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SIMPLE SOLUTIONS OF THREE EQUATIONS OF MATHEMATICAL PHYSICS

V. K. BELOSHAPKA

Abstract. In this paper, we consider three equations of mathematical physics for functions of two variables: the heat equation, the Liouville equation, and the Korteweg–de Vries (KdV) equation. We obtain complete lists of simple solutions for all three equations, that is, solutions of analytic complexity not exceeding one. All solutions of this type for the heat equation can be expressed in terms of the error function (Theorem 1) and form a 4-parameter family; for the Liouville equation, the answer is the union of a 6-parameter family and a 3-parameter family of elementary functions (Theorem 2); for the Korteweg–de Vries equation, the list consists of four 3-parameter families containing elementary and elliptic functions (Theorem 3).

§1. Introduction

The set of problems related to the possibility of representing analytic functions of several variables by superpositions of functions of fewer variables has been discussed in several works (see [1] and [2]). Consider a strictly increasing hierarchy of the complexity classes of analytic functions of two variables $z(x, y)$; this hierarchy is defined inductively using the function $(x + y)$:

$$Cl_0 \subset Cl_1 \subset Cl_2 \subset \cdots \subset Cl_n \subset \cdots .$$

Here Cl_0 is the class formed by functions of one variable (x or y) which is assigned complexity $N(z) = 0$; Cl_1 is the class of functions of the form $c(a(x) + b(y))$; they have complexity $N(z) \le 1$; Cl_{n+1} consists of functions of the form $C(A_n(x, y) + B_n(x, y))$, where C is a function of one variable and A_n and B_n are functions in Cl_n. Functions which are in Cl_n but do not belong to Cl_{n-1} have complexity $N(z) = n$. The condition $N(z) \le 1$ is equivalent to the fact that the germ locally representing z satisfies the differential relation

$$z'_x z'_y (z'''_{xxy} z''_y - z'''_{xyy} z'_x) + z''_{xy}((z'_x)^2 z''_{yy} - (z'_y)^2 z''_{xx}) = 0. \tag{1}$$

This homogeneous form of degree four is the numerator of the differential-rational expression $(\ln(z'_y/z'_x))''_{xy}$ (see [3]).

The pseudogroup $\mathcal{G} = \{z(x, y) \to c\,(z(a(x), b(y)))\}$, where (a, b, c) are germs of nonconstant analytic functions, acts on the family of germs of analytic functions of two variables. This pseudogroup preserves the analytic complexity of z, and therefore we call $\mathcal{G}$ the *gauge* pseudogroup. In accordance with the general definition, the stabilizer Stab_z of a function $z(x, y)$ consists of the families (a, b, c) for which $c(z(a(x), b(y))) = z(x, y)$. Let d_z be the dimension of the stabilizer Stab_z. As was proved in [4], if z is a function of two variables (that is, if neither partial derivative is identically zero), then d_z can take

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precisely three values: 0, 1, and 3. In addition, the equation $d_z = 3$ can only hold for functions of complexity 1, that is, for functions equivalent to $(x + y)$. This is a characterizing feature of functions of the first class. We may say that the first class consists of the functions z for which $d_z > 1$.

Thus, the search for the *simple solutions* of some equation, in other words, those belonging to the first class, can be regarded as the search for solutions with symmetry group in $\mathcal{G}$ whose dimension exceeds one. In [3], a description of simple solutions of Dirichlet's equation and the wave equation was obtained. Here we study the heat equation, the Liouville equation, and the Korteweg–de Vries equation.

§2. The heat equation

Let $z(x, y)$ be an analytic function satisfying the heat equation in the form

$$z'_y = z''_{xx}. \tag{2}$$

If z is a solution of (2) of complexity zero, then this function is either constant or linear in x. Our objective is to describe all solutions of the heat equation with complexity one.

Let G be the group of transformations acting on the functions as follows:

$$z(x, y) \to \delta z(\gamma\,(x - \alpha), \gamma^2(y - \beta)) + \varepsilon,$$

where $(\alpha, \beta, \gamma, \delta, \varepsilon)$ are complex constants, and γ and δ are nonzero. It is clear that this group takes solutions of the heat equation to solutions and does not change the complexity. Therefore, it is natural to give our description up to transformations in G.

We write out equation (2) for $z(x, y) = c(a(x) + b(y))$ (the subscripts are the orders of the derivatives):

$$a_1^2 c_2 + a_2 c_1 - b_1 c_1 = 0 \quad \text{or} \quad \frac{c_2}{c_1} = -\frac{a_2 - b_1}{a_1^2}.$$

But c_2/c_1 is a function of $(a(x) + b(y))$, and so we obtain

$$-b_2 a_1^2 - a_3 a_1 b_1 + 2a_2^2 b_1 - 2a_2 b_1^2 = 0. \tag{3}$$

Using the fact that a and b are nonconstant, we take $a_1 = A$ and $b_1 = B$ to be new independent variables; then $a_2 = P(A)$ and $b_2 = Q(B)$ are new unknown functions; here $a_3 = P'(A)P(A)$, and (3) becomes

$$-Q(B)A^2 - \left(\frac{d}{dA}P(A)\right) P(A)AB + 2\,P(A)^2 B - 2P(A)B^2 = 0.$$

This means that

$$-Q(B) = 2\,\frac{P(A)B^2}{A^2} + \frac{P(A)\left(A\frac{d}{dA}P(A) - 2P(A)\right)B}{A^2} = mB + nB^2.$$

Since Q is independent of A, it follows that the coefficients on the right-hand side (m of B and n of B^2) are both constant. Thus, we obtain two equations for $P(A)$:

$$\frac{P(A)}{A^2} = n \quad \text{and} \quad \left(A\frac{d}{dA}P(A) - 2P(A)\right) = m.$$

It follows from the first equation that $P(A) = nA^2$; we substitute this into the other equation and obtain $m = 0$, and thus $Q(B) = -2nB^2$. These relations are differential equations for $a(x)$ and $b(y)$. Solving them, we obtain

$$a(x) = -\frac{1}{n}\ln(C_1 x + C_2), \quad b(y) = \frac{1}{2n}\ln(C_3 y + C_4).$$

Then

$$t = a(x) + b(y) = \frac{1}{2n}\ln\frac{C_3 y + C_4}{(C_1 x + C_2)^2}.$$

Thus, up to transformations in G, we see that the solutions $z(x,y)$ we are interested in for the heat equation are of the form $z(x,y) = f(y/x^2)$. After making the substitution $y = tx^2$, the condition that $z(x,y) = f(y/x^2)$ satisfies (2) becomes

$$4\left(\frac{d^2}{dt^2}f(t)\right)t^2 - \left(\frac{d}{dt}f(t)\right) + 6\left(\frac{d}{dt}f(t)\right)t = 0. \tag{4}$$

The nonconstant solutions of this equation are given by

$$f(t) = C_1 + \operatorname{erf}\left(\frac{1}{2\sqrt{t}}\right)C_2.$$

We finally obtain the following theorem.

Theorem 1. *Up to transformations in G, every solution of the equation $z'_y = z''_{xx}$ with analytic complexity one is of the form*

$$z(x,y) = \operatorname{erf}\left(\frac{x}{2\sqrt{y}}\right),$$

where

$$\operatorname{erf}(T) = \frac{2}{\sqrt{\pi}}\int_0^T \mathrm{e}^{-t^2}\,dt$$

is the integral known in probability theory as the error function.

If we act on the function thus obtained by transformations in G, then we obtain the following 4-parameter family:

$$\delta\operatorname{erf}\left(\frac{x-\alpha}{2\sqrt{y-\beta}}\right) + \varepsilon.$$

There is an assertion which dates back to Kovalevskaya (1875, see [5]), claiming that all solutions to the heat equation that are analytic in both variables can be continued to entire functions with respect to the space variable x such that their order of growth does not exceed two. Note that, for every value of y, the function $\operatorname{erf}\left(\frac{x}{2\sqrt{y}}\right)$ obtained above is an entire function of order two. Thus, the maximum complexity of the solutions, from the point of view of the growth rate (order two), is realized on functions having the minimal complexity for functions of two variables, namely one.

§3. The Liouville equation

Consider the Liouville equation, that is, the equation $z''_{xy} = e^z$. As Liouville showed [6], all solutions of this equation are of the form

$$z = \ln\left(2\frac{a'(x)b'(y)}{(a(x)+b(y))^2}\right). \tag{5}$$

When speaking of analytic solutions, a and b are analytic here. It follows immediately from the formula that the analytic complexity $N(z)$ of a generic solution does not exceed two. The problem of describing solutions of complexity one (simple solutions) arises.

Theorem 2.

(a) *A function of the form* (5)*, where a and b are analytic functions, has complexity one in two cases:*

$$\text{(I)}\quad a(x) = \frac{kx+l}{mx+n},\quad b(y) = \frac{Ky+L}{My+N};$$

$$\text{(II)}\quad a(x) = ke^{\pm x} + l,\quad b(y) = Ke^{\pm y} - l.$$

All the constants are arbitrary, provided that the functions a and b are defined and nonconstant. The arrangement of the signs in the exponents for a and b in the second case are independent.

(b) *In all other cases, the complexity of* (5) *is equal to two.*

Proof. It is clear that the Liouville equation has no solutions depending on just one variable, that is, no solutions of complexity zero. Our immediate aim is to find all solutions of complexity one, that is, find all pairs $(a(x), b(y))$ satisfying condition (1). We can assume here that a and b are nonconstant.

Substituting (5) into (1), we obtain a differential fraction whose numerator is equal to zero (the subscripts are the orders of the derivatives),

$$\begin{aligned}(6)\qquad & -a_0^2a_1a_3b_2^2 + a_0^2a_2^2b_1b_3 - 2a_0a_1a_3b_2^2b_0 - 6a_0{a_2}^2b_2b_1^2 + 2a_0a_2^2b_1b_3b_0\\ &+ 6a_0a_2b_2^2a_1^2 - 4a_0a_1^2b_1b_3a_2 + 4a_0a_1a_3b_2b_1^2 - 6a_1^4b_2^2 + 4a_1a_3b_1^2b_2b_0\\ &+ 4a_1^4b_1b_3 + 6a_2b_2^2a_1^2b_0 - 6a_2^2b_2b_1^2b_0 - 4a_1^2b_1b_3a_2b_0 + 6a_2^2b_1^4\\ &- 4a_1a_3b_1^4 - a_1a_3b_2^2b_0^2 + a_2^2b_1b_3b_0^2 = 0,\end{aligned}$$

and the denominator is of the form $(a_2a_0+a_2b_0-2a_1^2)^2(b_2a_0+b_2b_0-2b_1^2)^2$. It can readily be seen that the last relation cannot vanish identically.

Solving (6) for b_3b_1, we obtain a differential fraction whose denominator is equal to $(a_2a_0+a_2b_0-2a_1^2)^2$ and does not vanish identically. Writing out the condition that this differential fraction does not depend on x, that is, equating the derivative with respect to x to zero, we obtain

$$\begin{aligned}(7)\qquad & -a_2^2b_2a_0^2 + a_0^2b_2a_1a_3 + 6a_0b_2a_2^3 - 3a_0b_2a_1a_2a_3 - 2a_0a_2a_1^2b_2\\ &- 2a_0a_2^2b_2b_0 + 2a_0a_1a_3b_2b_0 - 2a_0a_1a_3b_1^2 + 2a_0a_2^2b_1^2 - 2b_0a_2a_1^2b_2\\ &- 3b_2a_1a_2b_0a_3 + b_2a_1a_3b_0^2 - a_2^2b_2b_0^2 + 2a_1^4b_2 + 6b_0b_2a_2^3\\ &- 3a_1^2b_2a_2^2 - 2a_1a_3b_1^2b_0 + 2a_2^2b_1^2b_0 + 6a_1b_1^2a_2a_3 + 2a_2a_1^2b_1^2 - 9b_1^2a_2^3 = 0.\end{aligned}$$

In particular, this implies that, for a solution in general position, that is, for (a, b) not satisfying this relation, we have $N(z) = 2$.

Solving (7) for b_1^2/b_2, we obtain a fraction whose denominator

$$-2a_2^2b_0 - 6a_1a_2a_3 - 2a_2a_1^2 + 2a_1a_3b_0 - 2a_0a_2^2 + 2a_0a_1a_3 + 9a_2^3$$

can only vanish identically if $a_2 = 0$. Equating the derivative of this fraction with respect to x to zero, we obtain an expression which is a polynomial of degree 2 in b_0 with the following coefficients:

$$\begin{aligned} C_2 &= 27a_1^2a_2^4a_3 - 8a_0a_1^5a_2 - 2a_0^2a_1^4a_3 - 3a_0^2a_2^4a_3\\ &\quad - 27a_1^3a_2^4 - 4a_1^7 + 18a_1^5{a_2}^2 + 7a_0^2a_2^2a_3a_1^2\\ &\quad + 2a_0^2a_1^5 - 6a_0a_2^3a_3a_1^2 - 4a_0a_1^4a_3a_2 + 16a_1^6a_3\\ &\quad - 4a_0^2a_2^2a_1^3 + 18a_0a_2^3a_1^3 - 30a_1^4a_2^2a_3,\\ C_1 &= 18a_2^3a_1^3 + 14a_0a_2^2a_3a_1^2 - 8a_0a_2^2a_1^3 - 6a_0a_2^4a_3\\ &\quad - 4a_0a_1^4a_3 - 4a_1^4a_3a_2 - 6\,a_2^3a_3a_1^2 + 4a_0a_1^5 - 8a_1^5a_2,\\ C_0 &= -2a_1^4a_3 - 3a_2^4a_3 + 7a_2^2a_3a_1^2 + 2a_1^5 - 4a_2^2a_1^3.\end{aligned}$$

For a nonconstant $b_0 = b(y)$ to exist, it is necessary that all three coefficients be identically equal to zero. A necessary condition for the system $C_0 = C_1 = C_2 = 0$ to be solvable in terms of a_3 is that

$$a_2(a_1 - a_2)(a_1 + a_2)(-3a_2^2 + 2a_1^2) = 0.$$

Solving the differential equations, we see that the only possibilities for the function $a(x)$ are the following:

$$(1) \quad a(x) = \frac{kx+l}{mx+n}, \qquad\qquad (2) \quad a(x) = ke^{\pm x} + l.$$

Similarly, for $b(y)$ we have

$$(1) \quad b(y) = \frac{Ky+L}{My+N}, \qquad\qquad (2) \quad b(y) = Ke^{\pm y} + L.$$

Further, choosing the pairs satisfying (6) from among these pairs (a, b), we see that these pairs are of two types:

$$\begin{aligned} &(\mathrm{I}) \quad a(x) = \frac{kx+l}{mx+n}, \quad b(y) = \frac{Ky+L}{My+N}; \\ &(\mathrm{II}) \quad a(x) = ke^{\pm x} + l, \quad b(y) = Ke^{\pm y} - l. \end{aligned}$$

In both cases the constants are arbitrary, provided that the functions a and b are defined and nonconstant. In the second case, the signs in the exponents for a and b are independent.

This completes the proof of the theorem. □

§4. The Korteweg–de Vries equation

Let $z(x, y)$ be an analytic function satisfying the Korteweg–de Vries equation of the form $z'_y = Az'''_{xxx} + Bzz'_x$, where A and B are nonzero complex constants. The transformation $z(x, y) \to kz(l\,x, y)$ does not change the complexity of z and takes this equation to the equation $z'_y = l^3\,Az'''_{xxx} + lk\,Bzz'_x$. Choosing appropriate values of k and l, we can assume that the equation is of the form

$$z'_y = zz'_x + z'''_{xxx}. \tag{8}$$

We start by stating our theorem.

Theorem 3. *All solutions of equation* (8) *with analytic complexity one are contained in the following four* 3*-parameter families:*

$$\begin{aligned} z_1 &= \frac{l}{k} - 12k^2\,\wp(kx+ly+m), \\ z_2 &= \frac{l}{k} + 12\,\frac{k^2}{\cosh^2(kx+ly+m)} - 4\,k^2, \\ z_3 &= \frac{l}{k} - 12\,\frac{k^2}{(kx+ly+m)^2}, \\ z_4 &= \frac{kx+l}{m-ky}, \end{aligned}$$

where k ($\neq 0$), l, and m are complex constants, $\wp$ is the Weierstrass elliptic function, and cosh *is the hyperbolic cosine.*

From here, we can readily obtain all solutions that depend on one variable, that is, of complexity zero.

Corollary. *All solutions of equation* (8) *of analytic complexity zero are contained in the following four families:*

$$z_1 = -12k^2\,\wp(kx+m), \quad z_2 = \frac{12k^2}{\cosh^2(kx+m)} - 4k^2, \quad z_3 = -\frac{12}{(x+m)^2}, \quad z_4 = m.$$

Domrin has told the author that he has a conjecture concerning the rate of growth of solutions of the KdV equation (see also [5]). To formulate this conjecture, we have to represent the solutions in the form

$$z(x,y) = 12(\ln(\tau(x,y))''_{xx}.$$

Then the conjecture claims that every locally defined solution that is analytic (in both the variables) can be represented in the above form, where, for a fixed y, the function τ can be continued as an entire function of x whose order of growth does not exceed 3. For our four solutions, we obtain:

$$\begin{aligned}
z_1 &= \frac{l}{k} - 12k^2\,\wp(kx+ly+m), &\quad \tau_1 &= \exp\Big\{\frac{lx^2}{24k}\Big\}\sigma(kx+ly+m),\\
z_2 &= \frac{l}{k} + 12\,\frac{k^2}{\cosh^2(kx+ly+m)} - 4\,k^2, &\quad \tau_2 &= \exp\Big\{\frac{x^2(l-4k^3)}{24k}\Big\}\cosh(kx+ly+m),\\
z_3 &= \frac{l}{k} - 12\,\frac{k^2}{(kx+ly+m)^2}, &\quad \tau_3 &= (kx+ly+m)\exp\Big\{\frac{lx^2}{24k}\Big\},\\
z_4 &= \frac{kx+l}{m-ky}, &\quad \tau_4 &= \exp\Big\{\frac{x^2(kx+3l)}{72(m-ky)}\Big\},
\end{aligned}$$

where σ is the Weierstrass sigma function and cosh is the hyperbolic cosine, as above.

Since $\sigma(t)$ is an entire function of order 2, it follows that τ_1 always has order 2; further, τ_2 has order 2 for $l \neq 4k^3$ and order 1 for $l = 4k^3$; τ_3 has order 2 for $l \neq 0$ and order 0 for $l = 0$; τ_4 always has order 3.

The remainder of the paper is devoted to proving Theorem 3. We note that the proof is by exhausting the tree of possibilities. Each fork is determined by whether or not some differential polynomial condition (the discriminant set) holds. The whole analysis required significant computational efforts using the Maple system.

We first state the following readily verifiable assertion.

Lemma 4. *The group of transformations*

$$\mathcal{H} = \{z(x,y) \to k^2 z(k\,(x+C_1), k^3\,(y+C_2))\}, \quad k \neq 0,$$

takes the solutions of (8) *to solutions and does not change their analytic complexity.*

Lemma 5. *For every pair* $(k \neq 0, l)$*, all solutions of* (8) *of the form* $z = c(kx+ly)$ *are contained in the following list:*

$$\begin{aligned}
z_1 &= \frac{l}{k} - 12k^2\,\wp(kx+ly+C),\\
z_2 &= \frac{l}{k} + 12\frac{k^2}{\cosh^2(kx+ly+C)} - 4k^2,\\
z_3 &= \frac{l}{k} - 12\frac{k^2}{(kx+ly+C)^2}.
\end{aligned}$$

Note that Lemma 5 can be treated as a description of all solutions of (8) of the form $z = c(a(x)+b(y))$ under the additional condition that $a'' = b'' = 0$.

Proof of Lemma 5. By Lemma 4, it suffices to consider the case of solutions of the form $z = c(x+l\,y)$. Writing out equation (8) for a function of this form and using the notation $t = x+ly$, we obtain

$$c'''(t) + c(t)c'(t) + lc'(t) = 0.$$

If c is nonconstant, then let $A = c(t)$ be the new independent variable and $c' = P(A)$ be the new unknown function. We obtain the equation $P''P + (P')^2 + A + l = 0$ for P. All solutions of this differential equation (which are not identically zero) are of the form

$$(P(A))^2 = Q(A) = -\frac{1}{3l^2}A^3 + A^2 + C_1 A + C_2.$$

If the cubic polynomial $Q(A)$ has no multiple roots, then, making the change $c(t) = -12Z(t) + l$, we see that $Z(t)$ satisfies the differential equation

$$(Z')^2 = 4Z^3 - g_2 Z - g_3,$$

that is, $Z(t) = \wp(t)$ is the Weierstrass elliptic function. The corresponding solution for $z_1 = c(x + ly)$ is of the form

$$z_1 = -12\wp(x + ly + C) + l.$$

Let the polynomial $Q(A)$ have a single double root:

$$Q(A) = -\frac{1}{3}(A - m)^2(A - n);$$

then, by Vieta's theorem, the third root is $n = (2m - 3l)$. In this case, the differential equation for $c(t)$ has a solution in terms of elementary functions. After substituting $t = x + ly$, we obtain

$$z_2 = -3(m - l)\tan^2\left(\frac{\sqrt{(m-l)}}{2}(x + ly + C)\right) - 2m - 3l.$$

Up to renaming the constants and transformations in $\mathcal{H}$, this function coincides with z_2 in the statement of the lemma.

Finally, let the polynomial $Q(A)$ have a root of multiplicity three, that is,

$$Q(A) = -\frac{1}{3}(A - m)^3.$$

It follows from Vieta's theorem that $m = l$. Solving the equation, we obtain

$$z_3 = 1 - 12\,\frac{1}{(x + ly + C)^2}.$$

This completes the proof of Lemma 5. $\square$

The following auxiliary assertion can be proved immediately.

Lemma 6.

(1) *All solutions of the equation* $d_1(a) = a_3 a_1 - 2a_2^2 = 0$ *can be split into generic solutions,* $a(x) = m\ln(kx + l)$*, and special solutions,* $kx + l$*.*

(2) *All solutions of the equation* $d_2(a) = a_1 a_2 a_4 - a_1 a_3^2 - a_2^2 a_3 = 0$ *can be split into generic solutions,* $a(x) = m\ln(\cos(kx + l)) + n$*, and special solutions,* $kx^2 + lx + m$*.*

Now we weaken the condition $a'' = b'' = 0$ in Lemma 8. Namely, we seek a solution of (8) of the form $z = c(a(x) + b(y))$, where only one of the functions is linear: either $a'' = 0$ or $b'' = 0$.

Lemma 7.

(1) *If* $z = c(a(x) + b(y))$ *is a solution of* (8)*, then it follows from* $a_2 = 0$ *that* $b_2 = 0$*, and it follows from* $b_2 = 0$ *that* $a_2 = 0$*.*

(2) *The list in Lemma* 5 *contains all solutions of* (8) *of the form* $z = c(a(x) + b(y))$ *under the assumption that at least one of the inner functions, a or b, is linear.*

Proof.

FIRST CASE: let $a'' \neq 0$, and let $b'' = 0$. Since $b' \neq 0$, it follows that, in this case, the function z can be represented in the form $z = c(a(x) + y)$. Writing out equation (8) for this function, we obtain

$$c_3 a_1^3 + 3c_2 a_1 a_2 + 6c_0 c_1 a_1 + c_1 a_3 + c_1 = 0. \tag{9}$$

If we set $y = t - a(x)$ here, then c becomes a function of the variable t, independent of x. Expressing c_3 from (9) and expressing the condition that it is independent of x, we see that

$$-3a_1^2 c_2 a_3 + 6a_1 c_2 a_2^2 + 2a_1 a_2 c_1 c_0 - a_4 a_1 c_1 + 3a_2 c_1 a_3 + 3a_2 c_1 = 0. \tag{10}$$

Under the condition that $d_1(a) = a_1 a_3 - 2a_2^2 \neq 0$, we can use this relation to express c_2/c_1 and write out the condition for it being independent of x. We obtain

$$\begin{aligned}&\left(2a_4 a_1^3 a_2 - 2a_1^3 a_3^2 - 2a_1^2 a_2^2 a_3\right) c_0 - a_4^2 a_1^3 + 3a_4 a_1^2 a_2 a_3 + 6a_4 a_1 a_2^3\\ &+ a_1^3 a_3 a_5 - 2a_1^2 a_2^2 a_5 - 3a_1^2 a_3^3 - 6a_2^4 a_3 + 3a_4 a_1^2 a_2 - 3a_1^2 a_3^2 - 6a_2^4 = 0.\end{aligned}$$

Since c is nonconstant, this implies that

$$\begin{aligned}&d_2(a) = a_1 a_2 a_4 - a_1 a_3^2 - a_2^2 a_3 = 0\\ &d_3(a) = a_1^3 a_3 a_5 - a_4^2 a_1^3 - 2a_1^2 a_2^2 a_5 + 3a_4 a_1^2 a_2 a_3 - 3a_1^2 a_3^3\\ &\quad + 6a_4 a_1 a_2^3 - 6a_2^4 a_3 + 3a_4 a_1^2 a_2 - 3a_1^2 a_3^2 - 6a_2^4 = 0.\end{aligned} \tag{11}$$

If $d_1 = 0$, then, substituting $a(x) = m \ln(kx + l)$ into (9), we see that $c_1 = 0$. This is a contradiction. Let $d_2 = d_3 = 0$. Here d_2 has the differential order 4, and d_3 has the differential order 5. We differentiate d_2 with respect to x and obtain a relation of order 5, $dd_2 = 0$. Next we eliminate a_5 from d_3 and dd_2 and obtain a relation of order four, $\mathrm{res} = 0$, where res stands for the resultant of d_3 and dd_2 with respect to a_5. Eliminating a_4 from $\mathrm{res} = 0$ and $d_2 = 0$ we obtain

$$-3a_1^2 a_2^5 (a_1 a_3 - 2a_2^2) = 0,$$

which returns us to the case when $d_1 = 0$.

SECOND CASE: let $b'' \neq 0, a'' = 0$. Since $a' \neq 0$, it follows that in this case the function z can be represented in the form $z = c(x + b(y))$. Writing out equation (8) for this function and substituting $x = t - b(y)$, we obtain

$$c'''(t) + c(t)c'(t) + c'(t)b'(y) = 0.$$

Differentiating this relation with respect to y, we obtain $c'(t)b''(y) = 0$. A contradiction. This completes the proof of Lemma 7. $\square$

Lemma 8. *The solutions of* (8) *of the form* $z(x, y) = c(a(x) + b(y))$ *with the additional condition* $d_1(a) = 0$ *are of the form*

$$z = \frac{kx + l}{m - ky}.$$

Proof. Using transformations in the group $\mathcal{H}$, we can transform every solution of the form $z(x, y) = c(m \ln(kx + l) + b(y))$ to a solution of the form $z(x, y) = c(\ln(x) + b(y))$. Substituting this expression into (8) we obtain

$$c_1 b_1 x^3 + c_0 c_1 x^2 + 2c_1 - 3c_2 + c_3 = 0.$$

Now we solve for c_3 and write out the condition for this expression to give a function of $\ln(x) + b(y)$; we have

$$3b_1^2 x - b_2 x + 2c_0 b_1 = 0.$$

Differentiating this relation with respect to x, we see that

$$3b_1^2x - b_2x + 2c_1b_1 = 0,$$

which implies that $c_1 = c_0$, that is, $c(t) = \exp(t+\lambda)$. Substituting $z = \exp(\ln(x)+b(y)+\lambda)$ into the original equation, we obtain $b'(y) = \exp(\lambda)\exp(b(y))$. Hence, $b(y) = \lambda+\ln(C-y)$. Thus, taking the transformations in $\mathcal{H}$ into account, we see that $z = (kx+l)/(m-ky)$. This completes the proof of Lemma 8. □

Lemma 9. *There is no solution of* (8) *of the form* $z(x,y) = c(a(x)+b(y))$ *for which* $a_2 \neq 0$*,* $b_2 \neq 0$*, and* $d_2(a) = 0$*.*

Proof. According to Lemma 6, the solutions with $d_2(a) = 0$ split into two cases: the generic case, $a(x) = m\ln(\cos(kx+l)) + n$, and the special case, $a(x) = kx^2 + lx + m$. We first consider the special case.

Using transformations in $\mathcal{H}$, every solution of the form $z = c(kx^2+lx+m+b(y))$ can be transformed to a solution of the form $z = c(x^2+b(y))$. Substituting this solution into (8) we obtain

$$8c_3x^3 + 2c_0c_1x + 12c_2x - c_1b_1 = 0.$$

Expressing c_3, we write out the condition for this to be a function of $x^2 + b(y)$, giving

$$-2b_2c_1x^2 + 4c_0c_1b_1x + 24c_2xb_1 - 3c_1b_1^2 = 0.$$

Expressing c_2/c_1, we write out the condition for this to be a function of $x^2 + b(y)$, so that

$$-4b_3b_1x^4 + 4b_2^2x^4 - 4b_2b_1^2x^2 - 3b_1^4 = 0.$$

This implies that $b_1 = 0$, giving a contradiction.

Now let $a(x) = m\ln(\cos(kx+l)) + n$. Using transformations in $\mathcal{H}$, we can transform the corresponding solution to a solution of the form $z = c(\ln(\cos(x))+b(y))$. Substituting this solution into (8), we obtain

$$\begin{aligned}-2\sin^3(x)c_1c_0b_1 - 3\sin^2(x)\cos(x)c_1b_1^2 - 2\sin(x)\cos^2(x)c_1c_0b_1\\ - 3\cos^3(x)c_1b_1^2 + b_2\sin^2(x)\cos(x)c_1 + 6c_2b_1\sin^3(x)\\ - 4\sin^3(x)c_1b_1 + 6c_2\sin(x)\cos^2(x)b_1 - 4c_1b_1\cos^2(x)\sin(x) = 0.\end{aligned}$$

Using the expression for c_3, as it is a function of $\ln(\cos(x)) + b(y)$, we obtain

$$\begin{aligned}-2\sin^3(x)c_1c_0b_1 - 3\sin^2(x)\cos(x)c_1b_1^2 - 2\sin(x)\cos^2(x)c_1c_0b_1\\ - 3\cos^3(x)c_1b_1^2 + b_2\sin^2(x)\cos(x)c_1 + 6c_2b_1\sin^3(x)\\ - 4\sin^3(x)c_1b_1 + 6c_2\sin(x)\cos^2(x)b_1 - 4c_1b_1\cos^2(x)\sin(x) = 0.\end{aligned}$$

Expressing c_2/c_1, using the condition for it to be a function of $\ln(\cos(x))+b(y)$, we obtain

$$\begin{aligned}-3\sin^4(x)b_1^4 - 6\sin^2(x)\cos^2(x)b_1^4 - 3\cos^4(x)b_1^4\\ + 4b_2\sin^4(x)b_1^2 + 2b_2\sin^2(x)\cos^2(x)b_1^2 + b_2^2\sin^4(x) - b_3\sin^4(x)b_1 = 0.\end{aligned}$$

This implies that $b_1 = 0$, giving a contradiction. This completes the proof of Lemma 9. □

Now we turn to the general case. We assume that a_2, $d_1(a)$, $d_2(a)$, b_2, and c_1 are not identically zero. Write out the equation (8) for $z(x,y) = c(a(x)+b(y))$:

$$a_1^3c_3 + 3a_1a_2c_2 + a_1c_0c_1 + a_3c_1 + b_1c_1 = 0 \tag{12}$$

and express c_3 using (12). The condition that some function of two variables $\varphi(x,y)$ can be represented in the form $\rho(a(x)+b(y))$, where ρ is a function of a single variable, is the relation $V\varphi=0$, where V is a vector field of the form

$$V=b'(y)\frac{\partial}{\partial x}-a'(x)\frac{\partial}{\partial y}.$$

Writing out this condition for the expression for c_3 obtained above, we see that

$$\begin{aligned}&-3a_1^2a_3b_1c_2+6a_1a_2^2b_1c_2+2a_1a_2b_1c_0c_1+a_1^2b_2c_1-a_1a_4b_1c_1\\ &+3a_2a_3b_1c_1+3a_2b_1^2c_1=0.\end{aligned}\tag{13}$$

Since $d_1=a_3a_1-2a_2^2\neq 0$, we can use (13) to give an expression for c_2/c_1. We write out the condition $V(c_2/c_1)=0$ and obtain

$$\begin{aligned}&\left(-2a_1^3a_2a_4b_1^3+2a_1^3{a_3}^2b_1^3+2a_1^2a_2^2a_3b_1^3\right)c_0-b_3a_3a_1^5b_1\\ &+a_3{b_2}^2a_1^5+2b_3a_2^2a_1^4b_1-2\,{a_2}^2b_2^2a_1^4-a_4b_2a_1^4{b_1}^2\\ &+a_2a_3b_2a_1^3b_1^2-a_5a_3a_1^3b_1^3+a_4^2a_1^3b_1^3+4a_2^3b_2a_1^2b_1^2\\ &+2a_5a_2^2a_1^2b_1^3-3a_2a_3a_4a_1^2{b_1}^3-3a_2a_4a_1^2b_1^4+3a_3^3a_1^2b_1^3\\ &+3a_3^2a_1^2b_1^4-6a_2^3a_4a_1b_1^3+6a_2^4a_3b_1^3+6a_2^4b_1^4=0.\end{aligned}$$

Since $d_2=a_1a_2a_4-a_1a_3^2-a_2^2a_3\neq 0$, we can use this to express c_0 and to write out the condition $V(c_0)=0$; we have

$$\begin{aligned}\mathrm{Eq}_4&=a_1^7a_2a_4b_1^2b_4-4a_1^7a_2a_4b_1b_2b_3+3a_1^7a_2a_4b_2^3-a_1^7a_3^2b_1^2b_4\\ &+4a_1^7a_3^2b_1b_2b_3-3a_1^7a_3^2b_2^3-a_1^6a_2^2a_3b_1^2b_4+4a_1^6a_2^2a_3b_1b_2b_3\\ &-3a_1^6a_2^2a_3b_2^3+a_1^6a_2a_5b_1^3b_3-a_1^6a_2a_5b_1^2b_2^2-a_1^6a_3a_4b_1^3b_3\\ &+a_1^6a_3a_4{b_1}^2b_2^2-3a_1^5a_2a_3^2b_1^3b_3+3a_1^5a_2a_3^2b_1^2b_2^2+a_1^5a_3a_5b_1^4b_2\\ &-a_1^5a_4^2b_1^4b_2+2a_1^4a_2^2a_5b_1^4b_2-6a_1^4a_2a_3a_4b_1^4b_2-a_1^4a_2a_4a_6b_1^5\\ &+a_1^4a_2{a_5}^2b_1^5+2a_1^4a_3^3b_1^4b_2+a_1^4a_3^2a_6b_1^5-2a_1^4a_3a_4a_5b_1^5+a_1^4a_4^3b_1^5\\ &+3a_1^3a_2^3a_4b_1^4b_2-4a_1^3a_2^2a_3^2b_1^4b_2+a_1^3a_2^2a_3a_6b_1^5-6a_1^3a_2a_3^2a_5b_1^5\\ &+8a_1^3a_2a_3a_4^2b_1^5-3a_1^3a_3^3a_4b_1^5-3a_1^2a_2^4a_3b_1^4b_2-6a_1^2{a_2}^3a_4^2b_1^5\\ &+3a_1^2a_2^3a_5b_1^6-3a_1^2a_2^2a_3^2a_4{b_1}^5-12a_1^2a_2^2a_3a_4b_1^6+9a_1^2a_2a_3^4b_1^5\\ &+9a_1^2a_2{a_3}^3b_1^6+12a_1a_2^4a_3a_4{b_1}^5+6a_1a_2^4a_4b_1^6-6a_1a_2^3a_3^3b_1^5\\ &-6a_1a_2^3a_3^2b_1^6-6a_2^5a_3^2b_1^5-6a_2^5a_3b_1^6=0.\end{aligned}$$

Requiring that the expressions obtained for $c_3=\varphi$, $c_2/c_1=\chi$, and $c_0=\psi$ be functions of $a(x)+b(y)$ is not sufficient to ensure the existence of some $c=c_0$ satisfying (12). For this we need two more consistency conditions:

$$\mathrm{Eq}_5=\chi-\left(\frac{\varphi'_y}{b'}\right)'_y\Big/\varphi'_y=0,\quad \mathrm{Eq}_6=\psi-\frac{1}{b'}\left(\frac{1}{b'}\left(\frac{1}{b'}\,\varphi'_y\right)'_y\right)'_y=0,$$

which ensure that c_1, c_2, and c_3 are successive derivatives of c_0. The equation $\mathrm{Eq}_4=0$ has the differential order $(6,4)$, that is, six with respect to a and four with respect to b, $\mathrm{Eq}_5=0$ has the order $(5,5)$, and $\mathrm{Eq}_6=0$ has the order $(5,6)$.

Thus, we have proved the following lemma.

Lemma 10. *For every pair of functions $(a(x),b(y))$ such that $a_2\neq 0$, $d_1(a)\neq 0$, and $b_2\neq 0$, there is a nonconstant function $c(t)$ such that $z=c(a(x)+b(y))$ is a solution of (8) if and only if* $\mathrm{Eq}_4(a,b)=\mathrm{Eq}_5(a,b)=\mathrm{Eq}_6(a,b)=0$.

Now we make the change of variables $a_1 = A$, $a_2 = P(A)$, $b_1 = B$, $b_2 = Q(B)$ in Eq_4, where A and B are new independent variables and $P(A)$ and $Q(B)$ are new unknown functions. The expressions for d_1 and d_2 in the new variables take the form $d_1 = P(A)(AP'(A) - 2A)$ and $d_2 = P^3(A)(AP''(A) - P'(A))$. Denote the expressions for Eq_4, Eq_5, Eq_6 in the new variables by EQ_4, EQ_5, EQ_6, respectively. Note that EQ_4 has differential order 2, EQ_5 order 3, and EQ_6 order 4 with respect to Q. Denote the resultant of EQ_4 and $(\mathrm{EQ}\,4)'_A$ with respect to the variable Q'' by Res_1 and the resultant of EQ_4 and $(\mathrm{EQ}\,4)''_{AA}$ with respect to the variable Q'' by Res_2. Further, let $\mathrm{Res} = R(P, Q)$ be the resultant of Res_1 and Res_2 with respect to the variable Q'. This expression has the form

$$\mathrm{Res}(P, Q) = L(P)Q + M(P)B + N(P)B^2,$$

where $L(P)$, $M(P)$, and $N(P)$ are differential polynomials of order 5, 5, and 6, respectively. It follows immediately from the equality $\mathrm{Res}(P, Q) = 0$ that the following alternative holds.

Lemma 11. *If $(P(A), Q(B))$ is a solution of the equation*

$$\mathrm{EQ}_4(P(A), Q(B)) = \mathrm{EQ}_5(P(A), Q(B)) = \mathrm{EQ}_6(P(A), Q(B)) = 0,$$

then either $Q(B) = B(mB+n)$, where m and n are constant, or $L(P(A)) = M(P(A)) = N(P(A)) = 0$.

We start by considering the first possibility, that is, let $Q(B) = B(mB + n)$. The condition $b_2 \neq 0$ means that Q is not identically zero, and hence at least one of the constants is nonzero. Here the condition $Q'(B) = 2mB + n \neq 0$ holds.

Substitute $Q(B) = B(mB + n)$ into EQ_4, EQ_5, and EQ_6. We obtain differential polynomials depending on P; denote these polynomials by E_4, E_5, E_6, respectively. We have

$$\begin{aligned}
E_4 &= A^7Bm^3p_2 + A^7m^2np_2 - A^6Bm^3p_1 + A^6Bm^2p_0p_3 + 3A^6Bm^2p_1p_2 \\
&- A^6m^2np_1 + A^6mnp_0p_3 + 3A^6mnp_1p_2 - 3\,A^5Bm^2p_1^2 + A^5Bmp_0p_1p_3 - A^5Bmp_0p_2^2 \\
&+ 2A^5Bmp_1^2p_2 - A^4p_0^4p_2p_4 + A^4p_0^4p_3^2 - A^4p_0^3p_1p_2p_3 - 3A^4p_0^3p_2^3 \\
&- 3A^5mnp_1^2 + A^5np_0p_1p_3 - A^5np_0p_2^2 + 2A^5np_1^2p_2 + 2A^4Bmp_0^2p_3 + 2A^4Bmp_0p_1p_2 \\
&- 2A^4Bmp_1^3 + A^3p_0^4p_1p_4 + A^3p_0^3p_1^2p_3 + 12\,A^3p_0^3p_1p_2^2 + 2A^4np_0^2p_3 \\
&+ 2A^4np_0p_1p_2 - 2A^4np_1^3 + 3A^3Bmp_0^2p_2 - A^3Bmp_0p_1^2 - 6A^2{p_0}^4p_2^2 \\
&- 15A^2p_0^3p_1^2p_2 + 3\,A^3np_0^2p_2 - A^3np_0p_1^2 - 3A^2Bm{p_0}^2p_1 + 3A^2Bp_0^3p_3 \\
&+ 12Ap_0^4p_1p_2 + 6Ap_0^3p_1^3 - 3A^2np_0^2p_1 + 6ABp_0^3p_2 - 6p_0^4p_1^2 - 6B{p_0}^3p_1 = 0;
\end{aligned}$$

$$\begin{aligned}
E_5 &= (A^5m^2p_1 - 2A^4m^2p_0 + A^4mp_0p_2 + A^4mp_1^2 - A^3mp_0p_1 - 4A^2mp_0^2 \\
&+ 3A^2p_0^2p_2 - 6p_0^3)\ (A^4Bm^2 + 3A^3mp_0^2p_2 \\
&+ A^4mn + A^3Bmp_1 - 3A^2mp_0^2p_1 + A^2p_0^3p_3 + 3A^2p_0^2p_1p_2 \\
&+ A^3np_1 + 2A^2Bmp_0 - 3Ap_0^3p_2 - 3Ap_0^2p_1^2 + 2A^2np_0 \\
&+ 3p_0^3p_1 + 3Bp_0^2) = 0;
\end{aligned}$$

$$\begin{aligned}
E_6 &= (A^5m^2p_1 - 2A^4m^2p_0 + A^4mp_0p_2 + A^4mp_1^2 - A^3mp_0p_1 \\
&- 4A^2m{p_0}^2 + 3A^2p_0^2p_2 - 6p_0^3)(-2A^5m^2p_0^2p_2 + A^5Bm^2p_1 \\
&+ 2A^4m^2p_0^2p_1 + A^5mnp_1 - 2A^4Bm^2p_0 + A^4Bmp_0p_2 + A^4Bmp_1^2
\end{aligned}$$

$$
\begin{aligned}
&- 6A^3mp_0{}^3p_2 + A^3p_0^3p_1p_3 - A^3p_0^3p_2^2 + 2A^3p_0^2p_1^2p_2 - 2A^4mnp_0 \\
&+ A^4np_0p_2 + A^4np_1^2 - A^3Bmp_0p_1 + 6A^2mp_0^3p_1 - 2A^2p_0^4p_3 \\
&- 7A^2p_0^3p_1p_2 - 2A^2p_0^2p_1{}^3 - A^3np_0p_1 - 4A^2Bmp_0^2 + A^2Bp_0^2p_2 \\
&+ 6Ap_0^4p_2 + 8Ap_0^3p_1^2 - 4A^2np_0^2 + 2ABp_0^2p_1 \\
&- 6p_0^4p_1 - 6Bp_0^3) = 0.
\end{aligned}
$$

The polynomial E_4 is irreducible, $E_5 = e \cdot e_5$, and $E_6 = e \cdot e_6$, where e is a common factor

$$
e = A^5m^2p_1 - 2A^4m^2p_0 + A^4mp_0p_2 + A^4mp_1^2 - A^3mp_0p_1 - 4A^2mp_0^2 + 3A^2p_0^2p_2 - 6p_0^3,
$$

which has differential order 2, and e_5 and e_6 have order 3.

Let $e \neq 0$; then $E_4 = e_5 = e_6 = 0$. Eliminating p_3 from e_5 and e_6, we obtain

$$
R_{56} = 2A^3m^2p_0 + 3A^2mp_0p_1 - A^2Bm + Ap_0^2p_2 + Ap_0p_1^2 - A^2n - p_0^2p_1 - Bp_0 = 0.
$$

Differentiating with respect to A gives

$$
\begin{aligned}
(R_{56})'_A &= 2A^3m^2p_1 + 6A^2m^2p_0 + 3A^2mp_0p_2 + 3A^2mp_1^2 + 6Amp_0p_1 \\
&+ Ap_0^2p_3 + 4Ap_0p_2p_1 + Ap_1^3 - 2ABm - p_0p_1^2 - 2An - Bp_1 = 0.
\end{aligned}
$$

Eliminating p_3 from the equations $(R_{56})'_A = 0$ and $e_5 = 0$, we obtain $r_1 = 0$; eliminating p_3 from the equations $(R_{56})'_A = 0$ and $e_6 = 0$, we obtain $r_2 = 0$. Eliminating p_2 from the equations $R_{56} = 0$ and $r_1 = 0$, we obtain $r_3 = 0$. Finally, eliminating p_1 from r_1 and R_{56}, we obtain

$$
A^3p_0^2(Ap_1 - 2p_0)(mA^2 + p_0)(Bm + n) = 0,
$$

that is, $P(A) = -mA^2$. However, in this case, substituting this function into $e(P)$, we obtain zero. Therefore, the solutions of the system $E_4 = E_5 = E_6 = 0$ are reduced to the solution of $E_4 = e = 0$. Thus, we have the following result.

Lemma 12.

(a) *All solutions of the system* $E_4 = E_5 = E_6 = 0$ *are of the form*

$$
P(A) = -lA^2, \quad Q(B) = B(mB + n).
$$

(b) *The corresponding solutions of* $\mathrm{Eq}_4 = \mathrm{Eq}_5 = \mathrm{Eq}_6 = 0$ *are of the form*

$$
a(x) = l\ln(x + C_1) + c_2, \quad b(y) = -\frac{1}{m}\ln(m\exp(y + C_3) - 1) + C_4.
$$

(c) *There are no nonconstant functions* $c(t)$ *such that* $z = c(a(x) + b(y))$ *is a solution of* (8).

Now we look at the remaining possibility. We seek a solution of the system $L(P) = M(P) = N(P)$. Eliminating p_5 from $L = 0$ and $M = 0$, we obtain a reducible polynomial of differential order 4. Denote the factors of this polynomial that can vanish by R_1, R_2, and R_3. These factors have the following form:

$$\begin{aligned}R_1 &= 2A^3p_1p_3 - 3A^3p_2^2 - 4A^2p_0p_3 + 12A^2p_1p_2\\ &- 12Ap_0p_2 - 9Ap_1^2 + 12p_0p_1 = 0;\end{aligned}$$

$$\begin{aligned}R_2 &= A^5p_0p_2p_4 - A^5p_0p_3^2 + A^5p_1p_2p_3 + 3A^5p_2^3\\ &- A^4p_0p_1p_4 - 3A^4p_1^2p_3 - 6A^4p_1p_2^2 + 8A^3p_0p_1p_3\\ &- 6A^3p_0p_2^2 - 3A^3p_1^2p_2 - 8A^2p_0^2p_3 + 36A^2p_0p_1p_2\\ &+ 6A^2p_1^3 - 24Ap_0^2p_2 - 30Ap_0p_1^2 + 24p_0^2p_1 = 0;\end{aligned}$$

$$\begin{aligned}R_3 &= A^3p_0p_2p_4 - A^3p_0p_3^2 + A^3p_1p_2p_3 + 3A^3p_2^3 - A^2p_0p_1p_4\\ &- A^2p_1^2p_3 - 9A^2p_1p_2^2 + 9Ap_1^2p_2 - 3p_1^3 = 0.\end{aligned}$$

The polynomial R_1 has differential order 3, and R_2 and R_3 are of order 4.

Lemma 13.
(a) *There are no nontrivial solutions of the system* $L = M = N = R_1 = 0$.
(b) *There are no nontrivial solutions of the system* $L = M = N = R_2 = 0$.
(c) *There are no nontrivial solutions of the system* $L = M = N = R_3 = 0$.

Proof. The proof of the lemma is to eliminate variables in succession by calculating resultants and also deleting trivial factors in the resultants thus obtained.

1. We eliminate p_5 from the equations $L = 0$ and $(R_1)''_{AA} = 0$ and denote the result by $H = 0$.
2. We eliminate p_4 from the equations $H = 0$ and $(R_1)'_A = 0$ and denote the result by $H_1 = 0$.
3. We eliminate p_3 from the equations $H_1 = 0$ and $R_1 = 0$ and denote the result by $H_2 = 0$. Here

$$\begin{aligned}H_2 = A^4p_0p_2^2 + 6A^4p_1^2p_2 - 30A^3p_0p_1p_2 - 10A^3p_1^3\\ + 32A^2p_0^2p_2 + 53A^2p_0p_1^2 - 80Ap_0^2p_1 + 32p_0^3 = 0.\end{aligned}$$

4. We eliminate p_3 from the equations $H_1 = 0$ and $(H_2)'_A = 0$ and denote the result by $H_3 = 0$.
5. We eliminate p_2 from the equations $H_3 = 0$ and $H_2 = 0$ and denote the result by $H_4 = 0$. Here

$$\begin{aligned}H_4 &= -70253568A^9p_1^9 + 1395283968A^8p_0p_1^8 - 12200444928A^7p_0^2p_1^7\\ &+ 61620793344A^6p_0^3p_1^6 - 197967904768A^5p_0^4p_1^5 + 419085593600A^4p_0^5p_1^4\\ &- 583726145536A^3p_0^6p_1^3 + 514831679488A^2p_0^7{p_1}^2 - 260220846080Ap_0^8p_1\\ &+ 57230950400p_0^9 = 0.\end{aligned}$$

6. We eliminate p_2 from the equations $(H_4)'_A = 0$ and $H_2 = 0$ and denote the result by $H_5 = 0$. Here

$$\begin{aligned} H_5 &= 1103212141722206208A^{19}p_1^{19} - 47127416742629867520A^{18}p_0p_1^{18} \\ &+ 949249959445420572672A^{17}p_0^2p_1^{17} - 11983786221719720558592A^{16}p_0^3p_1^{16} \\ &+ 106306707637771054350336A^{15}p_0^4p_1^{15} - 704030499159252418953216A^{14}p_0^5p_1^{14} \\ &+ 3609492713687931533918208A^{13}p_0^6p_1^{13} - 14658410173981465507790848A^{12}p_0^7p_1^{12} \\ &+ 47843598571848865754906624A^{11}p_0^8p_1^{11} - 126586196852641571546333184A^{10}p_0^9p_1^{10} \\ &+ 272561521628794150809763840A^9p_0^{10}p_1^9 - 477462984612843785103081472A^8p_0^{11}p_1^8 \\ &+ 677459192105080195792240640A^7p_0^{12}p_1^7 - 771477939339944949012496384A^6p_0^{13}p_1^6 \\ &+ 694548641621792306115903488A^5p_0^{14}p_1^5 - 482838964862957556336689152A^4p_0^{15}p_1^4 \\ &+ 249801920472144918488285184A^3p_0^{16}p_1^3 - 90467314786526477326221312A^2p_0^{17}p_1^2 \\ &+ 20443125582599456471121920Ap_0^{18}p_1 - 2166876439506913643724800p_0^{19} = 0. \end{aligned}$$

7. The resultant of H_4 and H_5 with respect to p_1, after dividing by $(Ap_0)^{171}$, gives an integer with 277 digits. This proves part (a) of the lemma. Parts (b) and (c) can be proved in a similar way. □

This completes the proof of Theorem 3.

Note that the solutions of the Korteweg–de Vries equation occurring in the list given in Theorem 3 were previously known in some form [5]. However, Theorem 3 claims that it is impossible to extend the list in the framework of functions of complexity one.

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Department of Mechanics and Mathematics, Lomonosov Moscow State University
Email address: vkb@strogino.ru

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