Труды Москов. Матем. Общ.
Том 79 (2018), вып. 1

Trans. Moscow Math. Soc.
2018, Pages 85–100
https://doi.org/10.1090/mosc/285
Article electronically published on November 29, 2018

# THE SIMPLEST CRITICAL CASES IN THE DYNAMICS OF NONLINEAR SYSTEMS WITH SMALL DIFFUSION

S. A. KASHCHENKO

ABSTRACT. Systems of nonlinear equations of parabolic type provide models for many processes and phenomena. A special role is played by systems with relatively small diffusion coefficients. In investigating the dynamical properties of solutions, the diffusion coefficients being small leads to the appearance of infinite-dimensional critical cases in problems on the stability of solutions. In this paper we study the simplest and most important of these critical cases. Special nonlinear evolution equations are constructed which play the role of normal forms; their nonlocal dynamics determines the behaviour of solutions of the original system in a small neighbourhood of an equilibrium state. The importance of the renormalization procedure is demonstrated.

## INTRODUCTION

We study the system of equations

$$\frac{\partial u}{\partial t} = (\varepsilon D_1 + \varepsilon^2 D_2)\frac{\partial^2 u}{\partial x^2} + (A_0 + \varepsilon A_1 + \varepsilon^2 A_2)u + F(u) \tag{1}$$

with various classical boundary conditions:

- the periodic boundary condition

$$u(t, x + 2\pi) \equiv u(t, x), \tag{2}$$

- the Neumann boundary condition

$$\frac{\partial u}{\partial x}|_{x=0} = \frac{\partial u}{\partial x}|_{x=\pi} = 0, \tag{3}$$

- the Dirichlet boundary condition

$$u|_{x=0} = u|_{x=\pi} = 0. \tag{4}$$

Here $u \in \mathbb{R}^n, 0 < \varepsilon \ll 1$ is a small parameter and we assume that the real parts of all the eigenvalues of the matrix $D(\varepsilon) = \varepsilon D_1 + \varepsilon^2 D_2$ are positive and that the nonlinear vector function $F(u)$ is such that $F(u) = F_2(u, u) + F_3(u, u, u) + o(|u|^4)$, where the vector functions $F_{2,3}$ are linear with respect to each argument. It is convenient to choose the Sobolev space $\mathring{W}_2^2(\mathbb{R}^n)$ as the phase space. The Sobolev space $\mathring{W}_2^2(\mathbb{R}^n)$ is defined to be the space $W_2^2(\mathbb{R}^n)$ with the corresponding boundary condition (either (2), (3) or (4)).

We will investigate the behaviour of all solutions of the boundary value problems (1),(2), (1),(3) and (1),(4) with initial data in some sufficiently small neighbourhood (independent of $\varepsilon$) of the zero equilibrium state. We will examine all the similarities and differences in the dynamics of each of these boundary value problems. We note that systems of parabolic equations of the form (1) with boundary conditions (2), (3)

2010 *Mathematics Subject Classification.* Primary 35K67.
*Key words and phrases.* Quasinormal forms, asymptotic expansion, nonlinear dynamics, and small parameter.

or (4) are the basic models for many applied problems (see, for example, [1]–[17]. The behaviour of solutions of the linear system of equations

$$\frac{\partial u}{\partial t} = D(\varepsilon)\frac{\partial^2 u}{\partial x^2} + (A_0 + \varepsilon A_1 + \varepsilon^2 A_2)u, \tag{5}$$

with the corresponding boundary conditions, plays an important role in studying the dynamical properties of solutions to (1). Consider the matrix family

$$A(z) = A_0 - zD_1,$$

which depends on a nonnegative parameter $z$. If the real parts of all the eigenvalues of $A(z)$ are negative when $z \geqslant 0$, then the problem becomes trivial: all solutions of any of the boundary value problems under consideration for sufficiently small initial data (in the norm of $\mathring{W}_2^2(R^n)$), tend to zero as $t \to \infty$. If there exists $z_0 \geqslant 0$ for which the matrix $A(z_0)$ has eigenvalues with positive real part, then the problem becomes nonlocal: that is, a small neighbourhood (not depending on $\varepsilon$) of the zero equilibrium point does not contain attractors.

We will investigate the critical case, when for some $z_0 \geqslant 0$ the matrix $A(z_0)$ has eigenvalues with zero real part, while the real parts of all the eigenvalues of $A(z)$ for $z \geqslant 0$ but $z \neq z_0$ are negative. Notice that in [18]–[29] a method of normalization is proposed, with the help of which the boundary value problems under consideration reduce to special nonlinear boundary value problems that, as a rule, do not contain the parameter $\varepsilon$, and such that their nonlocal dynamics governs the local dynamics of the original boundary value problems for all small $\varepsilon$.

In this paper, we study the simplest and most interesting critical cases, when the matrix $A(z_0)$ has just one pair of purely imaginary eigenvalues and the real parts of all other eigenvalues are negative. A problem of this type was first studied in [7] for $z_0 = 0$. For this case, in §1 we give the corresponding normalized boundary value problems, which are special nonlinear equations of parabolic type. In §2 we investigate an important critical case when the corresponding parabolic operator obtained in §1 degenerates. We propose a new approach to constructing the normalized systems. Using it we can obtain nonlinear equations free of the parameter $\varepsilon$, of parabolic type of fourth order and with nonlinear terms of a special form.

In §§3 and 4 we assume that $z_0 > 0$, that is, the critical case is realized at asymptotically high modes of order $\varepsilon^{-1/2}$. In [18], [20]–[25] it was shown that in this situation the normalized system is described by a nonlinear hyperbolic equation. In §4, taking account of the terms of a higher order of smallness, renormalizing we can obtain a special nonlinear parabolic equation as the normalized equation and, from the structure of its solutions, deduce some conclusions about the behaviour of solutions of the original equation.

## 1. Bifurcation at low modes

Now we assume that $z_0 = 0$. We will show that, in contrast to the case $z_0 > 0$, bifurcation and the formation of structures occur at relatively low modes (as $\varepsilon \to 0$).

Thus, suppose that the matrix $A(z_0) = A_0$ has a pair of purely imaginary eigenvalues $\pm i\omega$ ($i\omega \neq 0$), so that $A_0 a = i\omega a$ ($a \neq 0$ is an eigenvector) and that all other eigenvalues of $A_0$ and all the eigenvalues of $A(z)$ for $z > 0$ have negative real parts. Let $A_0^* b = -i\omega b$, where $A_0^*$ is the transpose of $A_0$ and $(a, b) = 1$. Notice an important inequality

$$\operatorname{Re}(D_1 a, b) \geqslant 0, \tag{6}$$

which follows from the stated properties of $A(z)$. According to the algorithm in [18]–[26], we substitute

$$\begin{aligned} u =&\varepsilon^{1/2}(\xi(\tau,x)a\exp(i\omega t)+\bar{\xi}(\tau,x)\bar{a}\exp(-i\omega t)) \\ &+\varepsilon u_2(\tau,t,x)+\varepsilon^{3/2}u_3(\tau,t,x)+\dots \end{aligned} \tag{7}$$

into (1), where $\tau=\varepsilon t$ is slow time, the $\xi(\tau,x)$ are complex 'amplitudes' and the $u_j(\tau,t,x)$ are periodic in $t$ and satisfy the appropriate boundary conditions in $x$. First, we obtain $u_2(\tau,t,x)$ by collecting together the coefficients of like powers of $\varepsilon$ in the resulting formal identity. Next, the condition that the equation be solvable for $u_3(\tau,t,x)$ in the functional class under consideration yields the determining equation [18], [21] for $\xi(\tau,x)$:

$$\frac{\partial\xi}{\partial\tau}=(D_1a,b)\frac{\partial^2\xi}{\partial x^2}+(A_1a,b)\xi+d\xi|\xi|^2. \tag{8}$$

The boundary conditions for $\xi(\tau,x)$ are inherited from the boundary conditions for $u(t,x)$. Thus, for case (2) we have

$$\xi(\tau,x+2\pi)=\xi(\tau,x); \tag{9}$$

for case (3)

$$\frac{\partial\xi}{\partial x}|_{x=0}=\frac{\partial\xi}{\partial x}|_{x=\pi}=0; \tag{10}$$

and for case (4)

$$\xi|_{x=0}=\xi|_{x=\pi}=0. \tag{11}$$

The complex coefficient $d$ is called the Lyapunov value. An explicit expression for it was given in [18, 21]. We note that this is the same Lyapunov value as for the ordinary differential equation

$$\dot{u}=(A_0+\varepsilon A_1)u+F(u). \tag{12}$$

We now formulate the results (based on the Ansatz (7)) on the relationship, for small $\varepsilon$, between the solutions and dynamical properties of the original equation and the normalized equation (8) with the appropriate boundary conditions. In statements of this type we will assume that the coefficient $(D_1a,b)$ of the leading spatial derivative satisfies the strict inequality

$$\operatorname{Re}(D_1a,b)>0. \tag{13}$$

For the sake of definiteness, we restrict our attention to case (2) of periodic boundary conditions.

**Theorem 1.** *Suppose that condition* (13) *is satisfied, and let $\xi_0(\tau,x)$ be a solution of the boundary value problem* (8), (9) *that is bounded as $t\to\infty$, $x\in[0,2\pi]$. Then the boundary value problem* (1), (2) *admits an asymptotic solution $u_0(t,x,\varepsilon)$ with error $O(\varepsilon^{3/2})$ of the form*

$$u_0(t,x,\varepsilon)=\varepsilon^{1/2}(\xi_0(\tau,x)a\exp(i\omega t)+\bar{\xi}_0(\tau,x)\bar{a}\exp(-i\omega t))+\varepsilon u_2(\tau,t,x),\quad(\tau=\varepsilon t). \tag{14}$$

If, in addition, $\xi_0(\tau,x)$ is a solution of (8), (9) that is periodic in $\tau$ this theorem can be strengthened. To this end, assume that the following nondegeneracy condition holds:

$$\frac{\partial\xi_0}{\partial\tau}\not\equiv\text{const}\,\frac{\partial\xi_0}{\partial x}$$

and that the boundary value problem linearized at $\xi_0(\tau,x)$ has just two multipliers with absolute value 1. Then equation (14) defines a two-dimensional torus with the same stability as $\xi_0(\tau,x)$.

**Example 1.** *The dynamics of the logistic equation with a delay and small diffusion term.*

Consider the boundary value problem

$$\frac{\partial u}{\partial t} = \varepsilon^2 \frac{\partial^2 u}{\partial x^2} + ru[1 - u(t-T,x)], \quad u(t,x+2\pi) \equiv u(t,x). \tag{15}$$

Here $0 < \varepsilon \ll 1$, and the delay $T$ and the coefficient $r$ are positive. We study the behaviour of solutions of (15) in a small neighbourhood of the equilibrium state $u_0 = 1$ which is independent of $\varepsilon$.

The linearized equation at $u_0$ reads

$$\frac{\partial u}{\partial t} = \varepsilon^2 \frac{\partial^2 u}{\partial x^2} - ru(t-T,x)$$

and its characteristic quasi-polynomial is

$$\lambda = -\varepsilon^2 k^2 - r\exp(-\lambda T), \quad k = 0, \pm 1, \pm 2, \dots. \tag{16}$$

If $T < T_0 = \pi(2r)^{-1}$, then the real parts of all the roots of (16) are negative and separated from zero as $\varepsilon \to 0$, while if $T > T_0$, then there exists a root of (16) with positive real parts separated from zero as $\varepsilon \to 0$. Here we assume that the relation

$$T = T_0 + \varepsilon T_1 \tag{17}$$

holds for some constant $T_1$. Under condition (17), the real parts of infinitely many roots tend to zero as $\varepsilon \to 0$, and their imaginary parts tend to $\omega = r$.

Following the above algorithm, in (15) we set

$$\begin{aligned} u =& 1 + \sqrt{\varepsilon}(\xi(\tau,x)\exp(i\omega\tau) + \bar{\xi}(\tau,x)\exp(-i\omega\tau)) + \varepsilon u_2(\tau,t,x) \\ &+ \varepsilon^{3/2} u_3(\tau,t,x) + \dots, \end{aligned} \tag{18}$$

where $\tau = \varepsilon t$ and the functions $u_j(\tau,t,x)$ depend periodically on $t$ and $x$. After standard computations, we first find $u_2(\tau,t,x)$ and next, taking into account the solvability conditions on the equation with respect to $u_3(\tau,t,x)$ we arrive at the boundary value problem determining $\xi(\tau,x)$:

$$\frac{\partial \xi}{\partial \tau} = \left[\frac{1 - i\pi/2}{1 + \pi^2/4}\right]\frac{\partial^2 \xi}{\partial \xi^2} + \left[r_0 \frac{1 - i\pi/2}{1 + \pi^2/4} T_1\right]\xi - \frac{r_0[3\pi - 2 + i(\pi + 6)]}{10(1 + \pi^2/4)}|\xi|^2\xi.$$

In the next section we study the following case, which is important for applications:

$$\operatorname{Re}(D_1 a, b) = 0, \quad \text{that is,} \quad (D_1 a, b) = i\delta \quad (\operatorname{Im}\delta = 0). \tag{19}$$

## 2. Renormalization

Suppose that condition (19) holds. We assume that

$$\operatorname{Re}(A_1 a, b) = 0, \tag{20}$$

since otherwize the problem is either trivial or becomes nonlocal. We set $(A_1 a, b) = 0$ without loss of generality, because (20) makes it possible to eliminate the term $(A_1 a, b)\xi$ by using the Lyapunov change of variables $\xi \to \xi \exp((A_1 a, b)\tau)$.

After linearization at origin, equation (8) takes the form

$$\frac{\partial \xi}{\partial \tau} = i\delta \frac{\partial^2 \xi}{\partial x^2}. \tag{21}$$

Therefore the characteristic equation for (21) together with any of the boundary conditions (9)–(11) has infinitely many roots with zero real part. This means that the critical case of infinite dimensions is realized in the problem of the stability of the zero equilibrium state in (8). In particular, this means we have to find an analogue of (8) taking terms of order $\varepsilon$ into account.

The characteristic equation for (5), with the boundary conditions under consideration, is

$$det|A_0 + \varepsilon A_1 + \varepsilon^2 A_2 - \varepsilon k^2 D_1 - \varepsilon^2 k^2 D_2 - \lambda I| = 0, \tag{22}$$

where $k = 0, \pm 1, \pm 2$ in the case of (2); $k = 0, 1, 2, \ldots$ for (3) and $k = 1, 2, \ldots$ for (4). The asymptotics of those roots $\lambda_k(\varepsilon)$ and $\bar{\lambda}_k(\varepsilon)$ of (22), for which the real parts tend to zero as $\varepsilon \to 0$, can be written

$$\lambda_k(\varepsilon) = i\omega + \varepsilon\lambda_{k1} + \varepsilon^2\lambda_{k2} + \ldots, \tag{23}$$

and the eigenvectors $a_k(\varepsilon)$ corresponding to $\lambda_k(\varepsilon)$ are

$$a_k(\varepsilon) = a + \varepsilon a_{k1} + \varepsilon^2 a_{k2} + \ldots. \tag{24}$$

We now give expressions for $\lambda_{k1}, \lambda_{k2}$ and $a_{k1}$:

$$\begin{aligned}
\lambda_{k1} &= -k^2(D_1 a, b) = -i\delta k^2,\\
\lambda_{k2} &= k^2[(A_1 a_0, b) - (D_2 a, b)] + (A_2 a, b) - k^4(D_1 a_0, b),\\
a_{k1} &= k^2 a_0,
\end{aligned}$$

where $a_0$ denotes the vector solutions of the equation

$$(A_0 - i\omega I)a_0 = (D_1 - i\delta I)a,$$

such that $(a_0, b) = 0$.

From the above, we see that the normalized equation (the analogue of (8)) is

$$\frac{\partial \xi}{\partial \tau} = i\delta \frac{\partial^2 \xi}{\partial x^2} + \varepsilon[-\sigma \frac{\partial^4 \xi}{\partial x^4} + \varkappa \frac{\partial^2 \xi}{\partial x^2} + \alpha\xi] + d\xi|\xi|^2, \tag{25}$$

where $\sigma = (D_1 a_0, b), \varkappa = (D_2 a, b) - (A_1 a_0, b)$ and $\alpha = (A_2 a, b)$.

Note that the representation of $\xi(\tau, x)$ in the form of a Fourier series with respect to the corresponding eigenfunctions with coefficients $\xi_k(\tau)$ turns the linear part of (25) into a system of ordinary differential equations

$$\dot{\xi}_k = (\lambda_{k1} + \varepsilon\lambda_{k2} + \ldots)\xi_k,$$

and the coefficient $\sigma$ satisfies the inequality

$$\sigma \geqslant 0. \tag{26}$$

For $\varepsilon = 0$, the problem of the stability of the zero solution of (25) (with boundary condition (9), (10) or (11)) again gives the critical infinite dimensional case. We therefore apply the normalization algorithm once again. We introduce formal series (analogous to (7)). For the periodic boundary condition (9) we obtain

$$\begin{aligned}
\xi(\tau, x, \varepsilon) =& \varepsilon^{1/2}\Bigg( \sum_{k=-\infty}^{\infty} \eta_k(s) \exp(ikx - i\delta k^2\tau)\\
&+ \sum_{k=-\infty}^{\infty} \overline{\eta_k}(s) \exp(ikx - i\delta k^2\tau)\Bigg) + \varepsilon^{3/2} q_3(s, \tau, x) + \ldots, \ \ s = \varepsilon\tau,
\end{aligned} \tag{27}$$

where the ‘amplitudes’ $\eta_k(s)$ are to be determined, and the function $q_3(s, \tau, x)$ is periodic with respect to $\tau$ and $x$. For the Neumann boundary condition (10) we obtain

$$\xi(\tau, x, \varepsilon) = \varepsilon^{1/2} \left( \sum_{k=-\infty}^{\infty} \eta_k(s)(\cos kx) \exp(-i\delta k^2\tau) \right) + \varepsilon^{3/2} q_3(s, \tau, x) + \ldots, \tag{28}$$

where $q_3(s,\tau,x)=q_3(s,\tau,-x)$. Finally, for the Dirichlet boundary condition we have

$$\xi(\tau,x,\varepsilon)=\varepsilon^{1/2}\left(\sum_{k=-\infty,k\neq 0}^{\infty}\eta_k(s)(\sin kx)\exp(-i\delta k^2\tau)\right)+\varepsilon^{3/2}q_3(s,\tau,x)+\dots, \tag{29}$$

and $q_3(s,\tau,x)=-q_3(s,\tau,-x)$.

Now we substitute (27), (28) or (29) into (25). In doing this, we make essential use of the fact that the 'lower' resonances are absent; that is, the system of integer equations

$$k_1+k_2-k_3=k,\qquad k_1^2+k_2^2-k_3^2=k^2$$

admits only those triples as solutions in which two of the numbers are equal in modulus and have opposite sign.

After standard computations for each of the boundary value problems under consideration, we arrive at the infinite system of ordinary differential equations

$$\frac{\partial\eta_k}{\partial s}=(-\sigma k^4-\kappa k^2+\alpha)\eta_k+f_k(\eta_j), \tag{30}$$

where $k=0,\pm1,\pm2,\dots$ in case (9); $k=0,\pm1,\pm2,\dots$ in case (10) and $k=1,2,\dots$ in case (11).

Later in the paper, the central point is the definition of the function $\eta(s,x)$:

$$\eta(s,x)=\sum_k\eta_k(s)\exp(ikx) \tag{31}$$

with coefficients $\eta_k(s)$ as in (30). We introduce some more notation. Let $M(\varphi)$ denote the mean value of the function $\varphi(x)$:

$$M(\varphi(x))=\frac{1}{2\pi}\int_0^{2\pi}\varphi(x)dx.$$

Let $R(\varphi)$ denote the vector whose components are the Fourier coefficients of the function $\varphi(x)=\sum_{k=-\infty}^{\infty}\varphi_k\exp(ikx)$:

$$R(\varphi)=(\dots,\varphi_{-2}\exp(-2ix),\varphi_{-1}\exp(-ix),\varphi_0,\varphi_1\exp(ix),\varphi_2\exp(2ix),\dots).$$

In what follows we assume that multiplication of vectors is coordinate-wise. Then, for example,

$$(R^2(\eta),\bar{R}(\eta))=\sum_k\eta_k|\eta_k|^2\exp(ikx).$$

Consider the equation

$$\frac{\partial\eta}{\partial s}=-\sigma\frac{\partial^4\eta}{\partial x^4}+\varkappa\frac{\partial^2\eta}{\partial x^2}+\alpha\eta+2\eta M(|\eta|^2-(R^2(\eta),\bar{R}(\eta)) \tag{32}$$

with boundary conditions

$$\eta(s,x+2\pi)=\eta(s,x), \tag{33}$$

the equation

$$\frac{\partial\eta}{\partial s}=-\sigma\frac{\partial^4\eta}{\partial x^4}+\varkappa\frac{\partial^2\eta}{\partial x^2}+\alpha\eta+\eta M(|\eta|^2)-\frac{3}{4}(R^2(\eta),\bar{R}(\eta))-\frac{1}{4}M(\eta)|M(\eta)|^2 \tag{34}$$

with boundary conditions

$$\frac{\partial\eta}{\partial x}|_{x=0}=\frac{\partial\eta}{\partial x}|_{x=1}=0 \tag{35}$$

and the equation

$$\frac{\partial\eta}{\partial s}=-\sigma\frac{\partial^4\eta}{\partial x^4}+\varkappa\frac{\partial^2\eta}{\partial x^2}+\alpha\eta+\eta M(|\eta|^2)-\frac{3}{4}(R^2(\eta),\bar{R}(\eta)) \tag{36}$$

with boundary conditions

(37) $$\eta|_{x=0} = \eta|_{x=1} = 0.$$

Our main result is that the Fourier coefficients $\eta_k(s)$ of solutions to each of these boundary value problems satisfy systems of ordinary differential equations of the same form as (30). Thus, if conditions (19), (20) are fulfilled then the boundary value problems (32), (33); (34), (35) and (36), (37) play the role of the normal forms for the boundary value problems (8), (9); (8), (10) and (8), (11), respectively, and therefore for the original system (1) with the appropriate boundary conditions.

Now we can formulate a concrete statement which follows from the foregoing considerations.

**Theorem 2.** *Suppose that conditions* (19) *and* (20) *hold and that the boundary value problem* (32), (33) *(*(34), (35) *or* (36), (37)*) admits a solution* $\eta_0(s,x) = \sum \eta_{k0}(s)\exp(ikx)$ *that is bounded as* $s \to \infty$. *Then the boundary value problem* (25), (9) *(*(25), (10) *or* (25), (11), *respectively) admits an asymptotic solution* $\xi_0(\tau, x, \varepsilon)$ *with error* $O(\varepsilon^{3/2})$, *which takes the form*

$$\xi_0(\tau,x,\varepsilon) = \varepsilon^{1/2}(\sum_k \eta_{k0}(\varepsilon\tau)\exp(ikx - i\delta k^2\tau + \sum_k \bar{\eta}_{k0}(\varepsilon\tau)\exp(-ikx + i\delta k^2\tau))$$

*in case* (9), *the form*

$$\xi_0(\tau,x,\varepsilon) = \varepsilon^{1/2}(\sum_k \eta_{k0}(\varepsilon\tau)\cos(kx)\exp(-i\delta k^2\tau + \sum_k \bar{\eta}_{k0}(\varepsilon\tau)\cos(kx)\exp(-i\delta k^2\tau))$$

*in case* (10), *and*

$$\xi_0(\tau,x,\varepsilon) = \varepsilon^{1/2}(\sum_k \eta_{k0}(\varepsilon\tau)\sin(kx)\exp(-i\delta k^2\tau + \sum_k \bar{\eta}_{k0}(\varepsilon\tau)\cos(kx)\exp(-i\delta k^2\tau))$$

*in case* (11).

*Remark* 1. In a number of cases it is possible to prove that an exact solution of the original boundary value problem exists that is asymptotically close to the corresponding solutions of the normalized and renormalised boundary value problems, and also to show that these solutions have the same stability properties. However, under the hypotheses of Theorem 2, even assuming that the corresponding exact solution exists, generally speaking, the conclusion that the stability properties are inherited may be invalid. We can only assert that if the solution $\eta_0(s,x)$ is unstable, then so is $\xi_0(\tau,x,\varepsilon)$; however, if $\eta_0(s,x)$ is stable, this does not imply that the solution corresponding to $\xi_0(\tau,x,\varepsilon)$ is.

**Example 2.** As an example, we take a concrete pair of matrices $D_1$ and $A_0$ such that the conditions of Theorem 2 are satisfied.

Consider

$$D_1 = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \quad d_1, d_2 > 0, \quad A_0 = \begin{pmatrix} \alpha & 1 \\ -1 & \alpha \end{pmatrix}, \quad 0 < \alpha < 1.$$

Then

$$\omega = \sqrt{1-\alpha^2},\ a = (1, i\omega - \alpha) \text{ and } b = (2(1-\alpha^2) + 2i\omega)^{-1}(1, \alpha + i\omega).$$

The equality $\mathrm{Re}(D_1 a, b) = 0$ is equivalent to

$$\alpha_1 + \alpha_2[1 - 2\alpha^2 - 2\alpha] = 0.$$

Notice also that $\mathrm{Im}(D_1 a, b) \neq 0$.

## 3. Bifurcation at higher modes

Now let us assume that $z_0$ satisfies the inequality

$$z_0 > 0, \tag{38}$$

that is, the matrix $A(z_0) = A_0 - z_0 D_1$ has a single pair of purely imaginary eigenvalues $\pm i\omega$ ($\omega \neq 0$), and the real parts of all the other eigenvalues of this matrix, and also of all the eigenvalues of the family of matrices $A(z)$ at $z \geq 0$ and for $z \neq z_0$, are negative. Let

$$(A_0 - z_0 D_1)a = i\omega a \quad (a \neq 0), \qquad (A_0 - z_0 D_1)^* b = -i\omega b \;\text{ and }\; (a, b) = 1.$$

We let $\lambda(z)$ denote the eigenvalue of $A(z)$ which depends continuously on $z$ and takes the value $i\omega$ at $z = z_0$. The properties of $A(z)$ imply that

$$\operatorname{Re} \left.\frac{d\lambda(z)}{dz}\right|_{z=z_0} = 0, \qquad \operatorname{Re} \left.\frac{d^2\lambda(z)}{dz^2}\right|_{z=z_0} \geq 0. \tag{39}$$

It is natural to expect that bifurcations in this critical case occur for modes with numbers $k = k(\varepsilon)$ which are close to the quantity $(z_0\varepsilon^{-1})^{1/2}$, which is asymptotically large as $\varepsilon \to 0$. In this connection we introduce some more notation. Let $\theta = \theta(\varepsilon) \in [0, 1)$ denote the complements of the expression $(z_0\varepsilon^{-1})^{1/2}$ to an integer value, that is,

$$\theta(\varepsilon) = 1 + \left\{\sqrt{\frac{z_0}{\varepsilon}}\right\} - \sqrt{\frac{z_0}{\varepsilon}}.$$

We consider integer values $k = \pm k(\varepsilon)$ such that $k(\varepsilon) = k_0(\varepsilon) + n$, where $k_0(\varepsilon) = (z_0\varepsilon^{-1})^{1/2} + \theta$ and $n = 0, \pm 1, \pm 2, \cdots$. For $k = k(\varepsilon) + n$, the matrix $A_0 - \varepsilon k^2 D_1 - \varepsilon^2 k^2 D_2 + \varepsilon A_1 + \varepsilon A_2$ is of the form

$$A_0 - z_0 D_1 - 2(\varepsilon z_0)^{1/2}(\theta + n)D_1 - \varepsilon[(\theta + n)^2 D_1 + z_0 D_2 - A_1] + \cdots. \tag{40}$$

This matrix possesses infinitely many eigenvalues $\lambda_\varepsilon(n)$, $\overline{\lambda}_\varepsilon(n)$, and, for any $n$, these tend to the imaginary axis as $\varepsilon \to 0$. The following asymptotic expansions are valid:

$$\lambda_\varepsilon(n) = i\omega + \sqrt{\varepsilon}\lambda_{n1} + \varepsilon\lambda_{n2} + \cdots \tag{41}$$

and

$$a(n, \varepsilon) = a + \sqrt{\varepsilon} a_{n1} + \varepsilon a_{n2} + \cdots, \tag{42}$$

where $a(n, \varepsilon)$ is the corresponding eigenvector for (40). The coefficients in the series (41) and (42) can be written out explicitly. In particular,

$$\lambda_{n1} = -2i\sqrt{z_0}\gamma(\theta + n), \text{ where } \gamma = \operatorname{Im}(D_1 a, b).$$

It follows from (39) that $\operatorname{Re}(D_1 a, b) = 0$. For $\lambda_{n2}$ we have

$$\lambda_{n2} = -4z_0(\theta + n)^2(D_1 a_0, b) - i(\theta + n)^2\gamma - z_0(D_2 a, b) + (A_1 a, b),$$

and the vector $a_0$ is a solution of the system

$$(A(z_0) - i\omega I)a_0 = (D_1 - i\gamma I)a,$$

such that $(a_0, b) = 0$. Notice that the second condition in (39) implies that $\operatorname{Re}(D_1 a_0, b) \geq 0$. Below, we assume that strict inequality holds, so that

$$\operatorname{Re}\delta > 0, \text{ where } \delta = (D_1 a_0, b). \tag{43}$$

To study the dynamics of the boundary value problems under consideration, we use the algorithms developed in [18]–[26]. We introduce the formal series

$$\begin{aligned} u =& \varepsilon^{1/4}[a\exp(i\omega t)(\sum_n \xi_n(\tau,x,y) + \sum_n \eta_n(\tau,x,y)) + \overline{a}\exp(-i\omega t) \\ & \cdot(\sum_n \overline{\xi}_n(\tau,x,y) + \sum_n \overline{\eta}_n(\tau,x,y))] + \varepsilon^{1/2}u_2(\tau,t,x,y) + \varepsilon^{3/4}u_3(\tau,t,x,y) + \cdots . \end{aligned} \tag{44}$$

Here $\tau = \sqrt{\varepsilon}t$, $y = k(\varepsilon)x$, the dependence on $t$ and $y$ is periodic, and the structure of $\xi_n(\tau,x,y)$ and $\eta_n(\tau,x,y)$ depends on the choice of boundary conditions. For case (2) we have

$$\begin{aligned} &\xi_n(\tau,x,y) = \xi_n(\tau)\exp(iy + inx), \\ &\eta_n(\tau,x,y) = \eta_n(\tau)\exp(-iy + inx),\ n = 0, \pm1, \pm2, \cdots . \end{aligned} \tag{45}$$

For case (3) the relations are as follows:

$$\begin{aligned} &\xi_n(\tau,x,y) = \xi_n(\tau)\cos(y)\cdot\cos(nx),\ (n = 0, 1, \cdots), \\ &\eta_n(\tau,x,y) = \eta_n(\tau)\sin(y)\cdot\sin(nx),\ (n = 1, 2, \cdots), \end{aligned} \tag{46}$$

and for case (4)

$$\begin{aligned} &\xi_n(\tau,x,y) = \xi_n(\tau)\cos(y)\cdot\sin(nx),\ (n = 1, 2, \cdots), \\ &\eta_n(\tau,x,y) = \eta_n(\tau)\sin(y)\cdot\cos(nx),\ (n = 0, 1, \cdots). \end{aligned} \tag{47}$$

We substitute these formal series into (1), (2); (1), (3) or (1), (4), respectively, and equate the coefficients of like powers of $\varepsilon$ in the resulting formal identity. At the second step we find $u_2(\tau,t,x,y)$. The third step is to obtain an equation for $u_3(\tau,t,x,y)$. The condition for it to have a solution in the appropriate function class yields the final boundary value problem, to determine all the amplitudes $\xi_n(\tau,x,y)$ and $\eta_n(\tau,x,y)$. This problem plays the role of a normal form for the original boundary value problem. It is convenient to introduce some more notation. In (44), (45) set

$$\sum_n \xi_n(\tau,x,y) = \xi(\tau,x)\exp(iy),\ \sum_n \eta_n(\tau,x,y) = \eta(\tau,x)\exp(-iy).$$

Then $\xi(\tau,x)$ and $\eta(\tau,x)$ are periodic in $x$:

$$\xi(\tau,x+2\pi) \equiv \xi(\tau,x),\ \eta(\tau,x+2\pi) \equiv \eta(\tau,x). \tag{48}$$

If we set

$$\sum_n \xi_n(\tau,x,y) = \xi(\tau,x)\cos(y) \text{ and } \sum_n \eta_n(\tau,x,y) = \eta(\tau,x)\sin(y) \tag{49}$$

in (44), (46), then $\xi(\tau,x)$ and $\eta(\tau,x)$ satisfy the conditions

$$\left.\frac{\partial\xi}{\partial x}\right|_{x=0} = \left.\frac{\partial\xi}{\partial x}\right|_{x=\pi} = 0,\ \ \eta|_{x=0} = \eta|_{x=\pi} = 0. \tag{50}$$

Finally, if we set

$$\sum_n \xi_n(\tau,x,y) = \xi(\tau,x)\sin(y),\ \sum_n \eta_n(\tau,x,y) = \eta(\tau,x)\cos(y),$$

in (44), (47), we again obtain the conditions (50) for $\xi(\tau,x)$ and $\eta(\tau,x)$.

Now we write down the final boundary value problems which determine the amplitudes $\xi(\tau,x)$ and $\eta(\tau,x)$, which play the role of normal forms for equation (1) with the appropriate boundary conditions. In the case (1), (2) we obtain equations

$$\begin{aligned}\frac{\partial\xi}{\partial\tau} =& -2\sqrt{z_0}\gamma\frac{\partial\xi}{\partial x} - 2\sqrt{z_0}\gamma i\theta\xi\\ &+\varepsilon[(4z_0\delta - i\gamma)\frac{\partial^2\xi}{\partial x^2} + 2\theta(4i\delta z_0-\gamma)\frac{\partial\xi}{\partial x} + (\beta - i\theta^2\gamma - 4z_0\delta\theta^2)\xi]\\ &+\xi[R_1|\xi|^2 + R_2|\eta|^2],\end{aligned}\tag{51}$$

$$\begin{aligned}\frac{\partial\eta}{\partial\tau} =& 2\sqrt{z_0}\gamma\frac{\partial\eta}{\partial x} - 2\sqrt{z_0}\gamma i\theta\eta\\ &+\varepsilon[(4z_0\delta - i\gamma)\frac{\partial^2\eta}{\partial x^2} - 2\theta(4i\delta z_0-\gamma)\frac{\partial\eta}{\partial x} + (\beta - i\theta^2\gamma - 4z_0\delta\theta^2)\eta]\\ &+\eta[R_1|\xi|^2 + R_2|\eta|^2],\end{aligned}\tag{52}$$

with periodic boundary conditions (48). Here

$$\begin{aligned}R_1 =&(F_2(g_1,a) + F_2(a,g_1) + F_2(g_3,\overline{a}) + F_2(\overline{a},g_3)\\ &+F_3(a,a,\overline{a}) + F_3(a,\overline{a},a) + F_3(\overline{a},a,a),b),\\ R_2 =&(F_2(g_1,a) + F_2(a,g_1) + F_2(g_2,a) + F_2(a,g_2)\\ &+F_2(g_4,\overline{a}) + F_2(\overline{a},g_4) + 2(F_3(a,a,\overline{a}) + F_3(a,\overline{a},a) + F_3(\overline{a},a,a)),b).\end{aligned}$$

The vectors $g_{1,2,3,4}$ are given by the expressions

$$\begin{aligned}g_1 &= -A_0^{-1}(F_2(a,\overline{a}) + F_2(\overline{a},a)),\\ g_2 &= (4z_0D_1 - A_0)^{-1}(F_2(a,\overline{a}) + F_2(\overline{a},a)),\\ g_3 &= (2i\omega I + 4z_0D_1 - A_0)^{-1}F_2(a,a),\\ g_4 &= 2(2i\omega I - A_0)^{-1}F_2(a,a).\end{aligned}$$

The boundary conditions (50) correspond to

$$\begin{aligned}\frac{\partial\xi}{\partial\tau} =& 2\sqrt{z_0}\gamma\frac{\partial\xi}{\partial x} - 2\sqrt{z_0}\gamma i\theta\xi\\ &+\varepsilon[(4z_0\delta - i\gamma)\frac{\partial^2\xi}{\partial x^2} - 2\theta(4i\delta z_0-\gamma)\frac{\partial\xi}{\partial x} + (\beta - i\theta^2\gamma - 4z_0\delta\theta^2)\xi]\\ &+\xi[N_1|\xi|^2 + N_2|\eta|^2] + N_3\overline{\eta}\xi^2,\end{aligned}\tag{53}$$

$$\begin{aligned}\frac{\partial\eta}{\partial\tau} =& 2\sqrt{z_0}\gamma\frac{\partial\eta}{\partial x} - 2\sqrt{z_0}\gamma i\theta\eta\\ &+\varepsilon[(4z_0\delta - i\gamma)\frac{\partial^2\eta}{\partial x^2} + 2\theta(4i\delta z_0-\gamma)\frac{\partial\eta}{\partial x} + (\beta - i\theta^2\gamma - 4z_0\delta\theta^2)\eta]\\ &+\eta[N_4|\xi|^2 + N_5|\eta|^2] + N_6\overline{\eta}\xi^2.\end{aligned}\tag{54}$$

The same boundary value problem (53), (54), (50) also arises for boundary conditions (4). In this case

$$\begin{aligned}N_1 =&(F_2(a,g_1) + F_2(g_1,a) + F_2(a,g_2) + F_2(g_2,a)\\ &+F_2(\overline{a},g_4) + F_2(g_4,\overline{a}) + \frac{3}{4}f_0,b),\end{aligned}$$

$$N_2 = (F_2(a,g_1) + F_2(g_1,a) + 2F_2(\overline{a},g_3) + 2F_2(g_3,\overline{a}) + \frac{1}{2}f_0,b),$$

$$N_3 = (F_2(a,g_2) + F_2(g_2,a) + F_2(\overline{a},g_4) + F_2(g_4,\overline{a}) + \frac{1}{4}f_0,b),$$

$$N_4 = (F_2(a,g_1) + F_2(g_1,a) + 4F_2(\overline{a},g_3) + 4F_2(g_3,\overline{a}) + \frac{1}{2}f_0, b),$$

$$\begin{aligned} N_5 =&(F_2(a,g_1) + F_2(g_1,a) + F_2(a,g_2) + F_2(g_2,a) + \frac{3}{4}f_0, b) \\ &+ F_2(\overline{a},g_4) + F_2(g_4,\overline{a}) + 2F_2(\overline{a},g_3) + 2F_2(g_3,a), \end{aligned}$$

$$\begin{aligned} N_6 =&(F_2(a,g_2) + F_2(g_2,a) + F_2(\overline{a},g_4)F_2(g_4,\overline{a}) \\ &- 2F_2(\overline{a},g_3) - 2F_2(g_3,\overline{a}) + \frac{1}{4}f_0, b). \end{aligned}$$

## 4. Renormalization

All the boundary value problems presented in the previous section have the property that at $\varepsilon = 0$ the spectrum of their linear parts is purely imaginary. Thus the infinite-dimensional critical case is realized in the stability analysis of the zero equilibrium state. In this section, based on the approach developed in [18–26], we construct the boundary value problems which do not contain the parameter $\varepsilon$ and play the role of normal forms for these boundary value problems.

4.1. **Periodic boundary conditions.** Consider the boundary value problem (51), (52), (48). First, it is convenient to simplify it by using the transformations

$$\xi \to \xi \exp(-2i\sqrt{z_0}\gamma\theta\tau),\ \eta \to \eta \exp(-2i\sqrt{z_0}\gamma\theta\tau); \tag{55}$$

as a result, the second terms on both the right-hand side of (46) and of (47) vanish.

Expanding $\xi$ and $\eta$ formally gives

$$\begin{aligned} \xi &= \sqrt{\varepsilon}\sum_{k=-\infty}^{\infty} V_k(s)\exp(ik(x+\varkappa\tau) + \varepsilon^{3/2}V_3(s,\tau,x) + \cdots, \\ \eta &= \sqrt{\varepsilon}\sum_{k=-\infty}^{\infty} W_k(s)\exp(ik(x-\varkappa\tau) + \varepsilon^{3/2}W_3(s,\tau,x) + \cdots, \end{aligned} \tag{56}$$

where $\varkappa = z\sqrt{z_0}\gamma$, $s = \varepsilon\tau$. Set

$$V(s,x) = \sum_k V_k(s)\exp(ikx) \text{ and } W(s,x) = \sum_k W_k(s)\exp(ikx),$$

and substitute (56) into (51) and (52). Standard computations lead to the following system of equations for $V(s,x)$ and $W(s,x)$:

$$\begin{aligned} \frac{\partial V}{\partial s} =&(4z_0\delta - i\gamma)\frac{\partial^2 V}{\partial x^2} + 2\theta(4i\delta z_0 - \gamma)\frac{\partial V}{\partial x} \\ &+ (\beta - i\theta^2\gamma - 4z_0\delta\theta^2)V + V[R_1|V|^2 + R_2M(|W|^2)], \end{aligned} \tag{57}$$

$$\begin{aligned} \frac{\partial W}{\partial s} =&(4z_0\delta - i\gamma)\frac{\partial^2 W}{\partial x^2} + 2\theta(4i\delta z_0 - \gamma)\frac{\partial V}{\partial x} \\ &+ (\beta - i\theta^2\gamma - 4z_0\delta\theta^2)W + W[R_2M(|V|^2) + R_1|W|^2] \end{aligned} \tag{58}$$

with periodic boudary conditions

$$V(s,x+2\pi) \equiv V(s,x),\ W(s,x+2\pi) \equiv W(s,x). \tag{59}$$

The coefficients $R_j$ in (57), (58) are the same as in (51), (52).

Now we formulate our main result.

**Theorem 3.** *Suppose that, at some $\theta = \theta_0$, the boundary value problem* (57)–(59) *admits a solution $V_0(s,x)$, $W_0(s,x)$ that is bounded as $\tau \to \infty$, and let the sequence $\varepsilon_n \to 0$ be defined by the condition $\theta(\varepsilon) = \theta_0$. Then, for $\varepsilon = \varepsilon_n$, the boundary value problem* (51), (52), (48) *admits an asymptotic solution $\xi_0(\tau,x,\varepsilon)$, $\eta_0(\tau,x,\varepsilon)$ with error $O(\varepsilon^{3/2})$, such that*

$$\xi_0(\tau,x,\varepsilon) = \sqrt{\varepsilon} V_0(\varepsilon\tau, x + \varkappa\tau),$$
$$\eta_0(\tau,x,\varepsilon) = \sqrt{\varepsilon} W_0(\varepsilon\tau, x - \varkappa\tau).$$

**Example 3.** *The logistic equation with small diffusion and a small delay in the spatial variable.*

Consider the equation

$$\frac{\partial u}{\partial t} = \varepsilon^2 \frac{\partial^2 u}{\partial x^2} + ru[1 - u(t, x - \varepsilon h)] \tag{60}$$

with periodic boundary conditions

$$u(t, x + 2\pi) \equiv u(t,x). \tag{61}$$

Here $0 < \varepsilon \ll 1$, $r > 0$. We will investigate the behaviour of solutions of (60), (61) in a small neighbourhood (independent of $\varepsilon$) of the equilibrium state $u_0 \equiv 1$. The equation linearized at $u_0$ reads

$$\frac{\partial u}{\partial t} = \varepsilon^2 \frac{\partial^2 u}{\partial x^2} - ru(t, x - \varepsilon h). \tag{62}$$

The roots of its characteristic equation are $\lambda(z)$, where $z = \varepsilon k$ ($k = 0, \pm 1, \pm 2, \dots$), defined by the expression

$$\lambda(z) = -z^2 - r\exp(-ihz). \tag{63}$$

It follows that

$$\operatorname{Re}\lambda(z) = -z^2 - r\cos(hz).$$

We introduce some notation. Let $y_0$ denote the first positive root of the equation

$$y \cdot \tan(y) = -2$$

and let $h_0 = y_0(-r\cos(y_0))^{1/2}$. Then the following simple statement is true.

**Lemma 1.** *Let $0 < h < h_0$. If $\varepsilon$ is sufficiently small, then the real parts of all roots of* (63) *are negative and separated from zero as $\varepsilon \to 0$. If $h > h_0$ then there exists a root of* (63) *such that its real part is positive and separated from zero as $\varepsilon \to 0$.*

Here we assume that in (60) the following relation holds for some constant $h_1$:

$$h = h_0 + \varepsilon^2 h_1. \tag{64}$$

Let $z_0 = y_0 h_0^{-1}$, $\omega = r\sin(y_0)$ and $\omega_1 = rh\cos(y_0)$. Then

$$\operatorname{Re}\lambda(z_0) = 0,\ \frac{d}{dz}(\operatorname{Re}\lambda(z))\bigg|_{z=z_0} = 0,\ \lambda(z_0) = i\omega,\ \frac{d\lambda(z)}{dz}\bigg|_{z=z_0} = i\omega_1. \tag{65}$$

If (64) holds, the real parts of infinitely many roots of (63) tend to zero as $\varepsilon \to 0$, while their imaginary parts tend to $i\omega$. This means that the conditions described in §4 are realized.

We introduce the formal series

$$\begin{aligned} u =& \varepsilon^{1/2}(\xi(\tau,x)\exp(i\omega t + i(z_0\varepsilon^{-1} + \theta)x) + \overline{\xi}(\tau,x)\exp(-i\omega t + i(z_0\varepsilon^{-1} + \theta)x) \\ &+ \varepsilon u_2(\tau,t,x,y) + \varepsilon^{3/2} u_3(\tau,t,x,y), \end{aligned} \tag{66}$$

where $\tau = \sqrt{\varepsilon}t$ and the dependence on $t$ and $y = (z_0\varepsilon^{-1} + \theta)x$ in (66) is periodic. Substituting (66) into (60), standard computations lead to the boundary value problem which determines $\xi(\tau, x)$:

$$\begin{aligned}\frac{\partial\xi}{\partial\tau} =& \omega_1\frac{\partial\xi}{\partial x} + i\omega_1\theta\xi + \varepsilon[\frac{\partial^2\xi}{\partial x^2} + 2iz_0\theta\frac{\partial\xi}{\partial x}\\ &+ ((iz_0h_1 + \theta^2h_0^2)r\exp(-iy_0 - \theta^2)\xi] + d\xi|\xi|^2,\ \xi(\tau, x+2\pi) \equiv \xi(\tau, x),\end{aligned} \tag{67}$$

where $d = r^2[2\cos(y_0)(1 + \exp(-iy_0)) + (1 - \exp(3iy_0)) \cdot [2\omega + 4z_0^2 + r\exp(-2iy_0)]^{-1}]$.

In the example under consideration, this boundary value problem plays the role of the boundary value problem (48), (51), (52). For $\varepsilon = 0$, linearization about the zero solution of (67) yields an equation for which all solutions are periodic; taking this into account we have to normalize (67), that is, renormalize (60). Let

$$\xi = \varepsilon^{1/2}\eta(s, x + \omega_1\tau)\exp(i\omega_1\theta\tau) + \varepsilon\eta_2(s, x, \tau) + \varepsilon^{3/2}\eta_3(s, x, \tau) + \cdots, \tag{68}$$

where $s = \varepsilon\tau$, and the functions $\eta_{2,3}(s, x, \tau)$ are periodic in $x$ and $\tau$. Substituting (68) into (67) and making some standard computations, we obtain the final boundary value problem for $\eta(s, x)$:

$$\begin{aligned}\frac{\partial\eta}{\partial s} = \frac{\partial^2\eta}{\partial x^2} + 2iz_0\theta\frac{\partial\eta}{\partial x} + r[(iz_0h_1 + \frac{1}{2}\theta^2h_0^2)\exp(-iy_0) - \theta^2]\eta + d\eta|\eta|^2,\\ \eta(s, x+2\pi) \equiv \eta(s, x).\end{aligned} \tag{69}$$

Notice that due to the special form of the original equation, the boundary value problem (69) turns out to be much simpler than the boundary value problem (57), (58), (59).

4.2. **Boundary conditions** (50). First, we make the substitution (55) in (51), (52). Next, we consider the linear system with boundary conditions (50):

$$\frac{\partial\xi}{\partial\tau} = \varkappa\frac{\partial\eta}{\partial x},\ \frac{\partial\eta}{\partial\tau} = \varkappa\frac{\partial\xi}{\partial x}. \tag{70}$$

The formal expansions of the solutions of this system read

$$\xi = \sum_{k=0}^{\infty}\rho_k\cos(kx)\cdot\exp(ik\varkappa\tau),\ \eta = \sum_{k=0}^{\infty}i\rho_k\sin(kx)\cdot\exp(ik\varkappa\tau), \tag{71}$$

where the $\rho_k$ are arbitrary. The expressions (71) for $\xi$ and $\eta$ can be cast into the equivalent forms

$$\xi = \rho(\varkappa\tau + x) + \rho(\varkappa\tau - x),\ \eta = \rho(\varkappa\tau + x) - \rho(\varkappa\tau - x),$$

where $\rho(\tau)$ is a $2\pi$-periodic complex-valued function.

According to the algorithm described above, we introduce the formal expressions

$$\begin{aligned}\xi &= \varepsilon^{1/2}(\rho(s, \varkappa\tau + x) + \rho(s, \varkappa\tau - x)) + \varepsilon^{3/2}\xi_3(s, \tau, x) + \cdots,\\ \eta &= \varepsilon^{1/2}(\rho(s, \varkappa\tau + x) - \rho(s, \varkappa\tau - x)) + \varepsilon^{3/2}\eta_3(s, \tau, x) + \cdots.\end{aligned}$$

Here $s = \varepsilon\tau$, and the functions $\xi_3$ and $\eta_3$ are periodic with respect to $\tau$ and $x$. After some standard computations we obtain an equation to determine the amplitude $\rho(s, x)$

$$\begin{aligned}\frac{\partial\rho}{\partial s} = (4z_0\delta - i\gamma)\frac{\partial^2\rho}{\partial x^2} - 2\theta(4z_0\delta i - \gamma)\frac{\partial\rho}{\partial x} + (\beta - i\theta^2\gamma - 4z_0\delta\theta^2)\rho + (N_1 + N_2 + N_3)\rho|\rho|^2\\ + 2(N_1 - N_3)\rho M(|\rho|^2) + (N_1 - N_2 + N_3)\overline{\rho}M(|\rho|^2),\end{aligned} \tag{72}$$

$$\rho(s, x + 2\pi) \equiv \rho(s, x). \tag{73}$$

**Theorem 4.** *Suppose that there exists a value $\theta = \theta_0$ for which the boundary value problem* (72), (73) *admits a solution $\rho_0(s,x)$ that is bounded as $\tau \to \infty$, and suppose that the condition $\theta(\varepsilon) = \theta_0$ defines a sequence $\varepsilon_n \to 0$. Then, at $\varepsilon = \varepsilon_n$, the boundary value problem* (50), (53), (54) *admits an asymptotic solution $\xi_0(\tau,x,\varepsilon)$, $\eta_0(\tau,x,\varepsilon)$ with error $O(\varepsilon^{3/2})$, for which:*

$$\xi_0 = \varepsilon^{1/2}(\rho_0(\varepsilon\tau, \varkappa\tau + x) + \rho_0(s, \varkappa\tau - x)),$$
$$\eta_0 = \varepsilon^{1/2}(\rho_0(\varepsilon\tau, \varkappa\tau + x) - \rho_0(\varepsilon\tau, \varkappa\tau - x)).$$

*Remark* 2. In some situations it is possible to prove the existence of an exact solution of the original system that is asymptotically close to the constructed one and to determine the stability properties of this exact solution.

*Remark* 3. Interesting results on singularities in the dynamics of systems with small diffusion were obtained in [23, 25] in situations when the 'supercritical' term $\varepsilon A_1$ in (1) is replaced by $\varepsilon^\alpha A_1$, for $0 < \alpha < 1$, and also in cases with nonlinear terms depending on $\partial u/\partial x$. In [29], it was shown that in problems on a two-dimensional domain under a change in the space variable fundamentally new effects can arise.

## Conclusion

We have shown that the dimension of the critical cases that arise in the stability analysis of the class of singularly perturbed equations we look at is infinite. An effective constructive algorithm is developed for special nonlinear partial differential equations which play the role of normal forms. Their nonlocal dynamics determines the behaviour of the solutions of the original boundary value problems in a small neighbourhood of an equilibrium state. Theorems are formulated which give relations between the solutions of the original equations that are asymptotic with respect to the discrepancy and solutions of the constructed (normalized) boundary value problems.

All the constructions are similar for all the classical boundary value problems under consideration.

The necessity for renormalization is demonstrated, as well as the effectiveness of the corresponding constructions of normalized systems with universal nonlinearities.

We note that the investigation of the dynamics of higher modes (§§4 and 5) in normalized systems involves the 'inner' parameter $\theta = \theta(\varepsilon)$, which changes infinitely often between 0 and 1 as $\varepsilon \to 0$. This indicates that dynamic properties are highly sensitive to changes in the parameter $\varepsilon$, and that there is the possibility of an infinite process of direct and inverse bifurcations as $\varepsilon \to 0$.

It should be stressed that the numerical study of the original singularly perturbed problems presents considerable difficulties. The transition to the normalized problems greatly facilitates this task.

## References

[1] T. S. Akhromeeva, S. P. Kurdyumov, G. G. Malinetsky et al., *Nonstationary structures and diffusion chaos*, Nauka, Moscow,1992, 544 pp. [in Russian] MR1261183

[2] S. A. Gourley, J. W.-H. Sou and J. H. Wu, *Nonlocality of reaction-diffusion equations induced by delay: biological modeling and nonlinear dynamics*, J. Math. Sci. **124:4**(2004), 5119–5153. DOI: 10.1023/B:JOTH.0000047249.39572.6d. MR2129130

[3] H. Haken, *Synergetics: an introduction. 3rd ed.*, Springer Series in Synergetics; 1), Springer Verlag, Berlin, 1983. DOI: 10.1007/978-3-642-88338-5. MR2062548

[4] W. Huang, *Global dynamics for a reaction–diffusion equation with time delay*, J. Differential Equations **143:2**(1998), 293–326. DOI: 10.1006/jdeq.1997.3374. https://zbmath.org/0905.35093. MR1607956

[5] Y. Kuang, *Delay differential equations: with applications in population dynamics*, (Mathematics in science and engineering ; 191), Academic Press, Boston, 1993. 410 pp. MR1218880

[6] N. A. Kudryashov, *Methods of nonlinear mathematical physics*, MEPhI, Moscow, 2008, 352 pp. [in Russian]

[7] Y. Kuramoto and T. Tsuzuki, *On the formation of dissipative structures in reaction-diffusion systems*, Progr. Theor. Phys. **54:3**(1975), 687–699. DOI: 10.1143/PTP.54.687.

[8] Y. Kuramoto, *Chemical oscillations, waves, and turbulence*, Springer Series in Synergetics, vol. 19, Springer Verlag, Berlin,1984. 164 pp. DOI: 10.1007/978-3-642-69689-3. URL: https://zbmath.org/?q=an:0558.76051. MR762432

[9] V. F. Butuzov and N. T. Levashova, *On a system of reaction-diffusion-transfer type in the case of small diffusion and fast reactions*, Comput. Math. Math. Phys., **43:7** (2003), 962–974. MR2012786

[10] V. F. Butuzov, A. B. Vasil'eva and N. N. Nefedov, *Asymptotic theory of contrasting structures. A survey*, Autom. Remote Control, **58:7**(1997), 1068–1091. http://mi.mathnet.ru/eng/at/y1997/i7/p4 MR1615108

[11] H. Meinhardt, *The algorithmic beauty of sea shells*, Springer Verlag, Berlin, 1995. 215 pp. (The Virtual Laboratory). DOI: 10.1007/978-3-662-13135-0. MR1325695

[12] A. Rovinsky ,and A. Zhabotinsky, *Mechanism and mathematical model of the oscillating bromate-ferroin-bromomalonic acid reaction*, J. Phys. Chem. **88:25**(1984), 6081–6084.

[13] V. A. Vasiliev, Yu. M. Romanovsky and V. G. Yakhno, *Autowave processes* (Modern Problems of Physics), Nauka, Moscow, 1987, 240 pp. [in Russian]

[14] G. R. Ivanitsky, V. I. Krinsky and E. E. Sel'kov, *The mathematical biophysics of cells*, Theoretical and Applied Biophysics, Nauka, Moscow, 1978, 310 pp. [in Russian]

[15] Yu. M. Romanovsky, N. V. Stepanova and D. S. Chernavsky, *Mathematical biophysics*, Physics of Life Processes, Nauka, Moscow, 1984, 304 pp. MR756240

[16] Yu. M. Romanovsky, N. V. Stepanova and D. S. Chernavsky, *Mathematical modelling in biophysics*, Physics of life processes, Nauka, Moscow, 1975, 343 pp. MR756240

[17] R. J. Field and M. Burger (eds.), *Oscillations and traveling waves in chemical systems*, Mir, Moscow, 1988, 720 pp. MR965988

[18] S. A. Kashchenko, *On the quasi-normal forms for parabolic equations with small diffusion*, Dokl. Akad. Nauk SSSR, **299:5**(1988), 1049–1052. MR947229

[19] S. A. Kashchenko, *Local dynamics of two-component contrast structures near a bifurcation point*, Soviet Physics Doklady, **35**(1990), 420. MR1072879

[20] I. B. Bokolishvily, S. A. Kaschenko, G. G. Malinetskii et al., *Complex ordering and stochastic oscillations in a class of reaction-diffusion systems with small diffusion*, J. Nonlinear Science, **4:1**(1994), 545–562. DOI: 10.1007/BF02430645. http://link.springer.com/article/10.1007/BF02430645. MR1302353

[21] S. A. Kashchenko, *Normalization in the systems with small diffusion*, Internat. J. Bifur. Chaos in Appl. Sci. Engrg., **6:6**(1996), 1093–1109. DOI:10.1142/S021812749600059X. MR1409411

[22] S. A. Kashchenko, *Bifurcational features in systems of nonlinear parabolic equations with weak diffusion*, Internat. J. Bifur. Chaos in Appl. Sci. Engrg., **15:11**(2005), 3595–3606. DOI: 10.1142/S0218127405014258. MR2197751

[23] I. S. Kashchenko and S. A. Kashchenko, *Quasi-normal forms for parabolic systems with strong nonlinearity and weak diffusion*, Zh. Vychisl. Mat. Mat. Fiz., **52:8**(2012), 1482–1491. http://mi.mathnet.ru/zvmmf9696. MR3245239

[24] I. S. Kashchenko and S. A. Kashchenko, *Cooperative dynamics of strongly coupled distributed systems*, Doklady Mathematics, **85:1**(2012), 129–133. http://elibrary.ru/item.asp?id=17353318. MR2963703
addmr: Added MR number '3077454'.

[25] I. S. Kashchenko and S. A. Kashchenko, *Quasi-normal forms of two-component singularly perturbed systems*, Dokl. RAN, **447:4**(2012), 376–381. MR3077454

[26] I. S. Kashchenko and S. A. Kashchenko, *Local dynamics of two-component singularly perturbed systems of parabolic Type*, Internat. J. Bifur. Chaos in Appl. Sci. Engrg., **25:11**(2015), 1550142. DOI:10.1142/S0218127415501424. MR3416215

[27] S. A. Kashchenko, *Normal form for the Korteweg–de Vries–Burgers equation*, Dokl. RAN, **468:4**(2016), 383–386. DOI: 10.7868/S0869565216160052. MR3559770

[28] I. S. Kashchenko and S. A. Kaschenko, *Local dynamics of two-component singularly perturbed parabolic systems*, Trans. Moscow Math. Soc., **77**(2016), 55–68. MR3643964

[29] S. A. Kashchenko, *Spatial singularities of high-mode bifurcations of two-component systems with small diffusion*, Differ. Equ., **25:2**(1989), 193–199. MR994709

P. G. Demidov Yaroslavl State University, National Research Nuclear University MEPhI
*Email address*: kashch@uniyar.ac.ru

Translated by IAN MARSHALL
Originally published in Russian