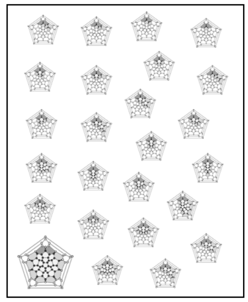
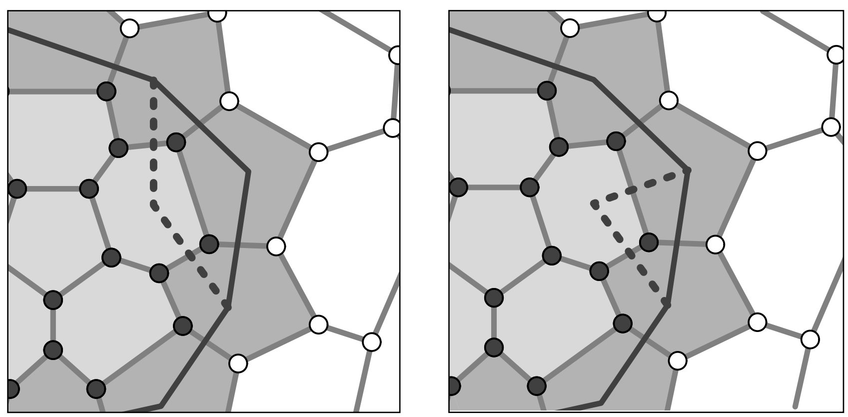
About the Cover

In preparing diagrams for Peter Sarnak's article in this issue on Ramanujan graphs, we decided that it would be an interesting exercise to verify that its expansion constant h is 1/4. I recall that if the graph has N nodes, then this constant is the minimum value of $|\partial X|/|X|$, where X varies over the subsets of nodes of size at most $N/2$. Thus a priori one might expect to have to look at close to 2^{80} subsets of nodes, and, indeed, it has been shown by M. Blum et al. (*Inform. Process. Lett.* **13** (1981)) that this is a very difficult problem. For the graph at hand, however, Sarnak was able to verify by hand that $h = 1/4$, and it was also possible to verify the calculation with a computer program that might work as well for more general 3-regular graphs. The basic idea of the program is to look at the possible *cut sets* separating X from its complement. There are two key observations that the program is based on. The first is that one need only look at connected subsets X, and in fact only at cut sets that are Jordan curves. The second observation is that in certain circumstances one need only look at cut sets that satisfy a kind of convexity condition at each vertex. The exact conditions ought to be clear from the accompanying diagrams, where the dashed lines cannot be the cut sets for a candidate X, since adjusting them in a simple way increases $|X|$ without decreasing $|\partial X|$. (The nodes in X are dark.)





It is straightforward and entirely practical to make up an algorithm that constructs all admissible cut sets. If $|X|$ for all of these is not greater than half the number of nodes, the convexity argument above shows that h can be calculated by perusing the list. Because of the symmetries in the graph at hand, it is necessary to consider only two types of cut sets, and the cover illustration is, in effect, the program output for one of these types. It shows all convex cut sets passing through the top two gray faces (up to mirror symmetry). The minimum value 1/4 is achieved in the large diagram at lower left, where $|X|$ is also the maximum value of 40. This graph is a Ramanujan graph. A result of Lipton and Tarjan (*SIAM J. Appl. Math.* **36** (1979)) implies that there are at most finitely many planar Ramanujan graphs. The largest ones known are 84:20, and 84:23 in the *Atlas of Fullerenes* by P. Fowler and D. Manolopoulos. I'd like to thank A. Gamburd for calling my attention to the graph used here, which he found in a paper by P. Frankin on the four-color problem, and also for telling me about Fullerenes.

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