|x|^k \exp(\alpha|x|^r)$$

for some $C_3 > 0$. We set $W(x) = \exp(\alpha|x|^r)$ and $G(z) = C_3 |\ln z|^\sigma$ if $z \geq 2$, where $\sigma = k/r > 1$. We obtain

$$LW(x, t) \leq C_3 - WG(W(x))$$

and $\eta(t) = C_4 \exp(-C_5 t^{-1/(\sigma-1)})$, where $C_4 > 0$ and $C_5 > 0$ depend only on C_3 , δ , and σ . Applying Corollary 2.3 we arrive at the inequality

$$\int_{\mathbb{R}^d} \exp(\alpha|x|^r) \mu_t(dx) \leq \gamma_1 \exp\left(\frac{\gamma_2}{t^{r/(k-r)}}\right),$$

where γ_1 and γ_2 depend only on C_1 , C_2 , δ , and σ .

Example 2.8. Let $r > 2$, $k > 2$, and $\alpha > 0$. Suppose that the inequality

$$\begin{aligned} \alpha r \operatorname{tr} A(x, t) + \alpha r(r-2)|x|^{-2}(A(x, t)x, x) \\ + \alpha r(b(x, t), x) + \alpha|x|^2 c(x, t) + \alpha^2 r^2 |x|^{r-2}(A(x, t)x, x) \leq C_1 - C_2|x|^k \end{aligned}$$

holds for all $(x, t) \in \mathbb{R}^d \times (0, T)$ and some numbers $C_1 > 0$ and $C_2 > 0$. Then

$$\alpha L|x|^r + \alpha^2 r^2 |x|^{2r-4}(A(x, t)x, x) \leq C_3 - C_3|x|^{k+r-2}.$$

Let $W(x) = \alpha|x|^r$ and $G(z) = C_3 \alpha^{-(1+\sigma)/\sigma} z^\sigma$, where $\sigma = \frac{k-2}{r} > 0$. Then

$$LW(x, t) + \left| \sqrt{A(x, t)} \nabla W(x) \right|^2 \leq C_3 - WG(W(x)).$$

Applying Corollary 2.3 with $\delta \in (0, 1)$ and $\eta(t) = C_4 t^{1/(\delta\sigma)}$, where C_4 depends only on C_3 , δ , and σ , for all $\beta > r/(k-2)$ we obtain the estimate

$$\int_{\mathbb{R}^d} \exp(\alpha t^\beta |x|^r) \mu_t(dx) \leq \gamma_1 \exp(\gamma_2 (t^{\beta-r/(k-2)} + t^{\beta+1})),$$

where the positive numbers γ_1 and γ_2 depend only on C_1 , C_2 , r , and β .

Note that the estimates in Examples 2.7 and 2.8 are independent of the initial condition. Applying the results of these examples to the transition probabilities $P(y, 0, t, dx)$ of a diffusion process with generator L we can obtain estimates that are uniform in y . As we already mentioned above, the first estimates of this type were obtained in [14] and [16] for semigroup kernels.

§ 3. LOCAL AND GLOBAL ESTIMATES OF SOLUTIONS

In this section we derive local and global L^p and L^∞ estimates for the densities of solutions. First we derive local estimates using Moser's iteration technique (see [15]), and then obtain global estimates using a suitable scaling.

Let $\mu = (\mu_t)_{t \in (0, T)}$ be a non-negative solution of equation (1.1).

We assume that $A = (a^{ij})$ is a symmetric matrix satisfying the following conditions:

(H1) for some $p > d + 2$, for every ball $U \subset \mathbb{R}^d$ and every closed interval $J \subset (0, T)$, we have

$$\sup_{t \in J} \|a^{ij}(\cdot, t)\|_{W^{1,p}(U)} < \infty,$$

$$0 < \lambda(U, J) := \inf\{(A(x, t)\xi, \xi) : |\xi| = 1, (x, t) \in U \times J\}.$$

Furthermore, we assume that

(H2) for the number $p > d + 2$ in condition (H1), for every ball $U \subset \mathbb{R}^d$ and every closed interval $J \subset (0, T)$, we have

$$b, c \in L^p(U \times J) \quad \text{or} \quad b, c \in L^p(U \times J, \mu).$$

By [11, Corollary 3.9] and [4, Corollary 2.2], conditions (H1) and (H2) guarantee the existence of a Hölder continuous density ϱ of a solution μ with respect to Lebesgue measure. Moreover, for every ball $U \subset \mathbb{R}^d$ and every closed interval $J \subset (0, T)$ we have the inclusion $\varrho(\cdot, t) \in W^{1,p}(U)$ and

$$\int_J \|\varrho(\cdot, t)\|_{W^{1,p}(U)}^p dt < \infty.$$

We set $B^i = b^i - \partial_{x_j} a^{ij}$. Then equation (1.1) can be rewritten in the form of a parabolic equation in divergence form

$$(3.1) \quad \partial_t \varrho = \operatorname{div}(A \nabla \varrho - B \varrho) + c \varrho,$$

which is understood in the sense of the integral identity

$$(3.2) \quad \int_0^T \int_{\mathbb{R}^d} [-\varrho \partial_t \varphi + (A \nabla \varrho, \nabla \varphi)] dx dt = \int_0^T \int_{\mathbb{R}^d} [(B, \nabla \varphi) \varrho + c \varrho \varphi] dx dt$$

for every function $\varphi \in C_0^\infty(\mathbb{R}^d \times (0, T))$.

Recall the following embedding theorem (see [2, Lemma 3.1] or [8]).

Lemma 3.1. *Let J be a closed interval in $(0, T)$ and suppose that a function $u(\cdot, t) \in W^{1,2}(\mathbb{R}^d)$ is such that the map $x \mapsto u(x, t)$ has compact support for almost all $t \in J$. Then there exists a number $C > 0$ depending only on the dimension d such that*

$$\|u\|_{L^{2(d+2)/d}(\mathbb{R}^d \times J)} \leq C \left(\sup_{t \in J} \|u(\cdot, t)\|_{L^2(\mathbb{R}^d)} + \|\nabla u\|_{L^2(\mathbb{R}^d \times J)} \right).$$

Note that we now *do not assume* that $c \leq 0$.

We set $c^+(x, t) = \max\{c(x, t), 0\}$.

The following lemma is a key result in this paper.

Lemma 3.2. *Let $m \geq 1$, let $U \subset \mathbb{R}^d$ be an arbitrary ball, and let $[s_1, s_2] \subset (0, T)$. Suppose that a function $\psi \in C_0^\infty(\mathbb{R}^d \times (0, T))$ is such that its support is contained in $U \times (0, T)$ and $\psi(x, s_1) = 0$ for all x . Then there exists a number $C(d) > 0$ depending only on d such that*

$$(3.3) \quad \left(\int_{s_1}^{s_2} \int_U |\varrho^m \psi|^{2(d+2)/d} dx dt \right)^{d/(d+2)} \leq 32C(d)m^2(1 + \lambda^{-1}) \\ \times \int_{s_1}^{s_2} \int_U \left(|\psi| \cdot |\psi_t| + \|A\| \cdot |\nabla \psi|^2 + |\sqrt{A^{-1}}B|^2 \psi^2 + c^+ \psi^2 \right) \varrho^{2m} dx dt,$$

where $\|A(x, t)\| = \max_{|\xi|=1} (A(x, t)\xi, \xi)$ and the number $\lambda = \lambda(U, [s_1, s_2])$ is defined above.

Proof. Let f be a smooth function on $[0, +\infty)$ such that $f \geq 0$, $f' \geq 0$, $f'' \geq 0$. Substituting the function $\varphi = f'(\varrho)\psi^2$ into the identity (3.2), for every $t \in [s_1, s_2]$ we obtain the inequality

$$\int_{\mathbb{R}^d} f(\varrho(x, t))\psi^2(x) dx - \int_{\mathbb{R}^d} f(\varrho(x, s_1))\psi^2(x) dx \\ + \frac{1}{3} \int_{s_1}^t \int_{\mathbb{R}^d} |\sqrt{A}\nabla \varrho|^2 f''(\varrho)\psi^2 dx d\tau \\ \leq \int_{s_1}^t \int_{\mathbb{R}^d} 2|\psi| \cdot |\psi_t| f(\varrho) + 3|\sqrt{A}\nabla \psi|^2 \frac{f'(\varrho)^2}{f''(\varrho)} + 3|\sqrt{A^{-1}}B|^2 \varrho^2 f''(\varrho)\psi^2 \\ + 2|(B, \nabla \psi)|\psi \varrho f'(\varrho) + c^+ \varrho f'(\varrho)\psi^2 dx d\tau.$$

In fact, it is enough to observe that

$$2(A\nabla \varrho, \nabla \psi)\psi f'(\varrho) \leq 3^{-1}|\sqrt{A}\nabla \varrho|^2 f''(\varrho)\psi^2 + 3|\sqrt{A}\nabla \psi|^2 \frac{f'(\varrho)^2}{f''(\varrho)}, \\ (B, \nabla \varrho)\varrho f''(\varrho)\psi^2 \leq 3^{-1}|\sqrt{A}\nabla \varrho|^2 f''(\varrho)\psi^2 + 3|\sqrt{A^{-1}}B|^2 \varrho^2 f''(\varrho)\psi^2.$$

We set $f(\varrho) = \varrho^{2m}$. Recall that $\psi(x, s_1) = 0$. We have

$$\sup_{t \in [s_1, s_2]} \int_{\mathbb{R}^d} \varrho^{2m}(x, t)\psi^2(x) dx + \frac{4m-2}{3m} \int_{s_1}^{s_2} \int_{\mathbb{R}^d} |\sqrt{A}\nabla(\varrho^m \psi)|^2 dx d\tau \\ \leq 32m^2 \int_{s_1}^{s_2} \int_{\mathbb{R}^d} \left(|\psi| \cdot |\psi_t| + |\sqrt{A}\nabla \psi|^2 + |\sqrt{A^{-1}}B|^2 \psi^2 + c^+ \psi^2 \right) \varrho dx d\tau.$$

Our assertion now follows from Lemma 3.1. $\square$

Theorem 3.3. *Let $p \geq 2(d+2)/d$ and suppose that U and U' are given balls in $\mathbb{R}^d$ such that $\overline{U'} \subset U$. Also let $[s_1, s_2] \subset (0, T)$. Then for all $s \in (s_1, s_2)$ there exists a number $C > 0$ depending only on U, U', s, s_1, d and p such that*

$$\|\varrho\|_{L^p(U' \times [s, s_2])} \leq C(1 + \lambda^{-1})^\gamma \int_{s_1}^{s_2} \int_U \left(1 + \|A\|^\gamma + |c^+|^\gamma + |\sqrt{A^{-1}}B|^{2\gamma} \right) \varrho dx dt,$$

where $\gamma = (d+2)/(2p')$, $p' = p/(p-1)$, and the numbers $\lambda = \lambda(U, [s_1, s_2])$ and $\|A\|$ are defined above.

Proof. We set $m = dp/(2(d+2))$ and

$$\alpha = 1 + \frac{4m}{(2m-1)d}, \quad \alpha' = 1 + \frac{(2m-1)d}{4m}, \quad \delta = \frac{4}{d(2m-1) + 4m}.$$

Note that $m \geq 1$. We set $\psi = \zeta(x)\eta(t)$, where $\zeta \in C_0^\infty(U)$, $\zeta(x) = 1$ for $x \in U'$, $0 \leq \psi \leq 1$, $\eta \in C_0^\infty((s_1, T))$, $\eta(t) = 1$ for $t \in [s, s_2]$, $0 \leq \eta \leq 1$, and

$$|\partial_t \eta(t)| \leq K\eta^{1-\delta}(t), \quad |\nabla \zeta(x)| \leq K\zeta^{1-\delta}(x)$$

for some $K > 0$ and all $(x, t) \in U \times [s_1, s_2]$. Note that K depends only on U , U' , s , and s_1 . Applying Lemma 3.2 we obtain

$$\begin{aligned} & \left(\int_{s_1}^{s_2} \int_U |\varrho^m \psi|^{2(d+2)/d} dx dt \right)^{d/(d+2)} \leq 32C(d)m^2(1 + \lambda^{-1}) \\ & \quad \times \int_s^{s_2} \int_U \left(|\psi| \cdot |\psi_t| + \|A\| \cdot |\nabla \psi|^2 + |\sqrt{A^{-1}}B|^2 \psi^2 + c^+ \psi^2 \right) \varrho^{2m} dx dt. \end{aligned}$$

Using Hölder's inequality with exponents α and α' we can bound the integral on the right-hand side of the last inequality by the following expression:

$$\begin{aligned} & K^2 \left(\int_{s_1}^{s_2} \int_U \left(1 + \|A\| + |\sqrt{A^{-1}}B|^2 + c^+ \right)^{\alpha'} \varrho^{2m} dx dt \right)^{1/\alpha'} \\ & \quad \times \left(\int_{s_1}^{s_2} \int_U |\varrho^m \psi|^{2(d+2)/d} dx dt \right)^{1/\alpha}. \end{aligned}$$

Applying the inequality $xy \leq \varepsilon x^\alpha + C(\alpha, \varepsilon)y^{\alpha'}$, with $\varepsilon > 0$ sufficiently small, we obtain the assertion of the theorem. $\square$

Theorem 3.4. *Let $\gamma > (d+2)/2$ and suppose that we are given balls U and U' in $\mathbb{R}^d$ such that $\overline{U'} \subset U$. Also let $[s_1, s_2] \subset (0, T)$. Then for all $s \in (s_1, s_2)$ there exists a number $C > 0$ depending only on U , U' , s , s_1 , d , and γ such that*

$$\|\varrho\|_{L^\infty(U' \times [s, s_2])} \leq C(1 + \lambda^{-1})^\gamma \int_{s_1}^{s_2} \int_U \left(1 + \|A\|^\gamma + |c^+|^\gamma + |\sqrt{A^{-1}}B|^{2\gamma} \right) \varrho dx dt,$$

where $\lambda = \lambda(U, [s_1, s_2])$ and $\|A\|$ are defined above.

Proof. If $\varrho \equiv 0$ on $U \times [s_1, s_2]$, then the assertion is trivial. Consider the case where $\varrho \not\equiv 0$. By multiplying the solution ϱ by the number

$$(1 + \lambda^{-1})^{-\gamma} \left(\int_{s_1}^{s_2} \int_U \left(1 + \|A\|^\gamma + |c^+|^\gamma + |\sqrt{A^{-1}}B|^{2\gamma} \right) \varrho dx dt \right)^{-1},$$

we can assume that

$$(1 + \lambda^{-1})^\gamma \int_{s_1}^{s_2} \int_U \left(1 + \|A\|^\gamma + |c^+|^\gamma + |\sqrt{A^{-1}}B|^{2\gamma} \right) \varrho dx dt = 1.$$

In this case, to prove the theorem it is sufficient to find a number C depending only on U , U' , s , s_1 , s_2 , d , and γ such that

$$\|\varrho\|_{L^\infty(U' \times [s, s_2])} \leq C.$$

Let $U = U(x_0, R)$, $U' = U(x_0, R')$, and $R' < R$. We set $R_n = R' + (R - R')2^{-n}$, $s_n = s - (s - s_1)2^{-n}$, and $U_n = U(x_0, R_n)$. We consider the system of nested cylinders

$$Q_n = U_n \times [s_n, s_2], \quad Q_0 = U \times [s_1, s_2].$$

For every n we define a function $\psi_n \in C_0^\infty(\mathbb{R}^d \times (0, T))$ in the same way as in the proof of Theorem 3.3; that is, $\psi(x, t) = 1$ for $(x, t) \in Q_{n+1}$, $0 \leq \psi \leq 1$, the support of ψ is contained in $U_n \times (s_n, T)$, and $|\partial_t \psi_n(x, t)| + |\nabla \psi_n(x, t)| \leq K^n$ for all $(x, t) \in \mathbb{R}^d$ and some number $K > 1$ depending only on the numbers s , s_1 , R , R' .

Applying Lemma 3.2 and Hölder's inequality with exponents γ and γ' we obtain

$$\left(\int_{Q_n} |\varrho^m \psi_n|^{2(d+2)/d} dx dt \right)^{d/(d+2)} \leq 32m^2 C(d, s) K^{2n} \left(\int_{Q_n} \varrho^{(2m-1)\gamma'+1} dx dt \right)^{1/\gamma'}.$$

We set

$$p_{n+1} = \beta p_n + (\gamma' - 1)\gamma'^{-1}, \quad p_1 = \gamma' + 1, \quad \beta = (d+2)d^{-1}\gamma'^{-1}.$$

We observe that $\beta^{n-1}p_1 \leq p_n \leq \beta^{n-1}(p_1 + 1)$. Setting $m = p_{n+1}d/(2d+4)$ we arrive at the inequality

$$\|\varrho\|_{L^{p_{n+1}}(Q_{n+1})} \leq C n^{\beta^{-n}} \|\varrho\|_{L^{p_n}(Q_n)}^{p_n/(p_n+\gamma'-1)},$$

where C depends only on K , d and γ . Note that $\sum_n n\beta^{-n} < \infty$. Furthermore, Theorem 3.3 gives an estimate for the norm $\|\varrho\|_{L^{p_1}(Q_1)}$ in terms of a constant depending only on the numbers p_1 , d , s , s_1 , U , and U_1 . Consequently, we obtain an estimate for the norms $\|\varrho\|_{L^{p_{n+1}}(Q_{n+1})}$ that is uniform in n , and this gives an L^∞ -estimate. $\square$

Remark 3.5. (i) Note that the constant C in Theorems 3.3 and 3.4 is independent of the number s_2 .

(ii) If $c \leq 0$, then all the estimates obtained above remain valid without the coefficient c on the right-hand side.

Corollary 3.6. *Let $\gamma > (d+2)/2$, $\kappa > 0$, and $t_0 \in (0, T)$. Then there exists $C > 0$ depending only on κ , t_0 , d , γ such that for all $(x, t) \in \mathbb{R}^d \times (t_0, T)$ the estimate*

$$\varrho(x, t) \leq C(1 + \lambda^{-1}(x, t))^\gamma \int_{t_0/2}^t \int_{U(x, \kappa)} \left(1 + \|A\|^\gamma + |c^+|^\gamma + |\sqrt{A^{-1}}B|^{2\gamma} \right) \varrho dy d\tau$$

holds, where

$$\lambda(x, t) = \inf \{ (A(y, \tau)\xi, \xi) : |\xi| = 1, (y, \tau) \in U(x, \kappa) \times [t_0/2, t] \}.$$

In particular, if $\mu_t(dx) = \varrho(x, t) dx$ is a sub-probability measure for almost all $t \in (0, T)$, the functions $\|A\|^\gamma$, $|c^+|^\gamma$, $|B|^{2\gamma}$ belong to $L^1(\mathbb{R}^d \times (t_0/2, T), \mu)$, and the function $\|A\|^{-1}$ is uniformly bounded, then $\varrho \in L^\infty(\mathbb{R}^d \times (t_0, T))$.

Proof. We shift the point x to 0 and apply Theorem 3.4 for the balls $U = U(x, \kappa)$, $U' = U(x, \kappa/2)$ and the points $s_1 = t_0/2$, $s = t_0$, $s_2 = t$. $\square$

Corollary 3.7. *Let $\gamma > (d+2)/2$ and $\Theta \in (0, 1)$. Then there exists a number $C > 0$ depending only on γ , d and Θ such that for all $(x, t) \in \mathbb{R}^d \times (0, T)$ the estimate*

$$\begin{aligned} \varrho(x, t) &\leq C(1 + \lambda^{-1}(x, t))^\gamma t^{-(d+2)/2} \\ &\quad \times \int_{\Theta t}^t \int_{U(x, \sqrt{t})} \left(1 + \|A\|^\gamma + t^{2\gamma}|c^+|^\gamma + t^{2\gamma}|\sqrt{A^{-1}}B|^{2\gamma} \right) \varrho dy d\tau \end{aligned}$$

holds, where

$$\lambda(x, t) = \inf \{ (A(y, \tau)\xi, \xi) : |\xi| = 1, (y, \tau) \in U(x, \sqrt{t}) \times [\Theta t, t] \}.$$

In particular, if $\mu_t(dx) = \varrho(x, t) dx$ is a sub-probability measure for almost all $t \in (0, T)$, the functions $\|A\|^\gamma$, $|c^+|^\gamma$, $|B|^{2\gamma}$ belong to $L^1(\mathbb{R}^d \times (0, T), \mu)$, and the function $\|A\|^{-1}$ is uniformly bounded, then there exists a number $\tilde{C} > 0$ such that

$$\varrho(x, t) \leq \tilde{C} t^{-d/2} \quad \text{for all } (x, t) \in \mathbb{R}^d \times (0, T).$$

Proof. To obtain the required estimate at a point (x_0, t_0) it suffices to make the change of coordinates $x \mapsto (x - x_0)/\sqrt{t_0}$ and $t \mapsto t/t_0$, and to apply Theorem 3.4 with $U = U(0, 1)$, $U' = U(0, 1/2)$ and $s_1 = \Theta$, $s = (1 + \Theta)/2$, $s_2 = 1$. $\square$

Corollary 3.8. *Let $\Phi \in C^{2,1}(\mathbb{R}^d \times (0, T))$ and $\Phi > 0$. Let*

$$\tilde{c} = c + (\partial_t \Phi + \operatorname{div}(A \nabla \Phi) + B \nabla \Phi) \Phi^{-1}, \quad \tilde{B} = B + \Phi^{-1} A \nabla \Phi.$$

Let $\gamma > (d+2)/2$ and $\Theta \in (0, 1)$. Then there exists a number $C > 0$ depending only on γ, d, Θ such that, for all $(x, t) \in \mathbb{R}^d \times (0, T)$,

$$\varrho(x, t) \leq C \Phi(x, t)^{-1} (1 + \lambda^{-1}(x, t))^\gamma \\ \times t^{-(d+2)/2} \int_{\Theta t}^t \int_{U(x, \sqrt{t})} \left(1 + \|A\|^\gamma + t^{2\gamma} |\tilde{c}^+|^\gamma + t^{2\gamma} |\sqrt{A^{-1}} \tilde{B}|^{2\gamma} \right) \Phi \varrho \, dy \, d\tau,$$

where λ is defined in the preceding corollaries. In particular, if

$$\sup_{t \in (0, T)} \int_{\mathbb{R}^d} \Phi(x, t) \varrho(x, t) \, dx < \infty,$$

the functions $\|A\|^\gamma \Phi$, $|\tilde{c}^+|^\gamma \Phi$, $|\tilde{B}|^{2\gamma} \Phi$ belong to $L^1(\mathbb{R}^d \times (0, T), \mu)$, and the function $\|A\|^{-1}$ is uniformly bounded, then there exists a number $\tilde{C} > 0$ such that

$$\varrho(x, t) \leq \tilde{C} t^{-d/2} \Phi(x, t)^{-1} \quad \text{for all } (x, t) \in \mathbb{R}^d \times (0, T).$$

Proof. It suffices to observe that the function $\Phi \varrho$ satisfies equation (3.1) with the new coefficients $\tilde{c}$ and $\tilde{B}$. $\square$

We consider two typical examples which apply the results we have obtained. Suppose that $c \leq 0$ and $\mu_t(dx) = \varrho(x, t) dx$ is a sub-probability solution of the Cauchy problem for equation (1.1) with initial condition ν such that $|c| \in L^1(\mu)$ and

$$\mu_t(\mathbb{R}^d) \leq \nu(\mathbb{R}^d) + \int_0^t \int_{\mathbb{R}^d} c(x, s) \mu_s(dx) \, ds.$$

We obtain upper estimates for the density ϱ in various situations.

Example 3.9. Let $\alpha > 0$, $r > 2$, and $k > r$. Suppose that $c \leq 0$ and

$$\alpha r |x|^{r-2} \operatorname{tr} A(x, t) + \alpha r(r-2) |x|^{r-4} (A(x, t)x, x) + \alpha^2 r^2 |x|^{2r-4} (A(x, t)x, x) \\ + \alpha r |x|^{r-2} (b(x, t), x) + c(x, t) \leq C - C|x|^k$$

for some $C > 0$ and all $(x, t) \in \mathbb{R}^d \times (0, T)$. Suppose also that the inequalities

$$C_1 \exp(-\kappa_1 |x|^{r-\delta}) \leq \|A(x, t)\| \leq C_2 \exp(\kappa_2 |x|^{r-\delta}), \\ |b^i(x, t)| + |\partial_{x_j} a^{ij}(x, t)| \leq C_3 \exp(\kappa_3 |x|^{r-\delta})$$

hold for all $(x, t) \in \mathbb{R}^d \times (0, T)$, where $C_1, C_2, C_3, \kappa_1, \kappa_2, \kappa_3$ are positive numbers and $\delta \in (0, r)$. Let $\alpha' \in (0, \alpha)$. Then the density ϱ satisfies the estimate

$$\varrho(x, t) \leq C_4 \exp(-\alpha' |x|^r) \exp(C_5 t^{-r/(k-r)})$$

for all $(x, t) \in \mathbb{R}^d \times (0, T)$ and some positive numbers C_4 and C_5 .

Proof. By Example 2.7 we have the inequality

$$\int_{\mathbb{R}^d} \exp(\alpha |x|^r) \mu_t(dx) \leq \gamma_1 \exp(\gamma_2 t^{-r/(k-r)})$$

for almost all $t \in (0, T)$ and some numbers γ_1 and γ_2 . We set $\Phi(x) = \exp(\alpha' |x|^r)$. We observe that $\tilde{c}^+ \leq \gamma_3$ for some number γ_3 and

$$(1 + \|A\|^\gamma + t^{2\gamma} |\sqrt{A^{-1}} \tilde{B}|^{2\gamma}) \Phi \leq \gamma_4 \exp(\alpha |x|^r)$$

for all $(x, t) \in \mathbb{R}^d \times (0, T)$. The required estimate is now given by Corollary 3.8. $\square$

Example 3.10. Let $r > 2$, $k > 2$, $\gamma > d + 2$, $\alpha > 0$, and $\beta > r/(k - 2)$. Suppose that the inequality

$$\alpha r \operatorname{tr} A(x, t) + \alpha r(r - 2)|x|^{-2}(A(x, t)x, x) + \alpha r(b(x, t), x) + |x|^2 c(x, t) + \alpha^2 r^2 |x|^{r-2}(A(x, t)x, x) \leq C - C|x|^k$$

holds for all $(x, t) \in \mathbb{R}^d \times (0, T)$ and some number $C > 0$. Suppose also that

$$C_1(1 + |x|^{m/\gamma})^{-1} \leq \|A(x, t)\| \leq C_2(1 + |x|^{m/\gamma}), \\ |b^i(x, t)|^{2\gamma} + |\partial_{x_j} a^{ij}(x, t)|^{2\gamma} \leq C_3(1 + |x|^m)$$

for all $(x, t) \in \mathbb{R}^d \times (0, T)$, where C_1, C_2, C_3 are positive numbers and $m \geq \gamma \max\{r - 1, r\beta^{-1}\}$. Let $\alpha' \in (0, \alpha)$. Then the density ϱ satisfies the estimate

$$\varrho(x, t) \leq C_4 t^{-(8m\beta + rd - 4\gamma r)/(2r)} \exp(-\alpha' t^\beta |x|^r)$$

for all $(x, t) \in \mathbb{R}^d \times (0, T)$ and some positive numbers C_4 and C_5 .

Proof. By Example 2.8,

$$\int_{\mathbb{R}^d} \exp(\alpha t^\beta |x|^r) \mu_t(dx) \leq \gamma_1$$

for almost all $t \in (0, T)$ and some number γ_1 . Note that $|x|^p \leq \gamma_2 t^{-\beta p/r} \exp(\varepsilon t^\beta |x|^r)$ for all $p \geq 1$ and $\varepsilon > 0$. Thus, we can apply Corollary 3.8 with $\Phi(x, t) = \exp(\alpha t^\beta |x|^r)$. $\square$

ACKNOWLEDGEMENTS

The author is deeply grateful to V. I. Bogachëv for fruitful discussions and remarks.

REFERENCES

- [1] V. I. Bogachëv, N. V. Krylov, and M. Rëkner, *Elliptic and parabolic equations for measures*, Uspekhi Mat. Nauk **64** (2009), no. 6, 5–116; English transl., Russian Math. Surveys **64** (2009), no. 6, 973–1078. MR2640966 (2011c:35592)
- [2] V. I. Bogachëv, M. Rëkner, and S. V. Shaposhnikov, *Global regularity and estimates for solutions of parabolic equations*, Teor. Veroyatn. Primen. **50** (2005), no. 4, 652–674; English transl., Theory Probab. Appl. **50** (2006), no. 4, 561–581. MR2331982 (2008f:60073)
- [3] V. I. Bogachëv, M. Rëkner, and S. V. Shaposhnikov, *Estimates for the densities of stationary distributions and transition probabilities of diffusion processes*, Teor. Veroyatn. Primen. **52** (2007), no. 2, 240–270; English transl., Theory Probab. Appl. **52** (2008), no. 2, 209–236. MR2742501 (2011i:60143)
- [4] V. I. Bogachav, M. Rëkner, and S. V. Shaposhnikov, *Positive densities of transition probabilities of diffusion processes*, Teor. Veroyatn. Primen. **53** (2008), no. 2, 213–239; English transl., Theory Probab. Appl. **53** (2009) no. 2, 194–215.
- [5] F. O. Porper and S. D. Èidel'man, *Two-sided estimates of the fundamental solutions of second-order parabolic equations and some applications of them*, Uspekhi Mat. Nauk **39** (1984), no. 3, 107–156; English transl., Russian Math. Surveys **39** (1984), no. 3, 119–178. MR747792 (86b:35078)
- [6] S. V. Shaposhnikov, *Regularity and qualitative properties of solutions of parabolic equations for measures*, Teor. Veroyatn. Primen. **56** (2011), no. 2, 318–350; English transl., Theory Probab. Appl. **56** (2012), no. 2, 252–279.
- [7] S. V. Shaposhnikov, *Estimates for solutions of parabolic equations for measures*, Dokl. Akad. Nauk **434** (2010), no. 4, 454–458; English transl., Dokl. Math. **82** (2010), no. 2, 769–772. MR2778983 (2011m:35394)
- [8] D. G. Aronson and J. Serrin, *Local behavior of solutions of quasilinear parabolic equations*, Arch. Rational Mech. Anal. **25** (1967), 81–122. MR0244638 (39:5952)
- [9] A. Aïbeche, K. Laidoune, and A. Rhandi, *Time dependent Lyapunov functions for some Kolmogorov semigroups perturbed by unbounded potentials*, Arch. Math. (Basel) **94** (2010), no. 6, 565–577. MR2653673 (2012g:47131)
- [10] V. I. Bogachev, G. Da Prato, and M. Röckner, *On parabolic equations for measures*, Comm. Partial Differential Equations **33** (2008), no. 1–3, 397–418. MR2398235 (2009a:35103)

- [11] V. I. Bogachev, N. V. Krylov, and M. Röckner, *On regularity of transition probabilities and invariant measures of singular diffusions under minimal conditions*, Comm. Partial Differential Equations **26** (2001), no. 11–12, 2037–2080. MR1876411 (2002m:60117)
- [12] V. I. Bogachev, M. Röckner, and S. V. Shaposhnikov, *On uniqueness problems related to the Fokker–Planck–Kolmogorov equation for measures*. Problems in mathematical analysis. No. 61. J. Math. Sci. (N.Y.) **179** (2011), no. 1, 7–47. MR3014097
- [13] K. Laidoune, G. Metafune, D. Pallara, A. Rhandi, *Global properties of transition kernels associated with second-order elliptic operators*, Parabolic Problems, 2011 (Progress Nonlinear Differential Equations Appl. vol. 80), Birkhäuser/Springer Basel AG, Basel, 2011, 415–432. MR3052589
- [14] G. Metafune, D. Pallara, and A. Rhandi, *Global properties of transition probabilities of singular diffusions*, Teor. Veroyatn. Primen. **54** (2009), no. 1, 116–148; English transl., Theory Probab. Appl. **54** (2010), no. 1, 68–96. MR2766650 (2011m:35144)
- [15] J. Moser, *A Harnack inequality for parabolic differential equations*, Comm. Pure Appl. Math. **17** (1964), 101–134. MR0159139 (28:2357)
- [16] C. Spina, *Kernel estimates for a class of Kolmogorov semigroups*, Arch. Math. (Basel) **91** (2008), no. 3, 265–279. MR2439601 (2009h:35182)

MOSCOW STATE UNIVERSITY

E-mail address: `starticle@mail.ru`

Translated by E. KHUKHRO

Originally published in Russian