The Mathematical Explorer: An Interactive Mathematics Book

Reviewed by David Austin

The Mathematical Explorer
Stan Wagon
Wolfram Research, Inc., 2001
$89.95 CD-ROM

Though electronic mathematical publishing is still in its infancy, the technology offers tantalizing possibilities. Certainly, TeX has democratized the way mathematical texts are produced, while PostScript and Adobe Acrobat, combined with the Internet, have given mathematicians more control over the distribution of their printed works. In spite of these revolutions, the format and content of what is delivered to the audience has changed little.

However, it is easy to imagine an electronic text containing hypertext links, the ability to search for keywords, and a more sophisticated use of color. Animations to illustrate dynamic concepts seem to be another reasonable expectation. While this kind of format is currently available on the World Wide Web, we have yet to see a text combining these features with the first-rate mathematical typesetting and ease of use that we are accustomed to in traditional books and are rightly hesitant to sacrifice.

We might also consider our last visit to our favorite online bookseller. The first page likely greeted us by name and contained personalized recommendations based upon previous visits. Essentially, the page has been constructed using our input, unwitting though it may be. Whether this is helpful or intrusive is for each of us to decide.

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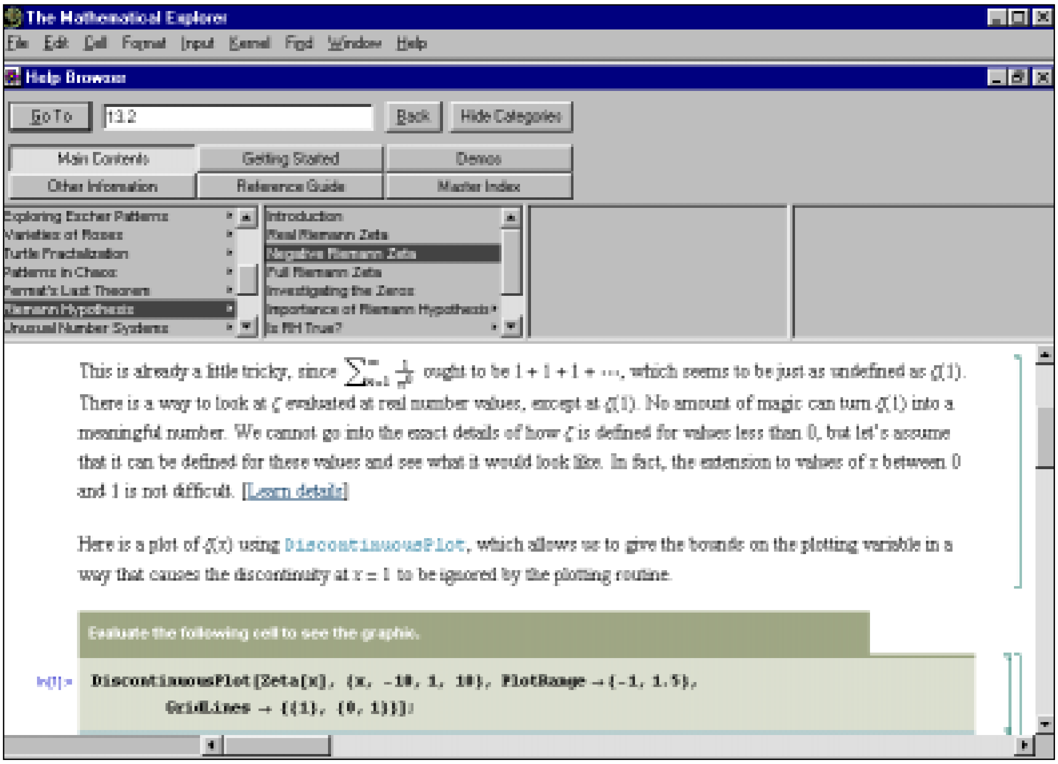
However, the idea of a mathematical text that is constructed in part by us, that allows us to generate examples, make conjectures, and discover results for ourselves is very appealing.

The ability to deliver such content challenges an author to think in new ways. Perhaps it is similar to taking a high school geology class on a field trip. Probably one would want to take the students to a place where they were certain to find something of interest with a little effort and something of great interest with more work. The students should have tools necessary for the job, say, a shovel and pick, but not so sophisticated that learning to use them detracts from their investigations. There should be a definite amount of time allowed for discovery at the end of which the students are called back together so their discoveries are studied and put into context.

The Mathematical Explorer

Since the ideas we understand best are those we discover ourselves, it is tempting to imagine using new technology to give mathematical readers an experience something like this field trip. Such is the promise of Stan Wagon's *The Mathematical Explorer*, an electronic "book" that includes a customized version of the *Mathematica* kernel together with a collection of *Mathematica* notebooks grouped into sections and chapters. While this is not a new means for delivering mathematical content—indeed, it is standard at some universities to distribute laboratory exercises to undergraduates in this way—the scope and intended audience of this book make it a novel and ambitious project.

Priced at $89.95, *The Mathematical Explorer* package includes a CD-ROM from which the book



may be installed on most Windows and Macintosh operating systems and a small booklet featuring installation and operating instructions. Upon opening the license agreement, one enters a new, not entirely friendly, age of reading—the owner is allowed to install *The Mathematical Explorer* on only one machine. As a result of the electronic format and the license agreement, the ability to read a book in the bathtub and then lend it to a friend is lost. After installing and opening the book, the reader is presented with an interface like that shown above.

With an advertised audience ranging from anyone with just "a little basic algebra and a sense of fun" to professional mathematicians, *The Mathematical Explorer* surveys a range of topics divided into fifteen roughly independent chapters: prime numbers, calculus, computations of π, properties of the catenary, code checking, encryption, recreational mathematics, Escher tilings, aesthetic qualities of parametric plots, fractals, chaos, Fermat's Last Theorem, the Riemann Hypothesis, unusual number systems, and the Four-Color Theorem. Clearly, some chapters aim to convey an impression of rather deep mathematics while others are more elementary and intend mainly to entertain. In between, we find applications of subjects likely familiar to an undergraduate mathematics major.

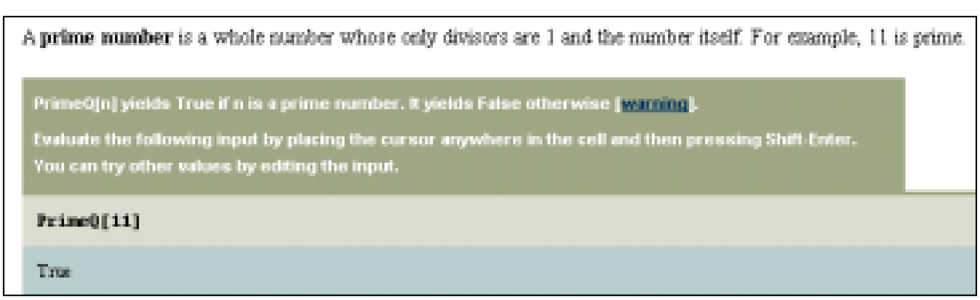
In no way is the treatment within a chapter meant to be exhaustive; the emphasis generally is on describing phenomena rather than explaining them. Instead, the goal is to convey some of the excitement of these mathematical ideas while allowing the reader to explore them using *Mathematica* commands embedded in the text. *The Mathematical Explorer* describes itself as "part text, part guide, part museum and completely fun." Indeed, it is most enjoyable when approached as a museum in which exhibits are chosen to present curious or even beautiful phenomena while others aim to familiarize the reader with important mathematics.

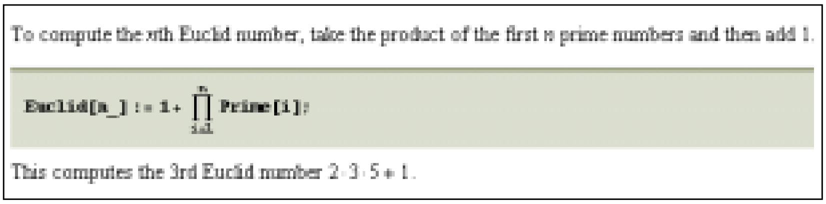
A Visit to the Museum

Let's take a quick tour through the first and perhaps strongest chapter, a pleasant excursion through prime numbers and their properties. The first section demonstrates one of the best uses of *Mathematica*. Here, the definition of a prime number is given and a *Mathematica* command, `PrimeQ`, that reports the primality of a given integer, is introduced. By editing the *Mathematica* command line, the reader is allowed to modify the integer and test the primality of other integers (see bottom left).

I modified this to ask whether 12 is prime, and after taking a few seconds to load the kernel, *Mathematica* told me that it was, in fact, not. This is a simple example, but the ability is useful. When presented with a new mathematical definition or construction, most readers likely think of other candidates that may fit the given definition. *Mathematica* gives us a chance to query the book with no intervening computation.

Not all *Mathematica* commands are meant to be edited, though: The use of *Mathematica* for displaying mathematical formulas appears next as we encounter Euclid's proof of the infinitude of prime numbers, one of the few proofs Wagon gives




To compute the *n*th Euclid number, take the product of the first *n* prime numbers and then add 1.

```
Euclid[n_] := 1 + ∏_{i=1}^{n} Prime[i];
```

This computes the 3rd Euclid number $2 \cdot 3 \cdot 5 + 1$.

in the book. This leads naturally to the introduction of Euclid numbers, conventionally defined by

$$e_{n+1} = 1 + \prod_{i=1}^{n} p_i$$

where p_i is the ith prime number. Wagon's definition appears as shown above.

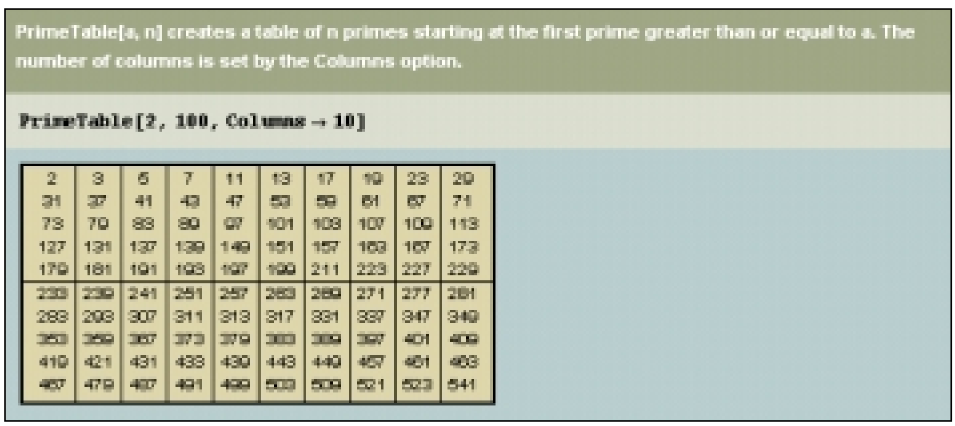
Here the definition is given, not in standard mathematical notation, but rather as a piece of *Mathematica* code. This is a general tendency: Displayed mathematical formulas are given in *Mathematica* command lines. Though it is not usually confusing, keeping track of two sets of notation can take some concentration, especially as new and highly specialized *Mathematica* commands begin to proliferate. For instance, algebraic manipulations are performed using commands such as `FullSimplify` and `FunctionExpand` whose meanings are not entirely clear without some investigation. (There is a reference for all of the commands included as an appendix together with hypertext links to places within the text where they are used.)

Following this, we see an intriguing result of electronic publishing. The largest prime Euclid number known at the time of publication was e_{2673}, and *Mathematica* shows us all 10,386 digits of this number. It is one thing to be told by an author that this number has 10,386 digits, but the sensation produced as the digits scroll by for five seconds is much more immediate. Of course, printing this number in a traditional book is not practical (it is about half the length of this review), but in its electronic format, *The Mathematical Explorer* can display it for no added cost. Many of the numbers Wagon presents to us are huge, and *The Mathematical Explorer* gives a good appreciation for the intellectual accomplishment of learning anything about them. Another illustration of this principle occurs in the chapter on Fermat's Last Theorem, primarily a discussion of Diophantine equations. Here, we are explicitly shown a lengthy polynomial in twenty-six variables for which the set of positive outputs resulting from nonnegative integral inputs is exactly the set of prime numbers. Even better, we are given some code to evaluate the polynomial, and we quickly see how difficult it is to generate prime numbers in this way.

Wagon makes the point that though the Euclid numbers will give new primes, few of the numbers are themselves prime, which he illustrates with a table of the first twenty Euclid numbers showing which ones are prime. He then suggests we explore by editing the *Mathematica* command line to change twenty to a larger number and see whether we find any other prime Euclid numbers. This is a typical kind of experiment that readers are encouraged to make. I edited this line to look at the first 100 Euclid numbers and saw that it is indeed rare to find a prime this way. However, this was not a particularly satisfying discovery; there was no thought required on my part and my understanding and appreciation of this fact was not increased by the experiment.

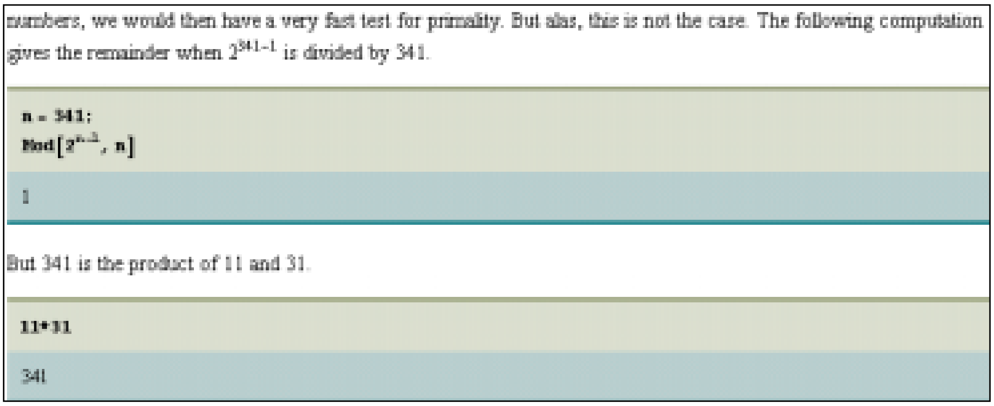
Another example of this kind of exploration occurs later in the chapter where we are given a command to build tables of prime numbers and asked to make conjectures about the behavior of primes (see below). These tables are a little difficult to digest and *Mathematica* could have helped by representing the information graphically, say, with the prime numbers indicated on a number line. In addition, most lay readers likely would benefit from a bit more guidance from Wagon; there is no indication of what kinds of things to look for. For instance, it would be helpful to be told to consider how the gaps between primes change as the primes grow larger. Wagon does not physically separate the region of the text in which the exploration occurs from the rest of the text, and it is consequently unclear how much effort we are encouraged to spend here. Finally, there is no summary given of the things we might have found. In fact, the next link takes us to a discussion of Gauss' discovery that the number of primes less than or equal to x is approximated by

$$\int_2^x \frac{1}{\ln(t)}\, dt.$$


PrimeTable[a, n] creates a table of n primes starting at the first prime greater than or equal to a. The number of columns is set by the Columns option.

```
PrimeTable[2, 100, Columns → 10]
```

2	3	5	7	11	13	17	19	23	29
31	37	41	43	47	53	59	61	67	71
73	79	83	89	97	101	103	107	109	113
127	131	137	139	149	151	157	163	167	173
179	181	191	193	197	199	211	223	227	229
233	239	241	251	257	263	269	271	277	281
283	293	307	311	313	317	331	337	347	349
353	359	367	373	379	383	389	397	401	409
419	421	431	433	439	443	449	457	461	463
467	479	487	491	499	503	509	521	523	541



numbers, we would then have a very fast test for primality. But alas, this is not the case. The following computation gives the remainder when 2^{341-1} is divided by 341.

```
n = 341;
Mod[2^(n-1), n]
```

```
1
```

But 341 is the product of 11 and 31.

```
11*31
```

```
341
```

Certainly, we are not expected to discover this approximation, though its qualitative content could be revealed by the exploration, and there is risk that the reader feels the time spent building tables has been wasted.

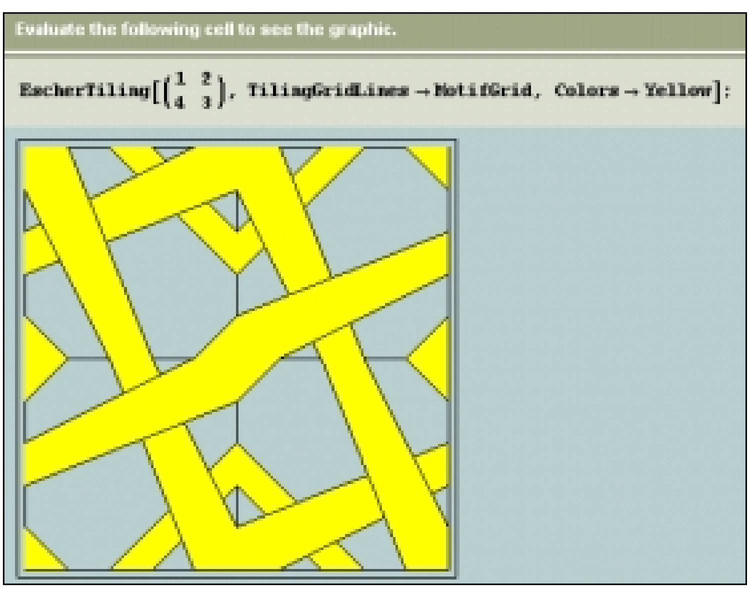
Most sections contain a few explorations, and the reader could spend considerable time with them. Generally, more creativity is needed in their design to make them more meaningful, and more discussion of the results of the explorations should be provided.

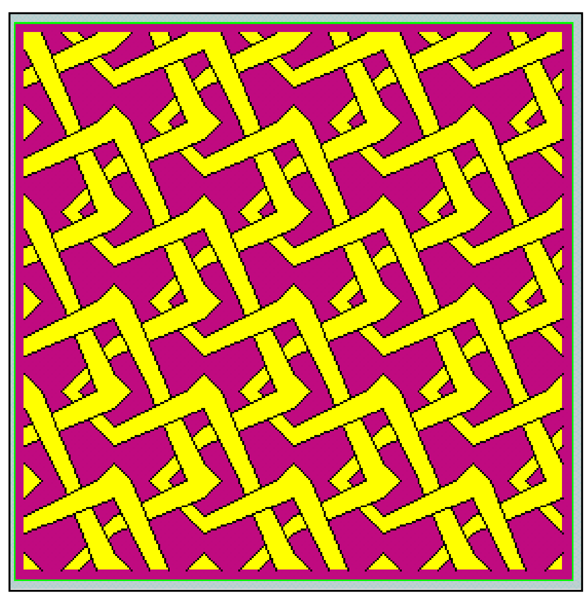
Later, a presentation of Mersenne primes and the question of their infinitude introduces us to some tests for primality and one "almost" test. In particular, we learn that if n is an odd prime, then $2^{n-1} \equiv 1 \pmod{n}$. Examples lead us to suspect that if n is composite, then $2^{n-1} \not\equiv 1 \pmod{n}$. However, $n = 341$ is a counterexample, as shown above.

This illustrates how *Mathematica* code is at times used as an authority that the reader is asked to trust. Typically, an author and reader have an implicit agreement that the author accurately presents the material and takes care of little details to unburden the reader. In several places, *The Mathematical Explorer* asks *Mathematica* to perform some lengthy computations that are only of interest for their output. We are told several times, in fact, that *The Mathematical Explorer* can prove a result for us, and a line of *Mathematica* code is given with the desired output. Whether this is more convincing than a simple statement by the author is arguable. Seen in this light, however, Wagon's discussion of the proof of the Four-Color Theorem and its implications for the nature of proof appears somewhat ironic.

The chapter on prime numbers is well organized and has a strong focus, and *Mathematica* makes a real contribution by showing us just what kind of numbers we are working with. Another interesting chapter concerns Escher's lesser known work on Euclidean tesselations. Escher began with a simple figure called a "motif", applied symmetries of the square to four copies of the motif, and then used these to build a 2×2 square (see bottom left). With this as a fundamental region, he tiled the plane to produce interesting, and sometimes beautiful, patterns (see bottom right). In this chapter, the graphics, built with the help of Rick Mabry, are quite good. By editing the *Mathematica* command line, we are able to modify the fundamental region and see what new effects are produced.

The chapter on Escher tilings seems like a good place for a mathematical discussion of symmetry. However, Wagon takes us in a more combinatorial direction, studying the problem of coloring the strands so that the colors appear continuous in the tesselation. Generally, the book tends to emphasize combinatorial mathematics, here at the expense of





a more appealing, geometrical study nicely set up by the graphics.

The chapter titled “Unusual Number Systems” also benefits from *Mathematica*’s abilities. Beginning with a straightforward discussion of representations of numbers in bases other than 10, it proceeds to describe continued fraction expansions. The reader is presented with opportunities to input a number and watch its continued fraction expansion come out. The ability to produce examples effortlessly is a good use of the technology, and we see clearly the property that the partial quotients of a continued fraction provide the best rational approximations for irrational numbers.

We also meet the so-called “harmonic” number system, in which a real number x is represented in the form

$$x = a + \frac{1}{2}\left(b + \frac{1}{3}\left(c + \frac{1}{4}(d + \cdots)\right)\right)$$

using integers where $0 \le b < 2$, $0 \le c < 3$, and so on. By looking at the representation of different numbers in this system, the reader quickly gains a working understanding. This theme reaches a crescendo with the introduction of the “spigot algorithm”, a surprising algorithm that utilizes a number system similar to the harmonic system to compute the digits of π using only integer arithmetic.

Finally, the chapter on Fermat’s Last Theorem is notable. Here we learn some of the history of the problem and the progress on a solution before Wiles’ work. Wagon wisely chooses not to attempt a sketch of Wiles’ proof and instead diverts us toward a historical discussion of Diophantine equations and various places in which they arise. This chapter has the effect of giving us the feeling that we have rubbed shoulders with a hard problem while still presenting some accessible mathematics.

Regrettably, the quality of the chapters varies considerably. For instance, the chapter on calculus is particularly poor. It is rather short and contains little that would lead one unfamiliar with the subject to much understanding. The writing is sloppy in this section, as it is in a few others, and ranges from merely careless—“We have already pointed out that the reciprocals of the integers diverge to infinity”—to confusing. Consider Wagon’s definition of tangent, the first time this word is used outside of the title of a reference: “Here the slope of the **tangent** [emphasis original] to the curve means that for a small enough interval, the tangent line is indistinguishable from the curve.”

Furthermore, this chapter fails to demonstrate that calculus is a useful subject. The only application of the derivative is to solving the problem of where best to stand when viewing a painting in an art gallery. Of course, *The Mathematical Explorer* draws the graph of the function we want to maximize. Why do we need derivatives to make life more complicated?

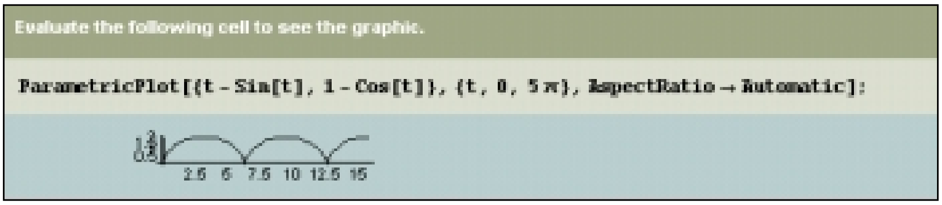
The chapter on code checking introduces us to roughly nine different methods, such as the UPC bar code, by which various kinds of identification numbers are verified. The sections describing these methods vary little and it seems as if so many different methods are included simply because it was easy to modify the *Mathematica* code to produce them. The reader who gives in to the temptation to skip the last sections will miss a surprising method that arises from an application of nonabelian groups and that was used to check the serial numbers of German currency.

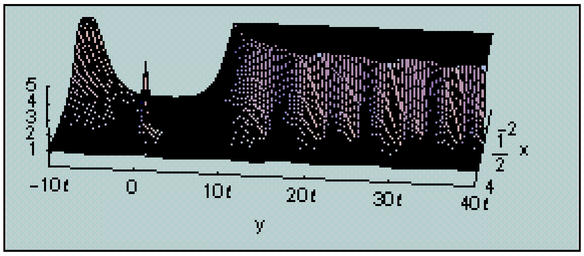
Most chapters contain at least a small amount of history or historical anecdotes. Also an appendix contains short biographies of many mathematicians from Archimedes to Wiles. Some of the anecdotes provide amusing counterpoint to the mathematical discussion.

As noted above, some of the writing is careless, frequently containing typographical errors, inaccurate statements, and repetitions. At times, the prose has a breathless quality that may not be appealing to all: “If the space-filling curve is not mindblowing enough, here is something that is totally outrageous.” Many sections would be better if they informed us as to where we are headed and summarized important ideas once we have seen them. Hypertext links are used inconsistently, sometimes leading to the next section and sometimes not.

Publishing with *Mathematica*

Though *Mathematica* adds useful computational features, its use as a publication tool appears to be somewhat limited. *The Mathematical Explorer* has fonts to produce mathematical notation, such as integral signs, square root symbols, and the Greek alphabet, with a quality only a little inferior to, say, Microsoft Word’s Equation Editor. However, it is cumbersome to navigate the text; the interface looks something like a web browser, and it is disappointing to realize how limited its functionality is compared to a typical browser. Moving about a page is performed by scrolling, with no page up or page down shortcuts. There is a “Back” button for returning the reader to the last page viewed, but no accompanying “Forward” button. Furthermore, the “Back” button always returns you to the top of the previous page rather than to the spot where you were reading. Finally, any modifications made to *Mathematica* commands are lost upon leaving that page. Imagine reading a book and wanting to look up a result in a previous section. By doing so, you lose any trace of the current section and how to return to it; when you do return, it is to the





beginning of the section and any marginal comments you have written have been erased.

Adding to this frustration (see figures above) is the fact that most figures are produced by *Mathematica* commands that the reader must evaluate by pressing "Return" while holding the "Shift" key down before seeing the figure. Of course, you have the flexibility to edit these figures, but there is a price to be paid: When you leave a page, your figures disappear. Furthermore, some of the figures depend on *Mathematica* commands that are given earlier on the page and that must be evaluated by the reader. This means that if you leave a page and return expecting to see a figure, you must reload a string of earlier *Mathematica* commands, and it is often not clear which commands are required.

Generally, the figures produced by *Mathematica* are inferior to illustrations in a traditional book. Of course, *Mathematica* is designed to handle a wide range of graphical requests. This means that the figures have not been customized in any meaningful way as they would be in a printed book.

Animations are especially bad. Usually the number of frames is too small and the time interval between them too large to create the illusion of motion. Worse, however, are the occasional animations created by taking a series of figures and laying them out vertically while the browser scrolls down at just the right speed so that one figure appears to replace its predecessor. Here the viewer has no control over the speed of the animation, and there is a distracting flicker as each new image is displayed. This particular style of animation is largely unusable, and the final chapter on the Four-Color Theorem especially suffers. In the most technical chapter of *The Mathematical Explorer*, Wagon explains Kempe's erroneous 1879 proof, demonstrating Kempe's algorithm for coloring a map with such an animation. Since an important part of the discussion in this chapter depends upon understanding this algorithm, the animation is a real hindrance to the reader's understanding.

Some good opportunities for animations are passed by. There is a section on the Brachistochrone Problem in which we can compare the time it takes for a ball to roll down the brachistochrone to the time it takes to roll down various other tracks. *Mathematica* here is used only to compute the amount of time it takes to reach the bottom of the different tracks. The idea could have been illustrated more effectively with an animation showing the two balls rolling down the tracks side by side. In addition, this would have given some insight into why the brachistochrone works.

The Economics of Electronic Publishing

The electronic format of *The Mathematical Explorer* raises some important issues concerning the cost required to produce a book. One appreciates immediately the frequent use of color, reflecting the fact that the marginal cost to include color electronically is almost nothing. Also, *The Mathematical Explorer* can afford to devote large chunks of space to figures and lists without worrying about filling up costly pages. However, the ability to add bulk to a book easily without necessarily increasing its content, as in the chapter on code checking, could be an undesirable byproduct.

There are other electronic books (see Devlin [D], for example) that provide high quality graphics, animations, and even movies, but generally these lack *The Mathematical Explorer*'s ability to respond to substantial requests made by the reader. In fact, the cost of adding interactivity to text is currently high, especially when the human effort of the author and reader is added to the financial cost. Clearly, a significant amount of work has gone into planning and creating the customized *Mathematica* commands used in *The Mathematical Explorer*. However, most of what the reader actually sees has suffered as a result: The writing appears hastily done and the figures are overly reliant on *Mathematica* for their creation.

Also, the reader can find editing the command line repeatedly in *The Mathematical Explorer* to be tedious and error prone. Consider Wagon's discussion of the derivative of a function by viewing its graph on an increasingly fine scale: To see each refinement, the reader needs to type in a new range. Experience with other software justifies the belief that there are easier ways to operate this demonstration by, say, clicking a button to see the next refinement.

Indeed, demonstrations like this exist on the Web written in the programming language Java, with notable efforts including Bogomolny's large collection of mathematical applets [B] and Joyce's edition of *Euclid's Elements* [J] illustrated in Java. This kind of work represents, in some sense, another end of a spectrum. Since Java offers only low-level mathematical and graphical capabilities compared to *Mathematica*, the cost to the author in creating Java applets is typically high. However, the rewards can be great: The illustrations and demonstrations that result can be tailored to very specific ends and delivered cheaply over the Web. Well-designed applets have a simpler, easier to use interface than *The Mathematical Explorer*, and the reader's explorations can be more carefully controlled and hence more productive. Furthermore, projects like WebMathematica [WM] and JavaMath [JM] offer the possibility of accessing a computational engine, like *Mathematica*, from within Java, while the New Typesetting System project [N] aims to rewrite $\TeX$ in Java. There is hope for an easily distributed, easy to use format that can support interactive content, high-level computation, and superb typesetting.

Summary

The Mathematical Explorer makes a valuable contribution to a discussion of electronic mathematical publishing. Indeed, it seems almost certain that some kind of high-powered computational engine will eventually be included in electronic texts. Ultimately though, one wishes for a more judicious use of *Mathematica*, or at least for it to be less visible. After all, we are reading a collection of *Mathematica* notebooks, and *The Mathematical Explorer* reveals *Mathematica*'s limited capabilities for displaying mathematics, producing illustrations, and easy navigation. Most users will be frustrated by the interface, expecting that some tasks could more easily be performed. Indeed, *The Mathematical Explorer* reflects the fact that no completely satisfactory means for delivering mathematics with interactive content currently exists.

Furthermore, the problem of how best to use *Mathematica* effectively within the text has not been completely solved. Many of the explorations either require too little thought, which leads to too little payoff, or lack clear objectives and summaries to give readers the pleasure of discovering mathematics. As it stands, this book would most likely appeal to proficient users of *Mathematica* or those who aspire to be. For others wishing to concentrate on the mathematical content, the embedded *Mathematica* can at times be distracting.

Finally, *The Mathematical Explorer* reminds us that books exist so that authors might communicate to their audience; any new tool requires fresh thought to serve that aim effectively. What is most valuable in an electronic book likely will not be much different than that in a traditional book: ease of use, familiar notation and conventions, high-quality typesetting and illustrations, clear exposition, and a style and format that help the reader take possession of new mathematical ideas.

References

[B] ALEX BOGOMOLNY, `http://www.cut-the-knot.com/`.

[D] KEITH DEVLIN, *An Electronic Companion to Calculus*, Cogito Learning Media, Inc., `http://www.cogitomedia.com/`.

[J] DAVID JOYCE, *Euclid's Elements*, `http://aleph0.clarku.edu/djoyce/java/elements/elements.html`.

[JM] JavaMath, `http://javamath.sourceforge.net`.

[N] New Typesetting System project, `http://nts.tug.org/`.

[W] Wolfram Research, `http://www.wolfram.com/`.

[WM] WebMathematica, `http://www.wolfram.com/products/webmathematica/`.