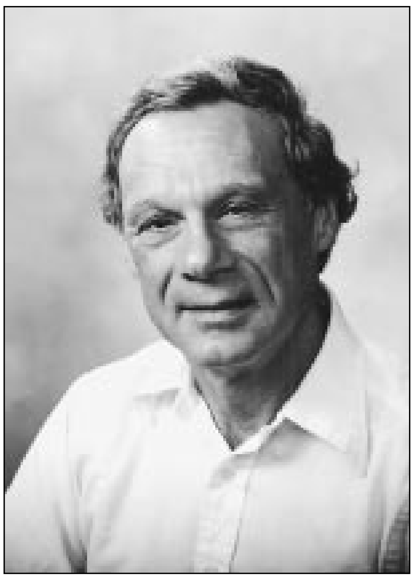
Julian D. Cole
(1925–1999)

George Bluman, L. Pamela Cook, Joe Flaherty, Jerry Kevorkian, Norman Malmuth, Robert O'Malley, Donald W. Schwendeman, and Marshall Tulin



Julian D. Cole

The applied mathematics community suffered a great loss with the death of one of its leaders, Julian D. Cole, on April 17, 1999, at the age of 74.

Julian was born in Brooklyn, New York, on April 2, 1925. He received his B.S. in engineering from Cornell and his Ph.D. from Caltech, where he worked with Hans Liepmann. During his career he was a faculty member at Caltech, scientific liaison officer in London for the Office of Naval Research, a faculty scholar at Boeing, a faculty member and department chair at UCLA, and a chaired professor at Rensselaer Polytechnic Institute. He received many awards for excellence throughout his career, including election to membership in the National Academy of Engineering and the National Academy of Sciences simultaneously in 1976. He was a fellow of the American Physical Society, the American Institute of Aeronautics and Astronautics, and the American Academy of Arts and Sciences. Among other awards he received the von Karman Prize (SIAM), the Award for Meritorious Civilian Service (Air Force), the Fluid Dynamics Award (AIAA), and the National Academy of Sciences Award in Applied Mathematics and Numerical Analysis.

The opening segment is adapted and reprinted, by permission, from an article that first appeared in SIAM News, *June 1999.*

Julian will be best remembered professionally for his seminal research and his books: *Perturbation Methods in Applied Mathematics* (1968, followed by revisions in 1981 and 1996); *Multiple Scale and Singular Perturbation Methods* (1996, with Kevorkian), *Transonic Aerodynamics* (1986, with Cook), and *Similarity Methods for Differential Equations* (1974, with Bluman). His professional career has been reviewed in some detail by Marshall Tulin as "Julian David Cole—A Scientific Biography" in *Mathematics Is for Solving Problems* (SIAM, 1996). This is a must read for any aspiring applied mathematician and for any devotee of Julian. It surveys the development and the significance of Julian's work, creating a profile of a career. It summarizes:

> …Julian's early experience in bringing mathematics forcefully to bear on important, current engineering problems took place at GALCIT, the very laboratory where von Karman translated his own tradition in America. The heart of that tradition was a deep commitment to the scientific formulation and mathematical solution of real engineering problems, rather than to the development of mathematics for its own sake…. Here the importance of historical and exemplary figures like Rayleigh and Kelvin in the [nineteenth] century and Timoshenko, Burgers, and von Karman in the first half of [the twentieth] century are crucial. In the second half of [the twentieth] century, Julian Cole certainly takes his place with his predecessors. Von Karman had clearly

> understood and publicized the vast and mostly uncharted territory of nonlinear phenomena looming ahead in engineering, and it was the heart of this territory in aerodynamics, transonics, which Julian conquered…. What sets Julian aside from many other engineers using theory is the depth and general applicability of the mathematical tools which he has devised for the solution of problems. In this respect, his body of work is exceptional.

Julian was notable for his patience, persistence, and integrity. He was accepting of others' faults, and a wise man. He thought about mathematics continually, and he was always ready for technical questions and discussions. He was a role model, a man whose mind was sharp and who was able to see through problems quickly, but yet a man who remained diffident and humble. He was not only a great researcher but even more a superb teacher. He was an absorbing lecturer. He was generous with his time and generous and open with his thoughts and ideas. He could get to the heart of any problem and distill it to its relevant and simplified components. It is no surprise then that Julian had so many collaborators and guided thirty-six Ph.D. theses to completion. Those of us who were lucky enough to have known and to have worked with Julian, including his students, are deeply indebted to him for our outlook on our work and on life.

Many of us have shared with Julian information gleaned from his voracious reading. Many of us have been to the racetrack with him and have learned how to read the *Daily Racing Form* and how to bet on horses. Some of us remember sitting through an initially frightening earthquake as Julian calmly and soothingly continued to lecture. Some remember a seminar (on the inner ear) in the Doctors' Lounge at the UCLA Hospital, with Julian in hospital patient's garb, until the nurses ejected us. (That was after an infamous attempt by Julian at motorcycling.) He enjoyed food, wine, and music, and most of us have shared a memorable meal with him. Julian was a hiker, a skier, and a runner who ran virtually everyday of his life. He was a devoted father who was proud of his children.

Despite his illness, typical of his love and dedication to his profession, Julian was teaching and initiating research collaborations until the time of his death. He was at that time the Margaret Darrin Distinguished Professor of Applied Mathematics at RPI, where his wife, Sue, had also been a faculty member in applied mathematics and where he had been for the last seventeen years.

—L. Pamela Cook, Joe Flaherty, and Marshall Tulin

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Jerry Kevorkian and Robert O'Malley

In the late 1940s Paco A. Lagerstrom and Julian D. Cole started a small research group on perturbation theory at the Guggenheim Aeronautical Laboratory at the California Institute of Technology (GALCIT). The main focus of the research was to understand the mathematical underpinnings for solutions of the Navier-Stokes equations for fluid flow in the limits as the Reynolds number tends to infinity (boundary-layer theory) and to zero (Stokes-Oseen flow). Many of the ideas that are now at the core of singular perturbations were developed by this group. In his narrative article "The development of perturbation theory at GALCIT" (*SIAM Review*, 1994), Cole generously credits colleagues for various crucial ideas. In fact, in many cases, the seminal contributions were uniquely his.

In collaboration with Lagerstrom and Leon Trilling, Cole found that weak shocks for a class of limiting solutions of the Navier-Stokes equations were described by what was later called the Burgers partial differential equation. Julian then found the exact solution of the nonlinear Navier-Stokes equation using an ingenious transformation to the linear diffusion equation that has since been widely used and generally known as the Cole-Hopf transformation. This was just the beginning of Julian's skillful use of similarity and invariance methods.

The nature of a limiting solution for a nonlinear equation is often physically, though not always mathematically, obvious. For example, in flow of a viscous fluid over a body, the effects of viscosity are concentrated in a layer adjacent to the body surface, and this (boundary) layer gets thinner as the viscosity decreases (i.e., as the Reynolds number approaches infinity). On the other hand, an observer at a fixed point in the stream experiences a progressively less viscous (outer) flow in this limit. The approximate equations for the outer flow follow trivially by ignoring the viscosity in the Navier-Stokes equations. The approximate equations governing the flow in the boundary layer may be derived physically on the basis of the relative importance of various terms in the Navier-Stokes equations. In fact, this was

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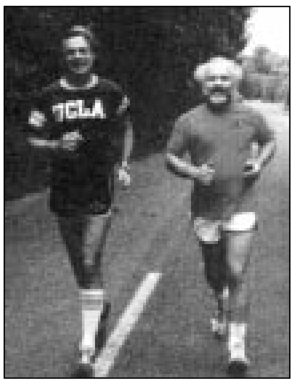
the point of view taken by L. Prandtl in his seminal seven-page report "On the motion of a fluid with very small viscosity", given as a ten-minute presentation at the Third International Congress of Mathematicians in Heidelberg in 1904. The idea that the boundary-layer limit is a systematic mathematical consequence of the Navier-Stokes equations—that it is, in fact, the leading term in the asymptotic expansion of the exact solution of the Navier-Stokes equations in the limit of the Reynolds number approaching infinity after a unique scaling of the variables—is due to Lagerstrom and Cole. To derive the partial differential equations that govern this limiting solution, one merely applies this rescaling and limit process to the Navier-Stokes equations. Moreover, this process generalizes to yield the higher-order terms in the asymptotic expansion. For the case of high Reynolds number flow, these expansions are called the inner (boundary layer) and outer (inviscid) expansions. Cole pointed out that a given equation has a finite number of such distinguished limits, each valid in a separate geometric domain, and that these are fundamental in computing a uniformly valid asymptotic solution to the boundary value problem.



Cole (on left) running with Jerry Kevorkian.

As explained by Van Dyke (*SIAM Review*, 1994), Prandtl's boundary-layer theory was ahead of its time and slow to catch on, even in the hydrodynamics community. By the 1930s Richard Courant, as head of the famous Mathematical Institute at Göttingen, sought to encourage a mathematical analysis of boundary-layer theory to be developed through interaction with Prandtl's own Aero- and Astronautics Institute, just down the street. The Nazi rise prevented such cooperation, however. Fittingly, Prandtl's flamboyant student Theodore von Karman was the founding director of GALCIT, and Julian Cole was awarded SIAM's von Karman prize in 1984. K.-O. Friedrichs provided the first significant mathematical analysis of the subject in the early 1940s at New York University, and he and Wolfgang Wasow introduced the term *singular perturbations* in a 1946 paper on nonlinear oscillations. Related mathematics was also developed in Russia by Tikhonov and Pontryagin, among others. (Indeed, our current mathematical understanding of matching is most rigorously presented in books by Vasil'eva, Oleinik, and Il'in.) The Caltech group interacted closely with Friedrichs and others both pure and applied at NYU, with analysts like Arthur Erdelyi at Caltech, and with fluid and aerodynamicists in Cambridge, England, and Cambridge, Massachusetts, among many others. Ultimately, contacts with the likes of George Carrier at Harvard, C.-C. Lin and Norman Levinson at MIT, Edward Fraenkel and Ian Proudman at Cambridge, Keith Stewartson in London, and Paul Germain in Paris put the GALCIT group at the center of a very significant interdisciplinary international effort in asymptotic analysis. Important European friendships were made while Cole served as a scientific liaison officer with the London Office of Naval Research from 1955 to 1957.

Cole called the expansion associated with a given distinguished limit a limit-process expansion, and the mathematical basis for connecting two adjacent expansions was an important issue that occupied him and other members of the GALCIT research group. For example, in studying high Reynolds number flow over a body, Prandtl had argued on physical grounds that the boundary-layer solution at an infinite distance from the body should agree with the inviscid flow on the body surface. This seemingly paradoxical requirement was shown to be the limiting form of a more general matching principle that was formulated by Saul Kaplun, a senior research fellow at GALCIT. Although Kaplun's ideas were important in establishing the general basis for matching of asymptotic expansions, Cole took a more pragmatic approach. He showed that once two adjacent distinguished limits and their associated expansions are derived, then matching can be implemented in terms of an intermediate variable and associated limit process valid in an overlap domain common to both.

Members of the GALCIT group met informally once a week to present their research and solicit feedback. At one of these meetings in 1959, Jerry Kevorkian, a graduate student working on the problem of satellite orbits in a thin atmosphere, highlighted the difficulty of calculating solutions valid for long times within the framework of limit process expansions. The classical perturbation methods used in celestial mechanics, which relied almost exclusively on the Hamiltonian formalism, did not apply to this dissipative motion. He argued that a regular perturbation expansion, where the time is fixed and the small parameter (measuring the small aerodynamic forces acting on the satellite) tends to zero, corresponds to an inner (or initially valid) expansion analogous to a boundary-layer expansion. This expansion is not valid for long times, however, and the breakdown in the solution is exhibited by the occurrence of mixed *secular terms*; these are products of linear and trigonometric functions that become unbounded for large time. Indeed, since the time of Lagrange those doing celestial mechanics had often been forced to develop methods based on casting out such mixed secular terms. No limit process expansion exists for long times because oscillatory terms have infinite frequency in this limit. Thus, although the solution depends on two time scales, a short time measuring the orbital period and a long time measuring the period over which orbital decay due to drag becomes significant, the orbital evolution over long times cannot be expressed in terms of a single scale as in an outer expansion. At this meeting Kaplun suggested an approach for deriving the slow evolution of the orbital elements based on their small change over one cycle. While Kevorkian was working on this idea (which proved to be feasible albeit very laborious), Cole casually remarked that it would be worthwhile to construct an expansion that depends on the two time scales simultaneously! This was the needed breakthrough that led to what is now known as the method of multiple scales for the solution of problems with cumulative perturbations. Kevorkian's 1961 Ph.D. thesis, later available as a Douglas Aircraft report and in an AMS 1966 publication *Space Mathematics*, outlined the necessary formalism and included a number of illustrative examples. In a layer-type problem, two or more limit process expansions, each involving its own independent variable (scale) and each valid in its own domain, arise in the asymptotic description of the solution; a uniformly valid approximation is then constructed by the appropriate superposition of elements of these expansions. In contrast, for a cumulative perturbation problem the asymptotic approximation of the solution is represented as a general function of the multiple scales at the outset, generalizing Lindstedt's strained coordinate method as presented in Poincaré's *Celestial Mechanics*. In the simplest case the solution depends on two scales only, the strained fast time and the slow time, and the expansions procedure is called "two-timing". The scope of this method has since been extended significantly. For ordinary differential equations it applies to rather general systems consisting of $N + M$ first-order equations for N fast phases and M slow variables. Such systems arise in many applications, including satellite dynamics, the dynamics of charged particles in accelerators, the motion of electrons in free electron lasers, and the modal evolution of a large class of nonlinear vibrating systems. The method also applies to a variety of partial differential equations that model problems where small cumulative perturbations affect the solution in the far field. Numerous examples occur for wave propagation in both homogeneous and heterogeneous media.

While on sabbatical leave at Harvard during 1963–64, Cole began a book, *Perturbation Methods in Applied Mathematics*, in which he gives a comprehensive account of the techniques mentioned above using examples taken from diverse areas. This book followed closely the first work on the subject, *Perturbation Methods in Fluid Mechanics* by Milton Van Dyke, another GALCIT graduate, devoted entirely to applications in fluid mechanics. Cole's 1968 book was revised and updated by Kevorkian and Cole in 1981 to include an account of the method of averaging and more recent work on multiple scale methods with applications in satellite dynamics as well as weakly nonlinear waves. The discussion of averaging, originally due to Krylov, Bogoliubov, and Mitropolsky, is particularly important, because averaging can sometimes be used to justify two-timing, as was first shown by John Morrison of Bell Laboratories in 1966. As this book demonstrates, however, multiple scales can be applied much more broadly than averaging, particularly for partial differential equations. Examples of singular perturbation problems in cell biology, an area that Cole and co-workers explored after he left Caltech, were also described.

Cole on the beach with Marshall Tulin.

Cole's own 1949 Caltech thesis, under Hans Liepmann, concerned transonic flow. That subject, like hypersonics, remained one of active research and collaboration by Cole until his death. Important early analytical work was done with his Caltech student Arthur Messiter, while some significant computational finite difference schemes were developed with Earll Murman in 1968 at the Boeing Scientific Research Laboratory. Much more recent practical work was done jointly with Norman Malmuth of the Rockwell International Science Center and with Pamela Cook of the University of Delaware. A hybrid of asymptotics and numerics, combined with considerable physical insight, is central to such transonic small disturbance theory. Most of this work is presented in Cole and Cook's 1986 book and in a 1993 SIAM book edited by Cook. At Rensselaer, Cole worked with and inspired much additional work by Zvi Rusak, Oleg Ryzhov, and Don Schwendeman. The recent use of hodograph methods by Cole and Schwendeman to numerically design airfoils is particularly impressive and masterful.

Julian Cole was particularly gifted in applying perturbation techniques to a wide spectrum of physical problems. Some were problems already explored by others, where he added valuable new perspectives. For example, in his paper on heat conduction in a heterogeneous medium (*SIAM J. on Appl. Math.*, 1995) he revisited a problem first discussed by Rayleigh to explain the origin of a curious 13/3 power occurring in the expansion of the effective conductivity of a cubical array of spheres in a matrix. However, most of Cole's work on singular perturbations dealt with fascinating new areas of applications such as cell biology, the mechanics of the inner ear (cochlea), and other examples that highlight the power and elegance of perturbation solutions. This interest and commitment to solving real problems is reflected in the title for the collection of papers celebrating his seventieth birthday: *Mathematics Is for Solving Problems*.

A more recent survey of perturbation methods is given in Kevorkian and Cole's 1996 book, *Multiple Scale and Singular Perturbation Methods*. In addition to a collection of new examples on layer-type problems, this book contains a unified and complete survey of multiple scale and averaging methods for ordinary differential equations. The discussion of multiple scale methods for partial differential equations is also considerably expanded. Together, Cole's three books on perturbation methods have been widely acclaimed as timely texts as well as reference works. Their broad appeal to students and research workers may be attributed to the systematic exposition of fascinating topics and the use of varied and challenging examples related to applications.

Donald W. Schwendeman

I knew Julian Cole as a colleague and as a good friend from 1987, at which time I joined the Department of Mathematical Sciences at Rensselaer Polytechnic Institute. Since then it was my privilege and good fortune to work with Julian on a number of interesting and challenging problems in applied mathematics and theoretical aerodynamics. Our main work together was in the areas of transonic and hypersonic flow, but Julian and I shared other interests, including mathematical problems from industry, semiconductor device modeling, and similarity methods, among others. By the time I began working with Julian, he was a well-known applied mathematician, having made a number of important contributions in the areas of perturbation theory, aerodynamics, group theory, and more. I, on the other hand, was an assistant professor just getting started in the business, but Julian enjoyed working with others and young people in particular. So, because of our shared interests and complimentary skills, we enjoyed a close collaboration, my recollections of which and connections to Julian's earlier work I share below.

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One of the first problems we worked on together involved shock-free inviscid flow past bodies of revolution. The main issue from the point of view of theoretical fluid dynamics is whether a body of revolution can be constructed so that in the presence of a uniform free-stream subsonic flow of a given Mach number M_∞ there is a local supersonic zone about the body that is free of shock waves. Such a flow would be free of wave drag due to shock waves and, through the "equal area rule", could be used as a guide to design airplanes in high-speed subsonic flow. Mathematically the problem can be formulated in terms of a nonlinear mixed-type partial differential equation assuming potential flow. Within the framework of small-disturbance theory, in the physical plane (x, r), where x is axial distance in the direction of the free-stream flow and r is radial distance, the body becomes a distribution of point sources along the x-axis given by $S(x)$ for $x \in [-1, 1]$. The source function provides a boundary condition for the Karman-Guderley equation,

$$(K - (\gamma + 1)\phi_x)\phi_{xx} + \phi_{\tilde{r}\tilde{r}} + (1/\tilde{r})\phi_{\tilde{r}} = 0,$$

where ϕ is a disturbance potential, $\tilde{r} = \delta r$ is the scaled radial distance (δ being the thickness of the body), and $K = (1 - M_\infty^2)/\delta^2 > 0$ is the transonic similarity parameter. The problem is to specify $S(x)$ for a given K such that the solution for $\phi(x, \tilde{r})$ in the region $\tilde{r} > 0$ has a smooth supersonic (hyperbolic) zone within the outer subsonic (elliptic) flow.

Julian suggested a method of solution in which the problem is formulated in the *hodograph* plane, where the velocity components become the independent variables and the partial differential equation for ϕ is transformed to a *nonlinear* Tricomi equation. In this way the unknown boundary between the elliptic and hyperbolic regions in the physical plane becomes a known boundary in the hodograph plane, and the smoothness of the solution can be determined by checking the sign of the Jacobian of the transformation. The boundary condition involving the source function moves to one at infinity, and the free stream becomes a singularity of known type at a point in the hodograph. A numerical formulation of the problem was devised, and source functions were found corresponding to a smooth shock-free flow in the physical plane. These results were the first shock-free solutions found for bodies of revolution.

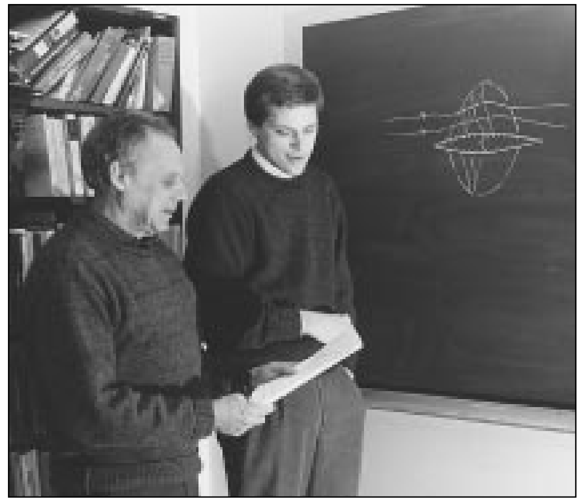
Another problem we pursued jointly concerned the design of optimal *critical* airfoils. An airfoil is critical when the flow is just sonic at some point or portion of the airfoil surface and the flow everywhere else is subsonic. The problem of optimal design is to construct a critical airfoil of a given thickness or area in a uniform inviscid flow with the highest free-stream (subsonic) Mach number. Additional constraints may be imposed, such as a given lift, a given angle at the tail of the airfoil, or a given curvature at the nose. For the nonlifting case without geometric constraints at the nose and tail, the optimal design was worked out by Gilbarg and Shiffman using a comparison theorem. Their design consists of flat vertical segments at the nose and tail connected by an arc on which the flow is exactly sonic. Again the problem is formulated conveniently in the hodograph plane (q, θ), where q is the flow speed and θ is its direction, so that the airfoil boundaries involving segments of known flow direction or speed become fixed boundaries in the hodograph. In the hodograph plane Chaplygin's equation,

$$\rho q((q/\rho)\psi_q)_q + (1 - M^2)\psi_{\theta\theta} = 0,$$

where ρ and M are the density and Mach number of the flow, respectively, is to be solved for the stream function $\psi(q, \theta)$ in the region $q \le c^*$ (c^* being the critical sound speed), subject to a free-stream singularity at $q = q_\infty$, $\theta = 0$. In the small-disturbance limit, Chaplygin's equation reduces to a Tricomi equation, and Julian found an *exact* solution to the relevant boundary-value problem. In this limiting case the vertical segments at the nose and tail reduce to points, and the whole airfoil becomes a sonic arc. Julian's solution specified the shape of this arc and a fixed value for the transonic similarity parameter K, which gives the limiting behavior of the thickness δ of an optimal critical airfoil as the free-stream Mach number M_∞ tends to 1. A numerical method of solution was used to determine optimal critical airfoils for the full, nonlifting case.

Extensions of this work to the lifting case involved the numerical solution of Chaplygin's equation on two Riemann sheets connected by a branch cut. The numerical implementation of this interesting and novel boundary-value problem was carried out by Mary Catherine Kropinski, Julian's and my graduate student at the time.

In both of these problems one sees the strong influence of Julian's earlier work in transonic flow, which began at Caltech in the late 1940s. Interest in this area of fluid dynamics at that time was spurred on by the desire to fly aircraft near the speed of sound and faster, the first supersonic flight having been achieved in 1947. In this effort, new fluid dynamics was observed about aircraft at transonic speeds, such as shocks and wave drag, and new mathematics was needed to explain and predict these observations. The essential difficulty from a mathematical point of view is that the equations of fluid dynamics for transonic flow are nonlinear. And, for example, in the case of steady potential flow the equations change type at sonic, being elliptic for subsonic flow and hyperbolic for supersonic flow. Julian played a leading role in the development of transonic small-disturbance theory, which was an important advance in the mathematical theory of transonic flow. His early work in this area was mostly analytical and required clever and sophisicated mathematical techniques, such as similarity and hodograph methods. Later, while at Boeing, he and Earll Murman developed the first type-sensitive numerical method for the transonic small-disturbance equations. This numerical method, which has now become a classic, enabled engineers to calculate and study the transonic flow about a wide range of airfoil shapes and played a key role in the development of modern computational fluid dynamics. Julian's important contribution to the theory of transonic flow is apparent from a long list of research reports and publications, much of which is compiled and expanded upon in his book *Transonic Aerodynamics*, coauthored with Pamela Cook.



Julian Cole (left) with Don Schwendeman at RPI.

Julian's and my collaboration was not limited to problems in transonic flow. The most recent problem we worked on involved the optimization of conical wings in hypersonic flow, where the flow speed is much faster than the speed of sound. The aim of this research effort was to design wing shapes that minimize (wave) drag for a given lift in inviscid hypersonic flow. The problem was formulated using hypersonic small-disturbance theory and assuming conical flow. Under these assumptions the Euler equations reduce to a system of first-order mixed-type nonlinear equations that take the form $\mathbf{f}_Y + \mathbf{g}_Z = \mathbf{h}$, where $\mathbf{f}$, $\mathbf{g}$, and $\mathbf{h}$ are vector functions involving the leading perturbations in density, velocity, and pressure in the limit as δ, the flow deflection, tends to 0 and M_∞, the free-stream Mach number, becomes large with $H = 1/(M_\infty \delta)^2$, the hypersonic similarity parameter, held fixed. The variables (Y, Z) are conical coordinates, $Y = y/(\delta x)$ and $Z = z/(\delta x)$, measuring (scaled) distance in the vertical and spanwise directions, respectively, for a fixed x. If one focuses attention on the shape of the compressive underside of the wing given by $Y = W(Z)$ for $0 \le Z \le \Lambda$ (assuming symmetry about $Z = 0$) and its contribution to the lift and drag, the problem is to find a wing shape function W that minimizes the figure of merit, $C_D/C_L^{3/2}$, for fixed H and Λ. Some analytical progress was made for certain special cases, but the problem required a numerical treatment in general.

Our work on this problem stemmed from Julian's early work in hypersonic flow, which began in the 1950s. In a typical problem, such as the one discussed above, there is a leading bow shock whose position is unknown and must be determined together with the disturbed flow between it and the wing. A mathematical description of this flow presents a difficult nonlinear problem. Hypersonic small-disturbance theory, which Julian helped develop, provides some simplification, but the equations remain nonlinear. Analytical progress is difficult and has been limited primarily to wings whose bow shock is attached (the so-called wave rider designs). For such wings, one of Julian's important contributions involved the use of similarity methods to study a family of power-law wing shapes in order to find the one that minimizes drag for a given lift. In view of Julian's early work, we were particularly interested in whether our optimal wing shapes would have attached or detached bow shocks, my numerical approach being able to handle both cases. As it turned out, our results showed that wings with attached bow shocks were preferred.

The results of the work discussed here amount to only a fraction of the contributions Julian has made throughout his career, but they are representative of the type of applied mathematics for which Julian was well known and is fondly remembered.

Norman Malmuth

My first interaction with Julian Cole was as his student at Caltech in 1957. I observed him in classes in applied mathematics and as my Ph.D. advisor during my five years at that institution. Besides applied math, I had the opportunity to study nonlinear gas dynamics with him, including transonic and hypersonic flow. I was impressed by his ability to get to the heart of the problem very quickly and express the issues in a very succinct mathematical way. What also amazed me was the ease with which he could derive things on the blackboard, even complex nonlinear mixed transonic flow. These skills demonstrated his deep understanding of the fundamental physics. In all my experience then and now, I have never seen anybody else who could build mathematical models with his skill. He was a pioneer in nonlinear gas dynamics who developed a beautiful revolutionary theory that embedded the classical ideas of Newton into a systematic approximation theory capable of accurately describing hypersonic flows. His work on optimum hypersonic wings, with his unique blend of combined asymptotics and numerics, provides new concepts that should make a sizable impact toward achieving 1–2-hour flights to the Pacific Rim instead of today's bone-crushing 10–15 hours. One very strong connection of his asymptotic mathematics is the link to modern computational fluid dynamics. Julian played a major role in embedding transonic small-disturbance theory into a systematic approximation framework through which successive refinements could be made. Julian integrated the far field boundary conditions into the numerics and was responsible for conceptualizing and extending the combined asymptotic and numerical scheme so obtained to slender bodies and lifting airfoils involving the interaction of the far field rectilinear vortex/wake and nonlinear effects with the near field. More importantly, Julian played a dominant role in applying the type-sensitive switch in the difference scheme that was backward in the hyperbolic regions and centered in the elliptic zones and had a degenerate special form on the parabolic lines. This not only provided stability but allowed shocks to be captured in the natural course of the calculations without having to be fitted into the field. Although some of the ideas had previously been explored by Lax, Magnus, and Yoshihara, the Cole-Murman breakthrough paper was the "Wright Flyer" of modern computational fluid dynamics, which is the backbone of aeronautical engineering today. It pervades all aerodynamic analysis and has made treatment of problems that were impossible fifty years ago a reality today.

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George Bluman

I came into contact with Julian Cole when beginning graduate study at Caltech in 1964. At first Julian came across as rather intimidating to new students, with his pointed questions and seemingly aloof manner. But there was a soft side to him, with his informal and down-to-earth manner. My close contact with Julian started with taking his graduate course AMA 201C on "Advanced Partial Differential Equations" in the spring quarter of 1966. Three eclectic topics were emphasized: dimensional analysis, similarity methods for ordinary differential equations (ODEs), and similarity methods for partial differential equations (PDEs), which Julian pointed out had "nothing to do with linearity."

Julian's interest in symmetry originated from similarity solutions arising from dimensional analysis considerations, or, more generally, scaling invariance, for various problems in fluid mechanics (boundary layer flows, transonic flow, etc.),

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especially as asymptotic solutions. His observation that the Burgers equation and the heat equation have similar symmetry properties motivated his 1951 work that introduced the celebrated Cole-Hopf transformation relating all solutions of these two equations. To find this transformation, Julian first used a differential substitution to obtain the integrated Burgers equation. The identical scaling invariance of the integrated Burgers equation and the heat equation led to his seeking a functional relationship between their solutions that yielded the Cole-Hopf transformation. The Cole-Hopf transformation, as a prototypical example, has sparked the development of systematic methods for finding transformations relating nonlinear PDEs to linear PDEs.

Julian was interested to learn that one could extend Lie's work for finding symmetries of ODEs to PDEs. In this way more general symmetries for PDEs could be found systematically, other than obvious ones such as scalings, translations, and rotations. In fact, Lie had done such an extension himself. Julian and I published a paper on finding symmetries and new similarity solutions of the heat equation arising from them. A novel idea introduced in this paper was the "nonclassical" method that generalized Lie's "classical" method of obtaining similarity (invariant) solutions through invariance of differential equations under Lie groups of point transformations. A Lie symmetry admitted by a PDE is a mapping admitted by its solution space. The "classical" invariant solutions are those solutions that map into themselves. The invariants of Lie symmetries yield specific functional ansatze for finding invariant solutions of a particular form with one less independent variable. The "nonclassical" method seeks "symmetries" leaving invariant an augmented system including the given PDE and the functional ansatz form for "its invariants". For any PDE this yields all functional ansatze for finding solutions of the particular form that can arise from Lie symmetries. Computationally, in the "classical" method the infinitesimals for Lie symmetries are first determined; they lead to the specific ansatze. In the "nonclassical" method the functional ansatz is the starting point with its associated "infinitesimals" substituted into Lie's determining equations for symmetries. Here the price that one pays is that the calculations to determine "infinitesimals" are now nonlinear. The "nonclassical" method came into prominence in a 1989 paper of Levi and Winternitz to clarify the Clarkson/Kruskal "direct" method for constructing new solutions for the Boussinesq equation. It has been demonstrated that the "nonclassical" method is computationally effective. Many new solutions for nonlinear PDEs have been so obtained.

In 1968 Julian and I began writing our book on similarity methods, which appeared in 1974. During this period Julian moved from Caltech to Boeing in Seattle and then to UCLA. Much of the material on ODEs was first laid out in Cole's AMA 201C notes. Meanwhile, Springer developed its series in Applied Mathematical Sciences, and our book became its thirteenth volume. It appeared to have led to a revival of interest in the West in group methods for differential equations. The book was written in an informal style, and a much updated and more polished version appeared in 1989.

Julian Cole always had an open mind to new approaches for solving problems. He had a remarkable intuition. His disdain of sophistication when uncalled for was legendary. His examination questions were applied mathematics at its best. His death is a great loss to the applied mathematics community.

Ph.D. Students of Julian Cole

Theodore Yao-tsu Wu, Caltech (1952)
Sune B. Berndt, Caltech (1955)
Shan Fu Shen, Caltech (1956)
Arthur F. Messiter, Caltech (1957)
W. Royce, Caltech (1959)
Meredith C. Gourdine, Caltech (1960)
Archibald D. MacGillivray, Caltech (1960)
Jerry Kevorkian, Caltech (1961)
Y. M. Lynn, Caltech (1961)
D. Hoult, Caltech (1962)
Norman D. Malmuth, Caltech (1962)
J. Petty, Caltech (1963)
William B. Bush, Caltech (1964)
K. Grimes, Caltech (1964)
George W. Bluman, Caltech (1967)
Kenneth Harstad, Caltech (1967)
Tse-Fou Zien, Caltech (1967)
H. Kabakow (with R. Feynman), Caltech (1968)
R. Schmulian, Caltech (1968)
J. F. Hamet, UCLA (1971)
F. Hendriks, UCLA (1972)
C. K. Feng, UCLA (1977)
Mark H. Holmes, UCLA (1978)
K. Kusunose, UCLA (1979)
E. Tse, UCLA (1979)
Fred Ziegler (with P. Cook), UCLA (1981)
S. Epps, UCLA (1982)
Gilberto Schleiniger, UCLA (1982)
Scott Rimbey, UCLA (1984)
J. J. Xu, RPI (1986)
V. Villamisar, RPI (1987)
Barbara A. Wagner, RPI (1989)
A. Kalocsai, RPI (1992)
Mary Catherine Kropinski (with D. Schwendeman), RPI (1993)
Ron Buckmire (with D. Schwendeman), RPI (1994)
Susan A. Triantafillou (with D. Schwendeman), RPI (1997)