# What Is Mathematics? An Elementary Approach to Ideas and Methods

*Reviewed by Brian E. Blank*

**What Is Mathematics? An Elementary Approach to Ideas and Methods**
*Richard Courant* and *Herbert Robbins*
*Second edition, revised by Ian Stewart*
*Oxford University Press, 1996*
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*$19.95, 566 pages*

Thirty-two years ago while browsing in my high school library, I happened upon an intriguingly titled expository book. I knew astonishingly little of the practice of mathematics at the time. I was aware that geometry derived from Thales, Pythagoras, and Euclid. Whether or not any geometric theorems were discovered after Euclid I could not say. That algebra and trigonometry were the results of conscious development would no more have occurred to me than the idea that the English language was the creation of dedicated professional linguists. I knew the names and work of many scientists—Copernicus, Kepler, and Galileo in astronomy; Darwin, Mendel, and Pasteur in biology; Boyle, Lavoisier, and Curie in chemistry; Archimedes, Newton, and Einstein in physics; Jenner, Harvey, and Koch in medicine; and many others, *none of whom were mathematicians*. Although my recreational reading of Hall and Knight had exposed me to an odd assortment of surnames, such as Venn and Horner, I knew of no first-rate scientist in the field of mathematics. Indeed, I did not really know that there was such a field.

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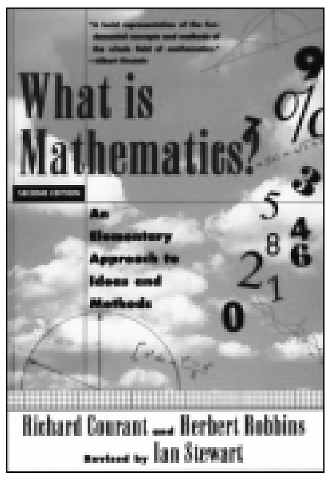


The book I pulled off the shelf that day did more than open my eyes; it changed my life. I graduated from high school later that year, armed with a twenty-dollar book certificate. It might have been prudent to save my award for the college textbooks that I would soon require. However, with no further access to my school library, I had a more pressing need. For less than nine dollars *What Is Mathematics?* by Richard Courant and Herbert Robbins became the first volume in my collection of mathematics books.

Only a sketchy summary of this well-known text need be given here. The first chapter is largely devoted to number theory. A chapter on real and complex numbers follows. The discussion includes Dedekind cuts, Cantor's theory of cardinal numbers, and Liouville's construction of transcendental numbers. The third chapter is concerned with field extensions and geometric constructions. It includes a thorough investigation of the impossible straightedge-and-compass constructions of classical Greek geometry (stopping short of a proof of Lindemann's Theorem). Chapters on projective geometry and topology come next. In preparation for the remainder of the book, Courant and Robbins continue with a chapter that introduces the reader to the rigorous definition of limit, reinforced by genuine proofs of the Intermediate and Extreme Value Theorems. Optimization and calculus constitute the final two chapters in a rather unusual ordering: the calculus of variations is introduced before the calculus, the definite integral before the derivative.

As is often the case with skilled exposition of deep subject matter, *What Is Mathematics?* can be read at different levels. The professional mathematician who has not yet encountered the book is likely to find new things in the chapters on projective geometry and optimization. Also rewarding are the numerous small details that illuminate material that is more old hat. For example, an excellent "proof" that all positive integers are equal warns the reader against improper use of mathematical induction. Later on, a quite different "proof" of the same absurdity illustrates the need for establishing the existence of an extremum before it is determined.

Of course, *What Is Mathematics?* was not written for the professional mathematician. Many of the topics that are covered in the first six chapters bring to mind the "transition" books that have sprouted up in the last decade. They are often intended for sophomore or junior mathematics majors and have titles such as *Introduction to Mathematical Reasoning*. In fact, Courant and Robbins foresaw that their book could serve as the text for "college courses of an unconventional type on the fundamental concepts of mathematics." In conceiving *What Is Mathematics?*, Courant hoped for a wide readership that would comprise a broad spectrum of educated laypersons. Accordingly, *What Is Mathematics?* "presupposes only knowledge that a good high school course could impart." In those days—the first edition appeared in 1941, followed by three revisions at two-year intervals—calculus was rarely if ever part of the high school curriculum.

In the sixty years during which *Courant and Robbins*, as the book is often called, has been in print, several distinguished scholars have had the opportunity to sing its praises. Reviewing the first edition, E. T. Bell described the work as "inspirational collateral reading." Hermann Weyl thought it "a work of high perfection." For Marston Morse it was simply "a work of art." Even Albert Einstein chimed in with high praise. Any subsequent reviewer with the slightest humility would be excused for feeling that his further say-so would only be superfluous. To counter the perception of immodesty, I should point out that I brought something to my first reading of *Courant and Robbins* that none of these learned scientists could boast: profound ignorance. Whereas Einstein found the text "easily understandable," I often struggled.

Though the prerequisites for *What Is Mathematics?* are minimal, the "certain degree of intellectual maturity" that is mentioned vaguely in the preface is likely to be a steep hurdle for the hypothetical layperson Courant hoped to capture as his reader. Twenty-nine pages into the book the Prime Number Theorem is stated. It is not proved, of course, but the statement alone is a tough swallow for the reader who has not yet seen the natural logarithm. (At this point Courant and Robbins introduce the logarithm as area under a curve, a concept that is not made precise until the last chapter.) The reader who persists soon comes to the Law of Quadratic Reciprocity, a theorem that is exceptionally beautiful to those who have an ample supply of that maturity thing, but is thoroughly bewildering to those who do not.

Did Courant and Robbins miscalculate? Not at all! As the preface says, a genuine comprehension of mathematics cannot be acquired through "painless entertainment." There are plenty of popular books that run away from every mathematical difficulty; the reader who seeks such a treatment has always been well served. *Courant and Robbins* has earned what ought to be a permanent place in the mathematical literature by conveying not only a treasure-trove of mathematical facts but also the ideas and methods behind them. To my mind the balance of intuition and rigor is just right for a popular book that intends to lay open the real substance of mathematics. In the matter of proofs I cannot think of a single misjudgment. Every proof that is included, no matter how difficult for the beginner, is there because it contains an idea that anyone who perseveres can master. There is no better way to begin the acquisition of intellectual maturity.

If the world was the way we wished, then *What Is Mathematics?* would have sold like nickel beer. Courant had ambitious expectations for a work he said "expresses my own personal views and aims more than any other of my publications." Although he considered the title "a little bit dishonest," he heeded Thomas Mann, who had experienced the effect a tantalizing title can have on marketability. The title notwithstanding, sales of *What Is Mathematics?* did not reach Courant's hopes. If Constance Reid's ballpark estimate of over 100,000 copies up until 1976 is at all accurate, then annual purchases by individuals could never have amounted to more than a trickle. (Oxford University Press did not respond to my request for a more recent figure.)

A disappointment to its senior author, *What Is Mathematics?* turned into a bitter blow for Herbert Robbins. In 1938 at the age of twenty-three, Robbins completed his dissertation in topology under the direction of Hassler Whitney. He came to New York University as an instructor one year later. Although he "had not the faintest acquaintance with or interest in either probability or statistics," he was assigned such a course when William Feller did not arrive there as planned. It marked the start of Robbins's long, distinguished career in mathematical statistics. After serving in the Navy during World War II, he taught at the University of North Carolina and then, from 1953 until his retirement in 1985, at Columbia University. After his first retirement Robbins taught at Rutgers University, Newark, until he retired again in 1997. Shortly before that second retirement Robbins's name was featured in news reports in conjunction with a remarkable software development. As a Harvard undergraduate in 1933 Robbins had conjectured that certain algebras, which came to be called Robbins algebras, are Boolean. Headlines were made in 1996 when the Robbins problem, which had thwarted all human effort, was proved automatically by a theorem-proving program developed at Argonne National Laboratory. Herbert Robbins died earlier this year at the age of eighty-six. (For further biographical details and a discussion of the contributions Robbins made to mathematical statistics, the interested reader may consult [8]. Transcriptions of two interviews, [1], [6], make for fascinating reading.)

Robbins was an excellent writer. His prose was sometimes provocative, often witty, always stylish. Serendipity brought him to New York University at the moment Courant was in need of help improving and amplifying the mimeographed notes that would become *What Is Mathematics?*. By Robbins's estimate the manuscript was only one quarter to one third written when he became involved. In an interview with Constance Reid he characterized his collaboration with Courant as "pretty close."

Mathematical collaboration is such a wonderful process that conventions have evolved to protect its sanctity. The basic principles are to not ask who did what and to assign each coauthor three quarters of the total credit due. In the case of *What Is Mathematics?* the temptation to violate the rules proved too great. The disparity in status between the two coauthors at the time of publication, the book's dedication (to the children of Courant), the preface (written and signed by Courant alone), the copyright (in Courant's name only) all prompted indiscreet questions. With justification Robbins became sick of the inquiries (for which he had a sharply humorous retort [10]). The relationship he had with Courant ended sordidly, with recriminations passing between the coauthors.

It is ironic really. One of the ten lessons Gian-Carlo Rota wished he had been taught is that mathematicians are more likely to be remembered for their expository work than for their original research [11]. Courant and Robbins may become the most convincing examples of this lesson. Their names have been inextricably linked for sixty years, and the bond will only grow stronger. I am reminded of Sir William Gilbert and Sir Arthur Sullivan. Gilbert penned many well-received plays, which are now known only to specialists of the Victorian theater. Sullivan composed many popular vocal and orchestral works: they are rarely heard nowadays. Yet working as a team, Gilbert and Sullivan were nonpareil. Their partnership was broken by a quarrel, but the work it produced is eternal. So too may the individual achievements of Courant and Robbins fade with the passing of time. So too does their joint effort seem destined to sparkle forever.

Even a classic can benefit from a sprucing up now and then, especially if it seeks to portray the existing state of an active field. *Courant and Robbins* was long overdue for an update when Oxford University Press brought out a second edition in 1996. This version is said to be by Courant and Robbins and *revised* by Ian Stewart. The latter part of that description is deceptive. Crack open the new edition: if you are familiar with the original, then you will be treated to a wave of nostalgia. All the old figures are there. Even the original typesetting has been preserved—quite a welcome change from the homogenized TeX of the present. Closer inspection, however, reveals something less welcome: the new edition is by and large a photographic reproduction. I could not find even one change in it. Stewart rationalizes this in his preface: "not a single word or symbol had to be deleted from this new edition." Really? After half a century could we not have had a correction to the name of Mr. Arch medes (p. 400)? Does anyone still use the term "dyadic system" in preference to "binary system" (pp. 8–9)? I suppose that there is nothing wrong with retaining the notation $C_i^n$ for the binomial coefficients, but it does seem dated now that mathematical notation has standardized on something else.

How, then, does the second edition differ from its predecessor? Stewart has added a thirty-seven-page new last chapter titled "Recent Developments". Without doubt, the mathematician will value Stewart's contribution. As a reviewer I cannot be so positive. My most serious complaints stem from the decision to leave the original text unaltered, reserving the necessary amendments for the new chapter. The modern reader still finds the statuses of Fermat's Last Theorem, the Four Color Problem, and the Continuum Hypothesis reported as they existed in 1947. There are no footnotes nor any forward references to Stewart's new chapter that alert the reader to the falsity of the statements he has read. This is particularly regrettable in a work that is not aimed at experts.

Given Stewart's lengthy record of successful popular exposition, I am surprised to find myself at odds with several aspects of his approach. I do not believe he focuses sharply enough on the hypothetical educated layperson that Courant and Robbins had in mind as their targeted reader. Summarizing over two thousand years of mathematical developments, Courant and Robbins deftly picked out the highlights and kept the discussion within the attention spans of their intended readers. By contrast, Stewart has been much less selective in telling us about the progress that has been made in less than sixty years. I fear the beginner will find the detail overwhelming in several places. That danger is especially great in the new discussions that receive inadequate foundation. For example, Courant and Robbins limit their treatment of the zeta function $\zeta(s)$ to (real) $s \geq 1$ (pp. 480–481). A few pages earlier they *hint* at analytic continuation when they *formally* substitute $z = ix$ into the power series for $\exp(z)$, (real $z$). I do not think the average reader will, on this basis, see so far as to extend the domain of the zeta function to complex $s$. Courant and Robbins must have come to the same conclusion, for they forbear mentioning the Riemann Hypothesis. Yet, without further preparation, Stewart rattles off its statement and, not stopping there, tells us what we will know should the (unstated) Generalized Riemann Hypothesis ever be proved true. Surely this is too much!

Although Stewart asserts in his preface that he has "made no attempt to introduce new topics that have recently come to prominence," he admits to bending his rules on occasion. Thus, the treatment of *dimension* by Courant and Robbins, fitting for its place in the chapter on topology, is used by Stewart as a pretext for a digression on Hausdorff dimension. Why? The original authors certainly knew about the concept and chose not to include it. This is a matter of the camel's nose slipping under the tent: once Hausdorff dimension enters, fractals inevitably follow. Granted, fractals can be amusing and even the source of serious mathematics. However, when given a gratuitous, superficial treatment, are they Courant-Robbins worthy?

There are some other places where old friends of *What Is Mathematics?* may wish that Stewart had used his author's license less freely. In the matter of Whitney's problem of an inverted pendulum on a train, a neat little riddle that attracted the attention of Littlewood [9, pp. 32–35], Stewart relates a challenge to the basic continuity assumption that underlies the solution given by Courant and Robbins. Ironically, that challenge itself relies on the assumption that the train can move in a way that is physically impossible, as Gillman has pointed out in his review [5]. (As long as I am mentioning the work of other reviewers, I would be remiss if I did not steer you to [3].)

It would be pointless and misleading to continue with further quibbles. It is hardly shocking to see that even an accomplished author can fall short of the standard Courant and Robbins set. On balance there can be no question that Stewart's contribution enhances *What Is Mathematics?*. Given that the original text is included verbatim in the new edition, Stewart's chapter may be regarded as a welcome bonus. At the time of this writing, you can still buy *What Is Mathematics?* for less than twenty dollars (although it is no longer clothbound, and you don't get as much change back); that makes it a real bargain.

Turning the yellowed pages of my first edition, seeing annotations in my own hand that I no longer remember making, considering the longevity of this classic, I am prompted to reflect on many things. The high school in which I discovered *What Is Mathematics?* was boarded up many years ago. The bookstore where I bought it has long since given way to the caffè latte trade. Now its two authors are dead. *Tempus fugit.* We mathematicians like to think that our subject has a permanent character that is rarely found elsewhere. As Hardy put it, "Greek mathematics is permanent, more permanent even than Greek literature. Archimedes will be remembered when Aeschylus is forgotten, because languages die and mathematical ideas do not." Hardy penned that thought as Courant and Robbins prepared for publication. For a long time it seemed like a safe bet.

Certainly Euclid's *Elements* has demonstrated that an inspired mathematical text can serve a very long time. In recent years, however, even Euclid has increasingly been charged with irrelevance. "In an age when computing power is abundant these maths are obsolete. At a minimum, it is time to transfer responsibility for teaching geometry to the history department. The problems for which geometry entered the schools have been either solved or taken over by other methods." So wrote the senior manager of a supercomputer company in a highly lauded glimpse of our digital future [2]. His is not an isolated voice. Though theorems once proved stay proved forever, appreciation of them can wane. What effect will shifts of interest have on *Courant and Robbins*? How will *What Is Mathematics?* hold up now that the good high school course that Courant spoke of is in danger of vanishing? How many students will be prepared to fight their way through *Courant and Robbins* in a time when education is so often confused with painless entertainment? I think that these questions are already being answered. For many reasons I find that Körner's *The Pleasures of Counting* [7] makes a more accessible recommendation for the sort of student I would have directed to *Courant and Robbins* twenty years ago. That in itself is not a bad thing: though it bears little resemblance to *Courant and Robbins*, Körner's book can fairly be described as "inspirational collateral reading." It allows the beginner to peek into the mathematician's art, and it *is* excellent. Do not take my word for it—see the *Notices* review [4]. That review urged the placement of *The Pleasures of Counting* in high school libraries where it can influence the talented student who might consider a career in mathematics. I would urge no less for *Courant and Robbins. What Is Mathematics?* should be in every high school library. You never know what life it might change when some curious student pulls it down from the shelf.

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