WHAT IS...

A Stack?

Dan Edidin

A Riemann surface of genus 1 is homeomorphic to the torus $T = S^1 \times S^1$. Therefore, a choice of a point to be the origin determines a group structure on the Riemann surface. An *elliptic curve* is a Riemann surface of genus 1 together with a choice of origin for the group structure. Although all elliptic curves are homeomorphic to the topological group $S^1 \times S^1$, they may have nonisomorphic complex structures. A natural question, called the problem of moduli, is to describe the space of all possible isomorphism classes of objects of a certain type. In this article we discuss this question for elliptic curves and explain how we are led to consider the notion of *stacks*.

We wish to construct a *moduli space* for elliptic curves. Points of the moduli space should correspond to isomorphism classes of elliptic curves. An elliptic curve can be expressed as a two-sheeted cover of the Riemann sphere branched at the set $\{0, 1, \infty, \lambda\}$, $\lambda \in \mathbb{C} - \{0, 1\}$. The rational function $j(\lambda) = \frac{2^8(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2}$ is an invariant of the curve, which was classically called the j-invariant. Direct calculation shows that the map $\mathbb{C} - \{0, 1\} \to \mathbb{C}$, $\lambda \mapsto j(\lambda)$ is a surjective map that is generically a 6:1 covering. Moreover, two elliptic curves are isomorphic if and only if they have the same j-invariant, so the isomorphism class of an elliptic curve is determined by a single complex number. A natural conclusion is that $\mathbb{C}$ is the moduli space of elliptic curves. To an extent this is true: $\mathbb{C}$ is called the *coarse moduli space* of elliptic curves. As we will see below, however, $\mathbb{C}$ lacks an important additional property and cannot be considered the true moduli space of elliptic curves.

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A *family* of elliptic curves over a base space B is a fibration $X \xrightarrow{\pi} B$ with a section $O: B \to X$ such that for every $b \in B$ the fiber $\pi^{-1}(b)$ is an elliptic curve with origin $O(b)$. Given a family of elliptic curves $X \xrightarrow{\pi} B$, define a classifying map $j_B: B \to \mathbb{C}$ by $b \mapsto j(\pi^{-1}(b))$. Because $\mathbb{C}$ is the coarse moduli space, $j_B(b) = j_B(b')$ if and only if the fibers $\pi^{-1}(b)$ and $\pi^{-1}(b')$ are isomorphic elliptic curves. Motivated by the concept of classifying space in topology, we require that a moduli space have a *universal family*. This means that if $\mathbb{C}$ were the moduli space of elliptic curves, there would exist a family of elliptic curves $\mathcal{E} \to \mathbb{C}$ such that every family would be obtained by pulling back the universal family via the map $j_B: B \to \mathbb{C}$. However, since every elliptic curve has an involution, there are nontrivial families of elliptic curves $X \xrightarrow{\pi} B$ such that $\pi^{-1}(b) \simeq E_0$ for all $b \in B$, where E_0 is a fixed elliptic curve. (Such a family is called *isotrivial*.) The classifying map $j_B : B \to \mathbb{C}$ is the constant map $b \mapsto j(E_0)$. This contradicts the existence of a universal family, because the classifying map $B \to \mathbb{C}$ associated to the trivial family $E_0 \times B \to B$ is also constant.

To obtain the moduli space of elliptic curves, we must define a new concept, that of a stack. The *stack of elliptic curves*, $\mathcal{M}$, is a category. Its objects are families of elliptic curves, and a morphism $(X' \xrightarrow{\pi'} B') \to (X \xrightarrow{\pi} B)$ is a pair of maps $X' \xrightarrow{e} X$, $B' \xrightarrow{\beta} B$ satisfying two conditions:

1. The diagram $\begin{array}{ccc} X' & \xrightarrow{e} & X \\ \pi' \downarrow & & \pi \downarrow \\ B' & \xrightarrow{\beta} & B \end{array}$ commutes.

2. X' is isomorphic to the pullback of X via the map $B' \xrightarrow{\beta} B$.

(Commutative diagrams satisfying condition (2) are called *cartesian*.) The subcategory of $\mathcal{M}$ corresponding to families over a fixed base B is called the *fiber* over B. Condition (2) says that the fibers of $\mathcal{M}$ are *groupoids*, that is, categories where all morphisms are isomorphisms. More generally, $\mathcal{M}$ is an example of an *algebraic stack*. An algebraic stack is a category fibered in groupoids which has a smooth covering by an affine variety, in a way which we explain below.

While this definition may look strange, we will see that there is a universal family of elliptic curves over the category $\mathcal{M}$. For any variety B we can construct a similar category $\underline{B}$: the objects are maps of varieties $T \xrightarrow{t} B$, and a morphism $(T' \xrightarrow{t'} B) \to (T \xrightarrow{t} B)$ is a map $T' \xrightarrow{f} T$ such that $t' = t \circ f$. It is relatively easy to show that the category $\underline{B}$ determines B, so we can identify B and $\underline{B}$. To give a family of curves $X \to B$ is equivalent to giving a functor (map of categories) $\underline{B} \to \mathcal{M}$. Let C be the category whose objects are families of elliptic curves with a (nowhere zero) section and morphisms defined as for $\mathcal{M}$. Forgetting the section defines a functor $C \to \mathcal{M}$. For any family of elliptic curves $X \to B$, the pullback of C via the corresponding map $\underline{B} \to \mathcal{M}$ is $\underline{X}$. Thus, $\mathcal{M}$ is the moduli space of elliptic curves and $C \to \mathcal{M}$ is the universal family.

Finally, let us see what it means to say that $\mathcal{M}$ has a smooth cover by an affine variety. Consider the Legendre family of elliptic curves $y^2 - x(x-1)(x-\lambda)$, where λ varies in $U = \mathbb{C} - \{0, 1\}$. The corresponding map $\underline{U} \to \mathcal{M}$ is a smooth cover in the following sense: given a map $B \to \mathcal{M}$, the map $\underline{U} \to \mathcal{M}$ pulls back to a 12:1 unbranched covering[1] of $\underline{B}$.

Current Trends

Deligne and Mumford introduced the term *stack* in their famous paper "The irreducibility of the space of curves of a given genus", proposing it as an English substitute for the French word *champ*, which had previously been used in nonabelian cohomology. (The choice of the word "stack" is somewhat puzzling, since *champ* means "field" in English. One possible explanation is that the "stacks" considered by Deligne and Mumford are mathematically related to a class of *champs* called *gerbes*. The French word *gerbe* can be translated either as "sheaf" or as "stack". Since the term *sheaf* was already in use, perhaps *stack* was the next logical choice.) In that paper the authors defined algebraic stacks and used them to prove the result of their title. The stacks they defined are now referred to as *Deligne-Mumford stacks*, and the term *algebraic stack* usually refers to a generalization given by M. Artin in the early 1970s. During the last ten years stacks have been widely used to prove theorems in algebraic geometry and related fields. For example, in the mid-1990s Kontsevich showed that Gromov-Witten invariants can be defined as integrals on the stack of *stable maps* of genus g. Last year Laurent Lafforgue won a Fields Medal for his proof of the Langlands conjecture for function fields. At the heart of the proof is his construction of compactifications of stacks of certain types of vector bundles, called *Drinfeld shtukas*, on curves defined over finite fields.

Further Reading

A nice introduction to algebraic stacks from the point of view of moduli of vector bundles was written by T. Gómez [2]. I wrote an article on the construction of the moduli space of curves [1] which also contains an introduction to algebraic stacks. The most comprehensive, and most technical, treatise on algebraic stacks is the book of G. Laumon and L. Moret-Bailly [3].

[1] *The number of inverse images of a point $b \in B$ corresponding to an elliptic curve with j-invariant $j(\lambda)$ is the cardinality of the set $\{\lambda, 1-\lambda, 1/\lambda, \lambda/(\lambda-1), (\lambda-1)/\lambda, 1/(1-\lambda)\}$ times the number of automorphisms of the curve. This number is always 12. For a general value of λ the set has six elements and the automorphism group has order 2.*

References

[1] D. Edidin, Notes on the construction of the moduli space of curves, *Recent Progress in Intersection Theory* (Bologna, 1997), Birkhäuser Boston, Boston, MA, 2000, pp. 85–113 (math.AG/9805101).

[2] T. L. Gómez, Algebraic stacks, *Proc. Indian Acad. Sci. Math. Sci.* **111** (2001), 1–31 (math.AG/9911198).

[3] G. Laumon and L. Moret-Bailly, *Champs Algébriques*, Springer, Berlin, 2000.