WHAT IS . . .

a Topos?

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Basic Examples

In the early 1960s Grothendieck chose the Greek word *topos* (which means "place") to denote a mathematical object that would provide a general framework for his theory of étale cohomology and other variants related to his philosophy of descent. Even if you do not know what a topos is, you have surely come across some of them. Here are two examples:

(a) The category of sheaves of sets on a topological space is a topos. In particular, the category of sets is a topos, for it is the category of sheaves of sets on the one point space. This topos, denoted $\{pt\}$, is called the *punctual topos*.

(b) Let G be a group. The category BG of G-sets, i. e., sets equipped with a left action of G, is a topos. For $G = \{1\}$, $BG = \{pt\}$.

What these categories have in common is that (i) they behave very much like the category of sets, and (ii) they possess a good notion of localization. In order to formalize (ii), Grothendieck conceived the idea of *sheaf on a site*, which generalizes the notion of sheaf on a topological space. That led him to the notion of topos, which encompasses (i) and (ii).

Sites and Toposes

Consider the category C of open subsets of a topological space X (the morphisms being the inclusions of open subsets). A sheaf of sets E on X is a contravariant functor $U \mapsto E(U)$ on C (with values in the category of sets) having the property that for any open cover $(U_i)_{i \in I}$ of U, a section s of E on U, i. e., an element of $E(U)$, can be identified via the restriction maps with a family of sections s_i of E on the U_i's which coincide on the intersections $U_i \cap U_j$. Now, let C be a category having finite projective limits. To give a *topology* (sometimes called a Grothendieck topology) on C means to specify, for each object U of C, families of maps $(U_i \to U)_{i \in I}$, called *covering families*, enjoying properties analogous to those of open covers of an open subset of a topological space, such as stability under base change and composition (see [SGA 4 II 1.3] for a precise definition). Once a topology has been chosen on C, C is called a *site*, and one can define a *sheaf of sets* on C in the same way as in the case in which C is the category of open subsets of a topological space: a sheaf of sets E on C is a contravariant functor $U \mapsto E(U)$ on C (with values in the category of sets) having the property that for any covering family $(U_i \to U)_{i \in I}$, a section s of E on U, i. e., an element of $E(U)$, can be identified via the "restriction" maps with a family of sections s_i of E on the U_i's that coincide on the "intersections" $U_i \times_U U_j$.

A *topos T* is a category equivalent to the category of sheaves of sets on a site C (which is then called a *defining site* for T). Here are some properties of toposes :

(1) A topos T admits finite projective limits: in particular, it has a final object, and it admits fibered products.

(2) If $(U_i)_{i \in I}$ is a family of objects of T, the sum $\coprod_{i \in I} U_i$ exists, is "disjoint", and commutes with base change.

(3) Quotients by equivalence relations exist and have the same good properties as in the category of sets.

A theorem of Giraud [SGA 4 IV 1.2] asserts that the converse is essentially true. Namely, if T is a category satisfying (1), (2), (3), and if moreover T

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satisfies a certain technical "smallness" condition, then T is a topos.

Several unequivalent sites may give rise to the same topos, as the case of $\{pt\}$ already shows : both the one point space and the category of sets, equipped with the topology defined by surjective families, are defining sites. Grothendieck liked to compare this with the fact that a group can be defined by generators and relations in many different ways. The site is some kind of system of generators and relations for the topos. And in the same way in which a group G can be defined by a set of generators that is G itself, a topos T can be defined by a site whose underlying category is T itself. Covering families are just epimorphic families. This topology is called the *canonical topology*. In the case of BG, the canonical topology is the topology defined by surjective families of G-maps.

Morphisms, Points, Cohomology

A continuous map of topological spaces $f : X \to Y$ defines a pair of adjoint functors (f^*, f_*) between the categories of sheaves of sets on X and Y. The *inverse image* functor f^* commutes (as a left adjoint) with inductive limits. It also commutes with *finite* projective limits. Now, if X, Y are toposes, one defines a *morphism* $f : X \to Y$ as a pair of adjoint functors $(f^* : Y \to X, f_* : X \to Y)$ such that f^* commutes with finite projective limits. If T is a topos, up to a unique isomorphism of functors, there is only one morphism from T to the punctual topos $\{pt\}$. On the other hand, a *point* of T is by definition a morphism from $\{pt\}$ to T. If T is defined by a topological space X, a (usual) point x of X defines a point of T, whose inverse image functor is the stalk functor $E \mapsto E_x$. But new phenomena occur. Deligne has constructed examples of toposes without points (he has also given criteria for the existence of "enough points") [SGA 4 IV 7, VI 9]. Moreover, if x and y are points of a topos T, there may exist nontrivial morphisms (of functors) from x to y. In the case of BG, for example, the forgetful functor from BG to $\{pt\}$ is the inverse image functor by a point of BG, whose group of automorphisms is G itself! This observation is at the root of Grothendieck's theory of the fundamental group in algebraic geometry. If X is a topological space and P is a G-torsor on X (i. e., a principal G-cover of X), then twisting a G-set E by P (i. e., forming the corresponding fiber space with fiber E over X) is the inverse image functor by a morphism $f_P : X \to BG$, and $P \mapsto f_P$ establishes a bijective correspondence between isomorphism classes of G-torsors on X and morphisms from X to BG. Thus BG plays the role of a classifying space.

If T is a topos, the direct image functor for the unique morphism $T \to \{pt\}$ associates to an object E of T the set of its "global sections" $\Gamma(T, E) = \mathrm{Hom}(e, E)$, where e is the final object of T. The category of abelian group objects of T admits enough injectives, and one can consider the derived functor of $\Gamma(T, -)$. This yields a common generalization of sheaf cohomology on topological spaces and group cohomology (the functor $\Gamma(BG, -)$ is "taking the invariants under G").

Toposes Arising from Algebraic Geometry

The most important example, which was the main motivation for Grothendieck and which is also the closest to geometric intuition, is the *étale topos* of a scheme X. It is the topos of sheaves on the étale site X_{et} of X. The underlying category of X_{et} is the category of schemes Y étale over X (i. e., étale morphisms $Y \to X$). Étale morphisms are the analogs, in algebraic geometry, of morphisms of complex analytic spaces that are analytically local isomorphisms. Covering families of X_{et} are surjective families $(Y_i \to Y)_{i \in I}$ of (étale) X-schemes. When X is the spectrum of a field k with absolute Galois group G, the étale topos of X is a variant of BG, namely the category of (discrete) sets endowed with a *continuous* action of G; cohomology in this case is the Galois cohomology of k. It is a miracle that the consideration of the cohomology of étale toposes with values in $\mathbb{Z}/n$, with n prime to the characteristics, has given rise to a *Weil cohomology* and, eventually, to the proof of the Weil conjectures, by Grothendieck and Deligne [D]. Variants of the étale topology, such as the so-called *fppf* topology, play an important role in certain moduli problems (Hilbert and Picard schemes, etc.) and also in the theory of group schemes and arithmetic applications.

Another interesting example is the *crystalline topos*, constructed by Grothendieck and Berthelot, which is crucial in differential calculus and the study of de Rham cohomology in positive or mixed characteristic. The comparison between crystalline cohomology and p-adic étale cohomology, sometimes called *p-adic Hodge theory* [P], is closely related to deep problems in arithmetic geometry.

Finally, let me mention that, in the wake of pioneering work of Lawvere in the late 1960s, a variant of the notion of topos, called *elementary topos*, has been extensively used in logic for the past thirty years.

References

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