How the Alternating Sign Matrix Conjecture Was Solved

David Bressoud and James Propp

Introduction

Perusing the four volumes of Muir's *The Theory of Determinants in the Historical Order of Development*, one might be tempted to conclude that the theory of determinants was well and truly beaten to death in the nineteenth century. In fact, the field is thriving, and it has continued to yield challenging problems of deceptive elegance and simplicity. The Alternating Sign Matrix Conjecture was one of the most notorious of these problems. For fifteen years it defied assaults by some of the world's best mathematicians; then in 1995 three distinct proofs appeared. The first, by Doron Zeilberger, drew on results and techniques from partition theory, symmetric functions, and constant term identities, with a pivotal role played by the partial difference operator philosophy and by computer algebra. Greg Kuperberg found the second proof, which relied on the machinery of statistical mechanics and in particular on the Yang-Baxter equation for the 6-vertex lattice model. The third proof, again by Zeilberger, expanded Kuperberg's approach to prove a more general result. It combined the Yang-Baxter equation with the q-calculus and its associated orthogonal polynomials, and it relied on the WZ-method of Herbert Wilf and Zeilberger. Wilf and Zeilberger would later receive the Steele Prize for this algorithmic approach to discovering and proving series identities (*Notices*, April 1998).

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These tools did not come from outside determinant theory; rather, the classical theory of determinants grew into nineteenth-century invariant theory, a field whose twentieth-century progeny include partition theory and the q-calculus, representation theory and symmetric functions, and statistical mechanics. The proofs of the Alternating Sign Matrix Theorem have served to strengthen ties between these fields and to suggest new avenues of research.

An **alternating sign matrix** (ASM) is a matrix of 0's, 1's, and -1's in which the entries in each row or column sum to 1 and the nonzero entries in each row or column alternate in sign. An example is

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

This generalization of the notion of permutation matrices was discovered by David Robbins and Howard Rumsey in the early 1980s, but to tell our story properly, we should begin with Charles Lutwidge Dodgson (better known as Lewis Carroll).

Dodgson devised a method of evaluating determinants called **condensation** that is eminently suited to hand-calculations. Recall that the determinant of an n-by-n matrix $(a_{i,j})$ is defined as

$$|a_{i,j}| = \sum_{\pi} (-1)^{\mathcal{I}(\pi)} \prod_{i=1}^{n} a_{i,\pi(i)},$$

1						
1	1					
2	3	2				
7	14	14	7			
42	105	135	105	42		
429	1287	2002	2002	1287	429	
7436	26026	47320	56784	47320	26026	7436

Figure 1. The counts of *n*-by-*n* ASMs with a 1 at the top of column *k*.

2/2					
2/3	3/2				
2/4	5/5	4/2			
2/5	7/9	9/7	5/2		
2/6	9/14	16/16	14/9	6/2	
2/7	11/20	25/30	30/25	20/11	7/2

Figure 2. The ratios of adjacent terms from Figure 1.

where π ranges over all permutations of $\{1, 2, \ldots, n\}$ and $\mathcal{I}(\pi)$ is the inversion number of π, i.e., the minimal number of transpositions of adjacent columns needed to turn the matrix representing π into the identity matrix. This formula is practical for 3-by-3 and perhaps 4-by-4 matrices, but for large matrices it is inefficient. Most mathematicians are familiar with Gaussian elimination as a more practical method of evaluating determinants by hand, but condensation is also useful and deserves to be better known. One starts with an n-by-n matrix and then successively computes an $(n-1)$-by-$(n-1)$ matrix, an $(n-2)$-by-$(n-2)$ matrix, etc., until one arrives at a 1-by-1 matrix whose sole entry is the determinant of the original n-by-n matrix. The rule for computing the k-by-k matrix $(n-1 \geq k \geq 1)$ is to take the k^2 2-by-2 connected subdeterminants of the $(k+1)$-by-$(k+1)$ matrix and divide them by the corresponding k^2 central entries of the $(k+2)$-by-$(k+2)$ matrix. (In the case $k = n-1$, no divisions are performed.) Although the use of division may seem like a liability, it actually provides a useful form of error checking for hand calculations with integer matrices: when the algorithm is performed properly (with extra provisos for avoiding division by 0), all the entries of all the intervening matrices are integers, so that when a division fails to come out evenly, one can be sure that a mistake has been made somewhere. The method is also useful for computer calculations, especially since it can be executed in parallel by many processors. The k-by-k matrix that one computes by this procedure has a natural interpretation: it is the matrix of determinants of the k^2 $(n-k+1)$-by-$(n-k+1)$ connected submatrices of the original matrix. The proof of this assertion makes use of one of Jacobi's matrix identities.

If one applies Dodgson condensation to the 3-by-3 matrix

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix},$$

one first obtains the 2-by-2 matrix

$$\begin{pmatrix} ae-bd & bf-ce \\ dh-eg & ei-fh \end{pmatrix},$$

and from this one finds the 1-by-1 matrix whose sole entry is

$$((ae^2i - aefh - bdei + bdfh) \\ - (bdfh - befg - cdeh + ce^2g))/e$$

or, upon collection of terms,

$$(1)aei + (-1)afh + (-1)bdi \\ + (0)bde^{-1}fh + (1)bfg + (1)cdh + (-1)ceg.$$

Six of these terms correspond to the six permutation matrices. For example, $(-1)afh$ is associated with the matrix with 1 in the same positions as occupied by a, f, and h above, with 0's elsewhere. In addition, there is an extra (vanishing) term $(0)bde^{-1}fh$ that can be associated with the matrix with 1's in the positions of b, d, f, and h and -1 in the position of e:

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

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If one does the same thing for the general 4-by-4 matrix, one finds that, in addition to the 24 monomials that make nonzero contributions to the determinant, there are also 18 monomials with vanishing coefficient. Each of these 42 monomials is associated with a 4-by-4 matrix of 0's, 1's, and -1's. In general, when Dodgson condensation is applied to an n-by-n matrix and all like monomials are gathered together, the terms in the final expression (taking the vanishing terms along with the nonvanishing ones) are associated with the n-by-n matrices of 0's, 1's, and -1's in which the nonzero entries in each row and column alternate in sign, beginning and ending with a +1. These are the alternating sign matrices (or ASMs) of order n, invented by Robbins and Rumsey in their study of Dodgson condensation.

It was simple curiosity that led Robbins and Rumsey, now joined by William Mills, to investigate the number of ASMs. Letting $\mathcal{A}_n$ denote the set of n-by-n ASMs and A_n the cardinality of $\mathcal{A}_n$, the three investigators found by computer calculation that the sequence A_n went

$$1, 2, 7, 42, 429, 7436, 218348,$$
$$10850216, 911835460, \ldots.$$

This was not a sequence any of them had seen before. The growth rate of the sequence and the absence of large prime divisors (e.g., $911835460 = 2^2 \cdot 5 \cdot 17^2 \cdot 19^3 \cdot 23$) suggested to Mills, Robbins, and Rumsey that there was a formula for A_n as a ratio of products of factorials. To find this formula, they divided the set of n-by-n ASMs into classes according to the position of the 1 in the first row. Their tallies yielded a triangular array in which the kth entry of the nth row is the number of n-by-n ASMs with a 1 in row 1, column k, as shown in Figure 1.

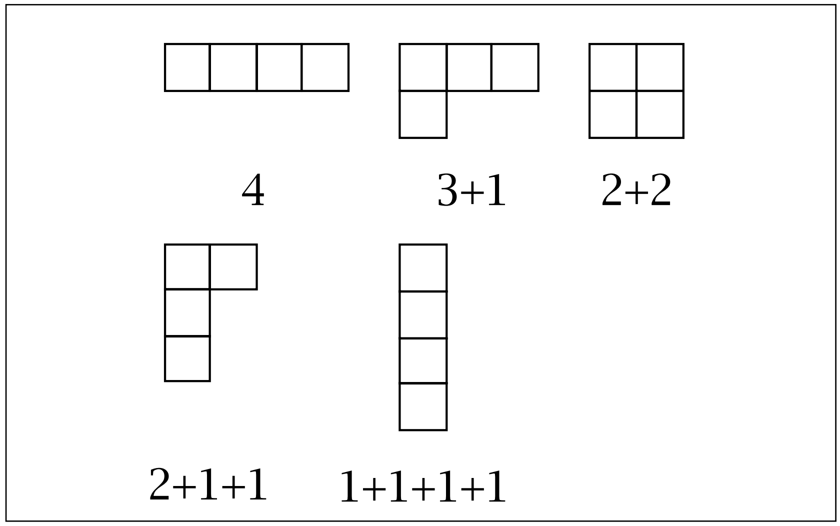


Figure 3. Young diagrams corresponding to partitions of 4.

Clearly the sum of the entries in each row is A_n, and it is not difficult to see as well that the first entry in each row must equal A_{n-1}. When Mills, Robbins, and Rumsey looked at ratios of horizontally adjacent entries, they discovered the remarkable pattern shown in Figure 2.

The nth row starts with $2/(n+1)$ and ends with $(n+1)/2$. The striking observation is that each ratio appears to arise from the two ratios diagonally above by adding numerators and adding denominators. Soon verified through $n = 20$, this became known as the **Refined ASM Conjecture**.

Using the fact that the first entry in each row is the sum of entries in the previous row, one can show that one consequence of the Refined ASM Conjecture is the formula

$$A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}.$$

This is the **ASM Conjecture**. It remained unproved until 1995 when an army of referees—88 people and one computer—pronounced as correct the latest version of the proof that Zeilberger had first proposed in 1992. The same year, Kuperberg produced a considerably simpler proof that relies on the Yang–Baxter equation for the 6-vertex model. By the end of that year, Zeilberger had adapted Kuperberg's proof to verify the Refined ASM Conjecture.

Descending Plane Partitions

When Mills, Robbins, and Rumsey told Richard Stanley about their conjecture, they were astonished to hear that the sequence $1, 2, 7, 42, 429, 7436, \ldots$ had recently arisen in research done by George Andrews on a seemingly unrelated problem in the theory of plane partitions.

To explain plane partitions, we jump back to the nineteenth century and describe Percival Alexander MacMahon's work, which generalized the notion of number-partitions whose study had been initiated by Euler and continued by Sylvester, Frobenius, and others. Euler had shown that the number of ways of representing the positive integer n as a sum of positive integers (without regard to order) equals the coefficient of q^n in the power-series expansion of the infinite product

$$\prod_{k=1}^{\infty} \frac{1}{1-q^k} = 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + 11q^6 + 15q^7 + \cdots.$$

In any partition of a number, it is customary to list the "parts", or summands, in nonincreasing order; thus, the five partitions of 4 are written as 4, 3 + 1, 2 + 2, 2 + 1 + 1, and 1 + 1 + 1 + 1. Partitions are frequently represented by means of Young diagrams; the Young diagrams of these five partitions are shown in Figure 3.

Each part in the partition is represented by a row of unit squares. These rows of squares are left- justified, and the lengths are weakly decreasing as one moves down. Figure 4 shows all the Young diagrams (including the empty partition of 0 at the upper left) that fit inside a 2-by-2 square. There is a unique lattice-path from the upper-right corner of the square to the lower-left corner that traces the lower-right outline of the Young diagram. In general, the partitions of integers less than or equal to mn in which there are at most n parts, and in which no part is larger than m, correspond to Young diagrams that fit inside an m-by-n rectangle, which in turn correspond to lattice paths that go from the upper-right corner of the rectangle to the lower-left corner by means of leftward and downward steps. Each such path corresponds to a way of interspersing m downward steps with n leftward steps, and elementary combinatorics tells us that the number of such paths is the binomial coefficient $(m+n)!/m!n!$.

MacMahon realized that these diagrammatic representations could be extended to three dimensions in a very natural way. Specifically, one can define 3-dimensional Young diagrams as as-

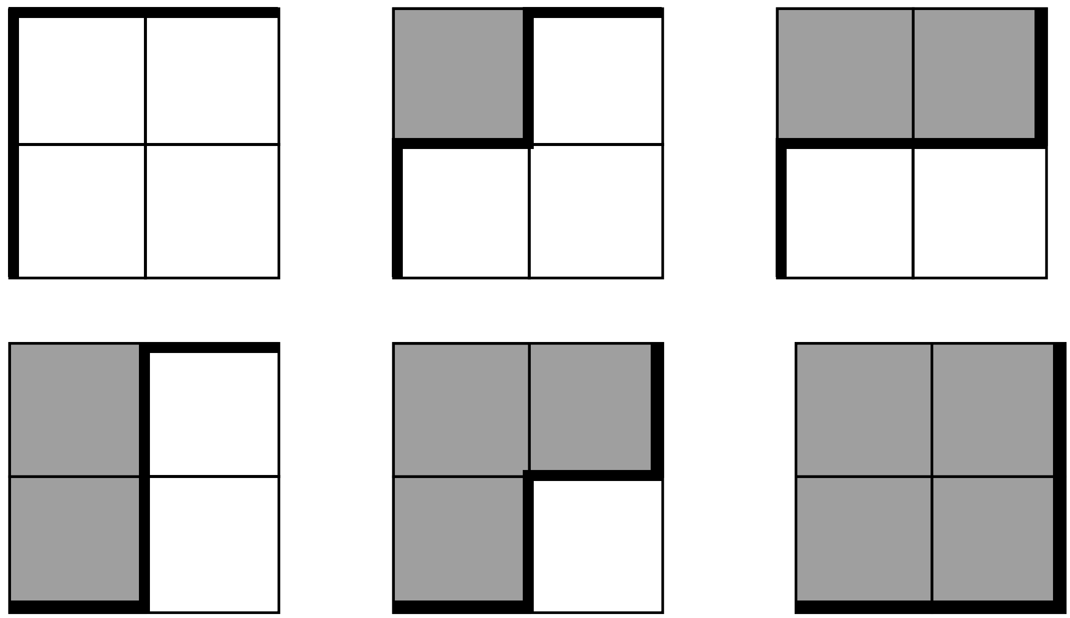


Figure 4. Young diagrams and lattice paths.

semblages of cubes inside an octant (as in Figure 5) such that every cube is "supported" on the three sides toward the bounding planes of the octant; to be supported on a particular side, a cube must be supported either by another cube that shares that face with it or by a bounding plane. These assemblages correspond to partitions of a number into parts arranged 2-dimensionally in a quadrant, as in the figure. Each vertical stack of cubes in part (a) of the figure is marked on its top face so that when we look straight down, we can read the number of cubes in that stack. When the assemblage of cubes is viewed from above, these numbers form the **plane partition** in part (b) of the figure.

MacMahon showed that the number of plane partitions of the number n is given by the coefficient of q^n in the power-series expansion of the infinite product

$$\text{(1)} \qquad \prod_{k=1}^{\infty} \frac{1}{(1-q^k)^k} = 1 + q + 3q^2 + 6q^3 + 13q^4 + \cdots .$$

He also found a formula for the number of plane partitions whose Young diagrams fit inside an a-by-b-by-c box; his formula was fairly complicated, but it is equivalent to the triple product

$$\text{(2)} \qquad \prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2}.$$

It should be mentioned that d-dimensional Young diagrams can be defined for larger integers d, but that the obvious generalizations of formulas (1) and (2) are wrong for every value of d larger than 3.

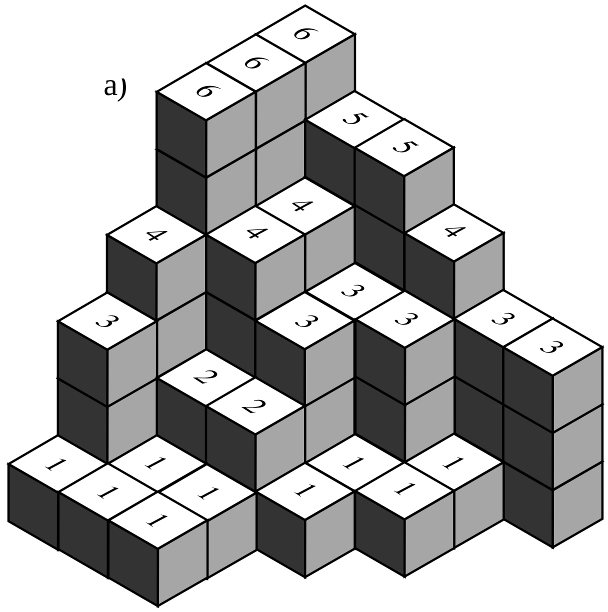
Earlier researchers had enumerated ordinary partitions whose Young diagrams are invariant under reflection in the diagonal axis, so it was natural for MacMahon to undertake an analysis of plane partitions with an analogous symmetry in their 3-dimensional representations. He did indeed discover a formula enumerating plane partitions with a single reflective symmetry; however, he did not give a proof, nor did he consider other sorts of symmetry. Starting in the 1960s, various researchers (notably Basil Gordon, Donald Knuth, Ian Macdonald, George Andrews, and Richard Stanley) sought to fill this gap by considering this and other symmetry classes of plane partitions. One class that proved challenging was the class of plane partitions whose solid Young diagrams are invariant under the rotation that cyclically permutes the x, y, and z axes. In 1979 Macdonald had formulated a conjecture for the number of cyclically symmetric plane partitions of a given integer (CSPPs for short) in an a-by-a-by-a box; specifically, he had proposed a product representation for the power series for which the coefficient of q^n is the number of CSPPs of n, but he had not been able to find a proof. In that same year, Andrews proved the $q = 1$ version of Macdonald's conjecture, that is, a formula for the total number of CSPPs that fit inside an a-by-a-by-a box.

One byproduct of Andrews's proof was a formula counting descending plane partitions. A **descending plane partition** (DPP) of order n is a 2-dimensional array of positive integers less than or equal to n such that the left-hand edges are successively indented, there is weak decrease across rows and strict decrease down columns, and the number of entries in each row is strictly less than the largest entry in that row. An example of order 7 (or greater) is

7	7	6	6	3	1
	6	5	4	2	
		3	3		
			2		.

There are seven DPPs of order 3. One of these is the empty DPP. Five of them consist of a single row: 2, 3, 31, 32, or 33. There is one with two rows: 33 above 2. Andrews had found a formula for the number of DPPs of order n, which he computed for small values of n, yielding the sequence $1, 2, 7, 42, 429, 7436, \ldots$. Thus it was natural that when Stanley heard about the work of Mills, Robbins, and Rumsey, he would recognize the sequence they had encountered. Stanley quickly verified that their conjectured formula for counting ASMs was essentially identical to Andrews's proved formula for DPPs. In this fashion, two lines of research—Dodgson's condensation algorithm and MacMahon's plane partitions—came together.

a)



b)

6	5	5	4	3	3
6	4	3	3	1	
6	4	3	1	1	
4	2	2	1		
3	1	1			
1	1	1			

Figure 5. The planar representation of a plane partition.

Mills, Robbins, and Rumsey tried to prove the ASM Conjecture by establishing a 1-to-1 correspondence between ASMs and descending plane partitions. ASMs have a natural parameter that marks the position of the 1 in the first row. What is the corresponding parameter for descending plane partitions? They conjectured that it is the number of times the integer n appears in the descending plane partition of order n.

Something unexpected happened. They discovered that this additional parameter was the key to a simple inductive proof of Andrews's formula for the number of descending plane partitions of order n. They translated this parameter to the problem of counting cyclically symmetric plane partitions. It simplified that proof and showed them how to prove Macdonald's original conjecture for the number of cyclically symmetric plane partitions of any integer inside any box. They had proved a significant outstanding conjecture, but not the one they had set out to prove. The ties between ASMs and plane partitions were now firmly established. They were about to be strengthened even more.

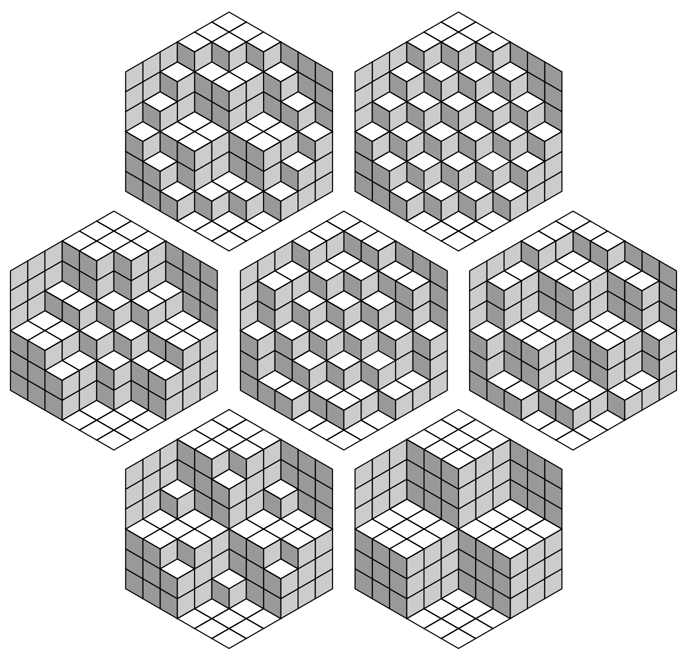


Figure 6. The seven TSSCPPs in a 6 x 6 x 6 box.

Symmetries of Plane Partitions

One of the first problems that Mills, Robbins, and Rumsey ran into in trying to elucidate the connection between ASMs and DPPs was that the group of symmetries of the square acts in a natural way on the set of ASMs, whereas there is no obvious nontrivial group action on the set of DPPs. The three researchers began to search for symmetries of DPPs that would mirror the symmetries of ASMs. Soon they discovered an involution on the set of descending plane partitions of order n that appeared to mimic vertical reflection of an ASM. Later they modified this involution so that it applied to the set of all cyclically symmetric plane partitions in an n-by-n-by-n box. If the solid Young diagram of a plane partition fits inside a box of given size, one can take the collection of cubes that are in the box but do *not* belong to the solid Young diagram. These determine another plane partition called the **complement**. If a plane partition in an n-by-n-by-n box is cyclically symmetric, so is its complement. The complement is in general different from the original plane partition but can in some cases be the same, in which case the plane partition is said to be **self-complementary**. Robbins looked at plane partitions whose Young diagrams fit inside an n-by-n-by-n box and that, in addition to being totally symmetric (that is, invariant under arbitrary permutations of the three axes), were also self-complementary. Figure 6 shows the solid Young diagrams associated with the seven totally symmetric self-complementary plane partitions (called TSSCPPs for short) whose solid Young diagrams fit inside a 6-by-6-by-6 box. When n is odd, there can be no TSSCPPs, since the number of cubes in a solid Young diagram and the number of cubes in its complement will necessarily have opposite parity. For n even, Robbins found that the number of TSSCPPs goes like $1, 2, 7, 42, 429, 7436, \ldots$. The sequence associated with ASMs had now appeared three times, each time arising from a combinatorial question that seemed unrelated to the others.

Mills, Robbins, and Rumsey noticed that one way to make the conjectural connection between ASMs and TSSCPPs appear more natural is to represent both sorts of objects in the form of triangular arrays. A monotone triangle of order n is a triangular array of numbers (n numbers on each side) with entries between 1 and n, with strict increase across rows and weak increase as one moves diagonally up or down to the right. There is a simple bijection between ASMs of order n and monotone triangles of order n. An example is given in Figure 7. In the triangle, the entries of row j, counted from the top, record the positions of the 1's in the vector formed by adding the top j rows of the matrix. Monotone triangles are also sometimes referred to as strict Gelfand patterns, and Zeilberger would later dub them "gog triangles".

In a slightly more involved fashion, the TSSCPPs in a $2n$-by-$2n$-by-$2n$ box are in 1-to-1 correspondence with order-n triangular arrays with entries 1 through n that increase weakly across rows and down columns and such that all entries in column j are less than or equal to j. An example for $n = 5$ is

$$\begin{array}{ccccc} 1 \\ 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 & 3 \\ 1 & 1 & 2 & 4 & 5. \end{array}$$

Zeilberger would later dub these "magog triangles".

The bottom row of a magog triangle is a weakly increasing sequence of the integers 1 through n, with the ith entry less than or equal to i. The northwest edge of a monotone or gog triangle is also a weakly increasing sequence of the integers 1 through n, with the ith entry less than or equal to i. Mills, Robbins, and Rumsey conjectured that the number of possible configurations for the bottom k rows of a magog triangle of order n (call this $M(n,k)$) is equal to the number of possible configurations for the first k diagonals of a gog triangle (call this $G(n,k)$). The case $k = n$ would imply that the number of ASMs of order n is equal to the number of TSSCPPs of order n.

The researchers proved the formula $G(n,k) = M(n,k)$ for $k = 2$ (the case $k = 1$ is the remark made at the beginning of the preceding paragraph), but their methods offered very little hope of yielding a proof for greater values of k. Zeilberger, hearing of the proof for $k = 2$, thought that a proof for general k might be within reach, but the amount of work that he foresaw was daunting. Furthermore, the reward for such efforts would not be a proof of the ASM Conjecture, but only a proof that the ASM Conjecture was equivalent to the TSSCPP conjecture. Therefore he did not pursue the problem.

Throughout the mid to late 1980s, articles appeared with conjectured formulas for plane partitions or ASMs that satisfied certain symmetry conditions. The best-known of these articles was Stanley's paper "A baker's dozen of conjectures concerning plane partitions". Some of the conjectures were subsequently proved, but many were not. In 1991 Robbins sought a broader audience for these problems with his *Mathematical Intelligencer* article "The story of 1, 2, 7, 42, 429, 7436, …", in which he exclaimed,

> These conjectures are of such compelling simplicity that it is hard to understand how any mathematician can bear the pain

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \longleftrightarrow \begin{array}{ccccccccc} & & & & 4 & & & & \\ & & & 2 & & 5 & & & \\ & & 1 & & 4 & & 5 & & \\ & 1 & & 3 & & 4 & & 5 & \\ 1 & & 2 & & 3 & & 4 & & 5. \end{array}$$

$$(0,0,0,1,0) + (0,1,0,-1,1) + (1,-1,0,1,0) = (1,0,0,1,1) \longleftrightarrow 1 \quad 4 \quad 5.$$

Figure 7. The correspondence between ASMs and monotone triangles.

of living without understanding why they are true.

The First Proof of the ASM Conjecture

By the time Robbins published his *Intelligencer* article, the succession of insights that would lead to the proof of the ASM Conjecture was well under way. The first contribution to the solution of the TSSCPP problem came from William Doran, then an undergraduate, who succeeded in translating an arbitrary TSSCPP into a set of lattice paths. Ira Gessel and Xavier Viennot had shown how to use determinants to count sets of lattice paths, but Doran's paths did not quite fit the Gessel-Viennot paradigm.

Soichi Okada had run into a similar problem a few years earlier when trying to count all totally symmetric plane partitions (plane partitions invariant under all permutations of the axes). He had realized that instead of trying to transform the problem directly into the evaluation of a determinant, the key was to translate it into the evaluation of a Pfaffian, an analogue of the determinant that applies to triangular arrays of numbers and that is a signed sum indexed by set partitions of $\{1,\ldots,n\}$ into pairs of elements. This is an approach to the enumeration of plane partitions that goes back to Basil Gordon in 1971.

John Stembridge realized that this would work for Doran's paths. The fact that the Pfaffian is the square root of the determinant of the corresponding skew symmetric matrix meant that the number of TSSCPPs could ultimately be expressed as a determinant. The matrix that emerged was skew symmetric with entries

$$H(i,j)=\sum_{2i-j<r\le 2j-i}\binom{i+j}{r}$$

with

$$0\le i<j\le n-1 \quad \text{for } n \text{ even},$$
$$1\le i<j\le n-1 \quad \text{for } n \text{ odd}.$$

Andrews then evaluated this determinant and confirmed the conjectured formula for the number of TSSCPPs in a box. Some highly unusual hypergeometric series appear in this problem, and Andrews relied on the WZ-method to prove the summation formulas that arose.

Emboldened by Andrews's solution of the TSSCPP problem, Zeilberger now tackled the problem of proving the formula $G(n,k)=M(n,k)$ by induction on n and k. A proof of this formula, combined with Andrews's proof of the formula for $M(n,n)$ conjectured by Robbins, would yield a proof of the formula for $G(n,n)$ and thus prove the ASM Conjecture.

Zeilberger began by expressing each of the quantities to be counted as the constant term of a Laurent series in k variables, $x_1,\ldots,x_k$. Using a technique he had learned from Stembridge and Dennis Stanton, Zeilberger divided each of these series by $x_1\cdots x_k$, shifted his attention to the residues at $x_1=\cdots=x_k=0$, and showed that these residues are left unchanged by the operator $g_{\sigma,S}$ that acts by first replacing each x_i for which $i\in S$ by $\bar{x}_i=1-x_i$ and then replacing each x_i by $x_{\sigma(i)}$. He then summed the images of these functions over all pairs (σ,S) where σ is a permutation and S is a subset of $\{1,\ldots,n\}$. Zeilberger needed to prove that the resulting rational functions had the same residues. In fact, he was able to prove that these rational functions were identical.

Zeilberger's proof was announced in 1992. Though essentially sound, it went through several revisions before it was finally accepted in 1995. The details of this proof are intricate. Zeilberger arranged them in a tree of lemmas, sublemmas, subsublemmas, through "sub^{7}lemmas". Many of these state that certain functions satisfy particular partial difference equations or boundary conditions. Some of them claim the invariance under $g_{\sigma,S}$ of various pieces of the final functions. All of this builds to the principal result that the sums over (σ,S) of $g_{\sigma,S}$ of each Laurent series are identical. Zeilberger recruited his eighty-nine referees, who were each given one sub^{i}lemma and asked to verify that it did, indeed, follow from the corresponding sub^{j}lemmas, $j>i$. The names of the referees were listed in the article, along with a brief biographical sketch of each. Many of the people who have already been mentioned here were among the referees; the article thus gives a snapshot of the principal players in the study of ASMs in the 1990s. It is likely that Zeilberger's approach could have been extended to prove the Refined ASM Conjecture, but no one had the courage to begin this daunting task. Fortunately, within a few months Greg Kuperberg had found a much simpler route to the proof of the ASM Conjecture, using the machinery of statistical mechanics.

Proofs via Statistical Mechanics

Kuperberg's work on the ASM Conjecture began around 1990 as an outgrowth of his work on enumeration of tilings in collaboration with Noam Elkies, Michael Larsen, and James Propp. On the one hand, the problem of counting domino tilings of certain plane regions known as Aztec diamonds had turned out to have connections with the theory of ASMs; on the other hand, the counting problem can be recast as a problem of counting "dimer configurations" and solved with the methods of statistical mechanics. Having become aware of such methods, Kuperberg proceeded to apply them to the problem of enumerating symmetry classes of plane partitions. We can view the TSSCPPs in Figure 6 as 2-dimensional hexagons filled with congruent parallelograms. Any plane partition inside a box translates visually into a 2-dimensional tiling problem. TSSCPPs are those tilings of a regular hexagon that are invariant under all symmetries

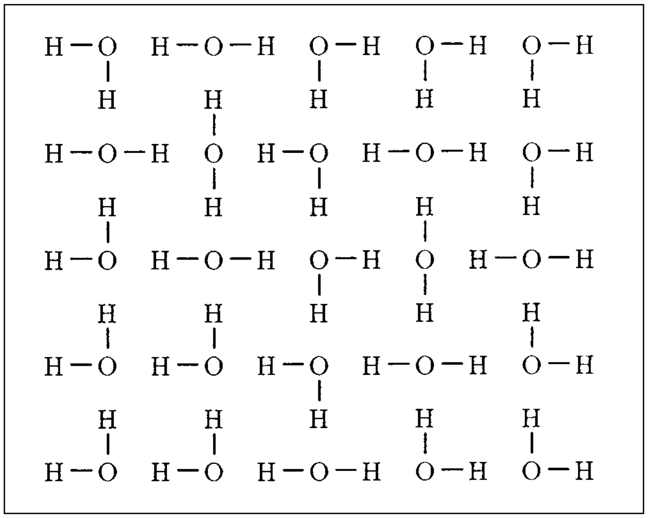


Figure 8. A patch of "square ice".

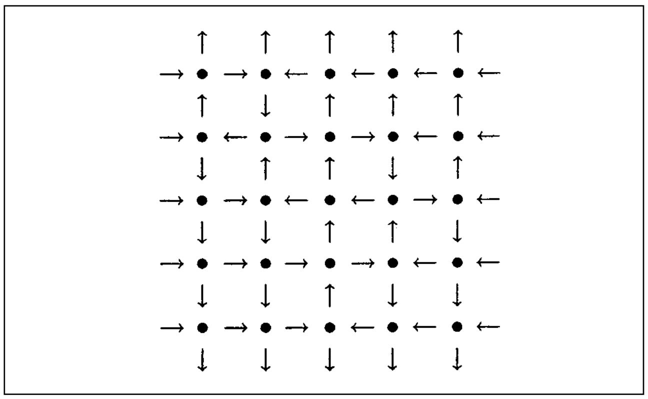


Figure 9. Figure 8 converted into a directed graph on a square lattice.

of the hexagon. Tilings of this kind can be viewed as states of 2-dimensional "dimer models". In his solution to the CSSCPP problem (enumerating cyclically symmetric, self-complementary plane partitions), Kuperberg made use of matrix methods developed by the statistical mechanician Pieter Willem Kasteleyn. It was natural for Kuperberg to turn next to the ASM problem to see whether statistical mechanics had anything to offer. He learned that physicists had independently been studying ASMs in another guise in connection with the study of the structure of ice.

The water molecules in actual ice crystals are arranged in a 3-dimensional lattice, but physicists substituted a 2-dimensional lattice (the square grid) to make the model more tractable. Figure 8 shows a patch of what is called "square ice". It corresponds to the ASM

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Horizontal molecules correspond to +1, vertical molecules to -1, and angled molecules to 0.

Physicists often represent such a square ice state as a directed graph on a square lattice in which each vertex has in-degree and out-degree two, as in Figure 9. The oxygen atoms are at the vertices, and the directed edges correspond to hydrogen atoms, directed toward the atom to which they are bonded. The fact that there are six possible configurations at each vertex gives this model its name, the **6-vertex model**.

Note that along the boundary of Figure 9 the arrows point inward along the left and right and outward along the top and bottom. This boundary condition is called the domain wall boundary condition for the 6-vertex model. States satisfying this boundary condition were studied by Vladimir Korepin in the early 1980s; they are the square-ice states that are equivalent to ASMs.

Physicists are interested in weighted sums taken over all possible configurations of given size and satisfying given boundary conditions. Few such state-sums can be expressed in closed form, but Anatoli Izergin (building on the earlier work of Korepin) found such a formula for the 6-vertex model with domain wall boundary conditions. That formula is equivalent to the following determinant evaluation:

$$\begin{aligned} &\det\left(\frac{1}{(x_i+y_j)(ax_i+y_j)}\right) \\ &\qquad \times \frac{\prod_{i,j=1}^{n}(x_i+y_j)(ax_i+y_j)}{\prod_{1\le i<j\le n}(x_i-x_j)(y_i-y_j)} \\ &= \sum_{A\in\mathcal{A}_n}\Big((-1)^{N(A)} \\ &\qquad \times a^{(n^2-n)/2-\mathcal{I}(A)}(1-a)^{2N(A)} \\ &\qquad \times \prod_{i=1}^{n} x_i^{N_i(A)} y_i^{N^i(A)} \prod_{\substack{1\le i,j\le n \\ a_{ij}=0}} (\alpha_{ij}x_i+y_j)\Big), \end{aligned}$$

a sum over ASMs where $N(A)$ (respectively $N_i(A)$, $N^i(A)$) is the number of -1's in A (respectively row i of A, column i of A), $\mathcal{I}(A)$ is the inversion number of A which is equal to $N(A)$ plus the number of southwest molecules (molecules with bonds to the hydrogen atoms to the left and below) in the corresponding patch of square ice, and α_{ij} is a if the corresponding molecule is southwest or northeast and is 1 otherwise. The key to proving this identity is knowing that the right side is a symmetric function in the x_i's and in the y_i's. This fact follows from the Yang–Baxter equation for the 6-vertex model. Kuperberg had learned from Vaughan Jones of the power of the Yang–Baxter equation, and this had led him to Korepin's work on the 6-vertex model.

Kuperberg's initial attempt to exploit this formula was stymied by the unavailability of a full write-up; although Izergin's article was in print, the book by Korepin, Nikolai Bogoliubov, and Izergin that gave a fuller account would not be published until 1993, and the two draft chapters that Kuperberg had were difficult to understand out of the context of the full book. Kuperberg therefore put the problem aside and returned to it only in 1995, after Zeilberger's proof had been fully validated.

In reexamining the Korepin–Izergin determinant formula, Kuperberg realized that with $x_j = e^{-\pi i/3}$, $y_j = 1$, and $a = e^{2\pi i/3}$, the right side of this equation becomes $(-3)^{(n^2-n)/2}$ times the number of n-by-n alternating sign matrices. Unfortunately, under this specialization, the left side behaves badly: both the determinant and the product in the denominator vanish. Kuperberg therefore needed to use some finesse on the left side. By approaching the desired specialization along an appropriate trajectory, he was able to show that the left side does indeed approach the desired value as the x_j's approach $e^{-\pi i/3}$ and the y_j's approach 1.

Kuperberg announced his proof and released a preprint in the summer of 1995. It is interesting to note that one of the techniques used in his article is Dodgson condensation, the very procedure whose study had led Mills, Robbins, and Rumsey to invent alternating sign matrices in the first place.

Philosophically, Kuperberg's proof is quite different from Zeilberger's: Kuperberg's proof is multiplicative, whereas Zeilberger's is additive. To explain this distinction with an analogy, we point out two different ways of obtaining an entry in Pascal's triangle. Under the additive approach, one obtains $\binom{n}{k}$ by adding $\binom{n-1}{k-1}$ and $\binom{n-1}{k}$ (the two entries in the row above). Under the multiplicative approach, one obtains $\binom{n}{k}$ by multiplying $\binom{n}{k-1}$ (the preceding entry in its row) by $(n-k)/k$. It seems fair to say that additive methods are more general and robust and give algebraically arduous proofs with very little combinatorial flavor; multiplicative methods are more fragile and specialized, but where they can be made to apply, they often give more elegant proofs.

After reading and absorbing Kuperberg's paper, Zeilberger proved the Refined ASM Conjecture by evaluating the limit of the left side with x_1 remaining indeterminate. His matrix evaluation uses the moments of the q-Legendre polynomials together with the fact that each monic polynomial in a family of orthogonal polynomials can be expressed as a ratio of determinants involving the moments. The Refined ASM Conjecture ultimately reduces to a cubic transformation formula for hypergeometric series. Zeilberger verified it using his WZ-method.

Conclusion

The study of ASMs has gone hand in hand with the study of symmetry classes of plane partitions, and ideas have traveled in both directions between the two sorts of problems. However, the connection is still somewhat mysterious; for instance, no natural bijection between ASMs of order n and TSSCPPs of order n is yet known. Results discovered in the study of ASMs and symmetrical plane partitions are finding applications in representation theory. Many of the formulas for counting plane partitions with various symmetries were special cases of character formulas for irreducible representations of the symmetric group. Results discovered in the pursuit of the ASM Conjecture have led to analogues for the other Weyl groups, and these insights are generating new problems and conjectures.

Although the ASM formula has now been proved, many intriguing problems remain. Some of the most tantalizing involve symmetry classes of ASMs. Just as one can enumerate the rhombus-tilings of a hexagon that are invariant under some symmetry-group that maps the hexagon to itself, so too can one enumerate the n-by-n ASMs that are invariant under some subgroup of the symmetry-group of the n-by-n square. Robbins has proposed some exact formulas for enumerating certain symmetry classes of ASMs, but, aside from the case in which the symmetry group is trivial (coinciding with the unconstrained case), none of these conjectures has been proved. Intriguingly, one of these symmetry-class enumerations gives rise to integers that are (empirically) intimately connected to the way certain polynomial analogues of the numbers A_n factor. Define the weight of an ASM as x to the power of the number of -1's in the matrix, and let $A_n(x)$ be the sum of the weights of all the ASMs of order n. $A_n(1)$ simply counts the number of alternating-sign matrices; Mills, Robbins, and Rumsey proved that $A_n(2) = 2^{n(n-1)/2}$; and Kuperberg proved a formula for $A_n(3)$ as a rational product of factorials. It does not appear that there exist similar nice formulas for $A_n(m)$ for larger values of m, since the resulting numbers have large prime factors. However, it appears that there exist polynomials $p_n(x)$ such that the polynomial $A_n(x)$ always factors as either $p_n(x)p_{n+1}(x)$ or $2p_n(x)p_{n+1}(x)$, according to whether n is odd or even. Furthermore, the coefficients of $p_n(x)$ appear always to be nonnegative integers. When n is odd, there is a conjectured interpretation of $p_n(x)$ as an enumeration of ASMs with a horizontal (or, equivalently, vertical) axis of bilateral symmetry; no such interpretation is known for when n is even.

The ASM Conjecture has served to cross-fertilize the various modern offspring of classical invariant theory, drawing attention to connections no one had recognized. The study of alternating-sign matrices should continue to bear fruit for many years to come—and to tantalize us with fruit that is just beyond our reach.

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