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ON THE SOLVABILITY OF A BOUNDARY VALUE PROBLEM IN p-ADIC STRING THEORY

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Abstract. This paper is devoted to the study and solution of a boundary value problem for a convolution-type integral equation with cubic nonlinearity. The above problem has a direct application to the p-adic theory of open-closed strings for the scalar tachyon field. It is shown that a one-parameter family of monotone continuous bounded solutions exists. Under additional conditions on the kernel of the equation, an asymptotic formula for the solutions thus constructed is established. Using these results, as particular cases we obtain Zhukovskaya's theorem on rolling solutions of the nonlinear equation in the p-adic theory of open-closed strings and the Vladimirov–Volovich theorem on the existence of a nontrivial solution between certain vacua.

The results are extended to the case of a more general nonlinear boundary value problem.

§1. Introduction

This paper is devoted to investigating the following boundary value problem for an integral equation of convolution type with a cubic nonlinearity (and some generalizations of this equation):

$$af^3(x) + (1-a)f(x) = \int_{\mathbb{R}} K_a(x-t)f(t)\,dt, \qquad x \in \mathbb{R}, \tag{1}$$

$$f(\pm\infty) = \lim_{x\to\pm\infty} f(x) = \pm 1, \tag{2}$$

where the unknown function $f(x)$ is odd and continuous on $\mathbb{R}$. In equation (1), $a \in (0,1]$ is a numerical parameter and the kernel $K(x) \equiv K_a(x)$ is an even function defined on the set $\mathbb{R}$ and satisfies the following conditions:

$$K \in L_1(\mathbb{R}) \cap C_M(\mathbb{R}), \tag{3a}$$

$$K(\tau) \downarrow \text{ with respect to } \tau \text{ on } [0,+\infty), \tag{3b}$$

$$K(x) > 0, \quad x \in \mathbb{R} \quad \text{and} \quad \int_{\mathbb{R}} K(x)\,dx = 1, \tag{3c}$$

where $C_M(\mathbb{R})$ is the space of bounded continuous functions on the set $\mathbb{R}$.

Problems (1)–(2) have an immediate application in the p-adic theory of open-closed strings for the scalar tachyon field (see [1]–[6]). In particular, when

$$K(x) \equiv K_a(x) = \frac{1}{\sqrt{4\pi a}} e^{-x^2/4a}, \quad x \in \mathbb{R}, \tag{4}$$

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problems (1)–(2) describe the motion (the rolling) of tachyons with respect to the time of open-closed strings (see [1] and [2]). Only real solutions of equation (1) are of physical interest.

A direct verification shows that equation (1) has the following trivial solutions (physically a vacuum):

$$f(x) \equiv 0, \quad f(x) \equiv \pm 1, \quad x \in \mathbb{R}.$$

The main objective of this paper is to construct nontrivial monotonically nondecreasing continuous and bounded solutions between the vacua $f(x) \equiv \pm 1$.

One of the main results in the paper is the following theorem.

Theorem 1. *Let the kernel K be an even function on the set $\mathbb{R}$ and satisfy conditions* (3a)–(3c)*. Then, for any parameter $a \in (0, 1]$, the boundary value problems* (1)–(2) *have a one-parameter family of continuous monotonically nondecreasing bounded solutions of the form $\{f_c(x)\}_{c\in\mathbb{R}}$, and $f_0(x)$ is an odd function. Moreover, if*

$$m(K) \equiv \int_0^\infty tK(t)\,dt < +\infty,$$

then for any $c \in \mathbb{R}$,

$$1 \pm f_c \in L_1(\mathbb{R}^\mp),$$

where $\mathbb{R}^- \equiv (-\infty, 0]$ and $\mathbb{R}^+ \equiv [0, +\infty)$.

As a special case, Theorem 1 implies the Zhukovskaya theorem on rolling solutions of the nonlinear equation (1) with a kernel of the form (4) (see [1]).

In the second part of the paper, the results we obtain are extended to the case of a "nonhomogeneous" integral equation of convolution type with cubic nonlinearity (see Theorem 2).

In the last part we study a more general boundary value problem for the following nonlinear integral equation of convolution type:

$$Q\big(f(x)\big) = \int_{\mathbb{R}} K(x-t)f(t)\,dt, \quad x \in \mathbb{R}, \tag{5}$$

with the boundary conditions

$$f(\pm\infty) = \pm\eta, \qquad \eta > 0, \tag{6}$$

where Q is an odd continuous function on $\mathbb{R}$ satisfying certain conditions (see Theorems 3 and 4). The results obtained generalize the Vladimirov–Volovich result on the existence of a nontrivial solution between certain vacua (see [5, Theorem 5]).

§2. The proof of Theorem 1

We break up the proof of the theorem stated above into the following steps. In the first three steps we establish lemmas which enable us to prove Theorem 1.

Step I. We first prove the following simple but important lemma.

Lemma 1. *If K is an even function on the set $\mathbb{R}$ satisfying conditions* (3a) *and* (3c)*, then, for every $a \in (0, 1]$, the characteristic equation*

$$\int_0^\infty K(t)e^{-pt}\,dt = \frac{2-a}{4} \tag{7}$$

has a unique positive solution $p_0 \equiv p_0(a)$.

Proof. Consider the function

$$\chi(p) \equiv \int_0^\infty K(t)e^{-pt}\,dt - \frac{2-a}{4}, \quad p \in \mathbb{R}^+.$$

Taking (3a) and (3c) into account, since the kernel K is even we can readily see that

$$1)\ \chi \in C(\mathbb{R}^+), \qquad 2)\ \chi(p) \downarrow \text{ with respect to } p \text{ on } \mathbb{R}^+,$$
$$3)\ \chi(0) = \frac{a}{4} > 0, \quad \chi(+\infty) = \lim_{p\to+\infty}\chi(p) = \frac{a-2}{4} < 0.$$

Hence, using the Bolzano–Cauchy theorem, we complete the proof of Lemma 1. □

Remark 1. In the table below, we present approximate values of $p_0 \equiv p_0(a)$ depending on the values of the parameter a when the kernel has the form (4). The calculations were carried out using the software Mathcad.

a	1	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
p_0	0.769	0.675	0.593	0.521	0.454	0.393	0.334	0.275	0.215	0.145

Now consider the well-known Lalesco kernel

$$K(x) = \frac{1}{2}e^{-|x|}, \qquad x \in \mathbb{R}.$$

Then the unique solution of the characteristic equation (7) is defined by the formula

$$p_0 \equiv p_0(a) = \frac{a}{2-a}, \qquad a \in (0,1]. \tag{8}$$

We choose a number $a \in (0,1]$ (and hence also the number $p_0 \equiv p_0(a)$) to use in the arguments below.

Step II. Note that the a priori estimate given below plays an important role in the presentation to follow. The following assertion holds.

Lemma 2. *Under the assumptions of Lemma* 1*, the following lower bound holds:*

$$\int_{-\infty}^x K(t)e^{p_0 t}\,dt + e^{2\,p_0\,x}\int_x^\infty K(t)e^{-p_0 t}\,dt \geq 1 - \frac{a}{2} \qquad \forall x \in \mathbb{R}^+. \tag{9}$$

Proof. Consider the function

$$I(x) = \int_{-\infty}^x K(t)\,e^{p_0 t}\,dt + e^{2p_0\,x}\int_x^\infty K(t)\,e^{-p_0 t}\,dt - 1 + \frac{a}{2}, \qquad x \in \mathbb{R}^+.$$

Taking Lemma 1 into account, since the kernel K is even, we see from (3a) and (3c) that

$$I(0) = \int_{-\infty}^0 K(t)e^{p_0 t}\,dt + \int_0^\infty K(t)e^{-p_0 t}\,dt - 1 + \frac{a}{2} = 2\int_0^\infty K(t)e^{-p_0 t}\,dt - 1 + \frac{a}{2} = 0,$$

$$I'(x) = K(x)e^{p_0 x} + 2p_0e^{2p_0 x}\int_x^\infty K(t)e^{-p_0 t}dt - K(x)e^{p_0 x} = 2p_0e^{2p_0 x}\int_x^\infty K(t)e^{-p_0 t}dt > 0$$

$\forall x \in \mathbb{R}^+$. Hence, we have $I(x) \geq I(0)$ for all $x \in \mathbb{R}^+$, which implies the assertion of the lemma. □

Step III. Using inequality (9), we will now prove the following lemma, which is the main lemma in this paper.

Lemma 3. *Let the conditions of Lemma 1 be valid. Then the following bound holds:*

$$
\begin{aligned}
(10)\qquad & 1-2\int_x^\infty K(t)dt-e^{-p_0x}\int_{-\infty}^x K(t)e^{p_0t}dt+e^{p_0\,x}\int_x^\infty K(t)e^{-p_0t}dt \\
& \geq \left(1-\frac{a}{2}\right)\left(1-e^{-p_0x}\right),\ \ x\in\mathbb{R}^+.
\end{aligned}
$$

Proof. By analogy with the proof of Lemma 2, if we consider the corresponding function

$$
h(x)\equiv 1-2\int_x^\infty K(t)dt-e^{-p_0x}\int_{-\infty}^x K(t)e^{p_0t}\,dt
$$

$$
+e^{p_0x}\int_x^\infty K(t)e^{-p_0t}\,dt-\left(1-\frac{a}{2}\right)\left(1-e^{-p_0x}\right),\ \ x\in\mathbb{R}^+,
$$

we can readily prove that

$$
h(0)=0 \quad\text{and}\quad h'(x)=p_0e^{-p_0x}\left(\int_{-\infty}^x K(t)e^{p_0t}\,dt+e^{2p_0x}\int_x^\infty K(t)e^{-p_0t}\,dt-1+\frac{a}{2}\right).
$$

Using Lemma 2, we see that $h'(x)\geq 0,\ x\in\mathbb{R}^+$. This yields

$$
h(x)\geq h(0)=0,\ \ x\in\mathbb{R}^+. \qquad \square
$$

Remark 2. Taking conditions (3a) and (3c) into account, we readily see from Lemma 3 that

$$
(11)\qquad \int_0^\infty (K(x-t)-K(x+t))\left(1-e^{-p_0t}\right)\,dt\geq\left(1-\frac{a}{2}\right)\left(1-e^{-p_0x}\right),\ \ x\in\mathbb{R}^+.
$$

Indeed, using conditions (3a) and (3c), after simple manipulations we obtain

$$
\int_0^\infty (K(x-t)-K(x+t))\left(1-e^{-p_0t}\right)dt
$$

$$
=1-\int_0^\infty (K(x-t)-K(x+t))\,e^{-p_0t}\,dt-2\int_x^\infty K(t)\,dt
$$

$$
=1-2\int_x^\infty K(t)dt-e^{-p_0x}\int_{-\infty}^x K(t)e^{p_0t}\,dt+e^{p_0x}\int_x^\infty K(t)e^{-p_0t}\,dt\geq\left(1-\frac{a}{2}\right)\left(1-e^{-p_0x}\right).
$$

Step IV. Consider the following auxiliary boundary value problem for a nonlinear integral equation with a sum-difference kernel on the semiaxis:

$$
\begin{aligned}
(12)\qquad a\varphi^3(x)+(1-a)\varphi(x) &= \int_0^\infty (K(x-t)-K(x+t))\,\varphi(t)\,dt,\quad x\in\mathbb{R}^+, \\
(13)\qquad \varphi(+\infty) &= \lim_{x\to+\infty}\varphi(x)=1
\end{aligned}
$$

with respect to the unknown function $\varphi(x)$ which is continuous and bounded on $\mathbb{R}^+$. For equation (12), we introduce the following successive approximations:

$$a\varphi_{n+1}^3(x) + (1-a)\varphi_{n+1}(x) = \int_0^\infty (K(x-t) - K(x+t))\,\varphi_n(t)\,dt, \quad x \in \mathbb{R}^+, \tag{14}$$
$$\varphi_0(x) \equiv 1, \quad n = 0,1,2,\dots, \quad x \in \mathbb{R}^+.$$

We will prove that the sequence $\{\varphi_n(x)\}_{n=0}^\infty$ is bounded, using induction on n:

$$\varphi_n(x) \le 1, \quad n = 0,1,2,\dots, \quad x \in \mathbb{R}^+. \tag{15}$$

In the case of $n = 0$, inequality (15) holds automatically. Suppose that (15) holds for some $n \in \mathbb{N}$. Then, using the readily verifiable inequality

$$K(x-t) \ge K(x+t), \quad (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+ \tag{16}$$

and conditions (3a) and (3c), we have

$$a\varphi_{n+1}^3(x) + (1-a)\varphi_{n+1}(x) \le 1 - 2\int_x^\infty K(t)\,dt \le 1,$$

and so

$$a\left(\varphi_{n+1}(x) - 1\right)\left(\varphi_{n+1}^2(x) + \varphi_{n+1}(x) + 1\right) + (1-a)\left(\varphi_{n+1}(x) - 1\right) \le 0$$

or

$$\left(\varphi_{n+1}(x) - 1\right)\left(a\varphi_{n+1}^2(x) + a\varphi_{n+1}(x) + 1\right) \le 0. \tag{17}$$

Note that

$$a\varphi_{n+1}^2(x) + a\varphi_{n+1}(x) + 1 \ge 1 - \frac{a}{4} > 0.$$

Using the last relation, from (17) we we see that (15) holds for $n+1$, and hence for every n.

Step V. In this step, applying induction on n we prove that the sequence is decreasing:

$$\varphi_n(x) \downarrow \text{ with respect to } n. \tag{18}$$

Obviously, the inequality $\varphi_1(x) \le \varphi_0(x)$ follows from (15). Suppose that (18) holds for some positive integer n. Then, taking (16) into account, we see from (14) that

$$a\varphi_{n+1}^3(x) + (1-a)\varphi_{n+1}(x) \le a\varphi_n^3(x) + (1-a)\varphi_n(x), \quad x \in \mathbb{R}^+. \tag{19}$$

Since

$$F(t) = at^3 + (1-a)t \uparrow \text{ with respect to } t \text{ on } \mathbb{R}$$

and $a \in (0,1]$, it follows from (19) that $\varphi_{n+1}(x) \le \varphi_n(x)$.

Step VI. Now we prove that all the elements of the sequence $\{\varphi_n(x)\}_{n=0}^\infty$ satisfy the following lower bound:

$$\varphi_n(x) \ge \frac{1 - e^{-p_0 x}}{\sqrt{2}}, \quad x \in \mathbb{R}^+,\ n = 0,1,2,\dots. \tag{20}$$

This bound holds for $n = 0$ by the definition of the zeroth approximation. Suppose that (20) holds for some $n \in \mathbb{N}$. Then, using Lemma 3 and taking (16) into account, we see from (14) that

$$a\varphi_{n+1}^3(x)+(1-a)\varphi_{n+1}(x)\geq\frac{1}{\sqrt{2}}\int\limits_0^\infty\left(K(x-t)-K(x+t)\right)\left(1-e^{-p_0t}\right)\,dt$$
$$\geq\frac{1}{\sqrt{2}}\left(1-\frac{a}{2}\right)\left(1-e^{-p_0x}\right)\geq\frac{1}{\sqrt{2}}\left((1-a)\left(1-e^{-p_0x}\right)+\frac{a}{2}\left(1-e^{-p_0x}\right)^3\right)$$
$$=(1-a)\,\frac{1-e^{-p_0x}}{\sqrt{2}}+a\left(\frac{1-e^{-p_0x}}{\sqrt{2}}\right)^3.$$

As $F(t)$ is monotonic, the inequality thus obtained gives the following bound:

$$\varphi_{n+1}(x)\geq\frac{1-e^{-p_0x}}{\sqrt{2}},\quad x\in\mathbb{R}^+.$$

Step VII. We will also use induction on n to prove that

$$\varphi_n\in C(\mathbb{R}^+),\quad n=0,1,2,\dots,\tag{21}$$

$$\varphi_n(x)\uparrow\text{ with respect to } x \text{ on } \mathbb{R}^+,\quad n=0,1,2,\dots.\tag{22}$$

We first look at the proof of assertion (22).

Let $x_1,x_2\in\mathbb{R}^+$, $x_1>x_2$, be arbitrary numbers. The inequality $\varphi_0(x_1)\geq\varphi_0(x_2)$ is obvious. Suppose that the inequality $\varphi_n(x_1)\geq\varphi_n(x_2)$ holds for some positive integer n. Then, representing the iterations (14) in the form

$$a\varphi_{n+1}^3(x)+(1-a)\varphi_{n+1}(x)=\int\limits_{-\infty}^{x}K(t)\,\varphi_n(x-t)dt-\int\limits_0^\infty K(x+t)\,\varphi_n(t)\,dt,$$
$$\varphi_0(x)\equiv 1,\quad n=0,1,2,\dots,\quad x\in\mathbb{R}^+,$$

and using (3b), we obtain

$$a\varphi_{n+1}^3(x_1)+(1-a)\varphi_{n+1}(x_1)=\int\limits_{-\infty}^{x_1}K(t)\,\varphi_n(x_1-t)\,dt-\int\limits_0^\infty K(x_1+t)\,\varphi_n(t)\,dt$$
$$\geq\int\limits_{-\infty}^{x_2}K(t)\,\varphi_n(x_2-t)\,dt-\int\limits_0^\infty K(x_2+t)\,\varphi_n(t)\,dt=a\varphi_{n+1}^3(x_2)+(1-a)\varphi_{n+1}(x_2).$$

Using this inequality together with the monotonicity of the function $F(t)$, we arrive at the inequality $\varphi_{n+1}(x_1)\geq\varphi_{n+1}(x_2)$.

The membership relation (21) follows immediately from the fact that both the kernels, K and F, are continuous; we have also used the fact that F is monotonic.

Step VIII. It follows from (15), (18), (20), and (21) that the sequence of continuous functions $\{\varphi_n(x)\}_{n=0}^\infty$ has pointwise limit as $n\to\infty$:

$$\lim_{n\to\infty}\varphi_n(x)=\varphi(x),\quad x\in\mathbb{R}^+,$$

where, by Beppo Levi's theorem (see [7]), the limit function $\varphi(x)$ satisfies equation (12). By (15) and (20), we obtain the following two-sided bound for $\varphi(x)$:

$$\frac{1-e^{-p_0x}}{\sqrt{2}}\leq\varphi(x)\leq 1,\quad x\in\mathbb{R}^+.\tag{23}$$

Now $\varphi \in M(\mathbb{R}^+)$ and $K \in L_1(\mathbb{R}) \cap C_M(\mathbb{R})$ by (3a), and so it is known that (see [8])

$$\int_0^\infty (K(x-t) - K(x+t))\,\varphi(t)\,dt \in C_M(\mathbb{R}^+). \tag{24}$$

Since the function F is continuous and monotonic, taking (24) into account, we see from (12) that $\varphi \in C_M(\mathbb{R}^+)$. Since the sequence of continuous functions $\{\varphi_n(x)\}_{n=0}^\infty$ tends to a continuous limit function $\varphi(x)$ on $\mathbb{R}^+$, it follows from Dini's theorem (see [9]) that the convergence of the sequence is uniform on every compact set in $\mathbb{R}^+$.

On the other hand, $\varphi_n(x) \uparrow$ with respect to x on $\mathbb{R}^+$, $\quad n = 0, 1, 2, \dots$ (see (22)), and hence the limit $\lim_{x\to+\infty} \varphi(x) \equiv \lambda$, $0 < \lambda < +\infty$, exists. In equation (12), passing to the limit as $x \to +\infty$ and taking the well-known limit relation (see [8])

$$\lim_{x\to+\infty} \int_0^\infty K(x-t)\,\varphi(t)\,dt = \int_{\mathbb{R}} K(\tau)\,d\tau \cdot \lim_{x\to+\infty} \varphi(x)$$

into account, we have: $a\lambda^3 + (1-a)\lambda = \lambda$. Since $\lambda > 0$, it follows from the last equation that $\lambda = 1$.

Step IX. Since the kernel K is even and the function F defined on $\mathbb{R}$ is odd, direct verification shows that the odd extension to $\mathbb{R}^-$ of the function φ,

$$f_0(x) \equiv \begin{cases} \varphi(x) & \text{for } x \geq 0, \\ -\varphi(-x) & \text{for } x < 0, \end{cases} \tag{25}$$

is a continuous and monotonic nondecreasing solution of the boundary value problems (1)–(2). Since the kernel K and the function φ are continuous, it follows from (12) and (25) that $f_0(0) = 0$.

Note that all possible shifts of the function $f_0(x)$ also satisfy the boundary value problems (1)–(2). Indeed, it is clear that the limit relations (2) hold for the one-parameter family of functions $\{f_c(x)\}_{c\in\mathbb{R}}$ of the form $f_c(x) = f_0(x+c)$. We will prove that, for every $c \in \mathbb{R}$, the function $f_c(x)$ in the family also satisfies equation (1). We have

$$\begin{aligned} \int_{\mathbb{R}} K(x-t) f_c(t)\,dt &= \int_{\mathbb{R}} K(x-t) f_0(t+c)\,dt = \int_{\mathbb{R}} K(x+c-\tau) f_0(\tau)\,d\tau \\ &= a f_0^3(x+c) + (1-a) f_0(x+c) = a f_c^3(x) + (1-a) f_c(x). \end{aligned}$$

Thus, the first part of the theorem is proved.

In the next two steps we will prove that if $m(K) < +\infty$, then

$$1 \pm f_c \in L_1(\mathbb{R}^\mp) \ \ \forall c \in \mathbb{R}.$$

Step X. First we prove by induction on n that

$$1 - \varphi_n \in L_1(\mathbb{R}^+), \ n = 0, 1, 2, \dots . \tag{26}$$

When $n = 0$, the inclusion (26) follows from (14). Let $1 - \varphi_n \in L_1(\mathbb{R}^+)$ for some $n \in \mathbb{N}$. Then, taking relations (3a)–(3c) and (15) into account, we see from (14) that

$$0 \le 1 - \varphi_{n+1}(x) \le (1 - \varphi_{n+1}(x)) \left(1 + a\varphi_{n+1}(x) + a\varphi_{n+1}^2(x)\right)$$
$$= 1 - a\varphi_{n+1}^3(x) - (1-a)\varphi_{n+1}(x) = \int_0^\infty K(x-t)\left(1 - \varphi_n(t)\right) dt + \int_x^\infty K(t)\, dt$$
$$+ \int_0^\infty K(x+t)\varphi_n(t)\, dt \le 2\int_x^\infty K(t)\, dt + \int_0^\infty K(x-t)\left(1 - \varphi_n(t)\right) dt \in L_1(\mathbb{R}^+),$$

because

$$0 \le \int_0^\infty \int_0^\infty K(x-t)\left(1 - \varphi_n(t)\right) dt dx \le \int_0^\infty (1 - \varphi_n(t)) dt \cdot \int_{\mathbb{R}} K(t) dt$$
$$= \int_0^\infty \left(1 - \varphi_n(t)\right) dt < +\infty$$

and

$$\int_0^\infty \int_x^\infty K(t) dt dx = \int_0^\infty tK(t) dt \equiv m(K) < +\infty.$$

Hence, $1 - \varphi_{n+1} \in L_1(\mathbb{R}^+)$.

Step XI. First, we note that there is a number $r > 0$ such that

$$\rho \equiv \int_{-\infty}^{r} K(t) dt < 1. \tag{27}$$

Inequality (27) follows immediately from (3c). Fix $r > 0$.

In the previous step we proved that

$$\begin{aligned}(1 - \varphi_{n+1}(x)) \left(1 + a\varphi_{n+1}(x) + a\varphi_{n+1}^2(x)\right) \\ \le 2\int_x^\infty K(t)\, dt + \int_0^\infty K(x-t)\left(1 - \varphi_n(t)\right) dt,\ n = 0, 1, 2, \ldots,\ x \in \mathbb{R}^+.\end{aligned} \tag{28}$$

Integrating both sides of (28) from 0 to ∞ with respect to x, we obtain

$$\int_0^\infty (1-\varphi_{n+1}(x))\left(1+a\varphi_{n+1}(x)+a\varphi_{n+1}^2(x)\right)dx$$

$$\leq 2\int_0^\infty\int_x^\infty K(t)dt\,dx+\int_0^\infty\int_0^\infty K(x-t)\,(1-\varphi_n(t))\,dt\,dx$$

$$\leq 2m(K)+\int_0^\infty\int_0^r K(x-t)\,(1-\varphi_{n+1}(t))\,dt\,dx+\int_0^\infty\int_r^\infty K(x-t)\,(1-\varphi_{n+1}(t))\,dt\,dx$$

$$= 2m(K)+\int_0^r (1-\varphi_{n+1}(t))\int_0^\infty K(x-t)dx\,dt+\int_r^\infty (1-\varphi_{n+1}(t))\int_0^\infty K(x-t)dxdt$$

$$\leq 2m(K)+\int_0^r (1-\varphi_{n+1}(t))\int_{-\infty}^t K(y)dy\,dt+\int_r^\infty (1-\varphi_{n+1}(t))\,dt$$

$$\leq 2m(K)+\rho\int_0^r (1-\varphi_{n+1}(t))\,dt+\int_r^\infty (1-\varphi_{n+1}(t))\,dt.$$

Therefore, by (20), it follows from this inequality that

$$\int_0^r (1-\varphi_{n+1}(x))\,dx+\left(1+\frac{a\left(1-e^{-p_0 r}\right)}{\sqrt{2}}+\frac{a\left(1-e^{-p_0 r}\right)^2}{2}\right)\int_r^\infty (1-\varphi_{n+1}(x))\,dx$$

$$\leq \int_0^\infty (1-\varphi_{n+1}(x))\left(1+a\varphi_{n+1}(x)+a\varphi_{n+1}^2(x)\right)dx$$

$$\leq 2m(K)+\rho\int_0^r (1-\varphi_{n+1}(x))\,dx+\int_r^\infty (1-\varphi_{n+1}(x))\,dx.$$

In turn, this implies that

$$(1-\rho)\int_0^r (1-\varphi_{n+1}(x))\,dx+\left(\frac{a\left(1-e^{-p_0 r}\right)}{\sqrt{2}}+\frac{a\left(1-e^{-p_0 r}\right)^2}{2}\right)\int_r^\infty (1-\varphi_{n+1}(x))\,dx$$
$$\leq 2m(K).$$

Hence,

$$\min\left\{(1-\rho)\,;\left(\frac{a\left(1-e^{-p_0 r}\right)}{\sqrt{2}}+\frac{a\left(1-e^{-p_0 r}\right)^2}{2}\right)\right\}\int_0^\infty (1-\varphi_{n+1}(x))\,dx\leq 2m(K)$$

or

$$\int_0^\infty (1-\varphi_{n+1}(x))\,dx\leq 2m(K)\left(\min\left\{(1-\rho)\,;\left(\frac{a\left(1-e^{-p_0 r}\right)}{\sqrt{2}}+\frac{a\left(1-e^{-p_0 r}\right)^2}{2}\right)\right\}\right)^{-1}.$$

Since $\varphi_n(x) \downarrow$ with respect to n and $\varphi_n(x) \leq 1$ for $n = 0, 1, 2, \ldots,$ and $x \in \mathbb{R}^+$ (see (18) and (15)), it follows from Lebesgue's theorem that $1 - \varphi \in L_1(\mathbb{R}^+)$ and

$$\int_0^\infty (1 - \varphi(x))\, dx \leq 2m(K) \left(\min\left\{ (1 - \rho)\,; \left(\frac{a\,(1 - e^{-p_0 r})}{\sqrt{2}} + \frac{a\,(1 - e^{-p_0 r})^2}{2} \right) \right\} \right)^{-1}.$$

Since $1 - \varphi \in L_1(\mathbb{R}^+)$, it follows immediately from the definitions of the functions f_c and f_0 (see formula (25)) that $1 \pm f_c \in L_1(\mathbb{R}^\mp)$ $\forall c \in \mathbb{R}$. Thus, *the proof of the theorem is complete.*

§3. The solvability of the corresponding "inhomogeneous" equation

Consider the corresponding "inhomogeneous" equation on the whole line:

$$a\Phi^3(x) + (1 - a)\Phi(x) = g(x) + \int_{\mathbb{R}} K(x - t)\Phi(t)\, dt, \qquad x \in \mathbb{R}, \tag{29}$$

with respect to the unknown function $\Phi(x)$, which is continuous and bounded. Here $a \in (0, 1]$ is a numerical parameter and the kernel K again satisfies conditions (3a)–(3c). We also make the following assumptions on the free term $g(x)$:

I): $g(x)$ is an odd bounded continuous function defined on $\mathbb{R}$,

II): $g(x) \geq 0, x \in \mathbb{R}^+$, and $g(x) \uparrow$ with respect to x on $\mathbb{R}^+$.

Taking condition I) on $g(x)$ and conditions (3a)–(3c) on $K(x)$ into account, it is easy to see that if $\Psi(x)$ is a continuous solution on $\mathbb{R}^+$ of the nonlinear integral equation of the form

$$a\Psi^3(x) + (1 - a)\Psi(x) = g(x) + \int_0^\infty (K(x - t) - K(x + t))\Psi(t)\, dt, \qquad x \in \mathbb{R}^+, \tag{30}$$

then

$$\Phi(x) \equiv \begin{cases} \Psi(x) & \text{for } x \in \mathbb{R}^+, \\ -\Psi(-x) & \text{for } x \in \mathbb{R}^- \end{cases}$$

is a continuous odd solution of equation (29).

Since $g(x)$ has properties I) and II) it follows that

$$\lim_{x \to +\infty} g(x) \equiv c_0 < +\infty, \quad g(x) \leq c_0, \quad x \in \mathbb{R}^+. \tag{31}$$

Consider the following characteristic equation:

$$a\tau^3 - a\tau = c_0 \tag{32}$$

with respect to $\tau \in \mathbb{R}^+$. First, we will show that equation (32) has a unique positive solution c_* for every $a \in (0, 1]$, and $c_* \geq 1$.

Consider the function

$$G(\tau) \equiv a\tau^3 - a\tau - c_0, \quad \tau \in \mathbb{R}^+. \tag{33}$$

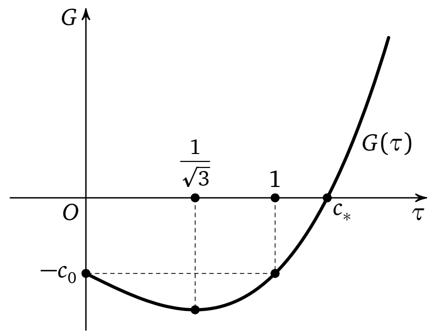
Obviously, $G(0) = G(1) = -c_0$ and $G(+\infty) = +\infty$,

$$G'(\tau) \leq 0 \ \text{ for } \ \tau \in \left[0, \frac{1}{\sqrt{3}}\right],$$

$$G'(\tau) \geq 0 \ \text{ for } \ \tau \in \left[\frac{1}{\sqrt{3}}, +\infty\right].$$

It follows from the properties of G listed above that there is a unique number $c_* \geq 1$ such that $G(c_*) = 0$ (see the figure), that is,

$$ac_*^3 - ac_* = c_0. \tag{34}$$



Consider the following successive approximations:

$$\begin{aligned} &a\Psi_{n+1}^3(x) + (1-a)\Psi_{n+1}(x) = g(x) + \int_0^\infty (K(x-t) - K(x+t))\Psi_n(t)\,dt, \\ &\Psi_0(x) \equiv c_*, \quad n = 0,1,2,\ldots, \quad x \in \mathbb{R}^+. \end{aligned} \tag{35}$$

First we will prove that

$$\Psi_n(x) \quad \downarrow \text{ with respect to } n. \tag{36}$$

To this end, we prove that $\Psi_1(x) \leq \Psi_0(x)$, $x \in \mathbb{R}^+$. Taking (31) and (34) and conditions (3a)–(3c) into account, we see from (35) that

$$a\Psi_1^3(x) + (1-a)\Psi_1(x) \leq c_0 + c_* = ac_*^3 - ac_* + c_* = ac_*^3 + (1-a)c_*. \tag{37}$$

As $F(t)$ is continuous and monotonic, it follows from (37) that $\Psi_1(x) \leq \Psi_0(x)$. Assuming that $\Psi_n(x) \leq \Psi_{n-1}(x)$, $x \in \mathbb{R}^+$, for some $n \in \mathbb{N}$ and using the monotonicity of the function $F(t)$ in (35) again, we see that $\Psi_{n+1}(x) \leq \Psi_n(x)$.

As in the proof of Theorem 1, we can verify that the following hold true:

$$1)\ \Psi_n(x) \geq \frac{1 - e^{-p_0 x}}{\sqrt{2}}, \quad n = 0,1,2,\ldots, \quad x \in \mathbb{R}^+, \tag{38}$$

where $p_0 = p_0(a)$ is the unique solution of the characteristic equation (7),

$$\begin{aligned} &2)\ \Psi_n \in C(\mathbb{R}^+) \text{ for } n = 0,1,2,\ldots, \\ &3)\ \Psi_n \uparrow \text{ with respect to } x \text{ on } \mathbb{R}^+, \quad n = 0,1,2,\ldots. \end{aligned}$$

The facts 1)–3) imply that the sequence $\{\Psi_n(x)\}_{n=0}^\infty$, converges pointwise and, by Beppo Levi's theorem, the limit function $\Psi(x) \equiv \lim_{n\to\infty} \Psi_n(x)$ satisfies equation (30). Since $K \in C(\mathbb{R})$, $g \in C(\mathbb{R})$ and $F \in C(\mathbb{R})$, it follows from (30) that $\Psi \in C(\mathbb{R})$. It follows immediately from fact 3) that $\Psi(x) \uparrow$ with respect to x on $\mathbb{R}^+$.

From (36) and (38), we conclude that $\Psi(x)$ satisfies the following two-sided inequality:

$$\frac{1 - e^{-p_0 x}}{\sqrt{2}} \leq \Psi(x) \leq c_*, \quad x \in \mathbb{R}^+.$$

Write $\mu = \lim_{x\to+\infty}\Psi(x) < +\infty$. Taking (31) and the limit relation

$$\lim_{x\to+\infty}\int_0^\infty K(x-t)\Psi(t)dt = \mu\cdot\int_{-\infty}^{+\infty}K(\tau)d\tau = \mu$$

into account, we see from (30) that $a\mu^3+(1-a)\mu = c_0+\mu$ or $a\mu^3 - a\mu = c_0$. Since $\mu > 0$, we see from (32) and (34) that $\mu = c_*$. We will now prove that if we assume in addition that $c_0 - g \in L_1(\mathbb{R}^+)$, then $c_* - \Psi \in L_1(\mathbb{R}^+)$. To this end, taking (31) and (3a)–(3c) into account, we represent the iterations (35) in the following form:

$$a(c_*^3 - \Psi_{n+1}^3(x)) + (1-a)(c_* - \Psi_{n+1}(x)) = c_* + c_0 - g(x)$$

$$-\int_0^\infty (K(x-t) - K(x+t))\Psi_n(t)dt, \quad \Psi_0(x)\equiv c_*, \quad n = 0,1,2,\dots, \quad x\in\mathbb{R}^+,$$

or

$$(c_* - \Psi_{n+1}(x))(ac_*^2 + ac_*\Psi_{n+1}(x) + a\Psi_{n+1}^2(x) + 1 - a)$$

$$= \int_0^\infty K(x-t)(c_* - \Psi_n(t))dt + \int_0^\infty K(x+t)\Psi_n(t)dt + c_0 - g(x)$$

$$+c_*\int_0^\infty K(x+t)dt, \quad \Psi_0(x)\equiv c_*, \quad n = 0,1,2,\dots, \quad x\in\mathbb{R}^+.$$

Since $c_* \geq 1$, we can prove (as in the proof of Theorem 1; see Steps X and XI) that $c_* - \Psi_n \in L_1(\mathbb{R}^+)$ and

$$\int_0^\infty (c_* - \Psi_n(x))dx \leq (2c_*m(K) + \int_0^\infty (c_0 - g(x))dx)\cdot q,$$

where

$$q \equiv \left(\min\left\{(1-\rho)\,;\left(\frac{a\,(1-e^{-p_0 r})}{\sqrt{2}} + \frac{a\,(1-e^{-p_0 r})^2}{2}\right)\right\}\right)^{-1}.$$

Hence, by Lebesgue's theorem, we can say that $c_* - \Psi \in L_1(\mathbb{R}^+)$, and the following bound holds:

$$\int_0^\infty (c_* - \Psi(x))dx \leq \left(2c_*m(K) + \int_0^\infty (c_0 - g(x))dx\right)\cdot q.$$

The arguments we have given above prove the following theorem.

Theorem 2. *Let K be an even function defined on $\mathbb{R}$ and satisfying conditions* (3a)–(3c). *In this case, if g has properties* I) *and* II), *then equation* (29) *has a continuous monotonic nondecreasing solution $\Phi(x)$ on $\mathbb{R}$, and $\lim_{x\to\pm\infty}\Phi(x) = \pm c_*$. Moreover, if in addition $m(K) < +\infty$ and $c_0 - g \in L_1(\mathbb{R}^+)$, then $c_* \pm \Phi \in L_1(\mathbb{R}^\mp)$.*

§4. Some generalizations of the above results

Consider the boundary value problems (5)–(6) under the assumption that the kernel function K that is defined on $\mathbb{R}$ is even and satisfies conditions (3a)–(3c) and that Q is a continuous function on $\mathbb{R}$ that is odd and satisfies the following conditions: there are numbers $\varepsilon \in (0,1)$ and $\xi \in (0,\eta)$ for which

(A): $0 \le Q(u) \le \varepsilon u, u \in [0,\xi]$,

(B): $Q(u) \uparrow$ with respect to u on the interval $[0,\eta]$ and $Q(\eta) = \eta$, where η stands for the first positive root of the equation $Q(u) = u$.

The following theorem holds.

Theorem 3. *Let the kernel K be an even function on $\mathbb{R}$ that satisfies* (3), *and let Q be an odd continuous function on $\mathbb{R}$ that satisfies* (A) *and* (B). *Then the boundary value problems* (5), (6) *have a one-parameter family of continuous bounded solutions of the form $\{f_c(x)\}_{c\in\mathbb{R}}$ that are monotonic nondecreasing, and $f_0(x)$ is an odd function.*

Proof.

Step I. Lemmas 4, 5, and 6 presented below are proved by analogy with the proofs of Lemmas 1, 2, and 3, respectively.

Lemma 4. *If K is an even function on $\mathbb{R}$ satisfying conditions* (3a) *and* (3c), *then the characteristic equation*

$$\int_0^\infty K(t)e^{-pt}\,dt = \frac{\varepsilon}{2}, \quad \varepsilon \in (0,1), \tag{39}$$

has a unique positive solution $p_* \equiv p_*(\varepsilon)$.

Lemma 5. *Under the assumptions of Lemma* 4, *the following lower bound holds:*

$$\int_{-\infty}^{x} K(t)e^{p_* t}\,dt + e^{2\,p_*\,x}\int_x^\infty K(t)e^{-p_* t}\,dt \ge \varepsilon \ \forall x \in \mathbb{R}^+. \tag{40}$$

Lemma 6. *Let the conditions of Lemma* 4 *hold. Then the following lower bound holds:*

(41)
$$1 - 2\int_x^\infty K(t)dt - e^{-p_* x}\int_{-\infty}^{x} K(t)e^{p_* t}\,dt + e^{p_*\,x}\int_x^\infty K(t)e^{-p_* t}\,dt \ge \varepsilon\left(1 - e^{-p_* x}\right), \quad x \in \mathbb{R}^+.$$

In particular, Lemma 6 readily implies the bound

$$\frac{1}{\varepsilon}\int_0^\infty (K(x-t) - K(x+t))\left(1 - e^{-p_* t}\right)\,dt \ge 1 - e^{-p_* x}, \ \ x \in \mathbb{R}^+. \tag{42}$$

Step II. Now, along with the boundary value problem (5), (6), we consider the following boundary value problem on the semiaxis:

$$Q\left(\varphi(x)\right) = \int_0^\infty (K(x-t) - K(x+t))\,\varphi(t)\,dt, \quad x \in \mathbb{R}^+, \tag{43}$$

$$\varphi(+\infty) = \lim_{x\to\infty} \varphi(x) = \eta > 0 \tag{44}$$

and introduce the following successive approximations for equation (43):

$$Q\left(\varphi_{n+1}(x)\right)=\int\limits_0^\infty (K(x-t)-K(x+t))\,\varphi_n(t)\,dt, \tag{45}$$
$$\varphi_0(x)\equiv\eta,\quad n=0,1,2,\ldots,\quad x\in\mathbb{R}^+.$$

Using the fact that Q is an odd continuous function on $\mathbb{R}$ with properties (A) and (B), it is easy to prove, similarly to the proof of Theorem 1, that the sequence of functions $\{\varphi_n(x)\}_{n=0}^\infty$ satisfies the following:

$$\varphi_n(x)\ \downarrow \text{ with respect to }\ n,\quad n=0,1,2,\ldots,\quad x\in\mathbb{R}^+; \tag{46}$$

$$\varphi_n\ \in C(\mathbb{R}^+),\quad n=0,1,2,\ldots; \tag{47}$$

$$\varphi_n(x)\ \uparrow \text{ with respect to } x\quad \text{on}\quad \mathbb{R}^+,\quad n=0,1,2,\ldots; \tag{48}$$

$$\varphi_n(x)\ \geq\ \xi\left(1-e^{-p_*x}\right),\ n=0,1,2,\ldots,\ x\in\mathbb{R}^+. \tag{49}$$

Relations (46)–(49) imply that the function sequence $\{\varphi_n(x)\}_{n=0}^\infty$ converges pointwise: $\lim_{n\to\infty}\varphi_n(x)=\varphi(x)$, and the function $\varphi(x)$ satisfies equation (43) by the continuity of the function Q and Beppo Levi's limit theorem. Taking (46) and (49) into account, we can say that the limit function $\varphi(x)$ satisfies the following two-sided inequality:

$$\xi\left(1-e^{-p_*x}\right)\leq\varphi(x)\leq\eta,\quad x\in\mathbb{R}^+. \tag{50}$$

Since $Q\in C(\mathbb{R})$ and $K\in C_M(\mathbb{R})$, we see from (43), taking (50) and conditions (A) and (B) into account, that $\varphi\in C(\mathbb{R}^+)$ and $\varphi(0)=0$. It also follows from (48) that

$$\varphi(x)\uparrow \text{with respect to } x\quad \text{on } \mathbb{R}^+. \tag{51}$$

Step III. Finally, our final step is to prove that

$$\lim_{x\to+\infty}\varphi(x)=\eta. \tag{52}$$

By (50) and (51), the limit $\lim_{x\to+\infty}\varphi(x)\equiv\lambda>0$ exists, and $\lambda\leq\eta$. However, as is well known,

$$\lim_{x\to+\infty}\int\limits_0^\infty (K(x-t)-K(x+t))\ \varphi(t)\,dt=\lambda\int\limits_{\mathbb{R}} K(\tau)\,d\tau=\lambda. \tag{53}$$

Using the continuity of the function Q and formula (53), we see from (43) that

$$\lim_{x\to+\infty}Q\left(\varphi(x)\right)=Q\left(\lim_{x\to+\infty}\varphi(x)\right)=\lambda\int\limits_{\mathbb{R}}K(\tau)\,d\tau=\lambda,$$

and hence

$$Q(\lambda)=\lambda. \tag{54}$$

In this case, using the fact that $\lambda\in(0,\eta]$ and the assumption that η is the first positive root of the equation $Q(u)=u$, we see from (54) that $\lambda=\eta$.

Similarly, as in the proof of Theorem 1, we can prove that the odd extension of the function φ to $\mathbb{R}^-$ (see formula (25)) and all possible shifts of the extension are continuous monotone nondecreasing solutions of the boundary value problems (5), (6). This completes the proof of the theorem. □

Theorem 4. *Let the kernel K satisfy the conditions of Theorem 2 and have finite moment of first order:*

$$m(K) \equiv \int_0^\infty tK(t)\,dt < +\infty.$$

Suppose, further, that Q is an odd continuous function on $\mathbb{R}$ satisfying condition (B)*. In this case, if the function Q satisfies the two-sided inequality (which is stronger than* (A)*)*

$$0 \le Q(u) \le \frac{au^3}{\eta^2} + (1-a)u, \quad a \in (0,1], \quad u \in [0,\eta],$$

which is stronger than (A)*, then $\eta \pm f \in L_1(\mathbb{R}^\mp)$.*

The proof is similar to that of the second part of Theorem 1.

Remark 3. It is of interest to note that Theorems 3 and 4 generalize the first ($\varepsilon = 1 - \frac{a}{2}$; $\xi = \frac{1}{\sqrt{2}}$; $\eta = 1$) and second parts of Theorem 1, respectively.

Remark 4. Theorem 3 implies, as a special case, the Vladimirov–Volovich existence theorem (see Theorem 5 in [5]) if we choose a function of the form (4) for the kernel K and take $Q(u) = u^p$, $a = 1$, $\varepsilon = \frac{1}{2}$, $\xi = \left(\frac{1}{2}\right)^{\frac{1}{p-1}}$ and $\eta = 1$, where $p > 2$ is an arbitrary odd number.

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