ON THE ORBIT SPACE OF AN IRREDUCIBLE REPRESENTATION OF THE SPECIAL UNITARY GROUP

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ABSTRACT. We prove that the quotient of an irreducible representation of a special unitary group of rank greater than 1 cannot be a smooth manifold.

§ 1. INTRODUCTION

This paper is a direct continuation of [1] and [2]. First, we recall three basic definitions playing a key role in those papers.

Definition. A continuous map between smooth manifolds is said to be *piecewise smooth* if it sends any smooth submanifold to a finite union of smooth submanifolds.

In particular, any proper smooth map between smooth manifolds is piecewise smooth.

Consider a differentiable action of a compact Lie group G on a smooth manifold M .

Definition. We shall say that the quotient M/G is *piecewise diffeomorphic* to a smooth manifold M' if the topological quotient M/G is homeomorphic to M' and the quotient map $M \rightarrow M'$ is piecewise smooth.

Definition. We shall say that the quotient M/G is a *smooth manifold* if it is piecewise diffeomorphic to a smooth manifold.

Let us now describe our problem.

Let V be a real vector space and $G \subset \mathbf{GL}(V)$ a compact linear group. As in [1] and [2], we want to know if the quotient V/G is a topological manifold and also if it is a smooth manifold. Following [1] and [2], we shall call a topological manifold simply a manifold.

Let $V_{\mathbb{C}}$ denote the complex space $V \otimes \mathbb{C}$, $\mathfrak{g}$ the linear Lie algebra $\mathrm{Lie} G \subset \mathfrak{gl}(V)$, and $\mathfrak{g}_{\mathbb{C}}$ the complex linear Lie algebra $\mathfrak{g} \otimes \mathbb{C} \subset \mathfrak{gl}(V_{\mathbb{C}})$.

At the present moment, the following two cases have been dealt with: $[\mathfrak{g}, \mathfrak{g}] = 0$ (see [1]) and $\mathfrak{g} \cong \mathfrak{su}_2$ (see [2]). In this paper, we consider the case when $\mathfrak{g} \cong \mathfrak{su}_{r+1}$, where $r = \mathrm{rk} \mathfrak{g} > 1$ and the linear Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(V)$ is irreducible.

Let R' denote the representation of a complex reductive Lie group dual to the representation R .

When speaking of indecomposable root systems we shall use the numbering of the simple roots adopted in [3, Table 1] and [4, Table 1], where $(i_1, \dots, i_m)$ denotes the subset of simple roots indexed by $i_1, \dots, i_m$, φ_i denotes the fundamental weight indexed by i , and $\varphi(\mathfrak{h})$ is the linear algebra corresponding to the irreducible representation of the complex simple algebra $\mathfrak{h}$ with highest weight φ .

The space V admits a G -invariant scalar product and hence may (and will) be considered as a Euclidean space on which G acts by orthogonal operators. Thus, $G \subset \mathbf{O}(V)$.

Let $\tilde{R}$ be the tautological representation $\mathfrak{g}_{\mathbb{C}} : V_{\mathbb{C}}$. We may have the following cases:

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- 1) $\tilde{R} = R$, where R is a faithful irreducible representation; and
- 2) $\tilde{R} = R + R'$, where R is a faithful irreducible representation.

In the latter case, V admits a $\mathfrak{g}$ -invariant complex structure, which naturally yields a (complex) representation $\mathfrak{g}_{\mathbb{C}} : V$ isomorphic to R .

Suppose now that $\mathfrak{g} \cong \mathfrak{su}_{r+1}$, where $r = \text{rk } \mathfrak{g} > 1$, and the linear Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(V)$ is irreducible.

In this paper we shall prove the following Theorems 1.1 and 1.2.

Theorem 1.1. *The quotient V/G can be a smooth manifold only in the following cases:*

- 1) *the representation $\tilde{R}$ of $\mathfrak{g}_{\mathbb{C}}$ coincides with the representation R and is isomorphic to one of the representations ad , R_{φ_2} ($r = 3$), $R_{2\varphi_2}$ ($r = 3$), and R_{φ_4} ($r = 7$);*
- 2) *$\tilde{R} = R + R'$ and the representation R is isomorphic (up to an outer automorphism of the algebra $\mathfrak{g}_{\mathbb{C}}$) to one of the representations R_{φ_1} , $R_{2\varphi_1}$, R_{φ_2} ($r > 3$), and R_{φ_3} ($r = 5$).*

Theorem 1.2. *If $G = G^0$ and the representation R is isomorphic (up to an outer automorphism of $\mathfrak{g}_{\mathbb{C}}$) to one of the representations ad , R_{φ_1} , $R_{2\varphi_1}$, R_{φ_2} ($r > 2$), $R_{2\varphi_2}$ ($r = 3$), R_{φ_3} ($r = 5$), and R_{φ_4} ($r = 7$), then V/G is not a manifold.*

Corollary 1.3. *If $G = G^0$, then the quotient V/G is not a smooth manifold.*

In §2, we introduce some notation used later in the text, and also prove some auxiliary results. In §3 we prove Theorem 1.1, and in §4, we prove Theorem 1.2.

§ 2. NOTATION AND USEFUL FACTS

In this section we mention a number of useful notation and results, including the ones taken from [1], [2], and [5] (all new assertions have proofs).

For a linear representation of a Lie group G (respectively, a Lie algebra $\mathfrak{g}$) in the space V , the stabilizer (respectively, the isotropy subalgebra) of a vector $v \in V$ will be denoted by G_v (respectively, by $\mathfrak{g}_v$).

In [5], for each indecomposable simple root system Π a certain subset $\partial\Pi \subset \Pi$ was defined.

All indecomposable simple root systems Π such that $\partial\Pi \neq \Pi$ and $\partial\Pi \subset \Pi$ are listed in [5, §4, Table 1]. For example, if $\Pi \cong A_r$, then

$$\partial\Pi = \begin{cases} (1, 2, r-1, r), & r \geq 6, \\ \Pi, & r < 6. \end{cases}$$

2.1. Representations of compact groups. Suppose we have a Euclidean space V , a compact Lie group G with Lie algebra $\mathfrak{g}$, a linear representation $G \rightarrow \mathbf{O}(V)$, and its differential – a representation $\mathfrak{g} : V$.

For any vector $v \in V$, let N_v denote the subspace $(\mathfrak{g}v)^{\perp} \subset V$. Clearly $\mathfrak{g}_v = \text{Lie } G_v$ and $G_v N_v = N_v$ ($v \in V$).

Lemma 2.1. *If V/G is a (smooth) manifold, then each quotient N_v/G_v , $v \in V$, is also a (smooth) manifold.*

Proof. See Lemma 2.3 in [2, §2]. □

Lemma 2.2. *Suppose $\dim \mathfrak{g} = 1$.*

- 1) *If V/G is a manifold, then $\dim(\xi V) \neq 2$ for any vector $\xi \in \mathfrak{g}$.*
- 2) *If V/G is a smooth manifold, then $\dim(\xi V) \leq 6$ for any vector $\xi \in \mathfrak{g}$.*

Proof. See Corollary 2.3 in [2, §2]. □

Lemma 2.3. *If $\mathfrak{g} \cong \mathfrak{su}_2$ and V/G is a smooth manifold, then the sum of the integer parts of halves of the dimensions of all irreducible components of the representation $\mathfrak{g} : V$ (with their multiplicities) is less than or equal to 4.*

Proof. See Theorem 1.1 in [2, §1]. □

Corollary 2.4. *If $\mathfrak{g} \cong \mathfrak{su}_2$ and V/G is a smooth manifold, then $\dim(\xi V) \leq 8$ for any $\xi \in \mathfrak{g}$.*

Corollary 2.5. *If $\text{rk } \mathfrak{g} = 1$ and V/G is a smooth manifold, then $\dim(\xi V) \leq \dim \mathfrak{g} + 5$ for any $\xi \in \mathfrak{g}$.*

Proof. Follows from Lemma 2.2 and Corollary 2.4. □

Corollary 2.6. *If $\text{rk } \mathfrak{g} = 1$ and V/G is a smooth manifold, then $\dim(\xi V) \leq \dim[\xi, \mathfrak{g}] + 6$ for any nonzero vector $\xi \in \mathfrak{g}$.*

Proof. Since $\text{rk } \mathfrak{g} = 1$, we have $\text{Ker}(\text{ad } \xi) = \mathbb{R}\xi$ for any nonzero vector $\xi \in \mathfrak{g}$, whence $\dim[\xi, \mathfrak{g}] = \dim \mathfrak{g} - 1$. Now use Corollary 2.5. □

Lemma 2.7. *Suppose that V/G is a smooth manifold. Then $\dim(\xi V) \leq \dim[\xi, \mathfrak{g}] + 6$ for any $v \in V$ such that $\text{rk } \mathfrak{g}_v = 1$ and for any nonzero $\xi \in \mathfrak{g}_v$.*

Proof. By Lemma 2.1, N_v/G_v is a smooth manifold. Applying Corollary 2.6 to the representation $G_v : N_v$, we have $\dim(\xi N_v) \leq \dim[\xi, \mathfrak{g}_v] + 6$ and, therefore,

$$\begin{aligned} \dim(\xi V) &= \dim(\xi(\mathfrak{g}v)) + \dim(\xi N_v) \\ &\leq \dim(\xi(\mathfrak{g}v)) + \dim[\xi, \mathfrak{g}_v] + 6 = \dim[\xi, \mathfrak{g}] + 6. \end{aligned} \quad \square$$

Lemma 2.8. *Suppose that V/G is a manifold. Then $\dim(\xi V) \neq \dim[\xi, \mathfrak{g}] + 2$ for any $v \in V$ such that $\dim \mathfrak{g}_v = 1$ and for any $\xi \in \mathfrak{g}_v$.*

Proof. By Lemma 2.1, N_v/G_v is a manifold. Applying Lemma 2.2 to the representation $G_v : N_v$, we have $\dim(\xi N_v) \neq 2$. Furthermore, $[\xi, \mathfrak{g}_v] = 0$ and $\dim(\xi(\mathfrak{g}v)) = \dim[\xi, \mathfrak{g}]$, whence

$$\begin{aligned} \dim(\xi V) &= \dim(\xi(\mathfrak{g}v)) + \dim(\xi N_v) \\ &= \dim[\xi, \mathfrak{g}] + \dim(\xi N_v) \neq \dim[\xi, \mathfrak{g}] + 2. \end{aligned} \quad \square$$

Lemma 2.9. *Let $v_1, \dots, v_k \in V$ be arbitrary vectors. If $(\mathfrak{g}v_i, v_j) = 0$ for all $i, j \in \{1, \dots, k\}$, $i \neq j$, then, in the subspace $\langle v_1, \dots, v_k \rangle \subset V$, there is a vector with stabilizer $\bigcap_{i=1}^k G_{v_i} \subset G$.*

Proof. We induct on $k \in \mathbb{N}$.

When $k = 1$ there is nothing to prove.

Now we prove the lemma for $k \in \mathbb{N} \setminus \{1\}$ assuming that it holds for $k - 1 \in \mathbb{N}$.

By the induction assumption, there is $v \in \langle v_1, \dots, v_{k-1} \rangle$ with stabilizer

$$G_v = \bigcap_{i=1}^{k-1} G_{v_i} \subset G.$$

By assumption, $(\mathfrak{g}v_i, v_k) = 0$ ($i = 1, \dots, k - 1$), and therefore $(\mathfrak{g}v, v_k) = 0$, $v_k \in N_v$. Hence, for some $v' = v + \varepsilon v_k \in \langle v_1, \dots, v_k \rangle \cap N_v$, $\varepsilon \in \mathbb{R}_{>0}$, we have $G_{v'} \subset G_v$, and, as a consequence,

$$G_{v'} = G_v \cap G_{\varepsilon v_k} = \bigcap_{i=1}^k G_{v_i}. \quad \square$$

Assertion 2.10. *If $G = \mathbf{SU}_m \times \mathbf{SU}_m$ and $V = \mathbb{R}^d \oplus \mathfrak{gl}_m(\mathbb{C})$ ($d \geq 0$, $m > 1$), and the action $G : V$ is given by $(g_1, g_2) : (x, y) \rightarrow (x, g_1 y g_2^{-1})$ ($g_1, g_2 \in \mathbf{SU}_m$, $x \in \mathbb{R}^d$, $y \in \mathfrak{gl}_m(\mathbb{C})$), then V/G is not a manifold.*

Proof. Set $v := E \in \mathfrak{gl}_m(\mathbb{C}) \subset V$. It is easy to see that

$$G_v = \{(g, g) : g \in \mathbf{SU}_m\} \subset G, \quad \mathfrak{g}v = \mathfrak{su}_m \subset \mathfrak{gl}_m(\mathbb{C}) \subset V$$

and

$$V = \mathbb{R}^d \oplus (\mathbb{C}E) \oplus \mathfrak{su}_m \oplus (i \cdot \mathfrak{su}_m).$$

This implies (since the groups $\mathbf{SU}_m$ and G_v are naturally isomorphic) that the representation $G_v : N_v$ is isomorphic to the direct sum of the trivial action $G_v : \mathbb{R}^{d+2}$ and the adjoint representation $\mathbf{SU}_m : \mathfrak{su}_m$. Hence,

$$\frac{N_v}{G_v} \cong \mathbb{R}^{d+2} \times \mathbb{R}_{\geq 0}^{m-1} \cong \mathbb{R}_{\geq 0}^{m+d+1}.$$

Thus, the quotient N_v/G_v is not a manifold and, by Lemma 2.1, neither is the quotient V/G . $\square$

Suppose that the representation $\mathfrak{g} : V$ is irreducible.

Let $\mathfrak{g}_{\mathbb{C}}$ denote the complex reductive Lie algebra $\tilde{\mathfrak{g}} \otimes \mathbb{C}$.

It is easy to see that there are a complex space $\tilde{V}$ and an irreducible representation $\mathfrak{g}_{\mathbb{C}} : \tilde{V}$ such that the representation $\mathfrak{g} : V$ is either the realification or a real form of the representation $\mathfrak{g} : \tilde{V}$.

Set $\delta := 1 \in \mathbb{R}$ when $\tilde{V} = V \otimes \mathbb{C}$ and $\delta := 2 \in \mathbb{R}$ when $\tilde{V} = V$.

We now fix a maximal commutative subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{t}_{\mathbb{C}} := \mathfrak{t} \otimes \mathbb{C}$ of $\mathfrak{g}_{\mathbb{C}}$. We then have a root system $\Delta \subset \mathfrak{t}_{\mathbb{C}}^*$ and the corresponding Weyl group $W \subset \mathbf{GL}(\mathfrak{t}_{\mathbb{C}}^*)$. We also fix a simple root system $\Pi \subset \Delta \subset \mathfrak{t}_{\mathbb{C}}^*$.

Let $\tilde{\Lambda} \subset \mathfrak{t}_{\mathbb{C}}^*$ be the set of weights of the irreducible representation $\mathfrak{g}_{\mathbb{C}} : \tilde{V}$, $\lambda \in \mathfrak{t}_{\mathbb{C}}^* \setminus \{0\}$ the highest weight of this representation relative to the simple root system $\Pi \subset \Delta \subset \mathfrak{t}_{\mathbb{C}}^*$, and Λ the subset $W\lambda \subset \tilde{\Lambda} \subset \mathfrak{t}_{\mathbb{C}}^*$. For a complex subspace $\mathfrak{a} \subset \mathfrak{t}_{\mathbb{C}}$ we set

$$\mathfrak{g}_{\mathbb{C}}(\mathfrak{a}) := \mathfrak{a} \oplus \left(\bigoplus_{\substack{\alpha \in \Delta, \\ h_{\alpha} \in \mathfrak{a}}} (\mathfrak{g}_{\mathbb{C}})_{\alpha} \right) \subset \mathfrak{g}_{\mathbb{C}}.$$

Finally, for a subset $\Omega \subset \mathfrak{t}_{\mathbb{C}}^*$, we set

$$\mathfrak{t}_{\mathbb{C}}^{\Omega} := \bigcap_{\omega \in \Omega} (\text{Ker } \omega) \subset \mathfrak{t}_{\mathbb{C}} \quad \text{and} \quad \mathfrak{g}^{\Omega} := \mathfrak{g}_{\mathbb{C}}(\mathfrak{t}_{\mathbb{C}}^{\Omega}) \cap \mathfrak{g} \subset \mathfrak{g}.$$

We then have

$$\mathfrak{g}^{\Omega_1 \cup \Omega_2} = \mathfrak{g}^{\Omega_1} \cap \mathfrak{g}^{\Omega_2} \subset \mathfrak{g} \quad (\Omega_1, \Omega_2 \subset \mathfrak{t}_{\mathbb{C}}^*).$$

Consider an arbitrary weight $\lambda' \in \Lambda$.

For any vector $v \neq 0$ in the (nontrivial) subspace $\tilde{V}_{\lambda'} \subset \tilde{V}$, the subalgebra $(\mathfrak{g}_{\mathbb{C}})_v$ of $\mathfrak{g}_{\mathbb{C}}$ coincides with the direct sum of all subspaces $\text{Ker } \lambda' \subset \mathfrak{t}_{\mathbb{C}}$ and $(\mathfrak{g}_{\mathbb{C}})_{\alpha}$ ($\alpha \in \Delta$, $\langle \lambda' | \alpha \rangle \geq 0$), and therefore $\mathfrak{g}_v = \mathfrak{g}^{\{\lambda'\}} \subset \mathfrak{g}$.

If $\tilde{V} = V \otimes \mathbb{C}$ and $2\lambda' \notin \Delta \cup (\Delta + \Delta)$, then $-\lambda' \in \Lambda$ and we also have

$$(\mathfrak{g}_{\mathbb{C}} \tilde{V}_{\lambda'}) \cap (\mathfrak{g}_{\mathbb{C}} \tilde{V}_{-\lambda'}) = 0, \quad (\mathfrak{g} \tilde{V}_{\lambda'}) \cap (\mathfrak{g} \tilde{V}_{-\lambda'}) = 0.$$

Therefore, for any nonzero vector v of the (nontrivial) subspace $(\tilde{V}_{\lambda'} \oplus \tilde{V}_{-\lambda'}) \cap V \subset V$, we have $\mathfrak{g}_v = \mathfrak{g}^{\{\lambda'\}} \subset \mathfrak{g}$.

This and Lemma 2.9 imply the following lemma.

Lemma 2.11. *Let $\Omega \subset \Lambda$ be a subset such that $(\Omega - \Omega) \cap \Delta = \emptyset$. Suppose that one of the following conditions is satisfied: 1) $\delta = 2$; 2) $\delta = 1$, $2\lambda \notin \Delta \cup (\Delta + \Delta)$ and $(\Omega + \Omega) \cap \Delta = \emptyset$. Then there is a vector $v \in V$ such that $\mathfrak{g}_v = \mathfrak{g}^\Omega \subset \mathfrak{g}$.*

Let $\|\cdot\|$ denote the order of a subset of the set $\mathring{\Lambda} \subset \mathfrak{t}_{\mathbb{C}}^*$ of weights with multiplicities of the representation $\mathfrak{g}_{\mathbb{C}} : \tilde{V}$.

Lemma 2.12. *Suppose that V/G is a smooth manifold. If $v \in V$, $\text{rk } \mathfrak{g}_v = 1$, and $\xi \in (\mathfrak{g}_v \cap \mathfrak{t}) \setminus \{0\}$, then*

$$\delta \cdot \|\{\lambda' \in \mathring{\Lambda} : \lambda'(\xi) \neq 0\}\| \leq \|\{\alpha \in \Delta : \alpha(\xi) \neq 0\}\| + 6.$$

Proof. By Lemma 2.7,

$$\delta \cdot \dim_{\mathbb{C}}(\xi \tilde{V}) = \dim(\xi V) \leq \dim[\xi, \mathfrak{g}] + 6 = \dim_{\mathbb{C}}[\xi, \mathfrak{g}_{\mathbb{C}}] + 6,$$

which implies the desired assertion. $\square$

Corollary 2.13. *Let $\Omega \subset \Lambda$ be an arbitrary subset. Let H denote the subspace $\langle \Omega \rangle_{\mathbb{C}} \subset \mathfrak{t}_{\mathbb{C}}^*$. If*

$$\begin{aligned} \dim_{\mathbb{C}}(\mathfrak{t}_{\mathbb{C}}^*/H) = 1; \quad \delta \cdot \|\mathring{\Lambda} \setminus H\| &> |\Delta \setminus H| + 6; \quad (\Omega - \Omega) \cap \Delta = \emptyset; \\ (\delta = 1) \quad \Rightarrow \quad &\left((2\lambda \notin \Delta \cup (\Delta + \Delta)) \text{ and } ((\Omega + \Omega) \cap \Delta = \emptyset) \right), \end{aligned}$$

then the quotient V/G is not a smooth manifold.

Proof. Follows from Lemmas 2.11 and 2.12. $\square$

Lemma 2.14. *Suppose the quotient V/G is a manifold and $\mathring{\Lambda} = \Lambda$. If $v \in V$, $\xi \in \mathfrak{t} \setminus \{0\}$, and $\mathfrak{g}_v = \mathbb{R}\xi$, then*

$$\delta \cdot \|\{\lambda' \in \Lambda : \lambda'(\xi) \neq 0\}\| \neq \|\{\alpha \in \Delta : \alpha(\xi) \neq 0\}\| + 2.$$

Proof. By Lemma 2.8,

$$\delta \cdot \dim_{\mathbb{C}}(\xi \tilde{V}) = \dim(\xi V) \neq \dim[\xi, \mathfrak{g}] + 2 = \dim_{\mathbb{C}}[\xi, \mathfrak{g}_{\mathbb{C}}] + 2,$$

which implies the desired assertion. $\square$

Corollary 2.15. *Let $\Omega \subset \Lambda$ be an arbitrary subset. Let H denote the subspace $\langle \Omega \rangle_{\mathbb{C}} \subset \mathfrak{t}_{\mathbb{C}}^*$. If*

$$\begin{aligned} \dim_{\mathbb{C}}(\mathfrak{t}_{\mathbb{C}}^*/H) = 1; \quad \mathring{\Lambda} &= \Lambda; \\ \delta \cdot |\Lambda \setminus H| &= |\Delta \setminus H| + 2; \quad (\Omega - \Omega) \cap \Delta = H^\perp \cap \Delta = \emptyset; \\ (\delta = 1) \quad \Rightarrow \quad &\left((2\lambda \notin \Delta \cup (\Delta + \Delta)) \text{ and } ((\Omega + \Omega) \cap \Delta = \emptyset) \right), \end{aligned}$$

then V/G is not a manifold.

Proof. Follows from Lemmas 2.11 and 2.14. $\square$

2.2. Polar representations.

Definition. A representation of a compact Lie group G in a real space V is *polar* if there is a subspace $V' \subset V$ such that $GV' = V$ and $V' \cap (T_v(Gv)) = 0$ ($v \in V'$).

Lemma 2.16. *The quotient space of an arbitrary nontrivial connected polar compact linear group is homeomorphic to a closed half-space.*

Proof. By the results of [6] (see Theorems 2.8, 2.9, and 2.10 in §2), the above quotient space is homeomorphic to a quotient space of a nontrivial finite linear group generated by (real) reflections. $\square$

Let G be a connected simply-connected compact semisimple Lie group, $\mathfrak{g}$ its tangent algebra, and $\theta \in \text{Aut}(G)$ a nontrivial involutive automorphism. Now consider the adjoint action $G^\theta : \mathfrak{g}$ and its restriction to the (clearly invariant) subspace $\mathfrak{g}^{-d\theta} \subset \mathfrak{g}$.

Lemma 2.17. *The quotient $\mathfrak{g}^{-d\theta}/G^\theta$ is homeomorphic to a closed half-space.*

Proof. By Theorem B of [7, §1], the Lie subgroup $G^\theta \subset G$ is connected. Since G is semisimple, the representation $G^\theta : \mathfrak{g}^{-d\theta}$ is nontrivial. This representation is also polar (see [8, §8, 8.5–8.6]). It remains to apply Lemma 2.16. $\square$

Remark. The assertion of Lemma 2.17 also follows from the results of [9].

2.3. Combinatorial inequalities. For natural numbers $n, k, m_1, \dots, m_k$ such that $m_1 + \dots + m_k = n$, set

$$\binom{n}{m_1, \dots, m_k} := \binom{n}{m_1} \cdot \binom{n-m_1}{m_2} \cdot \dots \cdot \binom{n-m_1-\dots-m_{k-2}}{m_{k-1}} = \frac{n!}{m_1! \cdot \dots \cdot m_k!} \in \mathbb{N}.$$

Assertion 2.18. *If $n, k, m_1, \dots, m_k \in \mathbb{N}$ and $m_1 + \dots + m_k = n$, then the inequality $\binom{n}{m_1, \dots, m_k} < n$ is possible only when $k = 1$ and $m_1 = n$.*

Proof. We have $\binom{n}{m_1} \leq \binom{n}{m_1, \dots, m_k} < n$, whence $m_1 = n$. $\square$

Assertion 2.19. *If $n, k, m_1, \dots, m_k \in \mathbb{N}$, $k > 1$, and $m_1 + \dots + m_k = n$, then the inequality $\binom{n}{m_1, \dots, m_k} \leq 2(n-1)$ is possible only in the following cases:*

- 1) $k = 2$, $\{m_1, m_2\} = \{1, n-1\}$;
- 2) $n = 4$, $k = 2$, $m_1 = m_2 = 2$.

Proof. Suppose $k > 2$. Then

$$0 < m_2 < n - m_1, \quad 0 < m_1 < n - m_2 \leq n - 1.$$

Therefore $\binom{n-m_1}{m_2} \geq n - m_1$, and also $\binom{n-1}{m_1} \geq n - 1$. Therefore,

$$2(n-1) \geq \binom{n}{m_1, \dots, m_k} \geq \binom{n}{m_1} \cdot (n - m_1 m_2) \geq \binom{n}{m_1} \cdot (n - m_1) = n \cdot \binom{n-1}{m_1} \geq n(n-1),$$

$(n-2)(n-1) \leq 0$, $n \leq 2 < k \leq m_1 + \dots + m_k = n$. This is a contradiction.

Assume now that $k = 2$ and $\{m_1, m_2\} \neq \{1, n-1\}$.

We have $m_1, m_2 \geq 2$ and $\binom{n}{m_1} \geq \binom{n}{2}$. Moreover, $n = m_1 + m_2 \geq 4$. Therefore,

$$2(n-1) \geq \binom{n}{m_1, m_2} = \binom{n}{m_1} \geq \binom{n}{2} = \frac{n(n-1)}{2} \geq \frac{4(n-1)}{2} = 2(n-1),$$

whence $n = 4$, and also

$$\binom{n}{m_1} = \binom{n}{2}, \quad \binom{4}{m_1} = \binom{4}{2}, \quad m_1 = m_2 = 2. \quad \square$$

2.4. The orbits of the Weyl group. Let $r > 1$ be a natural number. Setting $n := r + 1 \in \mathbb{N}$, we have $n \geq 3$.

Consider the Euclidean space $\mathbb{R}^n$ in which the standard basis $\{\varepsilon_i\}_{i=1}^n$ is orthonormal, an indecomposable root system $\Delta := \{\varepsilon_i - \varepsilon_j : i \neq j\} \subset \mathbb{R}^n$ of type A_r , its Weyl group $W \subset \mathbf{O}(\mathbb{R}^n)$, the Weyl chamber $C := \{x \in \mathbb{R}^n : x_1 \geq \dots \geq x_n\} \subset \mathbb{R}^n$, and the corresponding simple root system $\Pi := \{\alpha_1, \dots, \alpha_r\} \subset \Delta$ ($\alpha_i := \varepsilon_i - \varepsilon_{i+1}$, $i = 1, \dots, r$).

The lattices $P := \{\lambda \in \langle \Delta \rangle : \langle \lambda | \alpha \rangle \in \mathbb{Z} \forall \alpha \in \Delta\}$ and $Q := \langle \Delta \rangle_{\mathbb{Z}}$ in the subspace $\langle \Delta \rangle \subset \mathbb{R}^n$ satisfy the relations $WP = P$, $WQ = Q$, $Q \subset P$, and $|P/Q| = n < \infty$.

Let $\varphi_1, \dots, \varphi_r \in P$ be the fundamental weights relative to the simple root system $\Pi = \{\alpha_1, \dots, \alpha_r\} \subset \Delta$ (in the same order). Clearly $\{\varphi_i\}_{i=1}^r$ is a basis of the lattice $P \subset \langle \Delta \rangle$.

For brevity, the orbits of the action $W : P$ will be simply called *orbits*.

Let $\Lambda \subset P$ be an arbitrary orbit. Set $\mathring{\Lambda} := \text{conv}(\Lambda) \cap (\Lambda + Q) \subset P$. It is clear that:

- 1) $|\Lambda \cap C| = 1$;
- 2) $\Lambda - \Lambda \subset Q$;
- 3) $\mathring{\Lambda} \supset \Lambda$;
- 4) $W\mathring{\Lambda} = \mathring{\Lambda}$;
- 5) for any orbit $\Lambda' \subset \mathring{\Lambda}$, we have $\mathring{\Lambda}' \subset \mathring{\Lambda}$;
- 6) for any orbit $\Lambda' \subset \mathring{\Lambda}$, $\Lambda' \neq \Lambda$, we have $\mathring{\Lambda}' \cap \Lambda = \emptyset$.

Fix an orbit $\Lambda \subset P \setminus \{0\}$.

We have $\Lambda \cap C = \{\lambda\}$, where $\lambda \in (P \cap C) \setminus \{0\}$.

2.4.1. Main assertions. Let $\Pi' \subset \Pi \subset \mathbb{R}^n$ be an indecomposable simple root system of order $r - 2$.

It is easy to see that $H := \langle \{\lambda\} \cup \Pi' \rangle \subset \langle \Delta \rangle \subset \mathbb{R}^n$ is an $(r - 1)$ -dimensional subspace.

In this subsection, we prove the following Lemmas 2.20 and 2.21.

Lemma 2.20. *If $\lambda \notin \Delta$, $\lambda \neq \varphi_1, \dots, \varphi_r$, $|\mathring{\Lambda} \setminus H| \leq 4n$, and $(-\Lambda = \Lambda) \vee (|\mathring{\Lambda} \setminus H| \leq 2n)$, then either $\lambda \in \{2\varphi_1, 2\varphi_r, \varphi_1 + \varphi_{r-1}, \varphi_2 + \varphi_r\}$ or $(r = 3)$ and $(\lambda = 2\varphi_2)$.*

Lemma 2.21. *Suppose that $r \geq 8$, $\lambda = \varphi_j$, where $j \in \mathbb{N}$ and $3 \leq j \leq r - 2$, and also $\Pi' = \{\alpha_1, \dots, \alpha_{r-2}\} \subset \Pi$. Then $|\mathring{\Lambda} \setminus H| > 4n$.*

We have $\lambda \in (\langle \Delta \rangle \cap C) \setminus \{0\}$, $\lambda_1 > 0 > \lambda_n$, and therefore

$$\langle \Pi' \rangle = \{x \in \langle \Delta \rangle : x_p = x_q = 0\}, \quad H = \{x \in \langle \Delta \rangle : x_q = cx_p\},$$

where $(p, q) \in \{1, \dots, n\}^2$ is one of the pairs $(1, 2)$, $(1, n)$ and $(n, n - 1)$ (and, therefore, $\lambda_p \neq 0$), and c is the number $\frac{\lambda_q}{\lambda_p} \in \mathbb{R}$.

We begin with a proof of Lemma 2.20. First we need some auxiliary results.

Let $x \in P \cap C$ be a vector.

Set $K := \{x_1, \dots, x_n\} \subset \mathbb{R}$, $k := |K| \in \mathbb{N}$, $m(t) := |\{i \in \{1, \dots, n\} : x_i = t\}| \in \mathbb{N}$ ($t \in K$) and $m_0 := \max\{m(t)\}_{t \in K} \in \mathbb{N}$.

Proposition 2.22. *Suppose that $k \geq 3$. Then*

- 1) $|(Wx) \setminus H| \geq 2n - 2$;
- 2) if $m_0 \neq n - 2$, then $|(Wx) \setminus H| \geq 5(n - 2)$;
- 3) if $k \geq 4$, then $|(Wx) \setminus H| \geq 4n$.

Proof. Set

$$L(t') := \{t'' \in K \setminus \{ct'\} : (m(t') = 1) \Rightarrow (t'' \neq t')\} \subset K \quad (t' \in K),$$

$$L := \{(t', t'') \in K^2 : t'' \in L(t')\} \subset K^2,$$

and

$$K_0 := \{t \in K : ct \in K \setminus \{t\}, m(t) = 1\} \subset K.$$

It is clear that

$$(2.1) \quad \begin{aligned} \forall \tau \in L \quad d(\tau) &:= |\{y \in Wx : (y_p, y_q) = \tau\}| \geq 1; \quad |(Wx) \setminus H| = \sum_{\tau \in L} d(\tau); \\ \forall t \in K \quad |L(t)| &\geq k-2; \\ \forall t \in K \setminus K_0 \quad |L(t)| &> k-2. \end{aligned}$$

Hence

$$\begin{aligned} |L| &\geq k(k-2) + (k - |K_0|) \geq k(k-2), \\ |K_0| &\geq k(k-2) + k - |L| = k(k-1) - |L|. \end{aligned}$$

Let us show that

$$(2.2) \quad |L| \geq 4; \quad (|L| = 4) \Rightarrow (m_0 = n-2).$$

Suppose $|L| \leq 4$. Then

$$k(k-2) \leq |L| \leq 4 < 4 \cdot (4-2), \quad k = 3,$$

$$|K_0| \geq 6 - |L| \geq 2, \quad |\{t \in K : m(t) = 1\}| \geq |K_0| \geq 2.$$

Moreover, $n \geq |K| = k = 3$. Hence, $m_0 = n-2$. Now if $|L| \leq 3$, then $|K_0| \geq 6 - |L| \geq 3 = |K|$, $K_0 = K$,

$$\forall t \in K \quad ct \in K \setminus \{t\}, \quad ct \neq t, \quad t \neq 0, \quad c \neq 1, \quad c^3 \neq 1, \quad c^3 t \neq t; \quad K \subset \mathbb{R} \setminus \{0\};$$

$$\forall t \in K \quad ct \in K \setminus \{t\} \subset \mathbb{R} \setminus \{0\}, \quad c \neq 0,$$

and we thus have a bijective map $K \rightarrow K$, $t \rightarrow ct$, whose third power has no fixed points, whereas $|K| = 3$. This is a contradiction.

We have thus established (2.2).

Let us show that

$$(2.3) \quad l := |\{\tau \in L : d(\tau) < n-2\}| \leq 2; \quad (l > 0) \Rightarrow (m_0 = n-2).$$

Fix a number $t_0 \in K$ such that $m(t_0) = m_0$ (it does exist). We also set $L_0 := \{(t', t'') \in K^2 : t', t'' \neq t_0, t' \neq t''\} \subset K^2$.

Let $(t', t'') \in L$ be an arbitrary pair such that $d(t', t'') < n-2$. By Assertion 2.18 and inequalities $k \geq 3$, we have $k = 3$, $K = \{t, t', t''\} \subset \mathbb{R}$, $t' \neq t$, $t'' \neq t$, $t' \neq t''$, $m(t) = n-2 > d(t', t'') \geq 1 = m(t') = m(t'')$, $m_0 = n-2$, $t_0 = t \in K \setminus \{t', t''\}$, $(t', t'') \in L_0$, where $|L_0| = (k-1)(k-2) = 2$.

We have thus established (2.3).

It is clear that $n \geq |K| = k \geq 3$. Now, using (2.1)–(2.3), we have that

$$\begin{aligned} |(Wx) \setminus H| &\geq (n-2) \cdot (|L| - l) + l = (n-2) \cdot |L| - (n-3)l \\ &= 4(n-2) - 2(n-3) + (n-2) \cdot (|L| - 4) + (n-3) \cdot (2-l) \\ &\geq 4(n-2) - 2(n-3) = 2n-2, \end{aligned}$$

and, in the case $m_0 \neq n-2$, we also have $|L| \geq 5$ and $l = 0$, whence

$$|(Wx) \setminus H| \geq (n-2) \cdot |L| - (n-3)l = (n-2) \cdot |L| \geq 5(n-2).$$

Suppose $k \geq 4$. Then $n \geq k \geq 4$, $|L| \geq k(k-2) \geq 8$, and also $m_0 \leq n - (k-1) < n-2$. Therefore, $|(Wx) \setminus H| \geq (n-2) \cdot |L| \geq 8(n-2) = 4n + 4(n-4) \geq 4n$.

This finishes the proof. $\square$

Corollary 2.23. $|\Delta \setminus H| \geq 2n-2$.

Proof. We have $x := \varepsilon_1 - \varepsilon_n \in \Delta \cap C$ and also $|\{x_1, \dots, x_n\}| = 3$. Now use Proposition 2.22. $\square$

Proposition 2.24. *For any vector $x \in (P \cap C) \setminus \{0\}$, we have the inequality $|(Wx) \setminus H| \geq 2$, which may become an equality only in the following cases:*

- 1) $x_2 = \dots = x_n$;
- 2) $x_1 = \dots = x_{n-1}$;
- 3) $n = 4$, $x_1 = x_2$, $x_3 = x_4$.

Proof. Since the tautological representation $W : \langle \Delta \rangle$ is irreducible, we have

$$(2.4) \quad \sum_{y \in Wx} y = 0;$$

$$(2.5) \quad \langle Wx \rangle = \langle \Delta \rangle;$$

$$(2.6) \quad \bigcap_{w \in W} (wH) = 0.$$

By (2.4), if $|(Wx) \setminus H| \leq 1$, then $Wx \subset H$ and $\langle Wx \rangle \subset H$, contrary to (2.5). This proves that $|(Wx) \setminus H| \geq 2$.

Suppose that $|(Wx) \setminus H| = 2$.

By (2.6), there are elements $w_1, \dots, w_r \in W$ such that $\bigcap_{i=1}^r (w_i H) = 0$. We have

$$0 \notin Wx, \quad Wx = \bigcup_{i=1}^r ((Wx) \setminus (w_i H)),$$

$$|Wx| \leq \sum_{i=1}^r |(Wx) \setminus (w_i H)| = \sum_{i=1}^r |(Wx) \setminus H| = 2(n-1).$$

Now use Assertion 2.19. $\square$

Proposition 2.25. *Let $x \in (P \cap C) \setminus \{0\}$ be a vector and Λ' be the orbit $Wx \subset P$. Then $|\mathring{\Lambda}' \setminus H| \geq 2$, and the equality $|\mathring{\Lambda}' \setminus H| = 2$ is possible only in the following cases:*

- 1) $x_1 - x_2 = 1$, $x_2 = \dots = x_n$;
- 2) $x_1 = \dots = x_{n-1}$, $x_{n-1} - x_n = 1$;
- 3) $n = 4$, $x_1 = x_2$, $x_2 - x_3 = 1$, $x_3 = x_4$.

Proof. Proposition 2.24 and the relation $\mathring{\Lambda}' \supset \Lambda'$ imply the inequality $|\mathring{\Lambda}' \setminus H| \geq 2$, which may become an equality only in cases 1)–3) mentioned in the statement of Proposition 2.24, provided the $(W$ -invariant) subset $\mathring{\Lambda}' \setminus (\Lambda' \sqcup \{0\}) \subset P$ contains no orbits, i.e., when $\mathring{\Lambda}' \setminus \{0\} = \Lambda'$.

Suppose $|\mathring{\Lambda}' \setminus H| = 2$. Then, $\mathring{\Lambda}' \setminus \{0\} = \Lambda'$, and one of conditions 1)–3) from Proposition 2.24 is satisfied, whence $x \notin \Delta$. Furthermore, for any $\alpha \in \Delta$ such that $\langle x | \alpha \rangle > 0$ we have $x - \alpha \in \mathring{\Lambda}' \setminus \{0\} = \Lambda'$, which implies that $\langle x | \alpha \rangle = 1$. In particular, $\langle x | (\varepsilon_1 - \varepsilon_n) \rangle = 1$ and $x_1 - x_n = 1$.

This finishes the proof of the proposition. $\square$

Proposition 2.26. *Let $x \in P \cap C$ be a vector and Λ' be the orbit $Wx \subset P$. If $x \notin \Delta$ and $|\{x_1, \dots, x_n\}| \geq 3$, then we have the inequality $|\mathring{\Lambda}' \setminus H| \geq 2n$, which may become an equality only in the following cases:*

- 1) $x_1 = x_2$, $x_3 = \dots = x_{n-1}$, $x_2 - x_3 = x_{n-1} - x_n = 1$;
- 2) $x_2 = \dots = x_{n-2}$, $x_{n-1} = x_n$, $x_1 - x_2 = x_{n-2} - x_{n-1} = 1$.

Proof. Since $|\{x_1, \dots, x_n\}| \geq 3$, we have $x_1 - x_n > 1$. Thus, among all the pairs $(i_1, i_2) \in \{1, \dots, n\}^2$ such that $x_{i_1} - x_{i_2} > 1$, we can choose a pair (i_1, i_2) with smallest $i_2 - i_1$.

Clearly, $i_2 > i_1$. Moreover, if $i = i_1 + 1, \dots, i_2 - 1$ is an arbitrary number, then $x_{i_1} - x_i \leq 1$ and $x_i - x_{i_2} \leq 1$, whereas $(x_{i_1} - x_i) + (x_i - x_{i_2}) = x_{i_1} - x_{i_2} > 1$, whence $x_{i_1} - x_i = x_i - x_{i_2} = 1$. Thus,

$$(2.7) \quad \begin{aligned} & x_{i_1} - 1 \geq x_{i_2} + 1; \\ & (i_2 - i_1 > 1) \Rightarrow (x_{i_1} - 1 = x_{i_1+1} = \dots = x_{i_2-1} = x_{i_2} + 1). \end{aligned}$$

Set $\alpha := \varepsilon_{i_1} - \varepsilon_{i_2} \in \Delta$. Since $\langle x | \alpha \rangle = x_{i_1} - x_{i_2} > 1$ and $x \notin \Delta$ it follows that $y := x - \alpha \in \mathring{\Lambda}' \setminus (\Lambda' \sqcup \{0\})$. Let Λ'' denote the orbit $Wy \subset P$.

We have $x \in C$, $x_1 \geq \dots \geq x_n$, and, according to (2.7),

$$(2.8) \quad \begin{aligned} & x_1 \geq \dots \geq x_{i_1-1} > x_{i_1} - 1 \geq x_{i_1+1} \geq \dots \geq x_{i_2-1} \geq x_{i_2} + 1 > x_{i_2+1} \geq \dots \geq x_n; \\ & (i_2 - i_1 > 1) \Rightarrow (y_{i_1} = \dots = y_{i_2}), \end{aligned}$$

$y_1 \geq \dots \geq y_{i_1-1} > y_{i_1} \geq \dots \geq y_{i_2} > y_{i_2+1} \geq \dots \geq y_n$. Therefore, $y \in C$. Moreover, $\mathring{\Lambda}' \supset \Lambda' \sqcup \mathring{\Lambda}''$, and therefore,

$$|\mathring{\Lambda}' \setminus H| \geq |\Lambda' \setminus H| + |\mathring{\Lambda}'' \setminus H|.$$

Suppose $|\mathring{\Lambda}' \setminus H| \leq 2n$.

It follows from Propositions 2.22 and 2.25 that $|\mathring{\Lambda}' \setminus H| = 2n$, $y_1 = \dots = y_j$, $y_{j+1} = \dots = y_n$, $y_j - y_{j+1} = 1$, $j \in \{1, \dots, n-1\}$, and either $j \in \{1, n-1\}$ or $n = 4$ and $j = 2$.

Notice the following:

- 1) as we mentioned before, $y \neq 0$, and, by (2.8), $(i_1, i_2) \neq (1, n)$;
- 2) if $i_1 > 1$, then $y_{i_1-1} > y_{i_1}$, $j = i_1 - 1 < i_2 - 1 \leq n - 1$, $(j = 1) \vee ((n = 4) \text{ and } (j = 2))$;
- 3) if $i_2 < n$, then $y_{i_2} > y_{i_2+1}$, $j = i_2 > i_1 \geq 1$, $(j = n - 1) \vee ((n = 4) \text{ and } (j = 2))$.

Since $i_1 - 1 < i_1 < i_2$, exactly one of the following equalities $i_1 = 1$ and $i_2 = n$ holds.

We can have the following cases.

Case 1. $i_1 > 1$, $i_2 = n$, $j = i_1 - 1$, and $j = 1$.

We have $i_1 = 2$, $y_1 - 1 = y_2 = \dots = y_n$, $x_1 - 1 = x_2 - 1 = x_3 = \dots = x_{n-1} = x_n + 1$.

Case 2. $i_1 > 1$, $i_2 = n$, $j = i_1 - 1$, $n = 4$, and $j = 2$.

We have $i_1 = 3$, $y_1 = y_2 = y_3 + 1 = y_4 + 1$, $x_1 = x_2 = x_3 = x_4 + 2$.

Case 3. $i_1 = 1$, $i_2 < n$, $j = i_2$, and $j = n - 1$.

We have $i_2 = n - 1$, $y_1 = \dots = y_{n-1} = y_n + 1$, $x_1 - 1 = x_2 = \dots = x_{n-2} = x_{n-1} + 1 = x_n + 1$.

Case 4. $i_1 = 1$, $i_2 < n$, $j = i_2$, $n = 4$, and $j = 2$.

We have $i_2 = 2$, $y_1 - 1 = y_2 - 1 = y_3 = y_4$, $x_1 - 2 = x_2 = x_3 = x_4$.

Now use the fact that $|\{x_1, \dots, x_n\}| \geq 3$. □

Proposition 2.27. *Let $x \in P \cap C$ be a vector and let Λ' denote the orbit $Wx \subset P$. If*

$$(2.9) \quad |\{x_1, \dots, x_n\}| = 2,$$

$x_1 - x_n > 1$, and $|\mathring{\Lambda}' \setminus H| \leq 2n$, then one of the following holds:

- 1) $x_1 - x_2 = 2$, $x_2 = \dots = x_n$;
- 2) $x_1 = \dots = x_{n-1}$, $x_{n-1} - x_n = 2$;
- 3) $n = 4$, $x_1 = x_2 = 1$, $x_3 = x_4 = -1$.

Proof. By assumption, $x_1 = \dots = x_j$, $x_{j+1} = \dots = x_n$, and $x_j - x_{j+1} \geq 2$, where $j \in \{1, \dots, n-1\}$.

Set $\alpha := \alpha_j \in \Delta$. We have $\langle x | \alpha \rangle = x_j - x_{j+1} > 1$, and therefore $y := x - \alpha \in \mathring{\Lambda}' \setminus \Lambda'$. Moreover, $x \in C$, $x_1 \geq \dots \geq x_n$, $x_1 \geq \dots \geq x_{j-1} > x_j - 1 \geq x_{j+1} + 1 > x_{j+2} \geq x_n$, $y_1 \geq \dots \geq y_{j-1} > y_j \geq y_{j+1} > y_{j+2} \geq \dots \geq y_n$, $y \in C$.

Let Λ'' be the orbit $Wy \subset P$. It is easy to see that $\mathring{\Lambda}' \supset \Lambda' \sqcup \mathring{\Lambda}''$. Therefore, $2n \geq |\mathring{\Lambda}' \setminus H| \geq |\Lambda' \setminus H| + |\mathring{\Lambda}'' \setminus H|$. By Proposition 2.24, $|\Lambda' \setminus H| \geq 2$, which implies the inequality $|\mathring{\Lambda}'' \setminus H| \leq 2n - 2 < 2n$. By Proposition 2.26, we have that either $|\{y_1, \dots, y_n\}| \leq 2$ or $y \in \Delta$.

Suppose that $|\{y_1, \dots, y_n\}| \leq 2$.

For the integers $k_1 := |\{y_1, \dots, y_{j-1}\}|$, $k_2 := |\{y_j, y_{j+1}\}|$, $k_3 := |\{y_{j+2}, \dots, y_n\}|$ we have $k_1, k_3 \geq 0$, $k_2 \geq 1$, $(k_1 = 0) \Rightarrow (j = 1)$, and $(k_3 = 0) \Rightarrow (j = n - 1)$. Furthermore, $n - 1 > 1$, whence

$$k_1 + k_3 \geq 1; \quad (k_1 + k_3 = 1) \Rightarrow (j \in \{1, n - 1\}).$$

Moreover, $(k_1 + k_3) + k_2 = |\{y_1, \dots, y_n\}| \leq 2$ and $k_2 \geq 1$. Therefore, $k_1 + k_3 = 1$, $j \in \{1, n - 1\}$, and $k_2 = 1$, $y_j = y_{j+1}$, $x_j - x_{j+1} = 2$.

Suppose now that $y \in \Delta$.

By (2.9), $0 \notin \{x_1, \dots, x_n\}$. It now follows from the relations $x = y + \alpha \in \Delta + \Delta$ and $x \in C$ that

$$n \leq 4; \quad (n = 4) \Rightarrow (x_1 = x_2 = 1, x_3 = x_4 = -1).$$

Suppose $n = 3$. Then for $r = 2$,

$$(2.10) \quad \dim H = r - 1 = 1.$$

Furthermore,

$$\Lambda'' = \Delta, \quad \mathring{\Lambda}' \supset \Lambda' \sqcup \Delta,$$

$$|\Lambda'| + |\Delta| \leq |\Lambda' \cap H| + |\Delta \cap H| + |\mathring{\Lambda}' \setminus H| \leq |\Lambda' \cap H| + |\Delta \cap H| + 2n,$$

$$3 + n(n - 1) \leq |\Lambda' \cap H| + |\Delta \cap H| + 2n,$$

$$|\Lambda' \cap H| + |\Delta \cap H| \geq n(n - 3) + 3 = 3,$$

and, according to (2.10),

$$|\Lambda' \cap H|, |\Delta \cap H| \leq 2 < 3 \leq |\Lambda' \cap H| + |\Delta \cap H|,$$

$$|\Lambda' \cap H|, |\Delta \cap H| > 0, \quad \Lambda' \cap H, \Delta \cap H \neq \emptyset.$$

But, by (2.9), no root of $\Delta \subset \mathbb{R}^n$ is proportional to a vector from the orbit $\Lambda' = Wx \subset P$, contrary to (2.10).

Thus, the proposition is proved. $\square$

Corollary 2.28. *Let $x \in P \cap C$ be a vector and let Λ' denote the orbit $Wx \subset P$. If $x_1 - x_n > 2$, then $|\mathring{\Lambda}' \setminus H| > 2n$ and $|\Lambda' \setminus H| \geq 2$.*

Proof. Follows from Propositions 2.24, 2.26, and 2.27. $\square$

Proposition 2.29. *Let $x \in P \cap C$ be a vector, and let Λ' denote the orbit $Wx \subset P$. If $x \notin \Delta$, $x_1 - x_n > 1$, $-\Lambda' = \Lambda'$, and $|\mathring{\Lambda}' \setminus H| \leq 4n$, then $n = 4$, $x_1 = x_2 = 1$, and $x_3 = x_4 = -1$.*

Proof. We have $x_{n+1-i} = -x_i$ for any $i = 1, \dots, n$. Furthermore, since $n - 1 \geq 2$, we have $x_2 \geq x_{n-1} = -x_2$, $x_2 \geq 0$.

We can have the following cases.

Case 1. $x_1 - x_n > 2$ and $x_2 > 0$.

Case 2. $x_1 - x_n > 2$ and $x_2 = 0$.

Case 3. $x_1 - x_n = 2$.

First consider Case 1.

We have $x_1 = \frac{(x_1 - x_n)}{2} > 1$.

Set $\alpha := \varepsilon_1 - \varepsilon_n \in \Delta$. Because $\langle x | \alpha \rangle = x_1 - x_n > 2$ and $x \notin \Delta$, we have $y := x - \alpha \in \mathring{\Lambda}' \setminus (\Lambda' \sqcup \{0\})$. Let Λ'' denote the orbit $Wy \subset P$. We have $\Lambda'' \cap C = \{\tilde{y}\}$ and $\tilde{y} \in (P \cap C) \setminus \{0\}$. Moreover,

$$\mathring{\Lambda}' \supset \Lambda' \sqcup \Lambda'', \quad 4n \geq |\mathring{\Lambda}' \setminus H| \geq |\Lambda' \setminus H| + |\Lambda'' \setminus H|,$$

and, by Proposition 2.24, $|\Lambda' \setminus H|$ and $|\Lambda'' \setminus H| < 4n$.

By Proposition 2.22, $|\{x_1, \dots, x_n\}|, |\{\tilde{y}_1, \dots, \tilde{y}_n\}| < 4$, $|\{y_1, \dots, y_n\}| < 4$. Moreover, $\{x_1, \dots, x_n\} \supset \{x_1, x_2, x_{n-1}, x_n\} = \{\pm x_1, \pm x_2\}$, $\{y_1, \dots, y_n\} = \{y_1, x_2, \dots, x_{n-1}, y_n\}$, $y_1 = x_1 - 1 > 0$, $y_n = x_n + 1 = -x_1 + 1 = -y_1$, $\{y_1, \dots, y_n\} \supset \{\pm y_1, \pm x_2\}$. Thus, $|\{\pm x_1, \pm x_2\}| < 4$ and $|\{\pm y_1, \pm x_2\}| < 4$. On the other hand, $x_1, y_1, x_2 > 0$. This implies that $x_2 = x_1$ and $x_2 = y_1 = x_1 - 1$.

Thus Case 1 leads to a contradiction.

Now consider Case 2.

We have

$$x_{n-1} = -x_2 = 0, \quad x_2 = \dots = x_{n-1} = 0, \quad x = x_1(\varepsilon_1 - \varepsilon_n).$$

Furthermore, $x_1 = x_1 - x_2 \in \mathbb{Z}$, and also $x_1 = \frac{(x_1 - x_n)}{2} > 1$, whence $x_1 \geq 2$. Notice that $x_1 - 1 \geq 1 > 0 > -x_1$ and, as a consequence,

$$y := (x_1 - 1)\varepsilon_1 + \varepsilon_2 - x_1\varepsilon_n \in (\mathring{\Lambda}' \setminus \Lambda') \cap C \subset P.$$

The orbits $\Lambda'' := Wy \subset P$ and $-\Lambda'' \subset P$ are distinct, since $y_1 + y_n = (x_1 - 1) - x_1 \neq 0$. Therefore $\mathring{\Lambda}' \supset \Lambda' \sqcup (-\Lambda'') \sqcup \mathring{\Lambda}''$ and $4n \geq |\mathring{\Lambda}' \setminus H| \geq |\Lambda' \setminus H| + |\Lambda'' \setminus H| + |\mathring{\Lambda}'' \setminus H|$. Moreover,

- 1) $|\{x_1, \dots, x_n\}| = |\{\pm x_1, 0\}| = 3$, and, according to Proposition 2.22, $|\Lambda' \setminus H| \geq 2n - 2$;
- 2) $y_1 - y_n = (x_1 - 1) + x_1 = 2x_1 - 1 \geq 3$, and, by Corollary 2.28, $|\Lambda'' \setminus H| + |\mathring{\Lambda}'' \setminus H| > 2n + 2$.

Hence,

$$|\Lambda' \setminus H| + |\Lambda'' \setminus H| + |\mathring{\Lambda}'' \setminus H| > (2n - 2) + (2n + 2) = 4n.$$

This is a contradiction.

Finally, consider Case 3.

We have $x_1 = \frac{(x_1 - x_n)}{2} = 1$, $x = (\varepsilon_1 + \dots + \varepsilon_j) - (\varepsilon_{n+1-j} + \dots + \varepsilon_n)$, $j \in \mathbb{N}$, $j \leq \frac{n}{2}$. By assumption, $x \notin \Delta$, whence $j \geq 2$, $n \geq 4$. If $n = 4$, then $j = 2$, $x_1 = x_2 = 1$, and $x_3 = x_4 = -1$.

Suppose that $n > 4$. Set $y := (\varepsilon_1 + \varepsilon_2) - (\varepsilon_{n-1} + \varepsilon_n) \in P \cap C$. It is easy to see that $\mathring{\Lambda}' \supset (Wy) \sqcup \Delta$. Hence, $4n \geq |\mathring{\Lambda}' \setminus H| \geq |(Wy) \setminus H| + |\Delta \setminus H|$ and, according to Corollary 2.23, $|(Wy) \setminus H| \leq 4n - |\Delta \setminus H| \leq 4n - (2n - 2) = 2n + 2$. Furthermore, $n - 4 > 0$, $|\{y_1, \dots, y_n\}| = 3$ and, also, $n - 2 > 2$, $\max\{2, n - 4\} < n - 2$. This together with Proposition 2.22 implies that

$$|(Wy) \setminus H| \geq 5(n - 2) = (2n + 2) + 3(n - 4) > 2n + 2.$$

This is a contradiction.

The proposition is proved. □

Applying Propositions 2.26, 2.27 and 2.29 to the vector $\lambda \in (P \cap C) \setminus \{0\}$ and its orbit $\Lambda \subset P \setminus \{0\}$, we have Lemma 2.20. $\square$

Now let us prove Lemma 2.21.

By assumption, $j \in \{3, \dots, n-3\}$, $\lambda_1 = \dots = \lambda_j$, $\lambda_{j+1} = \dots = \lambda_n$, and $\lambda_j - \lambda_{j+1} = 1$. In particular, $\lambda_{n-1} = \lambda_n$, and therefore

$$H = \{x \in \langle \Delta \rangle : x_{n-1} = x_n\}, \quad |\Lambda \setminus H| = \binom{2}{1} \cdot \binom{n-2}{j-1}.$$

Moreover $2 \leq j-1 \leq (n-2)-2$, whence

$$\binom{n-2}{j-1} \geq \binom{n-2}{2} = \frac{(n-2)(n-3)}{2}.$$

Therefore,

$$|\Lambda \setminus H| \geq (n-2)(n-3) > n(n-5) = n(r-4) = 4n + n(r-8) \geq 4n.$$

Thus Lemma 2.21 is proved. $\square$

2.4.2. Additional results. Here we prove Propositions 2.30–2.33.

For arbitrary, pairwise distinct numbers $i_1, \dots, i_k \in \{1, \dots, n\}$, where $k = 1, \dots, n$, we set

$$\lambda_{(i_1, \dots, i_k)} := (\varepsilon_{i_1} + \dots + \varepsilon_{i_k}) - \frac{k}{n} \cdot (\varepsilon_1 + \dots + \varepsilon_n) \in P.$$

Furthermore, for arbitrary pairwise distinct numbers $i, i_1, i_2 \in \{1, \dots, n\}$ we set

$$\begin{aligned} \lambda_{(i_1, i_2)}^{(i)} &:= \lambda_{(i_1)} + (\varepsilon_{i_2} - \varepsilon_i) = \lambda_{(i_2)} + (\varepsilon_{i_1} - \varepsilon_i) \\ &= (\varepsilon_{i_1} + \varepsilon_{i_2} - \varepsilon_i) - \frac{1}{n} \cdot (\varepsilon_1 + \dots + \varepsilon_n) \in P. \end{aligned}$$

Proposition 2.30. *If $r > 2$ and $\lambda = \varphi_2 + \varphi_r$, then there are a subset $\Omega \subset \Lambda$ and a hyperplane $H \subset \langle \Delta \rangle$ such that $\langle \Omega \rangle = H$, $(\Omega - \Omega) \cap \Delta = \emptyset$ and $2 \cdot |\mathring{\Lambda} \setminus H| > |\Delta \setminus H| + 6$.*

Proof. We have $n = r + 1 \geq 4$, $\lambda = \lambda_{(1,2)}^{(n)}$, and $\mathring{\Lambda} = \Lambda \sqcup \{\lambda_{(1)}, \dots, \lambda_{(n)}\}$.

Set $\Omega := \{\lambda_{(i-1,i)}^{(i+1)} : i = 2, \dots, r\} \subset \Lambda$ and $H := \langle \Omega \rangle \subset \langle \Delta \rangle$. Let us prove that the subset $\Omega \subset \Lambda$ and the subspace $H \subset \langle \Delta \rangle$ satisfy the desired requirements.

An arbitrary vector of the form $(\varepsilon_{j-1} + \varepsilon_j - \varepsilon_{j+1}) - (\varepsilon_{i-1} + \varepsilon_i - \varepsilon_{i+1}) \in P$ ($2 \leq i < j \leq r$) has its $(i+1)$ -st coordinate equal to 2 when $j \leq i+2$, and six nonzero coordinates when $j > i+2$. Hence, $(\Omega - \Omega) \cap \Delta = \emptyset$.

The subspace $H' := H^\perp \cap \langle \Delta \rangle \subset \mathbb{R}^n$ is given by the equations $x_{i-1} + x_i = x_{i+1}$, where $i = 2, \dots, r$, and $x_1 + \dots + x_n = 0$. Moreover,

$$\dim H \leq |\Omega| = r - 1, \quad \dim H' = r - \dim H \geq 1.$$

Let $x \in H'$ be an arbitrary nonzero vector.

Let us show that $x_1, x_2, x_3, x_4 \neq 0$. We have $x_{i-1} + x_i = x_{i+1}$ ($i = 2, \dots, r$) and $x_1 + \dots + x_n = 0$, whence

$$(2.11) \quad \forall i = 2, \dots, r \quad x_{i-1}x_i \leq x_{i-1}x_i + x_i^2 = x_ix_{i+1};$$

$$(2.12) \quad \begin{aligned} 2x_3 + \sum_{i=4}^n x_i &= (x_1 + x_2 + x_3) + \sum_{i=4}^n x_i = \sum_{i=1}^n x_i = 0; \\ 4x_3^2 + 4 \cdot \sum_{i=4}^n x_3x_i + \sum_{i_1, i_2=4}^n x_{i_1}x_{i_2} &= 0. \end{aligned}$$

Suppose that there is a pair $(i_1, i_2) \in \{3, \dots, n\}^2$ such that $i_1 \leq i_2$ and $x_{i_1}x_{i_2} < 0$. Among all such pairs choose a pair (i_1, i_2) with smallest $i_1 + i_2$. We have $i_1 < i_2$. Furthermore, if $i_2 \geq 5$, then

$$3 \leq i_2 - 2 < i_2 - 1 < n \quad \text{and} \quad i_1 + (i_2 - 2) < i_1 + (i_2 - 1) < i_1 + i_2,$$

whence $x_{i_1}x_{i_2-2}, x_{i_1}x_{i_2-1} \geq 0$, $x_{i_1}(x_{i_2-2} + x_{i_2-1}) \geq 0$, and $x_{i_1}x_{i_2} \geq 0$, contrary to the assumption. Hence, $i_2 < 5$, $3 \leq i_1 < i_2 < 5$, and therefore $(i_1, i_2) = (3, 4)$ and $x_3x_4 < 0$. By (2.11), $x_1x_2 \leq x_2x_3 \leq x_3x_4 < 0$, $x_1, x_2, x_3, x_4 \neq 0$.

Suppose now that $x_{i_1}x_{i_2} \geq 0$ for all $i_1, i_2 \in \{3, \dots, n\}$ ($i_1 \leq i_2$). Then, according to (2.12), $x_3^2 = \dots = x_n^2 = 0$, $x_3 = \dots = x_n = 0$, $x_2 = x_4 - x_3 = 0$, $x_1 = x_3 - x_2 = 0$, $x_1 = \dots = x_n = 0$, whereas $x \neq 0$. This is a contradiction.

We have thus established that $\dim H' \geq 1$ and, moreover, $x_1, x_2, x_3, x_4 \neq 0$ for any nonzero vector $x \in H'$. As a consequence, $H' = \mathbb{R}x \subset \langle \Delta \rangle$ ($x \in \langle \Delta \rangle$, $x_1, x_2, x_3, x_4 \neq 0$), $\dim H' = 1$, $H = (H')^\perp \cap \langle \Delta \rangle = \{y \in \langle \Delta \rangle : (y, x) = 0\} \subset \mathbb{R}^n$, $\dim H = r - 1$.

It remains to show that $2 \cdot |\mathring{\Lambda} \setminus H| > |\Delta \setminus H| + 6$.

For any $i = 1, 2, 3, 4$, we have $(\lambda_{(i)}, x) = (\varepsilon_i, x) = x_i \neq 0$, whence $\lambda_{(i)} \in \mathring{\Lambda} \setminus H$. Therefore, $|\mathring{\Lambda} \setminus H| \geq |\Lambda \setminus H| + 4 > |\Lambda \setminus H| + 3$.

Let $I = \{i_1, i_2\} \subset \{1, \dots, n\}$ be an arbitrary two-element subset. Let us prove that there is a number $i \in \{1, \dots, n\} \setminus I$ such that $x_i \neq x_{i_1} + x_{i_2}$.

Suppose that $x_i = x_{i_1} + x_{i_2}$ for any $i \in \{1, \dots, n\} \setminus I$. Then

$$0 = \sum_{i=1}^n x_i = (x_{i_1} + x_{i_2}) + (n-2)(x_{i_1} + x_{i_2}) = (n-1)(x_{i_1} + x_{i_2}), \quad x_{i_1} + x_{i_2} = 0.$$

Thus, all numbers $x_i \in \mathbb{R}$ (where $i \in \{1, \dots, n\}$ and $i \neq i_1, i_2$) are zero, which is impossible since $x_1, x_2, x_3, x_4 \neq 0$.

According to the above, for all distinct numbers $i_1, i_2 \in \{1, \dots, n\}$ there is a number $i \in \{1, \dots, n\} \setminus \{i_1, i_2\}$ such that $x_i \neq x_{i_1} + x_{i_2}$ (and therefore $(\varepsilon_{i_1} + \varepsilon_{i_2} - \varepsilon_i, x) \neq 0$, $(\lambda_{(i_1, i_2)}^{(i)}, x) \neq 0$, $\lambda_{(i_1, i_2)}^{(i)} \in \Lambda \setminus H$). Hence,

$$|\Lambda \setminus H| \geq C_n^2, \quad |\mathring{\Lambda} \setminus H| > |\Lambda \setminus H| + 3 \geq C_n^2 + 3$$

and, therefore, $2 \cdot |\mathring{\Lambda} \setminus H| > 2 \cdot C_n^2 + 6 = |\Delta| + 6 \geq |\Delta \setminus H| + 6$. $\square$

Proposition 2.31. *If $r = 3$ and $\lambda = \varphi_1 + \varphi_2 + \varphi_3$, then*

- 1) $2\lambda \notin \Delta \cup (\Delta + \Delta)$;
- 2) *there are a subset $\Omega \subset \Lambda$ and a hyperplane $H \subset \langle \Delta \rangle$ such that $\langle \Omega \rangle = H$, $(\Omega + \Omega) \cap \Delta = (\Omega - \Omega) \cap \Delta = \emptyset$, and $|\mathring{\Lambda} \setminus H| > |\Delta \setminus H| + 6$.*

Proof. We have

$$\lambda = \frac{1}{2} \cdot (3(\varepsilon_1 - \varepsilon_4) + (\varepsilon_2 - \varepsilon_3)), \quad 2\lambda = 3(\varepsilon_1 - \varepsilon_4) + (\varepsilon_2 - \varepsilon_3) \notin \Delta \cup (\Delta + \Delta),$$

$$|\Lambda| = 24 \text{ and } |\Delta| = 12.$$

Set

$$\Omega := \left\{ \frac{1}{2} \cdot (3(\varepsilon_1 - \varepsilon_4) + (\varepsilon_2 - \varepsilon_3)), \frac{1}{2} \cdot (3(\varepsilon_1 - \varepsilon_2) + (\varepsilon_4 - \varepsilon_3)) \right\} \subset \Lambda.$$

As is easily seen,

$$(\Omega + \Omega) \cap \Delta = (\Omega - \Omega) \cap \Delta = \emptyset \quad \text{and} \quad H := \langle \Omega \rangle = \{x \in \langle \Delta \rangle : x_1 + 3x_3 = 0\} \subset \langle \Delta \rangle.$$

Furthermore,

$$\Lambda \cap H = \Omega \sqcup (-\Omega) \subset \Lambda, \quad |\Lambda \cap H| = 4, \quad |\Lambda \setminus H| = 20 > |\Delta| + 6 \geq |\Delta \setminus H| + 6. \quad \square$$

Proposition 2.32. *If $r = 6$ and $\lambda = \varphi_3$, then there are a subset $\Omega \subset \Lambda$ and a hyperplane $H \subset \langle \Delta \rangle$ such that $\langle \Omega \rangle = H$, $(\Omega - \Omega) \cap \Delta = \emptyset$, and $2 \cdot |\mathring{\Lambda} \setminus H| > |\Delta \setminus H| + 6$.*

Proof. By assumption, $\lambda = \lambda_{(1,2,3)}$ and $|\Lambda| = 35$.

Set

$$\Omega := \{\lambda_{(1,2,5)}, \lambda_{(3,4,5)}, \lambda_{(1,4,6)}, \lambda_{(2,3,6)}, \lambda_{(5,6,7)}\} \subset \Lambda.$$

We have

$$(\Omega - \Omega) \cap \Delta = \emptyset \quad \text{and} \quad H := \langle \Omega \rangle = \{x \in \langle \Delta \rangle : x_1 + x_3 = x_2 + x_4\} \subset \langle \Delta \rangle.$$

Hence, $|\Delta \setminus H| = n^2 - (3^2 + 2^2 + 2^2) = 32$. Moreover,

$$\Lambda \cap H = \{\lambda_{(i_1, i_2, i_3)} \in P : i_1 \in \{1, 3\}, i_2 \in \{2, 4\}, i_3 \in \{5, 6, 7\}\} \cup \{\lambda_{(5,6,7)}\} \subset \Lambda,$$

$$|\Lambda \cap H| = 13, \quad |\Lambda \setminus H| = 22, \quad 2 \cdot |\Lambda \setminus H| = 44 > |\Delta \setminus H| + 6. \quad \square$$

Proposition 2.33. *If $r = 7$ and $\lambda = \varphi_3$, then there are a subset $\Omega \subset \Lambda$ and a hyperplane $H \subset \langle \Delta \rangle$ such that $\langle \Omega \rangle = H$, $(\Omega - \Omega) \cap \Delta = \emptyset$ and $2 \cdot |\mathring{\Lambda} \setminus H| > |\Delta \setminus H| + 6$.*

Proof. We have $\lambda = \lambda_{(1,2,3)}$ and $|\Lambda| = |\Delta| = 56$.

Set

$$\Omega := \{\lambda_{(1,2,7)}, \lambda_{(3,4,7)}, \lambda_{(5,6,7)}, \lambda_{(4,5,8)}, \lambda_{(1,6,8)}, \lambda_{(2,3,8)}\} \subset \Lambda.$$

As is easily seen,

$$(\Omega - \Omega) \cap \Delta = \emptyset \quad \text{and} \quad H := \langle \Omega \rangle = \{x \in \langle \Delta \rangle : x_1 + x_3 + x_5 = x_2 + x_4 + x_6\} \subset \langle \Delta \rangle.$$

Furthermore,

$$\Lambda \cap H = \{\lambda_{(i_1, i_2, i_3)} \in P : i_1 \in \{1, 3, 5\}, i_2 \in \{2, 4, 6\}, i_3 \in \{7, 8\}\} \subset \Lambda,$$

$$\text{whence } |\Lambda \cap H| = 18, \quad |\Lambda \setminus H| = 38, \quad \text{and } 2 \cdot |\Lambda \setminus H| = 76 > |\Delta| + 6 \geq |\Delta \setminus H| + 6. \quad \square$$

§ 3. PROOFS OF THE MAIN RESULTS

This section is devoted to proving Theorem 1.1.

We return to the notation and the assumptions from §1.

Set $\delta := 1 \in \mathbb{R}$ if the representation R of the algebra $\mathfrak{g}_{\mathbb{C}}$ is orthogonal, and $\delta := 2 \in \mathbb{R}$ otherwise.

Fix a maximal commutative subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{t}_{\mathbb{C}} := \mathfrak{t} \otimes \mathbb{C}$ of $\mathfrak{g}_{\mathbb{C}}$. As a result, we have a root system $\Delta \subset \mathfrak{t}_{\mathbb{C}}^*$ and its Weyl group $W \subset \mathbf{GL}(\mathfrak{t}_{\mathbb{C}}^*)$. We have $\mathfrak{g} \cong \mathfrak{su}_{r+1}$, $r > 1$, and therefore $\Delta \subset \mathfrak{t}_{\mathbb{C}}^*$ is an indecomposable root system of type A_r . Moreover,

$$\langle \Delta \rangle = \{\lambda \in \mathfrak{t}_{\mathbb{C}}^* : \lambda(\mathfrak{t}) \subset i\mathbb{R}\} \subset \mathfrak{t}_{\mathbb{C}}^* = \langle \Delta \rangle \oplus i\langle \Delta \rangle.$$

Fix a simple root system $\Pi \subset \Delta$ and the corresponding Weyl chamber $C \subset \langle \Delta \rangle \subset \mathfrak{t}_{\mathbb{C}}^*$.

Let P and Q denote the lattices $\{\lambda \in \langle \Delta \rangle : \langle \lambda | \alpha \rangle \in \mathbb{Z} \ \forall \alpha \in \Delta\} \subset \langle \Delta \rangle$ and $\langle \Delta \rangle_{\mathbb{Z}} \subset \langle \Delta \rangle$, respectively. We have $Q \subset P$. Let $\lambda \in (P \cap C) \setminus \{0\}$ be the highest weight representation R of $\mathfrak{g}_{\mathbb{C}}$ relative to the simple root system $\Pi \subset \Delta \subset \mathfrak{t}_{\mathbb{C}}^*$.

Set $\Lambda := W\lambda \subset P$, $\mathring{\Lambda} := \text{conv}(\Lambda) \cap (\Lambda + Q) \subset P$, and $n := r + 1 \in \mathbb{N}$. It is easy to see that the set of weights of the representation R of $\mathfrak{g}_{\mathbb{C}}$ coincides with the subset $\mathring{\Lambda} \subset P$.

To prove Theorem 1.1 assume the opposite: suppose that V/G is a smooth manifold and the linear algebra $R(\mathfrak{g}_{\mathbb{C}})$ is not isomorphic to any of the linear algebras $\text{ad}(\mathfrak{g}_{\mathbb{C}})$,

$\varphi_1(A_r), \varphi_2(A_r), (2\varphi_1)(A_r)$ ($r > 1$), $(2\varphi_2)(A_3), \varphi_3(A_5)$ and $\varphi_4(A_7)$. The latter condition means that

$$(3.1) \quad \begin{aligned} \lambda &\notin \Delta \cup \{\varphi_1, \varphi_2, \varphi_{r-1}, \varphi_r, 2\varphi_1, 2\varphi_r\}; \\ (r=3) &\Rightarrow (\lambda \neq 2\varphi_2); \\ (r=5) &\Rightarrow (\lambda \neq \varphi_3); \\ (r=7) &\Rightarrow (\lambda \neq \varphi_4). \end{aligned}$$

Moreover, since the linear algebras $\varphi_j(A_r)$ and $\varphi_{n-j}(A_r)$ ($j = 1, \dots, r$), and also the linear algebras $(\varphi_1 + \varphi_{r-1})(A_r)$ and $(\varphi_2 + \varphi_r)(A_r)$, are isomorphic, we may (and will) assume without loss of generality that

$$(3.2) \quad (\lambda = \varphi_j, j = 1, \dots, r) \Rightarrow \left(j \leq \frac{n}{2}\right); \quad \lambda \neq \varphi_1 + \varphi_{r-1}.$$

Set $\Pi_\lambda := \{\alpha \in \Pi: \langle \lambda | \alpha \rangle \neq 0\} \subset \Pi$. Also, let $\mathcal{P}$ denote the family of all indecomposable simple root systems $\Pi' \subset \Pi \subset \mathfrak{t}_{\mathbb{C}}^*$ of order $r - 2$.

Assertion 3.1. *Suppose that $r > 2$. Then the simple root system Π coincides with the union of all of its subsets $\Pi' \in \mathcal{P}$.*

Proof. See 3.1 of [5, §3]. □

Lemma 3.2. *Let $\Pi' \in \mathcal{P}$ be a simple root system such that $(r > 2) \Rightarrow (\Pi_\lambda \cap \Pi' \neq \emptyset)$. If $\lambda \neq \varphi_2 + \varphi_r$, then at least one of the following conditions holds:*

$$(3.3) \quad \lambda \in \{\varphi_3, \dots, \varphi_{r-2}\}; \quad (\Pi' = (1, \dots, r-2) \subset \Pi) \Rightarrow (r < 8);$$

$$(3.4) \quad r > 2; \quad \Pi_\lambda \cap \Pi' = \{\alpha\} \subset \Pi, \quad \alpha \in \partial\Pi' \subset \Pi, \quad \langle \lambda | \alpha \rangle = 1.$$

Proof. It is clear that $H := \langle \{\lambda\} \cup \Pi' \rangle \subset \langle \Delta \rangle \subset \mathfrak{t}_{\mathbb{C}}^*$ is an $(r - 1)$ -dimensional (real) subspace and, therefore, the intersection of the kernels of all linear functions on that subspace is of the form $\mathbb{C}\xi \subset \mathfrak{t}_{\mathbb{C}}, \xi \in \mathfrak{t} \setminus \{0\}$.

We have $\Pi \cong A_r$ and $\Pi' \cong A_{r-2}$. As a consequence,

$$|\Delta| = r(r+1) \quad \text{and} \quad |\Delta \cap \langle \Pi' \rangle| = (r-2)(r-1).$$

Hence,

$$\begin{aligned} |\Delta \cap H| &\geq |\Delta \cap \langle \Pi' \rangle| = (r-2)(r-1), \\ |\Delta \setminus H| &\leq r(r+1) - (r-2)(r-1) = 4r-2. \end{aligned}$$

Suppose

$$(3.5) \quad \exists v \in V \quad \xi \in \mathfrak{g}_v, \text{ rk } \mathfrak{g}_v = 1.$$

By Lemma 2.12,

$$\delta \cdot |\overset{\circ}{\Delta} \setminus H| \leq |\Delta \setminus H| + 6 \leq 4r + 4 = 4n.$$

Moreover, $(\delta = 1) \Rightarrow (-\Lambda = \Lambda)$. Thus

$$|\overset{\circ}{\Lambda} \setminus H| \leq 4n; \quad (-\Lambda = \Lambda) \vee (|\overset{\circ}{\Lambda} \setminus H| \leq 2n).$$

Furthermore, according to (3.2), $\lambda \notin \{\varphi_1 + \varphi_{r-1}, \varphi_2 + \varphi_r\}$. Now, using Lemmas 2.20 and 2.21, and the relations (3.1), we have (3.3).

If (3.5) does not hold, then, by Lemma 3.4 of [5, §3], we have the relations (3.4). □

Corollary 3.3. *Let $\Pi' \in \mathcal{P}$ be a simple root system which does not satisfy (3.3). If $\lambda \neq \varphi_2 + \varphi_r$, then*

$$(3.6) \quad \begin{aligned} |\Pi_\lambda \cap \Pi'| &\leq 1, \quad \Pi_\lambda \cap (\Pi' \setminus (\partial\Pi')) = \emptyset, \quad \forall \alpha \in \Pi_\lambda \cap \Pi' \quad \langle \lambda | \alpha \rangle = 1; \\ ((r > 2) &\Rightarrow (\Pi_\lambda \cap \Pi' \neq \emptyset)) \Rightarrow r > 2. \end{aligned}$$

Corollary 3.4. *Let $\Pi' \in \mathcal{P}$ be a simple root system which does not satisfy (3.3). If $\lambda \neq \varphi_2 + \varphi_r$, then $r > 2$ and (3.6) holds.*

Suppose that $\lambda = \varphi_2 + \varphi_r$.

By (3.1), $\lambda \neq 2\varphi_r$, whence $r > 2$. Furthermore, the representation R of $\mathfrak{g}_{\mathbb{C}}$ is not self-adjoint, and therefore $\delta = 2$. Now, using Proposition 2.30 and Corollary 2.13, we obtain a contradiction with the fact that V/G is a smooth manifold.

Hence, $\lambda \neq \varphi_2 + \varphi_r$. By (3.2), $\lambda \notin \{\varphi_1 + \varphi_{r-1}, \varphi_2 + \varphi_r\}$.

Suppose $\lambda \notin \{\varphi_3, \dots, \varphi_{r-2}\}$.

None of the simple root systems $\Pi' \in \mathcal{P}$ satisfies (3.3). Moreover, as is easily seen, $\mathcal{P} \neq \emptyset$. Corollary 3.4 implies that $r > 2$ and that (3.6) holds for any simple root system $\Pi' \in \mathcal{P}$. Therefore,

- 1) for any $\alpha \in \Pi_{\lambda}$, we have $\langle \lambda | \alpha \rangle = 1$ (see Assertion 3.1);
- 2) in the Dynkin diagram of the simple root system Π , the path between the vertices corresponding to two distinct roots from the subset $\Pi_{\lambda} \subset \Pi$ has at least $r - 2$ edges.

Hence, either $\lambda \in \{\varphi_1, \dots, \varphi_r\} \cup \{\varphi_1 + \varphi_r, \varphi_1 + \varphi_{r-1}, \varphi_2 + \varphi_r\}$ or $r = 3$ and $\lambda = \varphi_1 + \varphi_2 + \varphi_3$. At the same time, $\lambda \notin \{\varphi_3, \dots, \varphi_{r-2}\} \cup \{\varphi_1 + \varphi_{r-1}, \varphi_2 + \varphi_r\}$. By (3.1), $r = 3$ and $\lambda = \varphi_1 + \varphi_2 + \varphi_3$. The representation R of $\mathfrak{g}_{\mathbb{C}}$ is orthogonal, and therefore $\delta = 1$. By Proposition 2.31 and Corollary 2.13, the quotient V/G is not a smooth manifold. On the other hand, V/G is a smooth manifold. This is a contradiction.

Thus, we have established that $\lambda \in \{\varphi_3, \dots, \varphi_{r-2}\}$.

We have $\lambda = \varphi_j$, $j \in \mathbb{N}$, $3 \leq j \leq r - 2$, whence $r \geq 5$. By (3.2), $j \leq \frac{n}{2}$.

If $r = 5$, then $j = 3$, $\lambda = \varphi_3$, contrary to (3.1).

Thus, $r \geq 6$, $3 \leq j \leq \frac{n}{2}$, and $\lambda = \varphi_j$. Moreover, by (3.1), $(r = 7) \Rightarrow (\lambda \neq \varphi_4)$.

Case 1. $r = 6$ and $\lambda = \varphi_3$.

The representation R of $\mathfrak{g}_{\mathbb{C}}$ is not self-adjoint. Therefore $\delta = 2$. Applying Proposition 2.32 and Corollary 2.13, we obtain a contradiction with the fact that V/G is a smooth manifold.

Case 2. $r = 7$ and $\lambda = \varphi_3$.

The representation R of $\mathfrak{g}_{\mathbb{C}}$ is not self-adjoint. Hence, $\delta = 2$. Using Proposition 2.33 and Corollary 2.13, we obtain a contradiction with the fact that V/G is a smooth manifold.

Case 3. $r \geq 8$.

The simple root system $\Pi' := (1, \dots, r - 2) \subset \Pi$, belonging to the family $\mathcal{P}$, does not satisfy (3.3) and, by Corollary 3.4, satisfies (3.6). In particular, $\Pi_{\lambda} \cap (\Pi' \setminus (\partial\Pi')) = \emptyset$. Moreover, $\Pi' \cong A_{r-2}$ and $r - 2 \geq 6$, whence $\partial\Pi' = (1, 2, r - 3, r - 2) \subset \Pi$ and $\Pi' \setminus (\partial\Pi') = (3, \dots, r - 4) \subset \Pi$. Furthermore,

$$j \leq \frac{n}{2} = n - \frac{n}{2} < n - \frac{r}{2} \leq n - 4 = r - 3, \quad 3 \leq j \leq r - 4,$$

and also $\Pi_{\lambda} = (j) \subset \Pi$. Hence, $\Pi_{\lambda} \cap (\Pi' \setminus (\partial\Pi')) = (j) \neq \emptyset$. We have obtained a contradiction. Thus we have a complete (“contrapositive”) proof of Theorem 1.1. $\square$

§ 4. PARTICULAR CASES

In this section, we prove Theorem 1.2.

As before, we shall use the notation and the assumptions from §1.

Set $n := r + 1 \in \mathbb{N}$.

Suppose the linear Lie group $G \subset \mathbf{GL}(V)$ is connected and the linear algebra $R(\mathfrak{g}_{\mathbb{C}})$ is isomorphic to one of the following linear algebras:

1) $\text{ad}(\mathfrak{g}_{\mathbb{C}})$;	5) $\varphi_2(A_3) \cong \varphi_1(D_3)$;
2) $\varphi_1(A_r)$ ($r > 1$);	6) $(2\varphi_2)(A_3) \cong (2\varphi_1)(D_3)$;
3) $(2\varphi_1)(A_r)$ ($r > 1$);	7) $\varphi_3(A_5)$;
4) $\varphi_2(A_r)$ ($r > 3$);	8) $\varphi_4(A_7)$.

First, consider cases 1), 2), 5), 6) and 8).

The following representations of the complex simple Lie groups are polar (see [6, §3]):

- the adjoint representation of an arbitrary complex simple Lie group;
- the representation $R_{\varphi_1} + R'_{\varphi_1}$ of the complex simple Lie group $\mathbf{SL}_n(\mathbb{C})$;
- the representation R_{φ_1} and $R_{2\varphi_1}$ of the complex simple Lie group $\mathbf{SO}_6(\mathbb{C})$;
- the representation R_{φ_4} of the complex simple Lie group $\mathbf{SL}_8(\mathbb{C})$.

Hence, in each of the cases 1), 2), 5), 6), and 8) the linear Lie group $G \subset \mathbf{GL}(V)$ is polar and, by Lemma 2.16, the quotient V/G is homeomorphic to a closed half-space (in particular, it is not a manifold).

Now we consider case 7).

Suppose that the linear group $G \subset \mathbf{GL}(V)$ is connected and the linear algebra $R(\mathfrak{g}_{\mathbb{C}})$ is isomorphic to the linear algebra $\varphi_3(A_5)$.

Consider an n -dimensional Hermitian space $\mathbb{E}$. It admits a direct sum decomposition into two orthogonal three-dimensional subspaces $\mathbb{E}_+$ and $\mathbb{E}_-$.

It is clear that the tautological representation $G : V$ is isomorphic to the natural action $\mathbf{SU}(\mathbb{E}) : \mathbb{E}^{\wedge 3}$. Without loss of generality we may assume that $G = \mathbf{SU}(\mathbb{E})$, $V = \mathbb{E}^{\wedge 3}$, and the tautological representation $G : V$ coincides with the natural action $\mathbf{SU}(\mathbb{E}) : \mathbb{E}^{\wedge 3}$.

As is easily seen, the space V decomposes into a direct sum of the subspaces $V_j := \mathbb{E}_+^{\wedge(3-j)} \wedge \mathbb{E}_-^{\wedge j}$ ($j = 0, 1, 2, 3$), and $V_0 = \mathbb{C}v$, $v \in V_0 \setminus \{0\}$. Furthermore, $G_v = \mathbf{SU}(\mathbb{E}_+) \times \mathbf{SU}(\mathbb{E}_-) \subset G$, where $\mathbf{SU}(\mathbb{E}_{\pm}) := \{g \in G : \mathbb{E}^g \supset \mathbb{E}_{\mp}\} \subset G$. As a result, $G_v V_j = V_j$ ($j = 0, 1, 2, 3$), $\mathfrak{g}v = (i\mathbb{R}v) \oplus V_1$, and $V = (\mathfrak{g}v) \oplus (\mathbb{R}v) \oplus V_2 \oplus V_3$, $V_3 \subset V^{G_v}$, $\dim_{\mathbb{C}} V_3 = 1$. Therefore, the representation $G_v : N_v$ is isomorphic to the direct sum of the trivial action $G_v : \mathbb{R}^3$ and the representation $G_v : V_2$. The latter is, in turn, isomorphic to the natural action $(\mathbf{SU}(\mathbb{E}_+) \times \mathbf{SU}(\mathbb{E}_-)) : (\mathbb{E}_+ \otimes \mathbb{E}_-^{\wedge 2})$ and, equivalently, to the natural action $(\mathbf{SU}(\mathbb{E}_+) \times \mathbf{SU}(\mathbb{E}_-)) : (\mathbb{E}_+ \otimes \mathbb{E}_-^*)$. Now, setting $d := m := 3 \in \mathbb{N}$ and applying Assertion 2.10 to the representation $G_v : N_v$, we have that the quotient of that representation is not a manifold. By Lemma 2.1, neither is the quotient V/G .

Now consider the remaining cases 3) and 4).

Henceforth, we shall assume that the linear group $G \subset \mathbf{GL}(V)$ is connected, and the linear algebra $R(\mathfrak{g}_{\mathbb{C}})$ is isomorphic to one of the linear algebras $(2\varphi_1)(A_r)$ ($r > 1$) and $\varphi_2(A_r)$ ($r > 3$).

Set $\delta := 1 \in \mathbb{R}$ (respectively, $\delta := -1 \in \mathbb{R}$) if the linear algebra $R(\mathfrak{g}_{\mathbb{C}})$ is isomorphic to the linear algebra $(2\varphi_1)(A_r)$ (respectively, the linear algebra $\varphi_2(A_r)$).

Suppose $\delta = -1$ and $n \in 2\mathbb{Z} + 1$.

The representation $R_{\varphi_2} + R'_{\varphi_2}$ of the complex simple Lie group $\mathbf{SL}_n(\mathbb{C})$ is polar (see [6, §3]). By Lemma 2.16, the quotient V/G is homeomorphic to a closed half-space and, therefore, is not a manifold.

Henceforth we shall assume that $(\delta = 1) \vee (n \in 2\mathbb{Z})$.

Consider an n -dimensional Hermitian space $\mathbb{E}$ with scalar multiplication $f_0(\cdot, \cdot)$, the complex space B of all bilinear forms on $\mathbb{E}$ and the subspace $B_{\delta} \subset B$ of all forms $f \in B$ such that $f(x, y) = \delta \cdot f(y, x)$ ($x, y \in \mathbb{E}$).

The tautological representation $G : V$ is isomorphic to the representation

$$(4.1) \quad \begin{aligned} \mathbf{SU}(\mathbb{E}) : B_{\delta}, \quad (gf)(x, y) &:= f(g^{-1}x, g^{-1}y) \in \mathbb{C} \\ (g \in \mathbf{SU}(\mathbb{E}), \quad f \in B_{\delta}, \quad x, y \in \mathbb{E}). \end{aligned}$$

There is a form $f_1 \in B_\delta$ such that the (fixed) orthonormal basis of the Hermitian space $\mathbb{E}$ is orthonormal when $\delta = 1$ and symplectic when $\delta = -1$. Moreover, there is clearly an antilinear operator $F: \mathbb{E} \rightarrow \mathbb{E}$ such that $F^2 = \delta E \in \mathbf{U}(\mathbb{E})$ and $f_1(x, y) = f_0(Fx, y)$ ($x, y \in \mathbb{E}$).

Let θ, θ_0 , and θ_1 be involutive automorphisms of the *real* Lie group $\mathbf{GL}_\mathbb{C}(\mathbb{E})$, uniquely determined by the relations

$$\theta(g)F = Fg \quad \text{and} \quad f_i(\theta_i(g)x, gy) = f_i(x, y) \quad (i = 0, 1, g \in \mathbf{GL}_\mathbb{C}(\mathbb{E}), x, y \in \mathbb{E}).$$

For arbitrary $g \in \mathbf{GL}_\mathbb{C}(\mathbb{E})$ and $x, y \in \mathbb{E}$, we have

$$f_0(F\theta_1(g)x, gy) = f_1(\theta_1(g)x, gy) = f_1(x, y) = f_0(Fx, y) = f_0(\theta_0(g)Fx, gy),$$

whence $F\theta_1(g) = \theta_0(g)F$. Therefore, $\theta_0 \equiv \theta \circ \theta_1$ and $\theta_0 \circ \theta_1 \equiv \theta_1 \circ \theta_0 \equiv \theta$. The real Lie algebra $\text{Lie}(\mathbf{GL}_\mathbb{C}(\mathbb{E})) = \mathfrak{gl}_\mathbb{C}(\mathbb{E})$ has commuting involutive automorphisms $\sigma_i := d\theta_i$ ($i = 0, 1$). It is easy to see that $f_i(\sigma_i(\xi)x, y) + f_i(x, \xi y) = 0$ ($i = 0, 1, \xi \in \mathfrak{gl}_\mathbb{C}(\mathbb{E}), x, y \in \mathbb{E}$). Hence, the automorphism σ_0 (respectively, σ_1) of the real Lie algebra $\mathfrak{gl}_\mathbb{C}(\mathbb{E})$ is antilinear (respectively, linear) over $\mathbb{C}$, and the subspace $(\mathfrak{gl}_\mathbb{C}(\mathbb{E}))^{-\sigma_1} \subset \mathfrak{gl}_\mathbb{C}(\mathbb{E})$ contains the subspace $\mathbb{C}E \subset \mathfrak{gl}_\mathbb{C}(\mathbb{E})$.

It is clear that $(\mathbf{GL}_\mathbb{C}(\mathbb{E}))^{\theta_0} = \mathbf{U}(\mathbb{E}) \subset \mathbf{GL}_\mathbb{C}(\mathbb{E})$. Furthermore, the map

$$S: \mathfrak{gl}_\mathbb{C}(\mathbb{E}) \rightarrow B, \quad (S(\xi))(x, y) := f_1(x, \xi y) \in \mathbb{C} \quad (\xi \in \mathfrak{gl}_\mathbb{C}(\mathbb{E}), x, y \in \mathbb{E})$$

is an isomorphism of complex linear spaces. Moreover, for all $g \in \mathbf{SU}(\mathbb{E})$, $\xi \in \mathfrak{gl}_\mathbb{C}(\mathbb{E})$, and $x, y \in \mathbb{E}$ we have

$$\begin{aligned} (S(\xi))(g^{-1}x, g^{-1}y) &= f_1(g^{-1}x, \xi g^{-1}y) = f_1(x, \theta_1(g)\xi g^{-1}y) \\ &= (S(\theta_1(g)\xi g^{-1}))(x, y); \\ \delta \cdot (S(\xi))(y, x) &= \delta \cdot f_1(y, \xi x) = f_1(\xi x, y) \\ &= -f_1(x, \sigma_1(\xi)y) = -(S(\sigma_1(\xi)))(x, y). \end{aligned}$$

This implies that

$$S^{-1}(B_\delta) = (\mathfrak{gl}_\mathbb{C}(\mathbb{E}))^{-\sigma_1} \subset \mathfrak{gl}_\mathbb{C}(\mathbb{E}),$$

and the representation (4.1) is isomorphic to the representation

$$(4.2) \quad \mathbf{SU}(\mathbb{E}) : (\mathfrak{gl}_\mathbb{C}(\mathbb{E}))^{-\sigma_1}, \quad g: \xi \rightarrow \theta_1(g)\xi g^{-1}.$$

As we already mentioned, the tautological representation $G: V$ is isomorphic to the representation (4.1), and therefore to the representation (4.2). Without loss of generality we may assume that $G = \mathbf{SU}(\mathbb{E}) \subset \mathbf{GL}_\mathbb{C}(\mathbb{E})$, $V = (\mathfrak{gl}_\mathbb{C}(\mathbb{E}))^{-\sigma_1} \subset \mathfrak{gl}_\mathbb{C}(\mathbb{E})$, and the tautological representation $G: V$ coincides with the representation (4.2).

The commuting automorphisms θ_0 and θ_1 of the Lie group $\mathbf{GL}_\mathbb{C}(\mathbb{E})$ preserve the commutant $\mathbf{SL}_\mathbb{C}(\mathbb{E})$ and, therefore, the subgroup $(\mathbf{SL}_\mathbb{C}(\mathbb{E}))^{\theta_0} = \mathbf{SL}_\mathbb{C}(\mathbb{E}) \cap \mathbf{U}(\mathbb{E}) = G$.

The commuting involutive automorphisms σ_0 and σ_1 of $\mathfrak{gl}_\mathbb{C}(\mathbb{E})$ preserve the center $\mathbb{C}E$ and the commutant $\mathfrak{sl}_\mathbb{C}(\mathbb{E})$, which allows us to decompose this algebra into a direct sum of σ_1 -invariant subspaces $\mathbb{C}E$, $(\mathfrak{sl}_\mathbb{C}(\mathbb{E}))^{\sigma_0}$, and $(\mathfrak{sl}_\mathbb{C}(\mathbb{E}))^{-\sigma_0}$. Furthermore, since $(\mathbf{SL}_\mathbb{C}(\mathbb{E}))^{\theta_0} = G \subset \mathbf{GL}_\mathbb{C}(\mathbb{E})$ we have that $(\mathfrak{sl}_\mathbb{C}(\mathbb{E}))^{\sigma_0} = \mathfrak{g} \subset \mathfrak{gl}_\mathbb{C}(\mathbb{E})$. Since $\mathbb{C}(\mathfrak{sl}_\mathbb{C}(\mathbb{E})) = \mathfrak{sl}_\mathbb{C}(\mathbb{E}) \subset \mathfrak{gl}_\mathbb{C}(\mathbb{E})$, and σ_0 is an antilinear over $\mathbb{C}$ automorphism of $\mathfrak{gl}_\mathbb{C}(\mathbb{E})$, we have

$$(\mathfrak{sl}_\mathbb{C}(\mathbb{E}))^{-\sigma_0} = i \cdot (\mathfrak{sl}_\mathbb{C}(\mathbb{E}))^{\sigma_0} = i\mathfrak{g} \subset \mathfrak{gl}_\mathbb{C}(\mathbb{E}).$$

By the above,

$$V = (\mathfrak{gl}_\mathbb{C}(\mathbb{E}))^{-\sigma_1} = (\mathbb{C}E)^{-\sigma_1} \oplus \mathfrak{g}^{-\sigma_1} \oplus (i\mathfrak{g})^{-\sigma_1} \subset \mathfrak{gl}_\mathbb{C}(\mathbb{E})$$

and $(\sigma_1 - \text{id}_{\mathfrak{g}})\mathfrak{g} = \mathfrak{g}^{-\sigma_1} \subset \mathfrak{g}$. Since

$$(\mathbb{C}E)^{-\sigma_1} = \mathbb{C}E \subset \mathfrak{gl}_{\mathbb{C}}(\mathbb{E}) \quad \text{and} \quad \sigma_1 \in \text{Aut}_{\mathbb{C}}(\mathfrak{gl}_{\mathbb{C}}(\mathbb{E})),$$

we have that

$$(4.3) \quad V = (\mathbb{C}E) \oplus \mathfrak{g}^{-\sigma_1} \oplus i(\mathfrak{g}^{-\sigma_1}) \subset \mathfrak{gl}_{\mathbb{C}}(\mathbb{E}).$$

It is clear that $v := E \in V$. Moreover (see (4.2)),

$$G_v = G^{\theta_1} \subset G, \quad \mathfrak{g}v = (\sigma_1 - \text{id}_{\mathfrak{g}})\mathfrak{g} = \mathfrak{g}^{-\sigma_1} \subset \mathfrak{g},$$

and the action $G_v : V$ is given by $g : \xi \rightarrow g\xi g^{-1}$. By (4.3), the representation $G_v : N_v$ is isomorphic to the direct sum of the trivial action $G^{\theta_1} : \mathbb{R}^2$ and the adjoint action $G^{\theta_1} : \mathfrak{g}^{-\sigma_1}$.

By Lemma 2.17, the quotient $\mathfrak{g}^{-\sigma_1}/G^{\theta_1}$ is homeomorphic to a closed half-space. Moreover, $N_v/G_v \cong \mathbb{R}^2 \times (\mathfrak{g}^{-\sigma_1}/G^{\theta_1})$ and, therefore, the quotient N_v/G_v is also homeomorphic to a closed half-space.

Thus we have established that the quotient N_v/G_v is not a manifold. By Lemma 2.1, neither is the quotient V/G .

Therefore, in each of cases 3) and 4) the quotient V/G is not a manifold.

Now Theorem 1.2 is fully proved. $\square$

Theorems 1.1 and 1.2 immediately imply Corollary 1.3.

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