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MULTIPOINT HERMITE–PADÉ APPROXIMANTS FOR THREE BETA FUNCTIONS

V. N. SOROKIN

Abstract. This paper is concerned with joint multipoint rational approximants with a common denominator for three beta functions. The limit distributions of the zeros of the denominators are obtained in terms of equilibrium logarithmic potentials and in terms of meromorphic functions on Riemann surfaces.

1. Introduction

Let r be a fixed natural number, and let $\alpha_1, \dots, \alpha_r$ be complex numbers pairwise incongruent modulo 1 and distinct from integers. Consider the family of functions

$$f_j(z) = \mathrm{B}(z, \alpha_j) = \frac{\Gamma(z)\Gamma(\alpha_j)}{\Gamma(z+\alpha_j)}, \quad j = 1, \dots, r, \tag{1.1}$$

where B is the beta function and Γ is the gamma function. These functions are meromorphic functions with simple poles at the points $z = -m$, where $m \in \mathbb{Z}_+$.

Let β be a fixed complex number. Given the functions (1.1), we pose the following multipoint Hermite–Padé approximation problem.

Problem 1.1. Given a nonnegative integer number n, find a nonzero polynomial Q_n of degree no greater than rn and polynomials $P_{n,1}, \dots, P_{n,r}$ of degree no greater than rn such that the following interpolation conditions are satisfied:

$$R_{n,j}(z) = Q_n(z) f_j(z) - P_{n,j}(z) = 0, \quad z = \beta, \beta+1, \dots, \beta + rn + n, \quad j = 1, \dots, r.$$

Problem 1.1 can be reduced to a system of $r(rn+n+1)$ homogeneous linear equations in $(r+1)(rn+1)$ unknown coefficients of polynomials, and this problem has a nontrivial solution. Problem 1.1 was studied for $r = 1$ in [1] and for $r = 2$ in [2]. From the arguments in these papers it follows that Problem 1.1 has a unique solution (up to normalization).

The papers [1] and [2] were mainly concerned with finding the limit measure for the distributions of the zeros of the polynomials Q_n after scaling. Namely, let

$$Q_n^*(z) = C_n Q_n(nz), \quad n \in \mathbb{Z}_+,$$

where C_n is a normalizing constant such that the leading coefficient of the polynomial Q_n^* is 1. We let Z_n denote the zero set of the polynomial Q_n^* and denote by λ_n the counting measure for these zeros; that is,

$$\lambda_n = \frac{1}{n} \sum_{\zeta \in \mathsf{Z}_n} \delta_\zeta,$$

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where δ_ζ is the unit measure at the point ζ.

In [1], [2] it was shown that the zero counting measures converge in the weak-$*$ topology,

$$\lambda_n \twoheadrightarrow \lambda. \tag{1.2}$$

The limit measures λ were characterized in terms of equilibrium problems of logarithmic potential theory (see [3] and [4]). Recall that the logarithmic potential of a positive finite Borel measure μ with support $\mathsf{S}(\mu)$ in the complex plane is defined as the Lebesgue integral

$$V^\mu(x) = \int \log \frac{1}{|x-t|}\, d\mu(t), \quad x \in \mathbb{C},$$

which can assume the value $+\infty$. In our setting, (1.2) is equivalent to the asymptotic formula

$$\Big(-\frac{1}{n}\Big) \log |Q_n^*(x)| \longrightarrow V^\lambda(x) \quad \text{as} \quad n \to \infty,$$

in which the convergence is uniform on compact subsets inside the domain $\mathbb{C} \setminus \mathsf{S}(\lambda)$.

The above results were proved using classical asymptotic methods. In both cases, new effects were discovered. For $r = 1$, Problem 1.1 can be reduced to orthogonality relations with variable weight with respect to a discrete measure with unbounded support. The measure λ, which satisfies an equilibrium condition in an external field and with a constraint, is supported on an infinite interval of the real line. For $r = 2$, the functions f_1 and f_2 form a Nikishin system (under certain restrictions on the parameters). Both measures involved in the definition of a Nikishin system are discrete and have unbounded support. This seems to be the first example of this type. Apart from the measure λ, the equilibrium problem involves the limit measure of the distribution of additional interpolation points. Both these measures have noncompact support and satisfy the constraints.

The purpose of this paper is to examine the solutions to Problem 1.1 for $r = 3$. This setting is not a mere generalization — in the case under consideration the functions f_1, f_2, f_3 no longer form a Nikishin system. This is a new effect, which we first discovered in [5]. It turns out that a part of the additional interpolation points escapes from the real line into the complex plane and the structure of the equilibrium problem changes.

To conclude this section, we give the solution to Problem 1.1 in an explicit form. We set

$$\begin{aligned}
\omega_n(x) &= (x-\beta)\dots(x-(\beta+rn+n)),\\
\omega_n^*(x) &= (x-(\beta+rn))\dots(x-(\beta+rn+n)),\\
g_\alpha(x) &= \frac{\Gamma(1-\alpha-x)}{\Gamma(1-x)},
\end{aligned}$$

and consider the difference operator $(\boldsymbol{\Delta} f)(x) = f(x+1) - f(x)$.

Proposition 1.1. *The following analogue of Rodrigues's formula holds:*

$$\frac{Q_n(x)}{\omega_n(x)} = \Big(\prod_{j=1}^{r} \mathcal{D}_{\alpha_j}^{(n)}\Big) \frac{1}{\omega_n^*(x)}, \quad n \in \mathbb{Z}_+, \tag{1.3}$$

where

$$\mathcal{D}_\alpha^{(n)} = \frac{1}{g_\alpha(x)} \frac{1}{n!}\, \boldsymbol{\Delta}^n g_{\alpha-n}(x). \tag{1.4}$$

Moreover, the operators in formula (1.3) *commute.*

The proof of Proposition 1.1 for arbitrary r paraphrases the proof given in [2] for $r = 2$ and is omitted.

2. Main results

Fix a number $x_* > 0$ and a pair of complex conjugates ζ_+ and ζ_- and connect the point x_* with the points ζ_+ and ζ_- by simple analytic arcs γ_+ and γ_- lying in the open upper and lower half-planes, respectively (apart from their common point x_*), and which are symmetric with respect to the real line. Set $\gamma = \gamma_+ \cup \gamma_-$. The class of all such curves γ will be denoted by $\daleth$. It is worth pointing out that in the definition of this class both the point x_* and the points $\zeta_\pm$ are parameters that can vary.

Fix a curve $\gamma \in \daleth$ and a number $\mathsf{t} \in (0, 1)$. We pose the following equilibrium problem in logarithmic potential theory.

Problem 2.1. Find four positive Borel measures λ_Δ, λ_F, λ_Γ, λ_T such that

(1) the supports of these measures satisfy the conditions

$$\mathsf{S}(\lambda_\Delta) \subset \Delta = (-\infty, 0], \qquad \mathsf{S}(\lambda_F) \subset F = [0, +\infty),$$
$$\mathsf{S}(\lambda_\Gamma) \subset \gamma_* = \gamma \cup [0, x_*], \quad \mathsf{S}(\lambda_T) \subset \Delta;$$

(2) the total variations of these measures are such that

$$\|\lambda_\Delta\| = 3, \quad \|\lambda_F\| = 1 + \mathsf{t}, \quad \|\lambda_\Gamma\| = 1 - \mathsf{t}, \quad \|\lambda_T\| = \mathsf{t};$$

(3) the following constraints are satisfied

$$\lambda_\Delta \leqslant \chi_\Delta, \qquad \lambda_P\big|_F \leqslant 2\chi_F, \quad \lambda_P = \lambda_F + \lambda_\Gamma,$$

where χ_Δ and χ_F are the classical Lebesgue measures on the intervals Δ and F, respectively;

(4) the following equilibrium conditions are satisfied for some constants w_Δ, w_F, w_Γ, w_T:

1° on the interval Δ:

$$W_\Delta = 2V^{\lambda_\Delta} - V^{\lambda_P} - V^{\lambda_\omega} \begin{cases} \leqslant w_\Delta \mid \mathsf{S}(\lambda_\Delta), \\ \geqslant w_\Delta \mid \Delta \setminus \mathsf{Z}_\Delta, \end{cases}$$

where

$$\mathsf{Z}_\Delta = \mathsf{S}(\lambda_\Delta) \setminus \mathsf{S}(\chi_\Delta - \lambda_\Delta)$$

is the saturation region of the measure λ_Δ and where the logarithmic potential of the classical Lebesgue measure λ_ω on the interval $[0, 4]$ plays the role of the external field $\phi = -V^{\lambda_\omega}$;

2° on the interval F:

$$W_F = -V^{\lambda_\Delta} + 2V^{\lambda_F} + V^{\lambda_\Gamma} - V^{\lambda_T} \begin{cases} \leqslant w_F \mid \mathsf{S}(\lambda_F), \\ \geqslant w_F \mid F_* \setminus \mathsf{Z}_F, \end{cases}$$

where $F_* = [x^*, +\infty)$, $x^* = \min\{x_*, \inf \mathsf{S}(\lambda_F)\}$, and Z_F is the saturation region of the measure λ_F, that is,

$$\mathsf{Z}_F = \mathsf{S}(\lambda_F) \setminus \mathsf{S}(2\chi_F - \lambda_P\big|_F);$$

3° on the set γ_*:

$$W_\Gamma = -V^{\lambda_\Delta} + V^{\lambda_F} + 2V^{\lambda_\Gamma} + V^{\lambda_T} \begin{cases} \leqslant w_\Gamma \mid \mathsf{S}(\lambda_\Gamma), \\ \geqslant w_\Gamma \mid \gamma_* \setminus \mathsf{Z}_\Gamma, \end{cases}$$

where Z_Γ is the saturation region of the measure λ_Γ, that is,

$$\mathsf{Z}_\Gamma = \mathsf{S}(\lambda_\Gamma) \setminus \mathsf{S}(2\chi_F - \lambda_P\big|_F);$$

4° on the support of the measure λ_T:

$$W_T = -V^{\lambda_F} + V^{\lambda_\Gamma} + 2V^{\lambda_T} \begin{cases} \leqslant w_T \mid \mathsf{S}(\lambda_T), \\ \geqslant w_T \mid \Delta. \end{cases}$$

Gonchar and Rakhmanov [6]–[9] studied equilibrium problems like Problem 2.1, but for measures with compact support. In [6]–[9], it was first shown that the solution of these problems exists and is unique. This result was then applied to study the asymptotics of various extremal polynomials. There are no such general results for measures with noncompact support. However, in what follows we exploit the opposite approach. We first use classical asymptotic methods to find the asymptotic behaviour of the polynomials and then show that the resulting limit measures are solutions to Problem 2.1. Below we will not look at the question of whether the solution of this problem is unique.

Proposition 2.1. *Problem* 2.1 *has a solution.*

Definition. A curve $\Gamma \in \beth$ will be called *extremal* if
(1) $\mathsf{S}(\lambda_\Gamma) = \Gamma_*$,
(2) $\mathsf{Z}_\Gamma = [0, x_*]$,
(3) the curve Γ has the *S-property*; that is,

$$\frac{\partial W_\Gamma}{\partial \vec{n}_+} = \frac{\partial W_\Gamma}{\partial \vec{n}_-}$$

on the curve Γ, where $\vec{n}_\pm$ are the unit normal vectors to two edges of this curve.

Proposition 2.2. *In the class $\beth$ there exists a unique extremal curve Γ together with the corresponding value of the parameter* t.

We skip the proof of the uniqueness of an extremal curve, simply referring the reader to [10] and [11].

Recall that we are dealing with the polynomials Q_n, which solve Problem 1.1 for $r = 3$. The next theorem is one of the main results in this paper.

Theorem 2.1. *The measure λ_Δ, involved in the solution of Problem* 2.1 *for the extremal curve Γ, is the limit measure for the distribution of the zeros of the scaled polynomials Q_n^*.*

We write out the limit measure in an explicit form. Consider the polynomial

$$\mathsf{Q}(x) = 2^8 \cdot 3^3 x^3 + 3^2 \cdot 7 \cdot 53 x^2 - 2^4 \cdot 5 \cdot 19 x + 2^5 \cdot 3^2. \tag{2.1}$$

This polynomial has one negative root and two complex conjugate roots, namely

$$x_0 = -0.815\ldots, \qquad x_\pm = 0.166\ldots \pm i \cdot 0.153\ldots.$$

Proposition 2.3. *The end-points $\zeta_\pm$ of the extremal curve Γ are the points $x_\pm$.*

We denote the open upper and lower half-planes by $\mathbb{C}_+$ and $\mathbb{C}_-$, respectively. The four-sheeted Riemann surface $\mathfrak{R}$ is constructed by gluing the following sheets:

$$\begin{aligned} \mathfrak{R}_\Delta &= \mathbb{C} \setminus (-\infty, x_0], \\ \mathfrak{R}_P &= \mathbb{C} \setminus ((-\infty, x_0] \cup \Gamma \cup [x_*, +\infty)), \\ \mathfrak{R}_F &= \mathbb{C}_+ \cup \mathbb{C}_-, \\ \mathfrak{R}_T &= \mathbb{C} \setminus ((-\infty, x_*] \cup \Gamma). \end{aligned}$$

The sheets $\mathfrak{R}_\Delta$ and $\mathfrak{R}_P$ are glued along the cut $(-\infty, x_0]$, the sheets $\mathfrak{R}_P$ and $\mathfrak{R}_F$ along the cut $[x_*, +\infty)$, the sheets $\mathfrak{R}_F$ and $\mathfrak{R}_T$ along the cut $(-\infty, x_*]$, and the sheets $\mathfrak{R}_T$ and $\mathfrak{R}_P$ along the cut Γ. The surface $\mathfrak{R}$ has genus zero.

On the Riemann surface $\mathfrak{R}$ consider the meromorphic function

$$\theta : \mathfrak{R} \longrightarrow \bar{\mathbb{C}}.$$

This surface is uniquely defined by its divisor and the normalization condition. The function θ has a third-order zero at the point $x = 0$ on the sheet $\mathfrak{R}_P$ and a first-order zero at the point $x = 4$ on the sheet $\mathfrak{R}_\Delta$. It also has a second-order pole at the point $x = 0$ on the sheet $\mathfrak{R}_\Delta$ and first-order poles at the point $x = 0$ on the sheets $\mathfrak{R}_F$ and $\mathfrak{R}_T$. The normalization condition is as follows: $\theta = 1$ at the point $x = \infty$.

The next theorem is the second main result in the paper.

Theorem 2.2. *The Markov function*

$$h_{\lambda_\Delta}(x) = \int \frac{d\lambda_\Delta(s)}{x - s}, \quad x \in \mathbb{C} \setminus \mathsf{S}(\lambda_\Delta),$$

of the limit measure λ_Δ can be written the form

$$h_{\lambda_\Delta}(x) = \log\Big\{\theta_\Delta(x) \cdot \frac{x}{x-4}\Big\}. \tag{2.2}$$

This function is holomorphic in the domain $\mathbb{C}\setminus\Delta$. The branch of the logarithm is chosen so as to have

$$h_{\lambda_\Delta}(x) \sim \frac{3}{x} \quad as \quad x \to \infty.$$

The support of the measure $\lambda_\Delta$is the whole of the interval Δ. Its saturation region is the interval $[x_0, 0]$.

Proposition 2.4. *The algebraic function θ satisfies the equation*

$$x(p+1)^4 + p(6p^2 + 4p + 1) = 0, \quad p = x(\theta - 1). \tag{2.3}$$

The proofs of the main results of the paper will be given in the next sections.

3. The first auxiliary problem

In accordance with Rodrigues's formula (1.3), we shall first examine the polynomials $\widetilde{Q}_n(\eta)$ defined by the formula

$$\frac{\widetilde{Q}_n(\eta)}{\widetilde{\omega}_n(\eta)} = \mathcal{D}_{\alpha_1}^{(n)} \frac{1}{\omega_n^*(\eta)}, \quad n \in \mathbb{Z}_+, \tag{3.1}$$

where

$$\begin{aligned} \omega_n^*(\eta) &= (\eta - (\beta + 3n)) \dots (\eta - (\beta + 4n)), \\ \widetilde{\omega}_n(\eta) &= (\eta - (\beta + 2n)) \dots (\eta - (\beta + 4n)). \end{aligned} \tag{3.2}$$

These polynomials solve the following problem.

Problem 3.1. Given a nonnegative integer number n, find a nonzero polynomial $\widetilde{Q}_n$ of degree at most n and a polynomial $\widetilde{P}_n$ of degree at most n such that the following interpolation conditions are satisfied:

$$\widetilde{R}_n(\eta) = \widetilde{Q}_n(\eta) f_{\alpha_1}(\eta) - \widetilde{P}_n(\eta) = 0, \quad \eta = \beta + 2n, \dots, \beta + 4n. \tag{3.3}$$

Let $\beta > 0$, and let $0 < \alpha_1 < 1$. (From the analysis that follows it will be clear that the final result is independent of the choice of these parameters.) The interpolation conditions (3.3) are equivalent to the orthogonality conditions with variable weight

$$\int_{-\infty}^0 \widetilde{Q}_n(\xi)\, \xi^l \,\frac{d\mu^{\alpha_1}(\xi)}{\widetilde{\omega}_n(\xi)} = 0, \quad l = 0, \dots, n-1,$$

where μ^{α_1} is a discrete measure with masses

$$\mu_m^{\alpha_1} = \frac{(1-\alpha_1)_m}{m!} \tag{3.4}$$

at the points $(-m)$, $m \in \mathbb{Z}_+$. Here

$$(p)_m = p(p+1)\dots(p+m-1)$$

is the Pochhammer symbol.

We set

$$\widetilde{Q}_n^*(\eta) = \widetilde{C}_n \widetilde{Q}_n(n\eta), \quad n \in \mathbb{Z}_+,$$

where $\widetilde{C}_n$ is a normalizing constant such that the leading coefficient of the polynomial $\widetilde{Q}_n^*$ is 1. We denote the zero counting measure of this polynomial by $\tilde{\lambda}_n$. We pose the following equilibrium problem.

Problem 3.2. Find a positive measure $\tilde{\lambda}$ such that

(1) the support of $\tilde{\lambda}$ lies on the interval $\Delta = (-\infty, 0]$; that is, $\mathsf{S}(\tilde{\lambda}) \subset \Delta$;

(2) the total variation of the measure $\tilde{\lambda}$ is 1; that is, $\|\tilde{\lambda}\| = 1$;

(3) the measure $\tilde{\lambda}$ satisfies the constraint

$$\tilde{\lambda} \leqslant \chi_\Delta, \tag{3.5}$$

where χ_Δ is the classical Lebesgue measure on the interval Δ;

(4) the logarithmic potential of the measure $\tilde{\lambda}$ satisfies the equilibrium conditions

$$\widetilde{W} = 2V^{\tilde{\lambda}} - V^{\lambda_{\tilde{\omega}}} \begin{cases} \leqslant \widetilde{w} | & \mathsf{S}(\tilde{\lambda}), \\ \geqslant \widetilde{w} | & \Delta \setminus (\mathsf{S}(\tilde{\lambda}) \setminus \mathsf{S}(\chi_\Delta - \tilde{\lambda})), \end{cases} \tag{3.6}$$

where $\widetilde{w}$ is some equilibrium constant and the logarithmic potential $(-V^{\lambda_{\tilde{\omega}}})$ of the classical Lebesgue measure $\lambda_{\tilde{\omega}}$ on the interval $[2, 4]$ plays the role of the external field.

The constraint appears as a result of scaling and passing to the limit in the separation theorem of Chebyshev–Markov–Stieltjes, and the external field arises from the variable weight.

We have the following results similar to those in [1].

Proposition 3.1. *Problem* 3.2 *has a solution.*

Proposition 3.2. *The limit* $\tilde{\lambda}_n \twoheadrightarrow \tilde{\lambda}$ *exists.*

Proposition 3.3. *The density of the measure* $\tilde{\lambda}$ *can be calculated by the formula*

$$\tilde{\lambda}'(\eta) = \begin{cases} 1, & -\dfrac{1}{12} < \eta < 0, \\ \dfrac{1}{\pi} \arccos \dfrac{2\eta^2 - 6\eta - 1}{2|\eta|\sqrt{(2-\eta)(4-\eta)}}, & -\infty < \eta < -\dfrac{1}{12}. \end{cases} \tag{3.7}$$

Proof. Using Cauchy's formula, we rewrite (3.1) as

$$\frac{\widetilde{Q}_n(\eta)}{\widetilde{\omega}_n(\eta)}\, g_{\alpha_1}(\eta) = \frac{1}{2\pi i} \int_l \frac{g_{\alpha_1 - n}(\xi)}{\omega_n^*(\xi)} \cdot \frac{d\xi}{\kappa_n(\xi - \eta)}, \tag{3.8}$$

where

$$\kappa_n(z) = z(z-1)\dots(z-n).$$

For definiteness, we assume that a point η lies in the upper half-plane. Then l is a closed contour lying in the upper half-plane and containing the points

$$\eta,\ \eta+1, \dots, \eta+n.$$

Scaling $\eta \mapsto n\eta$ and changing the variable $\xi \mapsto n\xi$ in formula (3.8), we obtain

$$\frac{\widetilde{Q}_n(n\eta)}{\widetilde{\omega}_n(n\eta)}\, g_{\alpha_1}(n\eta) = \frac{1}{2\pi i}\int_{l^*} \frac{g_{\alpha_1-n}(n\xi)}{\omega_n^*(n\xi)}\,\frac{n\,d\xi}{\kappa_n(n(\xi-\eta))}. \tag{3.9}$$

The contour l^* lies in the upper half-plane and encircles the interval $[\eta, \eta+1]$.

Letting $n \to \infty$, this gives

$$\Bigl(-\frac{1}{n}\Bigr)\log\frac{|\widetilde{\omega}_n(n\eta)|}{n^{2n+1}} \longrightarrow V^{\lambda_{\widetilde{\omega}}}(\eta), \quad \eta \in \mathbb{C}\setminus[2,4],$$
$$\Bigl(-\frac{1}{n}\Bigr)\log\frac{|\omega_n^*(n\xi)|}{n^{n+1}} \longrightarrow V^{\lambda_{\omega^*}}(\xi), \quad \xi \in \mathbb{C}\setminus[3,4],$$
$$\Bigl(-\frac{1}{n}\Bigr)\log\frac{|\kappa_n(nz)|}{n^{n+1}} \longrightarrow V^{\lambda_{\kappa}}(z), \quad z \in \mathbb{C}\setminus[0,1].$$

Here, $\lambda_{\widetilde{\omega}},\ \lambda_{\omega^*},\ \lambda_\kappa$ are the classical Lebesgue measures on the corresponding intervals. The complex potentials of these measures are as follows:

$$\mathcal{V}^{\lambda_{\widetilde{\omega}}}(\eta) = \int_2^4 \log\frac{1}{\eta - s}\,ds = (\eta-4)\log(\eta-4) - (\eta-2)\log(\eta-2) + 2,$$
$$\mathcal{V}^{\lambda_{\omega^*}}(\xi) = \int_3^4 \log\frac{1}{\xi - s}\,ds = (\xi-4)\log(\xi-4) - (\xi-3)\log(\xi-3) + 1,$$
$$\mathcal{V}^{\lambda_{\kappa}}(z) = \int_0^1 \log\frac{1}{z - s}\,ds = (z-1)\log(z-1) - z\log z + 1.$$

By Stirling's formula,

$$\Bigl(-\frac{1}{n}\Bigr)\log g_{\alpha_1-n}(n\xi) - (1-\pi i - \log n) \longrightarrow (\xi-1)\log(\xi-1) - \xi\log\xi \quad \text{as} \quad n\to\infty,$$

where the arguments of the numbers ξ and $\xi-1$ are taken on the interval $(0,\pi)$. The weak limit of the left-hand side of (3.9) is equal to

$$\lim_{n\to\infty}\Bigl(-\frac{1}{n}\Bigr)\log\Bigl|\int_{l^*}\exp\Bigl\{n\widetilde{S}(\xi,\eta)\Bigr\}d\xi\Bigr| \tag{3.10}$$

(up to a normalizing constant), where

$$\begin{aligned}\widetilde{S}(\xi,\eta) = (\xi-4)\log(\xi-4) - (\xi-3)\log(\xi-3) + (\xi-\eta-1)\log(\xi-\eta-1)\\ - (\xi-\eta)\log(\xi-\eta) - (\xi-1)\log(\xi-1) + \xi\log\xi.\end{aligned} \tag{3.11}$$

The limit in (3.10) will be found by the saddle-point method. To find the critical points of the function (3.11), we have

$$\frac{\partial}{\partial\xi}\,\widetilde{S}(\xi,\eta) = \log\frac{(\xi-4)(\xi-\eta-1)\xi}{(\xi-3)(\xi-\eta)(\xi-1)} = 0,$$

which is equivalent to the quadratic equation

$$\xi^2 - \xi - 3\eta = 0.$$

The algebraic function $\xi(\eta)$ has two second-order branch points, namely, $\eta = -1/12$ and $\eta = \infty$. Outside the cut $(-\infty, -1/12]$ we consider the single-valued branch

$$\xi_*(\eta) = \frac{1+\sqrt{1+12\eta}}{2}, \quad \text{where} \quad \sqrt{1+12\eta} > 0, \quad \eta > -\frac{1}{12}.$$

The point ξ_* gives the leading contribution to the asymptotic behavior of integral (3.10). The corresponding critical value is as follows:

$$\widetilde{S}_*(\eta) = \widetilde{S}(\xi_*(\eta),\eta). \tag{3.12}$$

So, the limit measure $\tilde{\lambda}$ exists and its potential is

$$V^{\tilde{\lambda}} = V^{\lambda_{\tilde{\omega}}} - \mathrm{Re}\widetilde{S}_* + \mathrm{const}.$$

The Markov function

$$\tilde{h}(\eta) = \int \frac{d\tilde{\lambda}(s)}{\eta - s}$$

of the measure $\tilde{\lambda}$ is found by the formula

$$\tilde{h} = \frac{\partial}{\partial \eta}\big(\mathrm{Re}\widetilde{S}_* - V^{\lambda_{\tilde{\omega}}}\big).$$

After some calculation we find that

$$\tilde{h}(\eta) = \log\Big\{\frac{\sqrt{1+12\eta} - 2\eta + 1}{\sqrt{1+12\eta} - 2\eta - 1} \cdot \frac{\eta - 2}{\eta - 4}\Big\}. \tag{3.13}$$

The function $\tilde{h}(\eta)$ is holomorphic in the domain $\mathbb{C}\backslash(-\infty, 0]$. The branch of the logarithm is specified by the condition

$$\tilde{h}(\eta) \sim \frac{1}{\eta}, \quad \eta \to \infty.$$

The points $\eta = 2$ and $\eta = 4$ are removable singular points of this function.

The density of the measure $\tilde{\lambda}$ is calculated by Sokhotskiĭ's formula

$$\tilde{\lambda}'(\eta) = \frac{1}{\pi}\,\mathrm{Im}\,\tilde{h}(\eta - i \cdot 0), \quad -\infty < \eta < 0.$$

If $-1/12 < \eta < 0$, then the number under the logarithm sign in (3.13) is negative. In view of the choice of the branch of the logarithm, the argument of this number is π. Therefore,

$$\tilde{\lambda}'(\eta) = 1, \quad -1/12 < \eta < 0.$$

In other words, the measure $\tilde{\lambda}$ attains constraint (3.5) on the interval $[-1/12, 0]$. If $-\infty < \eta < -1/12$, then the number under the logarithm is complex. Evaluating the argument of this number, we get formula (3.7).

We now verify the equilibrium conditions (3.6). To this end, we find the derivative of the potential $\widetilde{W}$ along the real line. We have

$$-\frac{d}{d\eta}\widetilde{W}(\eta) = \log\left|\left(\frac{\sqrt{1+12\eta} - 2\eta + 1}{\sqrt{1+12\eta} - 2\eta - 1}\right)^2 \cdot \frac{\eta - 2}{\eta - 4}\right|.$$

If $-\infty < \eta < -1/12$, then the expression under the logarithm is identically equal to 1; in other words, $\widetilde{W}'(\eta) = 0$. If $-1/12 < \eta < 0$, then $\widetilde{W}'(\eta) < 0$, because this expression is monotonic. An analysis of the behavior of the potentials at infinity shows that the equilibrium constant $\widetilde{w}$ is zero.

This completes the proofs of Propositions 3.1, 3.2, and 3.3. □

4. The second auxiliary problem

In this section, we shall be concerned with the polynomials $\widehat{Q}_n(t)$ defined by

$$\frac{\widehat{Q}_n(t)}{\widehat{\omega}_n(t)} = \mathcal{D}_{\alpha_2}^{(n)} \frac{\widetilde{Q}_n(t)}{\widetilde{\omega}_n(t)}, \quad n \in \mathbb{Z}_+, \tag{4.1}$$

where

$$\widehat{\omega}_n(t) = (t - (\beta + n)) \dots (t - (\beta + 4n)), \tag{4.2}$$

the polynomials $\widetilde{Q}_n$ and $\widetilde{\omega}_n$ are defined in (3.1) and (3.1), respectively, and the difference operator $\mathcal{D}_{\alpha_2}^{(n)}$ in (1.4). These polynomials solve the following problem.

Problem 4.1. Given a nonnegative integer n, find a nonzero polynomial $\widehat{Q}_n$ of degree at most $2n$ and polynomials $\widehat{P}_{n,1}$, $\widehat{P}_{n,2}$ of degree at most $2n$ such that the following interpolation conditions are satisfied:

$$\widehat{R}_{n,j}(t) = \widehat{Q}_n(t) f_j(t) - \widehat{P}_{n,j}(t) = 0, \quad t = \beta + n, \dots, \beta + 4n, \quad j = 1, 2.$$

Assume, for the time being, that the parameters satisfy the following conditions:

$$\beta > 0, \quad 0 < \alpha_1 < \alpha_2 < 1.$$

Then Problem 4.1 is equivalent to the orthogonality conditions

$$\int_{-\infty}^{0} \widehat{Q}_n(t)\, t^l \,\frac{d\mu^{\alpha_j}(t)}{\widehat{\omega}_n(t)} = 0, \quad l = 0, \dots, n-1, \qquad j = 1, 2,$$

where the measures μ^{α_j} are defined in (3.4). These measures form a Nikishin system [4], [12]–[14]; that is, their ratio (the Radon–Nikodým derivative)

$$u(t) = \frac{d\mu^{\alpha_2}(t)}{d\mu^{\alpha_1}(t)},$$

which is defined *ab initio* only at nonpositive integer points and which is equal to

$$u(t) = \frac{\Gamma(1-\alpha_1)}{\Gamma(1-\alpha_2)}\,\frac{\Gamma(1-t-\alpha_2)}{\Gamma(1-t-\alpha_1)},$$

is the Markov function of some measure. Namely,

$$u(t) = \int_0^{+\infty} \frac{d\nu^{\alpha_1,\alpha_2}(s)}{s-t},$$

where ν^{α_1,α_2} is a discrete measure with masses

$$\nu_k^{\alpha_1,\alpha_2} = \frac{1}{\mathrm{B}(1-\alpha_2, \alpha_2-\alpha_1)} \cdot \frac{(1+\alpha_1-\alpha_2)_k}{k!}$$

at the points

$$t_k = 1 - \alpha_2 + k, \qquad k \in \mathbb{Z}_+.$$

The corresponding equilibrium problem is a Nikishin problem.

We set

$$\widehat{Q}_n^*(t) = \widehat{C}_n \widehat{Q}_n(nt), \quad n \in \mathbb{Z}_+,$$

where the normalizing constant $\widehat{C}_n$ is chosen so that the leading coefficient of the polynomial $\widehat{Q}_n^*$ is equal to one. We denote the counting measure of the zeros of this polynomial by $\hat{\lambda}_n$. Consider the following problem.

Problem 4.2. Find two positive measures $\hat{\lambda}_\Delta$ and $\hat{\lambda}_F$ such that

(1) $\mathsf{S}(\hat{\lambda}_\Delta) \subset \Delta = (-\infty, 0]$, $\mathsf{S}(\hat{\lambda}_F) \subset F = [0, +\infty)$;

(2) $\|\hat{\lambda}_\Delta\| = 2$, $\|\hat{\lambda}_F\| = 1$;

(3) the constraints

$$\hat{\lambda}_\Delta \leqslant \chi_\Delta, \quad \hat{\lambda}_F \leqslant \chi_F \tag{4.3}$$

are satisfied, where χ_Δ and χ_F are the classical Lebesgue measures on the intervals Δ and F, respectively;

(4) the equilibrium conditions hold

(a) on the interval Δ,

$$\widehat{W}_\Delta = 2V^{\hat{\lambda}_\Delta} - V^{\hat{\lambda}_F} - V^{\lambda_{\widehat{\omega}}} \begin{cases} \leqslant \widehat{w}_\Delta \mid \mathsf{S}(\hat{\lambda}_\Delta), \\ \geqslant \widehat{w}_\Delta \mid \Delta \setminus (\mathsf{S}(\hat{\lambda}_\Delta) \setminus \mathsf{S}(\chi_\Delta - \hat{\lambda}_\Delta)), \end{cases} \tag{4.4}$$

where $\widehat{w}_\Delta$ is some equilibrium constant and the potential $(-V^{\lambda_{\widehat{\omega}}})$ of the classical Lebesgue measure on the interval $[1,4]$ plays the role of the external field;

(b) on the interval F,

$$\widehat{W}_F = 2V^{\hat\lambda_F} - V^{\hat\lambda_\Delta} \begin{cases} \leqslant \widehat{w}_F \mid \mathsf{S}(\hat\lambda_F), \\ \geqslant \widehat{w}_F \mid F \setminus (\mathsf{S}(\hat\lambda_F) \setminus \mathsf{S}(\chi_F - \hat\lambda_F)), \end{cases} \tag{4.5}$$

where $\widehat{w}_F$ is some equilibrium constant.

Proposition 4.1. *Problem* 4.2 *has a solution.*

Proposition 4.2. *The limit* $\hat\lambda_n \twoheadrightarrow \hat\lambda_\Delta$ *exists.*

Note that the measure $\hat\lambda_F$ is the limit measure of the distribution of scaled additional interpolation points (in other words, the zeros of the function $\widehat{R}_{n,1}/\widehat{\omega}_n$).

We write the measures $\hat\lambda_\Delta$ and $\hat\lambda_F$ in an explicit form. We set

$$t_\pm = \frac{1}{243}\,(-29 \pm 20\sqrt{10}). \tag{4.6}$$

We let $\mathfrak{K}$ denote the Riemann surface obtained by gluing the following three sheets:

$$\mathfrak{K}_\Delta = \mathbb{C} \setminus (-\infty, t_-], \tag{4.7}$$

$$\mathfrak{K}_* = \mathbb{C} \setminus ((-\infty, t_-] \cup [t_+, +\infty)), \tag{4.8}$$

$$\mathfrak{K}_F = \mathbb{C} \setminus [t_+, +\infty). \tag{4.9}$$

This is a surface of zero genus. On this surface we define the meromorphic function ϑ by its divisor. Namely, this function has a first-order zero at the point $t=4$ on the sheet $\mathfrak{K}_\Delta$ and a second-order zero at the point $t=0$ on the sheet $\mathfrak{K}_*$. It also has simple poles at the points $t=1$ and $t=0$ on the sheet $\mathfrak{K}_F$. We normalize the function ϑ by the condition $\vartheta(\infty)=1$. We denote the restriction of the function ϑ to the sheet $\mathfrak{K}_j$ by ϑ_j, where $j = \Delta,\ *,\ F$. We set

$$\hat h_\Delta(t) = \log\vartheta_\Delta(t) + \log\frac{t-1}{t-4}.$$

This function is holomorphic in the domain $\mathbb{C}\setminus\Delta$. The branches of the logarithm are specified by the condition

$$\hat h_\Delta(t) \sim \frac{2}{t}, \quad t\to\infty. \tag{4.10}$$

We set

$$\hat h_F(t) = -\log\vartheta_F(t). \tag{4.11}$$

This function is holomorphic in the domain $\mathbb{C}\setminus F$. The branch of the logarithm is specified by the condition

$$\hat h_F(t) \sim \frac{1}{t}, \quad t\to\infty. \tag{4.12}$$

Proposition 4.3. *The functions* $\hat h_\Delta$ *and* $\hat h_F$ *are Markov functions of the measures* $\hat\lambda_\Delta$ *and* $\hat\lambda_F$, *respectively.*

Proof. Arguing as in §3, we have to study the critical points of the function

$$\begin{aligned}\widehat{S}(\eta,t) &= \eta\log\eta - (\eta-1)\log(\eta-1) \\ &\quad + (\eta-t-1)\log(\eta-t-1) - (\eta-t)\log(\eta-t) + \widetilde{S}_*(\eta),\end{aligned}$$

where the function $\widetilde{S}_*(\eta)$ is defined by (3.12). We have

$$\frac{\partial}{\partial\eta}\widehat{S}(\eta,t)=\log\Big\{\frac{\eta(\eta-t-1)}{(\eta-1)(\eta-t)}\cdot\frac{\sqrt{1+12\eta}+1-2\eta}{\sqrt{1+12\eta}-1-2\eta}\Big\}=0,$$

which is equivalent to the cubic equation

$$\eta^3-2\eta^2+(t+1)\eta-t(3t+1)=0.$$

Its discriminant $243t^2+58t-13$ has the roots $t_\pm$ (see (4.6)). These points are second-order branch points of the algebraic function $\eta(t)$. At infinity this function behaves like

$$\eta\sim\sqrt[3]{3t^2}\quad\text{as}\quad t\to\infty.$$

So, the point at infinity is a third-order branch point for the function $\eta(t)$. We denote the branch that assumes positive values as $t\to+\infty$ by η_Δ. This branch is holomorphic in the domain (4.7). We denote the branch that assumes positive values as $t\to-\infty$ by η_F. This branch is holomorphic in (4.9). We denote the third branch by η_*. This branch is holomorphic in (4.8). So, the Riemann surface of the function η can be constructed by gluing the above three sheets.

The leading contribution to the asymptotics comes from the critical point of η_Δ. Setting

$$\widehat{S}_\Delta(t)=\widehat{S}(\eta_\Delta(t),t),\tag{4.13}$$

we have

$$V^{\hat\lambda_\Delta}=V^{\lambda_{\widehat\omega}}-\mathrm{Re}\widehat{S}_\Delta+\text{const.}$$

Next, let

$$\hat h_\Delta=\frac{\partial}{\partial t}(\mathrm{Re}\widehat{S}_\Delta-V^{\lambda_{\widehat\omega}}).$$

Then

$$\hat h_\Delta(t)=\log\Big\{\frac{\eta_\Delta(t)-t}{\eta_\Delta(t)-t-1}\cdot\frac{t-1}{t-4}\Big\}.\tag{4.14}$$

The function $\hat h_\Delta$ is holomorphic in the domain $\mathbb{C}\setminus\Delta$. The branch of the logarithm is specified in accordance with formula (4.10).

We let ϑ denote the algebraic function

$$\vartheta(t)=\frac{\eta(t)-t}{\eta(t)-t-1}.$$

This function has the same Riemann surface as the function $\eta(t)$. We denote its branches by ϑ_Δ, ϑ_*, ϑ_F .

Rewriting (4.14) as

$$\hat h_\Delta(t)=\log\vartheta_\Delta(t)+\log\frac{t-1}{t-4},$$

we have

$$\hat h_\Delta(t)=\int\limits_\Delta\frac{d\hat\lambda_\Delta(s)}{t-s},\quad t\in\mathbb{C}\setminus\Delta.$$

The derivative $\hat\lambda'_\Delta(t)$ can be found by Sokhotskiĭ's formula

$$\hat\lambda'_\Delta(t)=\frac{1}{\pi}\,\mathrm{Im}\,\hat h_\Delta(t-i\cdot 0),\quad -\infty<t<0.$$

This function is positive on the entire negative half-axis. For $t_-<t<0$ the function $\vartheta_\Delta(t)$ is negative. Therefore,

$$\hat\lambda'_\Delta(t)=1,\quad t\in[t_-,0].$$

So, $\hat{\lambda}_\Delta$ is a positive measure of total variation two on the interval Δ; this measure attains the constraint (4.3) on the interval $[t_-, 0]$.

Consider the function $\hat{h}_F$ defined by (4.11). This function is holomorphic in the domain $\mathbb{C} \setminus F$ and

$$\hat{h}_F(t) = \int_F \frac{d\hat{\lambda}_F(s)}{t-s}, \quad t \in \mathbb{C} \setminus F,$$

where

$$\hat{\lambda}'_F(t) = \frac{1}{\pi} \operatorname{Im} \hat{h}_F(t - i \cdot 0), \quad 0 < t < +\infty. \tag{4.15}$$

The function (4.15) is positive, and moreover,

$$\hat{\lambda}'_F(t) = 1, \quad 0 < t < t_+.$$

So, $\hat{\lambda}_F$ is a positive measure of total variation one on the interval F; this measure attains the constraint (4.3) on the interval $[0, t_+]$.

The function ϑ satisfies the equation

$$t^2(t-1)\vartheta^3 - 3t(t^2 - 2t - 1)\vartheta^2 + (3t^3 - 9t^2 - 3t + 1)\vartheta - t^2(t-4) = 0.$$

This function has the above divisor, and $\vartheta(\infty) = 1$. By Viéte's theorem,

$$\vartheta_\Delta \vartheta_* \vartheta_F = \frac{t-4}{t-1}.$$

The graph of the function ϑ on the real line is depicted in Figure 4.1 (the figure shows only the topological behavior). At the origin the function behaves as follows:

$$\vartheta_*(t) \sim -4t^2, \quad \vartheta_\Delta(t) \sim \frac{\varkappa_+}{t}, \quad \vartheta_F(t) \sim \frac{\varkappa_-}{t}, \quad \text{where} \quad \varkappa_\pm = \frac{3 \pm \sqrt{13}}{2}.$$

The derivative ϑ' has the following divisor. It has first-order zeros at the point $t = 0$ on the sheet $\mathfrak{K}_*$ and at the point $t = 104/243$ on the sheet $\mathfrak{K}_\Delta$; it has a sixth-order zero at the branch point at infinity. This function has second-order poles at $t = 0$ on the sheets $\mathfrak{K}_\Delta$ and $\mathfrak{K}_F$, and at $t = 1$ on the sheet $\mathfrak{K}_\Delta$; it also has first-order poles at the branch points t_+ and t_-.

We will verify the equilibrium conditions (4.4). To do this, we calculate the derivative of the function $\widehat{W}_\Delta$ along the real line. We have

$$-\widehat{W}'_\Delta = \operatorname{Re}\Big\{2\hat{h}_\Delta - \hat{h}_F + \log\frac{t-4}{t-1}\Big\} = \log\Big|\vartheta_\Delta^2 \vartheta_F \frac{t-1}{t-4}\Big|$$
$$= \log\Big|\frac{\vartheta_\Delta^2 \vartheta_F \vartheta_*}{\vartheta_*} \cdot \frac{t-1}{t-4}\Big| = \log\Big|\frac{\vartheta_\Delta}{\vartheta_*}\Big|.$$

If $-\infty < t < t_-$, then ϑ_Δ and ϑ_* are complex conjugates, and hence, $\widehat{W}'_\Delta = 0$. So, the function $\widehat{W}_\Delta$ is constant on this interval. If $t_- < t < 0$, then $\widehat{W}'_\Delta < 0$, because the functions ϑ_Δ and ϑ_* are monotonic on this interval. So, on the interval $(t_-, 0)$ the function $\widehat{W}_\Delta$ is decreasing.

Now we verify the equilibrium conditions (4.5). For the derivative, we have

$$-\widehat{W}'_F = \operatorname{Re}\{2\hat{h}_F - \hat{h}_\Delta\} = -\log\Big|\frac{\vartheta_F^2 \vartheta_\Delta \vartheta_*}{\vartheta_*} \frac{t-1}{t-4}\Big| = \log\Big|\frac{\vartheta_*}{\vartheta_F}\Big|.$$

As before, we see that the function $\widehat{W}_F$ is increasing on the interval $(0, t_+)$ and is constant on the interval $(t_+, +\infty)$. From the behavior of the potentials at infinity it follows that the equilibrium constants $\widehat{w}_\Delta$ and $\widehat{w}_F$ are both zero. □

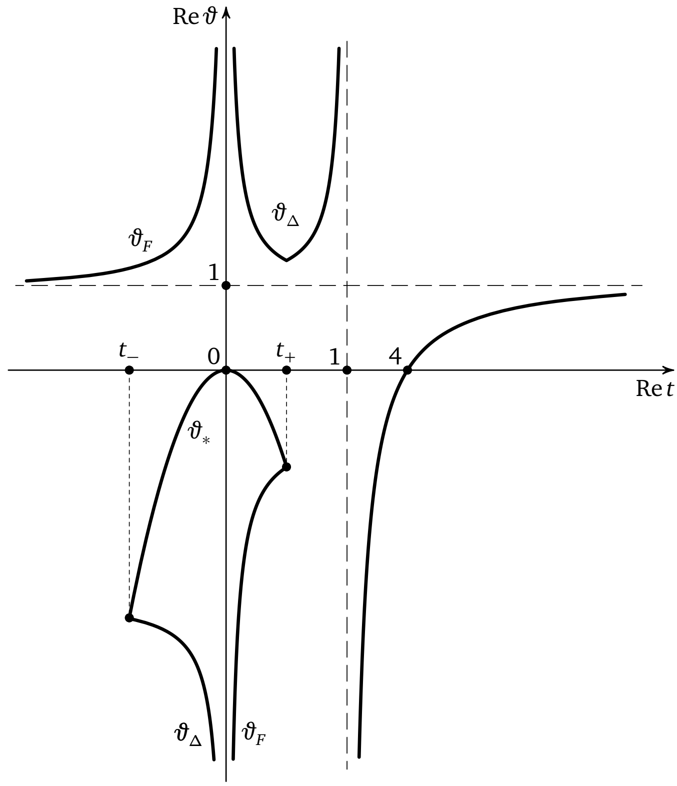

FIGURE 4.1

5. PROOFS OF THE MAIN RESULTS

In view of (1.3) we have

$$\frac{Q_n}{\omega_n} = \mathcal{D}^{(n)}_{\alpha_3} \frac{\hat{Q}_n}{\hat{\omega}_n},$$

where

$$\omega_n(x) = (x-\beta)\dots(x-(\beta+4n)),$$

and the polynomials $\hat{Q}_n$ and $\hat{\omega}_n$ are defined by (4.1) and (4.2), respectively; the difference operator $\mathcal{D}^{(n)}_{\alpha_3}$ is defined by (1.4). A similar analysis to before leads us to study the critical points of the function

$$S(t,x) = t\log t - (t-1)\log(t-1) + (t-x+1)\log(t-x+1) - (t-x)\log(t-x) + \hat{S}_\Delta(t),$$

where the function $\hat{S}_\Delta(t)$ is defined by (4.13). The equation $\frac{\partial S}{\partial t} = 0$ is equivalent to the fourth-order equation

$$t^4 - 3t^3 + (x+3)t^2 + (x^2-2x-1)t - x(3x^2+x-1) = 0. \tag{5.1}$$

The algebraic function $t(x)$ behaves at infinity as

$$t(x) \sim \sqrt[4]{3x^3}, \quad x \to \infty.$$

Hence, the point at infinity is a fourth-order branch point. The discriminant of equation (5.1) is (2.1). The roots x_0 and $x_\pm$ of the discriminant are second-order branch points. The Riemann surface $\mathfrak{R}$ of the function t was constructed in §2. We denote the branch of the function t which is holomorphic on the sheet $\mathfrak{R}_\Delta$ by t_Δ. The corresponding critical value is

$$S_\Delta(x) = S(t_\Delta(x), x).$$

Hence, the logarithmic potential of the limit measure for the distributions of the zeros of the scaled polynomials Q_n^* is given by

$$V^{\lambda_\Delta} = V^{\lambda_\omega} - \mathrm{Re} S_\Delta + \text{const}.$$

That this limit measure exists follows from the saddle-point method. As a corollary, for the Markov function of the limit measure we get formula (2.2), where

$$\theta(x) = \frac{t(x) - x}{t(x) - x - 1}. \tag{5.2}$$

Excluding the variable t from equations (5.1) and (5.2), we arrive at the algebraic equation satisfied by the function θ, namely,

$$\begin{aligned} x^4\theta^4 + (-4x^4 + 4x^3 + 6x^2)\theta^3 + (6x^4 - 12x^3 - 12x^2 + 4x)\theta^2 + \\ + (-4x^4 + 12x^3 + 6x^2 - 4x + 1)\theta + x^3(x-4) = 0. \end{aligned} \tag{5.3}$$

In (2.3) equation (5.3) is written in a compact form. The function θ has the same Riemann surface as the function t.

The algebraic function θ behaves as follows at infinity:

$$\theta(x) = 1 - \frac{1}{x} + \frac{1}{x}\sqrt[4]{\frac{3}{x}} + O(x^{-3/2}), \quad x \to \infty. \tag{5.4}$$

The point $x = \infty$ is a fourth-order branch point.

Next, we examine the behavior of the function θ at the origin. At $x = 0$ the branch θ_P has a third-order zero, namely,

$$\theta_P(x) \sim 4x^3, \quad x \to 0.$$

The branch θ_Δ has a second-order pole at the origin, namely,

$$\theta_\Delta(x) \sim -\frac{6}{x^2}, \quad x \to 0.$$

The two remaining branches θ_F and θ_T have first-order poles at the origin, namely,

$$\theta(x) \sim \frac{c_\pm}{x}, \quad x \to 0, \quad \text{where} \quad c_\pm = \frac{-2 \pm \sqrt{2}i}{6}.$$

The branch θ_Δ has a simple zero at the point $x = 4$,

$$\theta_\Delta(x) \sim \frac{2^6}{5^2 \cdot 7} \cdot (x - 4), \quad x \to 4.$$

Figure 5.1 shows the topological behavior of the section of the graph of the function θ by the real plane.

We will describe in more detail how the single-valued branches θ_J of the function θ are singled out on the sheets $\mathfrak{R}_J$ of the Riemann surface $\mathfrak{R}$, where $J = \Delta,\ P,\ F,\ T$. By $\varphi = \arg x$ we denote the branch of the argument for any of the branches of θ_J which is taken in formula (5.4) in evaluating the root $\sqrt[4]{x}$.

By θ_Δ we denote the branch of the function θ for which $\varphi = 0$ as $x \to +\infty$. This branch is a meromorphic function on the sheet $\mathfrak{R}_\Delta$. It has a simple zero at the point $x = 4$ and a second-order pole at the point $x = 0$. We set

$$\mathsf{h}_\Delta(x) = \log\Big\{\theta_\Delta(x) \cdot \frac{x}{x-4}\Big\}.$$

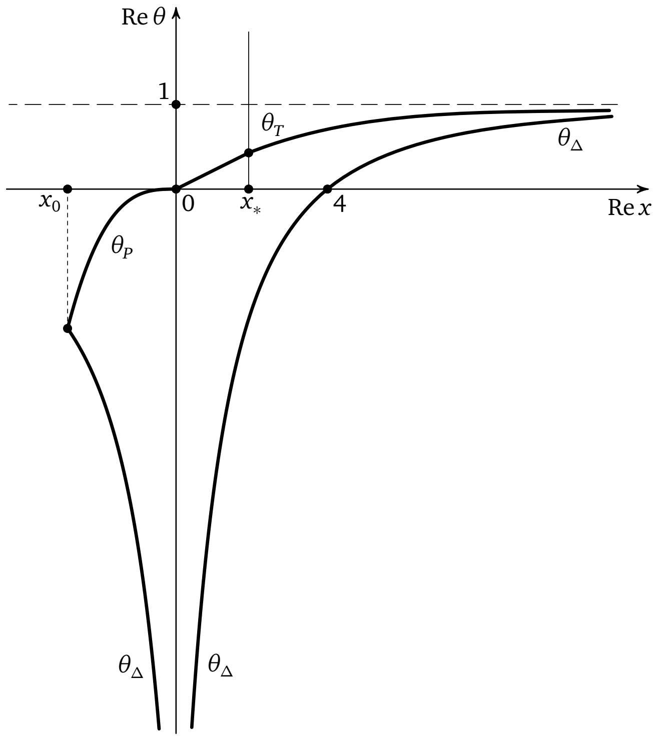

FIGURE 5.1

This function is holomorphic in the domain $\mathbb{C}\setminus\Delta$. The branch of the logarithm is chosen so as to have $\mathsf{h}_\Delta(\infty)=0$. Then

$$\mathsf{h}_\Delta(x)\sim\frac{3}{x}\quad\text{as}\quad x\to\infty.$$

The function h_Δ is a Markov function of some positive measure λ_Δ : $\mathsf{h}_\Delta = h_{\lambda_\Delta}$. The support of this measure is the entire interval Δ. The total variation of the measure is three. The measure λ_Δ is absolutely continuous with respect to the classical Lebesgue measure χ_Δ on the interval Δ. Its density can be calculated by Sokhotskiĭ's formula

$$\lambda'_\Delta(x)=\frac{1}{\pi}\operatorname{Im}\mathsf{h}_\Delta(x-i\cdot 0),\quad -\infty<x<0.$$

If $x_0<x<0$, then the number $\theta_\Delta(x)\cdot\frac{x}{x-4}$ is negative, and in accordance with the above choice of the branch of the logarithm, it has argument $\mp\pi$ on the upper and lower edges of the cut $[x_0,0]$, respectively. Therefore, $\lambda'_\Delta(x)=1$ for $x_0<x<0$. The measure λ_Δ satisfies constraint (3) of Problem 2.1, and the constraint is attained on the interval $[x_0,0]$, which means it is the saturation region of this measure.

Since, for the branch θ_Δ, the argument φ is $\pm\pi$ as $x\to-\infty$ on the upper and lower edges of the cut $(-\infty,x_0]$, respectively, the branch θ_P is specified by the condition $\varphi=\mp\pi$ as $x\to-\infty$. This branch is holomorphic on the sheet $\mathfrak{R}_P$. It has a third-order zero at the point $x=0$. The sheets $\mathfrak{R}_\Delta$ and $\mathfrak{R}_P$ are glued along the cut $(-\infty,x_0]$.

We set

$$\mathsf{h}_P(x)=\log\theta_P(x).$$

This is a piecewise analytic function which is holomorphic in the domains $\mathbb{C}_+ \setminus \Gamma_+$ and $\mathbb{C}_- \setminus \Gamma_-$. The branches of the logarithms are chosen so as to have $\mathsf{h}_P(\infty) = 0$. Hence

$$\mathsf{h}_P(x) \sim -\frac{1}{x}, \quad x \to \infty.$$

The function h_P is a Markov function of the difference of two positive measures,

$$\mathsf{h}_P = h_{\lambda_P} - h_{\lambda_\Delta}.$$

The measure λ_Δ was already defined above. The support of the measure λ_P is the set $\mathsf{S}(\lambda_P) = F \cup \Gamma$. The total variation of this measure is two. Note that the measure λ_P is positive on Γ whenever Γ is an extremal curve. In accordance with our choice of the branches of the logarithms, the argument of this number is $\mp 2\pi$. By Sokhotskiĭ's formula

$$\lambda_P'(x) = 2, \quad 0 < x < x_*.$$

On the interval F the measure λ_P satisfies constraint (3). The constraint is attained on the interval $[0, x_*]$. This interval is the saturation region of the measure λ_P.

Note that the generalized potentials $W_\Delta, W_F, W_\Gamma, W_T$ are continuous on the entire plane. Consider the function

$$\log \frac{\theta_\Delta(x)}{\theta_P(x)} = -\log \frac{x}{x-4} + 2h_{\lambda_\Delta}(x) - h_{\lambda_P}(x). \tag{5.5}$$

We will calculate the derivative of the function W_Δ along the real axis, taking it as the real part of the complex derivative of the complexification of this function. In view of (5.5), we have

$$W_\Delta'(x) = \log\left|\frac{\theta_P(x)}{\theta_\Delta(x)}\right|, \quad -\infty < x < 0.$$

If $-\infty < x < x_0$, then $\theta_P(x)$ and $\theta_\Delta(x)$ are are complex conjugates, and

$$\left|\frac{\theta_P(x)}{\theta_\Delta(x)}\right| = 1.$$

Therefore, $W_\Delta'(x) = 0$. The function W_Δ is constant on the interval $(-\infty, x_0]$. It is easily shown that $W_\Delta' < 0$ on the interval $(x_0, 0)$; i.e., the function W_Δ is decreasing on this interval. This proves the equilibrium conditions (Δ) of Problem 2.1. We have $W_\Delta(\infty) = 0$, and hence the equilibrium constant w_Δ is zero.

Since for the branch θ_P the argument φ is $\mp 2\pi$ as $x \to +\infty$, we specify the branch θ_F by the condition $\varphi = \pm 2\pi$ as $x \to +\infty$. This is a piecewise analytic function which is holomorphic in the half-planes $\mathbb{C}_\pm$. This sheet $\mathfrak{R}_F$ is glued to the sheet $\mathfrak{R}_P$ along the cut $[x_*, +\infty)$.

Consider the function

$$\mathsf{h}_F(x) = \log \theta_F(x).$$

It is a piecewise analytic function which is holomorphic in the half-planes $\mathbb{C}_\pm$. The branches of the logarithm are chosen so as to have $\mathsf{h}_F(\infty) = 0$. Then

$$\mathsf{h}_F(x) \sim -\frac{1}{x}, \quad x \to \infty.$$

The function h_F is a Markov function of the difference of two positive measures,

$$\mathsf{h}_F = h_{\lambda_T} - h_{\lambda_F}.$$

The densities of these measures can be found using Sokhotskiĭ's formula. The support of the measure λ_F is the entire interval F, and the support of the measure λ_T is the interval Δ. We denote the total variation of the measure λ_T by $\mathsf{t} = \|\lambda_T\|$. Then $\|\lambda_F\| = 1 + \mathsf{t}$. On the interval $[x_*, +\infty)$ the measures λ_F and λ_P are equal. We set

$$\lambda_\Gamma = \lambda_P - \lambda_F.$$

Then λ_Γ is a positive measure with support

$$\mathsf{S}(\lambda_\Gamma) = \Gamma_* = \Gamma \cup [0, x_*]$$

and total variation $\|\lambda_\Gamma\| = 1 - \mathsf{t}$.

As before, we have

$$W_F'(x) = \log\left|\frac{\theta_F(x)}{\theta_P(x)}\right|, \quad x \in F.$$

If $x_* < x < +\infty$, then $\theta_F(x)$ and $\theta_P(x)$ are complex conjugates, and $|\theta_F/\theta_P| = 1$. Therefore, $W_F' = 0$; i.e., the function W_F is constant on the interval $[x_*, +\infty)$. It is easily shown that $W_F' > 0$ on the interval $(0, x_*)$; that is, the function W_F is increasing on this interval. This proves the equilibrium conditions (F). We have $W_F(\infty) = 0$, and hence the equilibrium constant w_F is zero.

For the branch θ_F, the argument φ is $\pm 3\pi$ as $x \to -\infty$. Hence, we specify the branch θ_T by the condition $\varphi = \mp 3\pi$ as $x \to -\infty$. This branch is holomorphic on the sheet $\mathfrak{R}_T$.

We set

$$\mathsf{h}_T(x) = \log \theta_T(x).$$

This function is also holomorphic on the sheet $\mathfrak{R}_T$. The branch of the logarithm is specified by the condition $\mathsf{h}_T(\infty) = 0$. Then

$$\mathsf{h}_T(x) \sim -\frac{1}{x}, \quad x \to \infty.$$

The function h_T is a Markov function of the sum of two negative measures

$$\mathsf{h}_T = -h_{\lambda_T} - h_{\lambda_\Gamma}.$$

The measures λ_T and λ_Γ were already defined above in terms of other branches of the algebraic function θ.

We have

$$W_T'(x) = \log\left|\frac{\theta_T(x)}{\theta_F(x)}\right|, \quad x \in \Delta.$$

The numbers $\theta_T(x)$ and $\theta_F(x)$ are complex conjugates on the whole of Δ; furthermore, $|\theta_T(x)/\theta_F(x)| = 1$. Therefore, $W_T' = 0$ on Δ so that W_T is constant on this interval. The equilibrium conditions (T) are proved. We have $W_T(\infty) = 0$, and hence the equilibrium constant w_T is zero.

Let $\mathcal{W}_\Gamma$ be the complexification of the generalized potential W_Γ. Then

$$\mathcal{W}_\Gamma' = \log \frac{\theta_T}{\theta_P}.$$

We let s denote the natural parameter of the curve Γ and τ, the unit tangent vector to this curve. Then on this curve the derivative of the generalized potential W_Γ along the curve Γ looks like

$$\frac{d}{ds} W_\Gamma = \operatorname{Re}\left\{\tau \log \frac{\theta_T}{\theta_P}\right\}.$$

The following conditions are equivalent in view of the Cauchy–Riemann conditions:

1°. $\dfrac{dW_\Gamma}{ds} = 0 \ \Gamma$.

2°. *The measure* λ_Γ *is positive.*

3°. *The curve* Γ *has the S-property* (*the curve is extremal*).

4°. *The curve* Γ *lies in the set on which the moduli of two critical values of the function* $\exp S$ *are equal* (*the saddle-point method*).

So, the equilibrium conditions (Γ) are proved. Moreover, the curve Γ is a trajectory of the corresponding quadratic differential.

This completes the proof of the main results of the paper.

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Faculty of Mechanics and Mathematics, Moscow State University
Email address: `vladimirs1957@gmail.com`

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