Book Review

The Invention of Infinity: Mathematics and Art in the Renaissance

Reviewed by Anthony Phillips

The Invention of Infinity: Mathematics and Art in the Renaissance
J. V. Field
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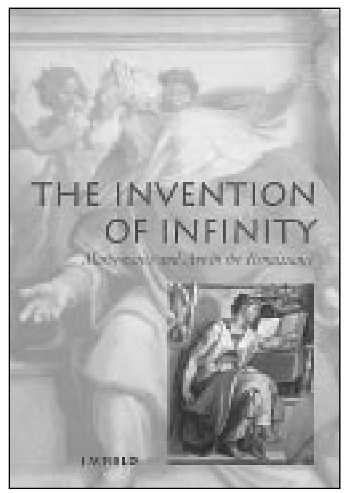
It seems to be a property of human perception that a two-dimensional graphic pattern must be a picture of something. The earliest graphics we know are representations. The picture may be more or less faithful or schematic, on one or the other side of what Gombrich calls "the great divide which runs through the history of art and sets off the few islands of illusionist styles…from the vast ocean of 'conceptual' art" ([3], p. 9). Since our minds perceive reality as three-dimensional, a basic geometrical problem for the illusionistic tradition is how to render three-dimensional objects and the surrounding space on a two-dimensional surface. J. V. Field's *The Invention of Infinity* tells the story of how this problem was discovered as an explicit geometric problem by the artist/mathematicians of the Italian Renaissance, how its solution became part of the standard artistic curriculum, and how its purely mathematical aspects were generalized and developed during the next two hundred years into what we now know as projective geometry.

The problem is generally labelled by the term *perspective*. In its simplest form, "one-point per-



spective", a perspective image in a frame looks like a window into a three-dimensional world. The image is completely defined by the size of the window and the position of the viewer's eye: each detail of the world visible beyond the frame appears at the spot where the straight line joining it to the eye crosses the plane of the picture. *The Invention of Infinity* shows us a famous woodcut by Dürer, dated 1525, which materializes the concept ingeniously: the eye is a hook on the wall, the straight lines are strings, etc. A mathematical consequence is that all the lines perpendicular to the picture-plane must appear in the picture to converge to a single point. This is the "vanishing point", the foot in the picture-plane of the perpendicular dropped from the eye, so to speak. Using Cartesian coordinates reduces to an exercise the computation of all the aspects of this projection; however, when artists at the beginning of the fifteenth century needed a more realistic rendering of space on surface, Descartes (1596–1650) was two hundred years away. They had only what was left of classical geometry.

(The relation between perspective and binocular vision often causes confusion. Two eyes give

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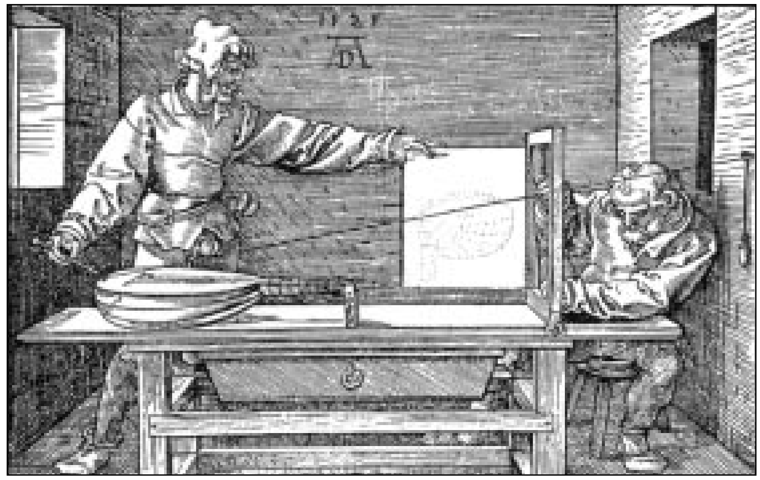
us "stereoscopic" input that, for nearby objects, delivers an immediate sensation of volume and distance. Leonardo da Vinci, as quoted in Gombrich's *Art and Illusion* ([3], p. 98), complains about the impossibility of reproducing this sensation graphically and adds: "…it is impossible for a painting to look as rounded as a mirror image…except if you look at both with one eye only." One-point perspective should be viewed from far away or *with one eye at a time.*)

A camera automatically produces a one-point perspective image of its target. In this age of the photograph it may be difficult to realize that the process had to be invented. It is then useful to think back to childhood. Kids learn the trick of drawing a square and two adjacent parallelograms to get something that "looks like" a cube. They do not "see" the parallelograms any more than the untrained eye sees the gamut of colors it takes to render a faithful image of a yellow ball in the sunlight. The brain automatically rectifies and corrects. There is a striking passage in Piaget and Inhelder's study of how children think about space ([6], p. 211). A six-year-old has drawn a gate in the distance as large as a gate up close. When asked, "Does it seem smaller or not, at the end, down there?" the child answers, "It is not smaller." Similarly, since God is greater than man, it was natural for the medieval artist to paint a huge Virgin and Child surrounded by pint-sized prophets and saints, just as it was natural for the Egyptian tomb painter to show the master twice as tall as his peasants (see illustrations in [3]). Far from being natural, perspective is a calculated illusion, giving the brain false clues so it will construct a virtual reality.

The Invention of Infinity describes the defining event, which took place around 1415 just inside the main doorway of the Florence cathedral. Filippo Brunelleschi (1377–1446) set up a demonstration in which his fellow citizens were invited to look out across the way and to compare their view of the Baptistry with the view, through a peephole, of his picture of that view. Presumably the views were similar enough to excite admiration and astonishment; this would have been the first objective, systematic exploration of the illusion of depth. In fact, Brunelleschi did not need any special geometrical knowledge to construct his image: all he had to do was look at the Baptistry through a peephole himself and make an accurate drawing of what he saw. His fundamental innovation was the *idea* of an illusionistic rendering of space, much in the spirit of Giotto's revolutionary naturalism in the treatment of figures some hundred years before. Another, perhaps more important one was his discovery of the *eye* as a point in space from which a picture was seen. This made perspective into a problem in solid geometry. Beyond his "demo" in the cathedral, which is fairly well documented [5] although the picture itself is lost, we have no trace of these discoveries. Nevertheless, the appearance of one-point perspective in the bas-relief of St. George and the Dragon (Orsanmichele, Florence) executed by Brunelleschi's friend Donatello in 1417 makes it plausible that the method was indeed invented by Brunelleschi before that date.



Albrecht Dürer woodcut, 1525, depicting an instrument being used to draw the perspective image of a lute.

The first surviving written instructions for the geometric construction of perspective images are due to Alberti (1404–1472), another Florentine polymath. Gadol [2] gives a comprehensive survey of this and the rest of his extraordinarily various activity. His *Della Pittura* [1], written in 1435–36, contains an algorithm for drawing a one-point perspective image of a checkerboard pavement given the position of the eye relative to the frame. This special case was of great use to painters, because it allowed them to draw a convincing floor on which figures could be placed, with relative positions readable from the checkerboard "coordinates". Moreover, since the floor can be raised or lowered, the algorithm allows the complete projection of a three-dimensional grid.

At that time there was a recipe in use for drawing checkerboard floors in perspective. By having the columns taper off towards a vanishing point and making each row two thirds of the depth of the row in front of it, one could produce an illusion of depth. Alberti describes this construction and criticizes it as inaccurate because *it does not locate the position of the eye* or, as he calls it, "the point of the visual pyramid" ([1], p. 57; in fact the decrease in row width with distance is inverse-quadratic, not exponential). Alberti probably refers to the calculation of the correct position of the eye by reversing the construction, which is possible if the perspective has been correctly drawn. More details of this backwards construction are given below.

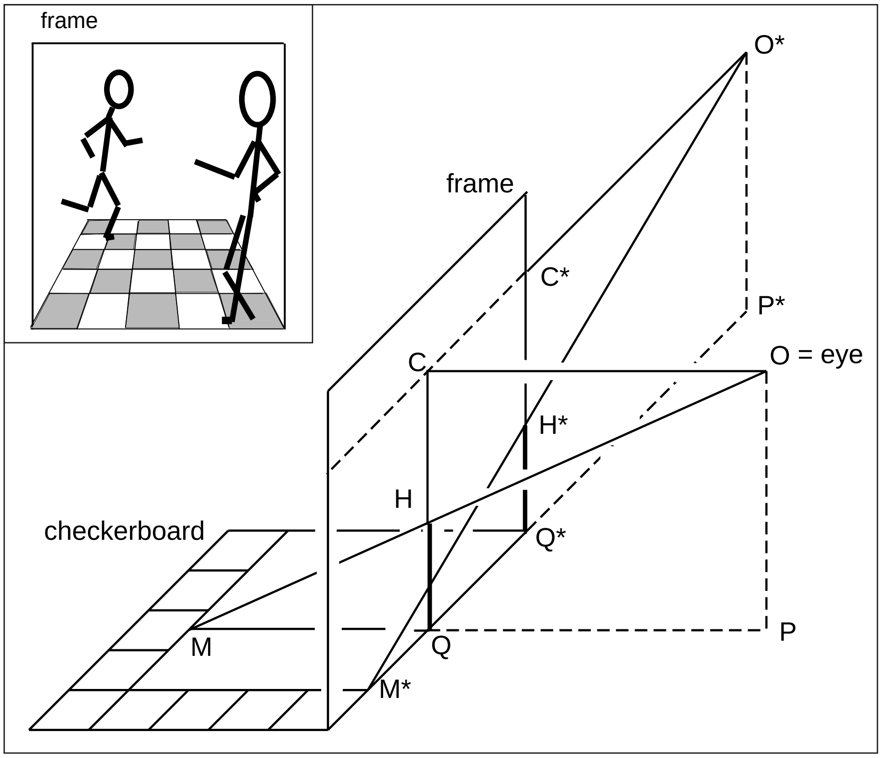


Figure 1. Alberti's construction: unstarred letters represent points in the vertical plane through the eye O, perpendicular to the picture-plane and perpendicular to the plane of the checkerboard. Starred letters represent additional points in the picture plane. C is the vanishing point; O*C* = OC. The base of the fifth *row* as seen by the eye is at the same height as the intersection point with the picture-frame of the line from O* to the inside edge of the fifth *column*. Inset, a completed construction.

Alberti's algorithm, which is not at all obvious, is shown in Figure 1. The checkerboard is horizontal and abuts the edge of the (vertical) frame. A point O^* is drawn in the picture-plane (to the right in this illustration) on a level with the vanishing-point C and such that the horizontal distance O^*C^* to the frame is equal to the distance from the eye to C. As seen from O, the lines in the checkerboard which are perpendicular to the frame appear as lines through C. Each line parallel to the bottom of the frame projects to a horizontal line in the picture. The height of that line is found by intersecting the right-hand edge of the frame with the line from O^* to the foot of the corresponding vertical line (the correspondence is given by a ninety-degree rotation to the left). Why does this work? Alberti does not tell us, and Field's explanation (on pp. 26–27) is unsatisfactory. The construction depends crucially on the rotational and translational symmetry of a checkerboard. This allows the cone of vision and the entire polygon OCHMQP to be swung around into the plane of the picture and allows a three-dimensional construction to be collapsed to two.

The construction itself may be independently due to Brunelleschi; it appears in the design of the vaulted ceiling in the large fresco *The Trinity* (Santa Maria Novella, Florence), executed by Masaccio in 1426. Vasari (1511–1574), whose *Lives of the Artists* [7] is the standard early reference on Italian Renaissance art, tells us (Vol. I, p. 272) that "In particular he [Brunelleschi] taught Masaccio the painter, then a youth and his close friend, who did honour to his instructor, as appears in the buildings which occur in his works." About *The Trinity* he says (Vol. I, p. 265): "But the most beautiful thing besides the figures is a barrel vault represented in perspective and divided into squares full of bosses, which gradually diminish so realistically that the building seems hollowed in the wall." The illusion derives its power from geometry: applying the inverse checkerboard construction to the vault locates the ideal viewing position as the height of the eye of a person looking up from across the aisle.

The next arresting character in the narrative is Piero della Francesca (c. 1412–1492). Piero, one of the greatest painters of the Renaissance, wrote mathematical treatises. Three survive, one of which is *De Prospectiva pingendi* (On perspective in painting, c. 1480). In a general discussion of Piero's mathematics, *The Invention of Infinity* reproduces illustrations from this work showing how he locates points by pairs of "rulers", one horizontal, one vertical. For example, the position of one corner of a column capital is indicated by a tickmark on one of the horizontal rulers and a similarly labeled tickmark on the corresponding vertical ruler. These pictures are fascinating because they show explicitly how coordinates were used before coordinate systems existed.

In *De Prospectiva* Piero gives a refinement of Alberti's construction and a diagrammed proof that it really works (*Della Pittura* has no figures). The verticals are drawn using the vanishing point as before. Piero constructs the top edge of the perspective floor by Alberti's method, but then locates the horizontals by intersecting the verticals with a diagonal, as shown in Figure 2. Alberti had already introduced the diagonal as a check on his construction: "If one straight line contains the diagonal of several quadrangles described in the picture, it is an indication to me whether they are drawn correctly or not" ([1], p. 57). This use of the projection-invariance of the relation between diagonal, horizontals, and verticals is the first hint of what we now call *projective geometry*. A further note: the diagonal, extended upwards, meets the horizon (the horizontal through the vanishing point) at O^{**}. Reasoning with similar triangles shows that $AB/O^*C^* = BH'/H'C^* = AB/O^{**}C$, with points labeled as in Figure 2. It follows that $O^{**}C = O^*C^*$ and therefore equals the distance to the eye. This is the easiest way to locate the ideal viewing position. On the other hand, one can short-cut the construction by first locating O** using O**C = OC and joining O** to the opposite lower corner of the frame. This further and final simplification of Alberti's algorithm is traditionally called the "distance point construction".

The Invention of Infinity's first five chapters tell the story of perspective in the Italian Renaissance. The second half of the book is denser with mathematics. It follows the geometry of perspective through the sixteenth century and well into the seventeenth, with Commandino (1509–1575), Benedetti (1530–1590), Danti (1536–1586), Guidobaldo del Monte (1545–1607), Kepler (1571–1630), and finally Desargues (1591–1661). Desargues is given the special attention he deserves as the man who brought perspective into pure mathematics by his recognition of the importance of projective invariance: "…Desargues is not looking at what is changed by perspective, as artists were, but at what is not changed." We are shown how he implemented this principle in his unified study of conics. He is also recognized as "the first mathematician to get the idea of infinity properly under control." As Field puts it,"He uses the concept in a completely precise mathematical way." Here is the culmination, in geometry at least, of the invention referred to in the title.

This is a handsome book on a wonderful topic. It covers a period when mathematics seems to have been part of the natural efflorescence of genius, and it links two neighborhoods, fifteenth-century Florence and seventeenth-century France, when an unusual amount of genius walked the earth. This was a singular period in another way: mathematics has traditionally taken its new problems and new material from the physical sciences; *The Invention of Infinity* tells the story of a significant transfusion where the donor was art. And since the artists involved were among the greatest of their period, we can witness milestones of this transfer in details of artistic masterpieces. For mathematicians it makes an irresistible package. For the nonmathematical audience to which this book is presumably addressed, there is a problem. They can learn that Renaissance mathematics evolved hand in hand with Renaissance art, but if they want to penetrate the details of that mathematics, they will have a hard time. Field concentrates on the what-did-they-know-and-when-did-they-know-it questions, which are certainly significant. But when the original proof is unintelligible, incorrect, or nonexistent, readers are left to their own devices. Today's mathematicians can fill the gaps without much difficulty; for the general reader the book would have been improved by a complete and careful account, explicitly anachronistic when necessary, of each new piece of mathematics.

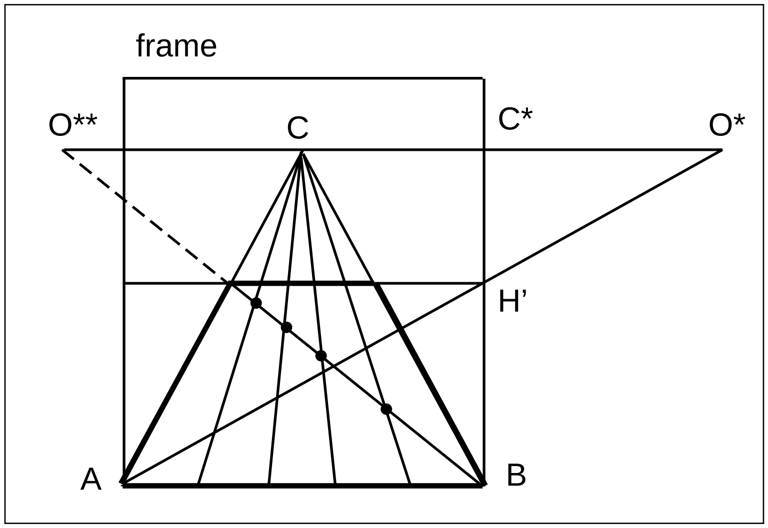


Figure 2. Piero della Francesca's version of the perspective pavement construction: C, C*, O* as before. Here H′ (at the height of the top edge of the floor) is located using O* as in Alberti's method. Then the heights of the other transversals are located by the diagonal.

Field presents the history of these mathematical ideas in great detail, with many quotations of texts and diagrams by the participants. There is a wealth of excellent illustration going well beyond what is necessary to elucidate the text and helping to give a visual impression of an entire intellectual era; this is especially true of the sections on the Italian Renaissance. There is also a wealth of fascinating and instructive information. We get to compare Piero della Francesca's sober perspective *palazzo* with the wildly baroque perspective dome painted by Andrea del Pozzo in 1685, just as we can compare Piero's humble mathematics—the problems in his *Abacus treatise* are given in terms of the reckoning of debts, the weight of fish, the worth of bolts of cloth—with Desargues's confident search for maximum generality.

The Invention of Infinity grew out of a series of lectures and in many places has preserved the irreverent tone and breezy, donnish mock-pedantry that must have made those lectures hugely entertaining. "Leonardo of Pisa (c. 1170–c. 1240, sometimes called Fibonacci)"—O.K. But the irreverence can get out of hand. Partway through the Sistine Chapel decoration, some of the scaffolding was taken down. We are told that the view from the floor led Michelangelo (that scorner of mathematical constructions!) to enlarge the scale of the figures after work was resumed, and Field indulges in a bit of fun at the master's expense (p. 117). The real story is probably not so simple. I refer readers to the "Michelangelo" entry in the *Grove Dictionary of Art*. The irreverence is pointed up by the publishers' (probably unwittingly) shameless use of some glorious details of the Sistine Chapel ceiling for the book's dust jacket.

Sometimes the breeziness can lead to misapprehension. Federico da Montefeltro, Alberti's friend and Piero's patron, deserves better than his treatment in this text, where (p. 77) he and "the Duke of Urbino" seem to have independent existences. His "Duchess"—she never was—fares worse. In addition, the book takes an authoritative tone on linguistic matters which is not always justified. Readers unfamiliar with the history of the modern Italian language may be puzzled by the many references to "Tuscan". This is like saying that Shakespeare wrote in "Elizabethan". In fact, Alberti's *Della Pittura* is still held up as an example of *pura ed efficace lingua* (pure and effective language; see the introduction to the Italian edition of [1]). The comments tossed off on Italian style and spelling are most always incorrect, while the author's own Italian spelling is unreliable. And mock-pedantry is only funny when it is dead right. We are told that Descartes's *Geometry* was published as "*Geometrie* [*sic*, no accents]," but then we are told that Pascal wrote *L'Esprit de la Géométrie*—impossible. (We are further told that this work "reads like a declaration of love in a tragedy by Racine," which is just dead wrong.) These are annoying but minor shortcomings in a very engaging book that could have been more useful if its mathematical potential had been more fully and more carefully exploited.

References

[1] Leon Battista Alberti, *On Painting*, translated by J. R. Spencer, Yale University Press, New Haven, 1966. Edition in Italian: *Il Trattato della Pittura e I Cinque Ordini Architettonici*, R. Carabba, Lanciano, 1913.

[2] Joan Gadol, *Leon Battista Alberti*, University of Chicago Press, Chicago and London, 1969.

[3] E. H. Gombrich, *Art and Illusion: A Study in the Psychology of Human Perception*, 2nd edition, Princeton University Press, 1961.

[4] ____ , *Meditations on a Hobby Horse and Other Essays on the Theory of Art*, Phaidon Press, London, 1963.

[5] Antonio Manetti, *The Life of Brunelleschi*, Pennsylvania State University Press, University Park, PA, 1970.

[6] Jean Piaget and Bärbel Inhelder, *La Représentation de l'Espace chez l'Enfant*, Presses Universitaires de France, Paris, 1948. This work is available in English: *The Child's Conception of Space*, Routledge & Kegan Paul, London, 1963. The passage quoted appears on p. 177 in a different translation.

[7] Giorgio Vasari, *Lives of the Most Eminent Painters, Sculptors & Architects*, translated by A. B. Hinds, Dutton (Everyman Library), New York, 1963.